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Silvestru Sever Dragomir (Melbourne and Johannesburg)

APPROXIMATING THE VOLUME INTEGRAL BY A SURFACE INTEGRAL VIA THE DIVERGENCE THEOREM

Abstract. By utilising the *divergence theorem* for n-dimensional integrals, we provide some error estimates for approximating the integral on a body B, a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B, by an integral on the surface ∂B and some other simple terms. Some examples in the 3-dimensional case are also given.

1. Introduction. Consider a closed, bounded and convex subset D of $\mathbb{R}^2$. Denote by

$$A_D := \iint_D dx\,dy$$

the *area* of D and by $(\overline{x_D}, \overline{y_D})$ the *centre of mass* of D, where

$$\overline{x_D} := \frac{1}{A_D} \iint_D x\,dx\,dy, \qquad \overline{y_D} := \frac{1}{A_D} \iint_D y\,dx\,dy.$$

Let $f = f(x,y)$ be a function of two variables. We assume that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions at a point $(u,v) \in D$:

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(u,v)\right| \leq L_1|x-u| + K_1|y-v| \tag{1.1}$$

and

$$\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(u,v)\right| \leq L_2|x-u| + K_2|y-v| \tag{1.2}$$

for any $(x,y) \in D$, where L_1, K_1, L_2 and K_2 are given positive constants.

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In the recent paper [7] we established the following result on approximating the double integral by a contour integral:

THEOREM 1. *Let ∂D be a simple, closed counterclockwise oriented curve bounding a region D and let f be defined on an open set containing D and having continuous partial derivatives on D. Assume that $(u,v) \in D$ and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ satisfy the Lipschitz type conditions* (1.1) *and* (1.2). *Then for any $\alpha, \beta \in \mathbb{C}$ we have*

$$\begin{aligned}(1.3)\quad &\Bigg| \frac{1}{A_D} \iint_D f(x,y)\,dx\,dy \\ &\quad - \frac{1}{2A_D} \oint_{\partial D} [(\beta - y) f(x,y)\,dx + (x-\alpha) f(x,y)\,dy] \\ &\quad - \frac{1}{2}\frac{\partial f}{\partial x}(u,v)(\alpha - \overline{x_D}) - \frac{1}{2}\frac{\partial f}{\partial y}(u,v)(\beta - \overline{y_D}) \Bigg| \\ &\le \frac{L_1}{2A_D} \iint_D |\alpha - x|\,|x-u|\,dx\,dy + \frac{K_1}{2A_D} \iint_D |\alpha - x|\,|y-v|\,dx\,dy \\ &\quad + \frac{L_2}{2A_D} \iint_D |\beta - y|\,|x-u|\,dx\,dy + \frac{K_2}{2A_D} \iint_D |\beta - y|\,|y-v|\,dx\,dy.\end{aligned}$$

In particular,

$$\begin{aligned}(1.4)\quad &\Bigg| \frac{1}{A_D} \iint_D f(x,y)\,dx\,dy \\ &\quad - \frac{1}{2A_D} \oint_{\partial D} [(\overline{y_D} - y) f(x,y)\,dx + (x-\overline{x_D}) f(x,y)\,dy] \Bigg| \\ &\le \frac{L_1}{2A_D} \iint_D |\overline{x_D} - x|\,|x-u|\,dx\,dy + \frac{K_1}{2A_D} \iint_D |\overline{x_D} - x|\,|y-v|\,dx\,dy \\ &\quad + \frac{L_2}{2A_D} \iint_D |\overline{y_D} - y|\,|x-u|\,dx\,dy + \frac{K_2}{2A_D} \iint_D |\overline{y_D} - y|\,|y-v|\,dx\,dy\end{aligned}$$

and

$$\begin{aligned}(1.5)\quad &\Bigg| \frac{1}{A_D} \iint_D f(x,y)\,dx\,dy \\ &\quad - \frac{1}{2}\frac{\partial f}{\partial x}(u,v)(x_{f,\partial D} - \overline{x_D}) - \frac{1}{2}\frac{\partial f}{\partial y}(u,v)(y_{f,\partial D} - \overline{y_D}) \Bigg| \\ &\le \frac{L_1}{2A_D} \iint_D |x_{f,\partial D} - x|\,|x-u|\,dx\,dy + \frac{K_1}{2A_D} \iint_D |x_{f,\partial D} - x|\,|y-v|\,dx\,dy \\ &\quad + \frac{L_2}{2A_D} \iint_D |y_{f,\partial D} - y|\,|x-u|\,dx\,dy + \frac{K_2}{2A_D} \iint_D |y_{f,\partial D} - y|\,|y-v|\,dx\,dy,\end{aligned}$$

where

$$x_{f,\partial D} := \frac{\oint_{\partial D} x f(x,y)\,dy}{\oint_{\partial D} f(x,y)\,dy} \quad \textit{and} \quad y_{f,\partial D} := \frac{\oint_{\partial D} y f(x,y)\,dx}{\oint_{\partial D} f(x,y)\,dx}$$

provided the denominators are not zero.

For other integral inequalities for multiple integrals see [3]–[15].

In this paper, motivated by the above results and by utilising the famous *divergence theorem* for n-dimensional integrals, we provide some error estimates for approximating the integral on a body B, a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B, by an integral on the surface ∂B and some other simple terms. Some examples for the 3-dimensional case are also given.

2. Some preliminary facts. Let B be a bounded open set in $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let $F = (F_1, \ldots, F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal on ∂B. Then the divergence theorem (see for instance [16]) states that

$$\int_B \operatorname{div} F\,dV = \int_{\partial B} F \cdot \mathbf{n}\,dA, \tag{2.1}$$

where

$$\operatorname{div} F = \nabla \cdot F = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k},$$

dV is the volume element in $\mathbb{R}^n$ and dA is the surface area element on ∂B.

If $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_n)$, $x = (x_1, \ldots, x_n) \in B$ and we write dx for dV, we can rewrite (2.1) more explicitly as

$$\sum_{k=1}^{n} \int_B \frac{\partial F_k(x)}{\partial x_k}\,dx = \sum_{k=1}^{n} \int_{\partial B} F_k(x) n_k(x)\,dA. \tag{2.2}$$

By taking the real and imaginary parts, we can extend the above equality to complex-valued functions F_k, $k \in \{1, \ldots, n\}$, defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector through 90° (in the correct direction so that it points outwards). The quantity $t\,ds$ can be written as (dx_1, dx_2) along the surface, so that

$$\mathbf{n}\,dA := \mathbf{n}\,ds = (dx_2, -dx_1).$$

Here t is the tangent vector along the boundary curve and ds is the arc-length element.

From (2.2) we get, for $B \subset \mathbb{R}^2$,

$$\int_B \frac{\partial F_1(x_1,x_2)}{\partial x_1}\, dx_1\, dx_2 + \int_B \frac{\partial F_2(x_1,x_2)}{\partial x_2}\, dx_1\, dx_2 = \int_{\partial B} F_1(x_1,x_2)\, dx_2 - \int_{\partial B} F_2(x_1,x_2)\, dx_1, \tag{2.3}$$

which is Green's theorem in the plane.

If $n = 3$ and if ∂B is described as a level-set of a function of three variables, i.e. $\partial B = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid G(x_1,x_2,x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\operatorname{grad} G$. We shall use the case where $G(x_1,x_2,x_3) = x_3 - g(x_1,x_2)$ for (x_1,x_2) in D, a bounded domain in $\mathbb{R}^2$ for some differentiable function g on D, and

$$B = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1,x_2)\}.$$

Also assume that B is bounded in $\mathbb{R}^3$. Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{(1 + g_{x_1}^2 + g_{x_2}^2)^{1/2}}, \qquad dA = (1 + g_{x_1}^2 + g_{x_2}^2)^{1/2}\, dx_1\, dx_2$$

and

$$\mathbf{n}\, dA = (-g_{x_1}, -g_{x_2}, 1)\, dx_1\, dx_2.$$

From (2.2) we get

$$\begin{aligned}\int_B \left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1} + \frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2} + \frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3} \right) dx_1\, dx_2\, dx_3 \\ = -\int_D F_1(x_1,x_2,g(x_1,x_2)) g_{x_1}(x_1,x_2)\, dx_1\, dx_2 \\ - \int_D F_2(x_1,x_2,g(x_1,x_2)) g_{x_2}(x_1,x_2)\, dx_1\, dx_2 \\ + \int_D F_3(x_1,x_2,g(x_1,x_2))\, dx_1\, dx_2,\end{aligned} \tag{2.4}$$

which is the Gauss–Ostrogradsky theorem in space.

Following Apostol [1], consider a surface described by the vector equation

$$r(u,v) = x_1(u,v)\vec{i} + x_2(u,v)\vec{j} + x_3(u,v)\vec{k} \tag{2.5}$$

where $(u,v) \in [a,b] \times [c,d]$.

If x_1, x_2, x_3 are differentiable on $[a,b] \times [c,d]$, we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x_1}{\partial u}\vec{i} + \frac{\partial x_2}{\partial u}\vec{j} + \frac{\partial x_3}{\partial u}\vec{k},$$
$$\frac{\partial r}{\partial v} = \frac{\partial x_1}{\partial v}\vec{i} + \frac{\partial x_2}{\partial v}\vec{j} + \frac{\partial x_3}{\partial v}\vec{k}.$$

Their *cross product* $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants.* In fact, we have [1, p. 420]

$$
\begin{aligned}
(2.6) \qquad \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} &= \begin{vmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix} \vec{i} + \begin{vmatrix} \frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u} \\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix} \vec{k} \\
&= \frac{\partial(x_2, x_3)}{\partial(u, v)} \vec{i} + \frac{\partial(x_3, x_1)}{\partial(u, v)} \vec{j} + \frac{\partial(x_1, x_2)}{\partial(u, v)} \vec{k}.
\end{aligned}
$$

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T = [a, b] \times [c, d]$. The area of ∂B denoted $A_{\partial B}$ is defined by the double integral [1, pp. 424–425]

$$
\begin{aligned}
(2.7) \qquad A_{\partial B} &= \int_a^b \int_c^d \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\, dv \\
&= \int_a^b \int_c^d \sqrt{\left(\frac{\partial(x_2, x_3)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(x_3, x_1)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(x_1, x_2)}{\partial\, (u, v)} \right)^2}\, du\, dv.
\end{aligned}
$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on $T = [a, b] \times [c, d]$ and let $f : \partial B \to \mathbb{C}$ be bounded. The surface integral of f over ∂B is defined by [1, p. 430]

$$
\begin{aligned}
(2.8) \qquad \iint_{\partial B} f\, dA &= \int_a^b \int_c^d f(x_1, x_2, x_3) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\, dv \\
&= \int_a^b \int_c^d f(x_1(u, v), x_2(u, v), x_3(u, v)) \\
&\quad \times \sqrt{\left(\frac{\partial(x_2, x_3)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(x_3, x_1)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(x_1, x_2)}{\partial(u, v)} \right)^2}\, du\, dv.
\end{aligned}
$$

If $\partial B = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to ∂B at each regular point of the surface. At each such point there are two unit normals, $\mathbf{n}_1$ which has the same direction as N, and $\mathbf{n}_2$ in the opposite direction. Thus

$$
\mathbf{n}_1 = \frac{N}{\|N\|} \quad \text{and} \quad \mathbf{n}_2 = -\mathbf{n}_1.
$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also F be a vector field defined on ∂B and assume that the surface integral

$$\iint_{\partial B} (F \cdot \mathbf{n})\, dA,$$

called the flux surface integral, exists. Here $F \cdot \mathbf{n}$ is the dot or inner product.

We can write [1, p. 434]

$$\iint_{\partial B} (F \cdot \mathbf{n})\, dA = \pm \int_a^b \int_c^d F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) du\, dv$$

where "+" is used if $\mathbf{n} = \mathbf{n}_1$, and "−" if $\mathbf{n} = \mathbf{n}_2$.

If

$$F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3)\vec{i} + F_2(x_1, x_2, x_3)\vec{j} + F_3(x_1, x_2, x_3)\vec{k}$$

and

$$r(u,v) = x_1(u,v)\vec{i} + x_2(u,v)\vec{j} + x_3(u,v)\vec{k} \quad \text{where} \quad (u,v) \in [a,b] \times [c,d]$$

then the flux surface integral for $\mathbf{n} = \mathbf{n}_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned} \iint_{\partial B} (F \cdot \mathbf{n})\, dA = & \int_a^b \int_c^d F_1(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_2, x_3)}{\partial(u,v)}\, du\, dv \\ & + \int_a^b \int_c^d F_2(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_3, x_1)}{\partial(u,v)}\, du\, dv \\ & + \int_a^b \int_c^d F_3(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_1, x_2)}{\partial(u,v)}\, du\, dv. \end{aligned} \tag{2.9}$$

The sum of the double integrals on the right is often written more succinctly as [1, p. 435]

$$\begin{aligned} \iint_{\partial B} F_1(x_1, x_2, x_3)\, dx_2 \wedge dx_3 + \iint_{\partial B} F_2(x_1, x_2, x_3)\, dx_3 \wedge dx_1 \\ + \iint_{\partial B} F_3(x_1, x_2, x_3)\, dx_1 \wedge dx_2. \end{aligned}$$

Let $B \subset \mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface ∂B, and let $\mathbf{n}$ be the unit outer normal to ∂B. If F is a continuously differentiable vector field defined on B, we have the Gauss–Ostrogradsky identity

$$\iiint_B (\operatorname{div} F)\, dV = \iint_{\partial B} (F \cdot \mathbf{n})\, dA. \tag{GO}$$

If we express

$$F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3)\vec{i} + F_2(x_1, x_2, x_3)\vec{j} + F_3(x_1, x_2, x_3)\vec{k},$$

then (2.4) can be written as

(2.10)
$$\iiint_B \left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1} + \frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2} + \frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3} \right) dx_1\, dx_2\, dx_3$$
$$= \iint_{\partial B} F_1(x_1,x_2,x_3)\, dx_2 \wedge dx_3 + \iint_{\partial B} F_2(x_1,x_2,x_3)\, dx_3 \wedge dx_1 + \iint_{\partial B} F_3(x_1,x_2,x_3)\, dx_1 \wedge dx_2.$$

3. Some perturbed identities. For the body B we consider the coordinates of the *centre of gravity*
$$G\left(\overline{x_{B,1}}, \ldots, \overline{x_{B,n}}\right)$$
defined by
$$\overline{x_{B,k}} := \frac{1}{V(B)} \int_B x_k\, dx, \qquad k \in \{1,\ldots,n\},$$
where
$$V(B) := \int_B dx$$
is the volume of B.

We have the following identity of interest:

THEOREM 2. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in an open neighbourhood of B, and with complex values. If $\alpha_k, \beta_k, \delta_k \in \mathbb{C}$ for $k \in \{1,\ldots,n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then*

(3.1)
$$\frac{1}{V(B)} \int_B f(x)\, dx = \sum_{k=1}^n \frac{1}{V(B)} \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx + \sum_{k=1}^n \delta_k (\beta_k - \alpha_k \overline{x_{B,k}}) + \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\, dA.$$

We also have

(3.2)
$$\frac{1}{V(B)} \int_B f(x)\, dx = \sum_{k=1}^n \alpha_k \frac{1}{V(B)} \int_B (\gamma_k - x_k) \left(\frac{\partial f(x)}{\partial x_k} - \delta_k \right) dx + \sum_{k=1}^n \alpha_k \delta_k (\gamma_k - \overline{x_{B,k}}) + \sum_{k=1}^n \alpha_k \frac{1}{V(B)} \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x)\, dA$$

for all $\gamma_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$, *and in particular*

$$\begin{aligned}(3.3)\quad & \frac{1}{V(B)}\int_B f(x)\,dx \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B (\gamma_k - x_k)\left(\frac{\partial f(x)}{\partial x_k} - \delta_k\right)dx \\ &\quad + \frac{1}{n}\sum_{k=1}^{n}\alpha_k\delta_k(\gamma_k - \overline{x_{B,k}}) + \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-\gamma_k)f(x)n_k(x)\,dA.\end{aligned}$$

Proof. Let $x = (x_1, \dots, x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k)f(x), \qquad k \in \{1, \dots, n\}.$$

Then

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}, \qquad k \in \{1, \dots, n\}.$$

If we sum this equality over k from 1 to n we get

$$\begin{aligned}(3.4)\quad \sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k} &= \sum_{k=1}^{n}\alpha_k f(x) + \sum_{k=1}^{n}(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k} \\ &= f(x) + \sum_{k=1}^{n}(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\end{aligned}$$

for all $x = (x_1, \dots, x_n) \in B$.

Now, if we integrate (3.4) over $(x_1, \dots, x_n) \in B$ we get

$$(3.5)\quad \int_B\left(\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k}\right)dx = \int_B f(x)\,dx + \sum_{k=1}^{n}\int_B\left[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\right]dx.$$

By the divergence theorem (2.2) we also have

$$(3.6)\quad \int_B\left(\sum_{k=1}^{n}\frac{\partial F_k(x)}{\partial x_k}\right)dx = \sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k)f(x)n_k(x)\,dA,$$

and by making use of (3.5) and (3.6) we derive

$$\begin{aligned}\int_B f(x)\,dx + \sum_{k=1}^{n}\int_B\left[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\right]dx \\ = \sum_{k=1}^{n}\int_{\partial B}(\alpha_k x_k - \beta_k)f(x)n_k(x)\,dA,\end{aligned}$$

which gives the representation

$$\int_B f(x)\,dx = \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\,dx + \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA. \tag{3.7}$$

Now, observe that

$$\begin{aligned}
&\int_B (\beta_k - \alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k} - \delta_k\right) dx \\
&\qquad = \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\,dx - \delta_k \int_B (\beta_k - \alpha_k x_k)\,dx \\
&\qquad = \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\,dx - \delta_k(\beta_k V(B) - \alpha_k V(B)\overline{x_{B,k}}) \\
&\qquad = \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\,dx - V(B)\delta_k(\beta_k - \alpha_k \overline{x_{B,k}}),
\end{aligned}$$

which for summation over $k \in \{1, \dots, n\}$ provides

$$\begin{aligned}
&\sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k} - \delta_k\right) dx \\
&\qquad = \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\,dx - V(B)\sum_{k=1}^n \delta_k(\beta_k - \alpha_k \overline{x_{B,k}}),
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}\,dx \\
&\quad = \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k} - \delta_k\right) dx + V(B)\sum_{k=1}^n \delta_k(\beta_k - \alpha_k \overline{x_{B,k}}).
\end{aligned}$$

From (3.7) we then get

$$\begin{aligned}
\int_B f(x)\,dx &= \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k} - \delta_k\right) dx \\
&\quad + V(B)\sum_{k=1}^n \delta_k(\beta_k - \alpha_k \overline{x_{B,k}}) + \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA,
\end{aligned}$$

which on division by $V(B)$ produces the desired result (3.1).

The identity (3.2) follows from (3.1) for $\beta_k = \alpha_k \gamma_k$, $k \in \{1, \dots, n\}$. ■

The following particular cases are of interest:

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\frac{1}{V(B)}\int_B f(x)\,dx = \sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_B(\overline{x_{B,k}}-x_k)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx + \sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_{\partial B}(x_k-\overline{x_{B,k}})f(x)n_k(x)\,dA, \tag{3.8}$$

and in particular

$$\frac{1}{V(B)}\int_B f(x)\,dx = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B(\overline{x_{B,k}}-x_k)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx + \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-\overline{x_{B,k}})f(x)n_k(x)\,dA. \tag{3.9}$$

This follows from (3.1) on taking $\beta_k=\alpha_k\overline{x_{B,k}}$, $k\in\{1,\dots,n\}$.

For a function f as in Theorem 2 above, we define the points

$$x_{\partial B,f,k} := \frac{\int_{\partial B}x_k f(x)n_k(x)\,dA}{\int_{\partial B}f(x)n_k(x)\,dA}, \qquad k\in\{1,\dots,n\},$$

provided that all denominators are nonzero.

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\frac{1}{V(B)}\int_B f(x)\,dx = \sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_B(x_{\partial B,f,k}-x_k)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx + \sum_{k=1}^{n}\delta_k\alpha_k(x_{\partial B,f,k}-\overline{x_{B,k}}) \tag{3.10}$$

and in particular

$$\frac{1}{V(B)}\int_B f(x)\,dx = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_B(x_{\partial B,f,k}-x_k)\left(\frac{\partial f(x)}{\partial x_k}-\delta_k\right)dx + \frac{1}{n}\sum_{k=1}^{n}\delta_k(x_{\partial B,f,k}-\overline{x_{B,k}}). \tag{3.11}$$

This follows from (3.1) on taking $\beta_k=\alpha_k x_{\partial B,f,k}$, $k\in\{1,\dots,n\}$, and observing that

$$\sum_{k=1}^{n}\alpha_k\int_{\partial B}(x_k-x_{\partial B,f,k})f(x)n_k(x)\,dA=0.$$

4. Some inequalities for bounded partial derivatives. Let B be a bounded closed subset of $\mathbb{R}^n$ $(n \ge 2)$ with smooth (or piecewise smooth) boundary ∂B. Now, for $\phi, \Phi \in \mathbb{C}$, define the following sets of complex-valued functions:

$$\bar{U}_B(\phi,\Phi) := \{f : B \to \mathbb{C} \mid \operatorname{Re}[(\Phi - f(x))(\overline{f(x)} - \overline{\phi})] \ge 0 \text{ for each } x \in B\},$$
$$\bar{\Delta}_B(\phi,\Phi) := \left\{f : B \to \mathbb{C} \;\middle|\; \left|f(x) - \frac{\phi+\Phi}{2}\right| \le \frac{1}{2}|\Phi - \phi| \text{ for each } x \in B\right\}.$$

The following representation result may be stated.

Proposition 1. *For any distinct $\phi, \Phi \in \mathbb{C}$, the sets $\bar{U}_B(\phi,\Phi)$ and $\bar{\Delta}_B(\phi,\Phi)$ are nonempty, convex and closed and*

$$\bar{U}_B(\phi,\Phi) = \bar{\Delta}_B(\phi,\Phi). \tag{4.1}$$

Proof. We observe that for any $w \in \mathbb{C}$ we have the equivalence

$$\left|w - \frac{\phi+\Phi}{2}\right| \le \frac{1}{2}|\Phi - \phi| \iff \operatorname{Re}[(\Phi - w)(\overline{w} - \overline{\phi})] \ge 0.$$

This follows from the equality

$$\frac{1}{4}|\Phi - \phi|^2 - \left|w - \frac{\phi+\Phi}{2}\right|^2 = \operatorname{Re}[(\Phi - w)(\overline{w} - \overline{\phi})],$$

which holds for any $w \in \mathbb{C}$.

The equality (4.1) is thus a simple consequence of this fact. ∎

On making use of the properties of complex numbers we can also state that:

Corollary 3. *For any distinct $\phi, \Phi \in \mathbb{C}$, we have*

$$\begin{aligned}\bar{U}_B(\phi,\Phi) = \{f : B \to \mathbb{C} \mid (\operatorname{Re}\Phi - \operatorname{Re} f(x))(\operatorname{Re} f(x) - \operatorname{Re}\phi)\\ + (\operatorname{Im}\Phi - \operatorname{Im} f(x))(\operatorname{Im} f(x) - \operatorname{Im}\phi) \ge 0 \textit{ for each } x \in B\}.\end{aligned} \tag{4.2}$$

Now, if we assume that $\operatorname{Re}(\Phi) \ge \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \ge \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_B(\phi,\Phi) := \{f : B \to \mathbb{C} \mid \operatorname{Re}\Phi \ge \operatorname{Re} f(x) \ge \operatorname{Re}\phi \text{ and}\\ \operatorname{Im}\Phi \ge \operatorname{Im} f(x) \ge \operatorname{Im}\phi \text{ for each } x \in B\}.\end{aligned} \tag{4.3}$$

One can easily observe that $\bar{S}_B(\phi,\Phi)$ is closed, convex and

$$\emptyset \ne \bar{S}_B(\phi,\Phi) \subseteq \bar{U}_B(\phi,\Phi). \tag{4.4}$$

Theorem 3. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n \ge 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in an open neighbourhood of B and with complex values. Assume that there exist $\phi_k, \Phi_k \in \mathbb{C}$, $\phi_k \ne \Phi_k$*

for $k \in \{1, \ldots, n\}$ *and such that* $\frac{\partial f}{\partial x_k} \in \bar{\Delta}_B(\phi_k, \Phi_k)$ *for* $k \in \{1, \ldots, n\}$*. If* $\alpha_k, \beta_k \in \mathbb{C}$ *for* $k \in \{1, \ldots, n\}$ *with* $\sum_{k=1}^n \alpha_k = 1$*, then*

$$\begin{aligned}(4.5)\qquad &\Bigg| \frac{1}{V(B)} \int_B f(x)\,dx - \sum_{k=1}^n \frac{\phi_k + \Phi_k}{2}(\beta_k - \alpha_k \overline{x_{B,k}}) \\ &\qquad - \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA \Bigg| \\ &\qquad \le \frac{1}{2} \sum_{k=1}^n |\Phi_k - \phi_k| \frac{1}{V(B)} \int_B |\beta_k - \alpha_k x_k|\,dx.\end{aligned}$$

We also have

$$\begin{aligned}(4.6)\qquad &\Bigg| \frac{1}{V(B)} \int_B f(x)\,dx - \sum_{k=1}^n \frac{\phi_k + \Phi_k}{2} \alpha_k \left(\gamma_k - \overline{x_{B,k}}\right) \\ &\qquad - \sum_{k=1}^n \alpha_k \frac{1}{V(B)} \int_{\partial B} \left(x_k - \gamma_k\right) f(x) n_k(x)\,dA \Bigg| \\ &\qquad \le \frac{1}{2} \sum_{k=1}^n |\Phi_k - \phi_k|\; |\alpha_k| \frac{1}{V(B)} \int_B |\gamma_k - x_k|\;dx\end{aligned}$$

for all $\gamma_k \in \mathbb{C}$*,* $k \in \{1, \ldots, n\}$*, and in particular*

$$\begin{aligned}(4.7)\qquad &\Bigg| \frac{1}{V(B)} \int_B f(x)\,dx - \frac{1}{n} \sum_{k=1}^n \frac{\phi_k + \Phi_k}{2}(\gamma_k - \overline{x_{B,k}}) \\ &\qquad - \frac{1}{n} \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x)\,dA \Bigg| \\ &\qquad \le \frac{1}{2n} \sum_{k=1}^n |\Phi_k - \phi_k| \frac{1}{V(B)} \int_B |\gamma_k - x_k|\,dx.\end{aligned}$$

Proof. By using identity (3.1) for $\delta_k := \frac{\phi_k + \Phi_k}{2}$, $k \in \{1, \ldots, n\}$, we get

$$\begin{aligned}(4.8)\qquad &\frac{1}{V(B)} \int_B f(x)\,dx \\ &= \sum_{k=1}^n \frac{1}{V(B)} \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \frac{\phi_k + \Phi_k}{2} \right) dx \\ &\quad + \sum_{k=1}^n \frac{\phi_k + \Phi_k}{2} (\beta_k - \alpha_k \overline{x_{B,k}}) + \sum_{k=1}^n \frac{1}{V(B)} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA.\end{aligned}$$

Since $\frac{\partial f}{\partial x_k} \in \bar{\Delta}_B(\phi_k, \Phi_k)$ for $k \in \{1, \ldots, n\}$, (4.8) yields

$$
\begin{aligned}
&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}(\beta_k-\alpha_k\overline{x_{B,k}})\right.\\
&\qquad\qquad\left.-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(\alpha_k x_k-\beta_k)f(x)n_k(x)\,dA\right|\\
&\qquad\le\sum_{k=1}^{n}\frac{1}{V(B)}\left|\int_B(\beta_k-\alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k}-\frac{\phi_k+\Phi_k}{2}\right)dx\right|\\
&\qquad\le\sum_{k=1}^{n}\frac{1}{V(B)}\int_B\left|(\beta_k-\alpha_k x_k)\left(\frac{\partial f(x)}{\partial x_k}-\frac{\phi_k+\Phi_k}{2}\right)\right|dx\\
&\qquad\le\sum_{k=1}^{n}\frac{1}{2}|\Phi_k-\phi_k|\frac{1}{V(B)}\int_B|\beta_k-\alpha_k x_k|\,dx,
\end{aligned}
$$

which proves (4.5). The rest is obvious. ∎

Corollary 4. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
(4.9)\qquad&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_{\partial B}(x_k-\overline{x_{B,k}})f(x)n_k(x)\,dA\right|\\
&\qquad\le\frac{1}{2}\sum_{k=1}^{n}|\alpha_k|\,|\Phi_k-\phi_k|\frac{1}{V(B)}\int_B|\overline{x_{B,k}}-x_k|\,dx,
\end{aligned}
$$

and in particular

$$
\begin{aligned}
(4.10)\qquad&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-\overline{x_{B,k}})f(x)n_k(x)\,dA\right|\\
&\qquad\le\frac{1}{2n}\sum_{k=1}^{n}|\Phi_k-\phi_k|\frac{1}{V(B)}\int_B|\overline{x_{B,k}}-x_k|\,dx.
\end{aligned}
$$

This follows from (4.6) by taking $\gamma_k=\overline{x_{B,k}}$, $k\in\{1,\dots,n\}$.

Corollary 5. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
(4.11)\qquad&\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}\alpha_k(x_{\partial B,f,k}-\overline{x_{B,k}})\right|\\
&\qquad\le\frac{1}{2}\sum_{k=1}^{n}|\Phi_k-\phi_k|\,|\alpha_k|\frac{1}{V(B)}\int_B|x_{\partial B,f,k}-x_k|\,dx,
\end{aligned}
$$

and in particular

$$(4.12)\quad \left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{\phi_k+\Phi_k}{2}(x_{\partial B,f,k}-\overline{x_{B,k}})\right| \le \frac{1}{2n}\sum_{k=1}^{n}|\Phi_k-\phi_k|\frac{1}{V(B)}\int_B|x_{\partial B,f,k}-x_k|\,dx.$$

This follows from (4.6) by taking $\gamma_k = x_{\partial B,f,k}$, $k\in\{1,\dots,n\}$, and observing that

$$\sum_{k=1}^{n}\alpha_k\frac{1}{V(B)}\int_{\partial B}(x_k-x_{\partial B,f,k})f(x)n_k(x)\,dA=0.$$

5. Inequalities for Lipschitzian partial derivatives. We assume that the partial derivatives $\frac{\partial f}{\partial x_k}$, $k\in\{1,\dots,n\}$, satisfy the Lipschitz type conditions at the point $u=(u_1,\dots,u_n)\in D$:

$$(5.1)\quad \left|\frac{\partial f(x)}{\partial x_k}-\frac{\partial f\,(u)}{\partial x_k}\right|\le\sum_{j=1}^{n}L_{k,j}|x_j-u_j|$$

for any $x=(x_1,\dots,x_n)\in D$, where $L_{k,j}$, $k,j\in\{1,\dots,n\}$, are given positive constants.

THEOREM 4. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n\ge 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in an open neighbourhood of B, and with complex values. Assume that for $u\in B$ there exist $L_{k,j}$, $k,j\in\{1,\dots,n\}$, such that the Lipschitz condition (5.1) holds for $k\in\{1,\dots,n\}$. If α_k, $\beta_k\in\mathbb{C}$ for $k\in\{1,\dots,n\}$ with $\sum_{k=1}^{n}\alpha_k=1$, then*

$$(5.2)\quad \left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{\partial f(u)}{\partial x_k}(\beta_k-\alpha_k\overline{x_{B,k}}) -\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(\alpha_k x_k-\beta_k)f(x)n_k(x)\,dA\right| \le\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B|\beta_k-\alpha_k x_k|\;|x_j-u_j|\,dx.$$

We also have

$$(5.3)\quad \left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\alpha_k\frac{\partial f(u)}{\partial x_k}(\gamma_k-\overline{x_{B,k}}) -\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k\,(x_k-\gamma_k)\,f(x)n_k(x)\,dA\right| \le\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}|\alpha_k|\frac{1}{V(B)}\int_B|\gamma_k-x_k|\,|x_j-u_j|\,dx$$

for all $\gamma_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$, *and in particular*

$$\text{(5.4)}\quad \left| \frac{1}{V(B)} \int_B f(x)\,dx - \frac{1}{n} \sum_{k=1}^{n} \frac{\partial f(u)}{\partial x_k} (\gamma_k - \overline{x_{B,k}}) - \frac{1}{n} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} \alpha_k (x_k - \gamma_k) f(x) n_k(x)\,dA \right| \le \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} L_{k,j} \frac{1}{V(B)} \int_B |\gamma_k - x_k|\,|x_j - u_j|\,dx.$$

Proof. If we write the equality (3.1) for $\delta_k = \frac{\partial f(u)}{\partial x_k}$, $k \in \{1, \dots, n\}$, we obtain

$$\frac{1}{V(B)} \int_B f(x)\,dx = \sum_{k=1}^{n} \frac{1}{V(B)} \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \frac{\partial f(u)}{\partial x_k} \right) dx + \sum_{k=1}^{n} \frac{\partial f(u)}{\partial x_k} (\beta_k - \alpha_k \overline{x_{B,k}}) + \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA.$$

Therefore

$$\begin{aligned} &\left| \frac{1}{V(B)} \int_B f(x)\,dx - \sum_{k=1}^{n} \frac{\partial f(u)}{\partial x_k} (\beta_k - \alpha_k \overline{x_{B,k}}) - \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x)\,dA \right| \\ &\le \sum_{k=1}^{n} \frac{1}{V(B)} \left| \int_B (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \frac{\partial f(u)}{\partial x_k} \right) dx \right| \\ &\le \sum_{k=1}^{n} \frac{1}{V(B)} \int_B \left| (\beta_k - \alpha_k x_k) \left(\frac{\partial f(x)}{\partial x_k} - \frac{\partial f(u)}{\partial x_k} \right) \right| dx \\ &= \sum_{k=1}^{n} \frac{1}{V(B)} \int_B |\beta_k - \alpha_k x_k| \left| \frac{\partial f(x)}{\partial x_k} - \frac{\partial f(u)}{\partial x_k} \right| dx \\ &\le \sum_{k=1}^{n} \frac{1}{V(B)} \int_B |\beta_k - \alpha_k x_k| \sum_{j=1}^{n} L_{k,j} |x_j - u_j|\,dx \qquad \text{(by (5.1))} \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} L_{k,j} \frac{1}{V(B)} \int_B |\beta_k - \alpha_k x_k|\,|x_j - u_j|\,dx \end{aligned}$$

and the inequality (5.2) is proved. ■

Corollary 6. *With the assumptions of Theorem 4 we have*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k(x_k-\overline{x_{B,k}})f(x)n_k(x)\,dA\right| \\ \le \sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}|\alpha_k|\frac{1}{V(B)}\int_B|\overline{x_{B,k}}-x_k|\,|x_j-u_j|\,dx, \tag{5.5}$$

and in particular

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k-\overline{x_{B,k}})f(x)n_k(x)\,dA\right| \\ \le \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B|\overline{x_{B,k}}-x_k|\,|x_j-u_j|\,dx. \tag{5.6}$$

Corollary 7. *With the assumptions of Theorem 4 we have*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\alpha_k\frac{\partial f(u)}{\partial x_k}(x_{\partial B,f,k}-\overline{x_{B,k}})\right| \\ \le \sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}|\alpha_k|\frac{1}{V(B)}\int_B|x_{\partial B,f,k}-x_k|\,|x_j-u_j|\,dx, \tag{5.7}$$

and in particular

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\frac{1}{n}\sum_{k=1}^{n}\frac{\partial f(u)}{\partial x_k}(x_{\partial B,f,k}-\overline{x_{B,k}})\right| \\ \le \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B|x_{\partial B,f,k}-x_k|\,|x_j-u_j|\,dx. \tag{5.8}$$

Corollary 8. *With the assumptions of Theorem 4 we have*

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx-\sum_{k=1}^{n}\alpha_k\frac{\partial f(u)}{\partial x_k}(u_k-\overline{x_{B,k}}) \right.\\ \left. -\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k(x_k-u_k)f(x)n_k(x)\,dA\right| \\ \le \sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}|\alpha_k|\frac{1}{V(B)}\int_B|u_k-x_k|\,|x_j-u_j|\,dx, \tag{5.9}$$

and in particular

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n}\frac{\partial f(u)}{\partial x_k}(u_k - \overline{x_{B,k}}) - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - u_k)f(x)n_k(x)\,dA\right| \leq \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B |u_k - x_k|\,|x_j - u_j|\,dx. \tag{5.10}$$

REMARK 1. With the assumptions of Theorem 4 and for $G = (\overline{x_{B,1}},\dots,\overline{x_{B,n}}) \in B$ if there exist $M_{k,j} > 0$, $k,j \in \{1,\dots,n\}$, such that the Lipschitz conditions

$$\left|\frac{\partial f(x)}{\partial x_k} - \frac{\partial f(G)}{\partial x_k}\right| \leq \sum_{j=1}^{n} M_{k,j}|x_j - u_j| \tag{5.11}$$

hold for $k \in \{1,\dots,n\}$, then

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx - \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\alpha_k(x_k - \overline{x_{B,k}})f(x)n_k(x)\,dA\right| \leq \sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}|\alpha_k|\frac{1}{V(B)}\int_B |\overline{x_{B,k}} - x_k|\,|x_j - \overline{x_{B,j}}|\,dx, \tag{5.12}$$

and in particular

$$\left|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{x_{B,k}})f(x)n_k(x)\,dA\right| \leq \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}L_{k,j}\frac{1}{V(B)}\int_B |\overline{x_{B,k}} - x_k|\,|x_j - \overline{x_{B,j}}|\,dx. \tag{5.13}$$

It is well known that if a function g has bounded partial derivatives on B, which is assumed also to be convex, then for all $x, y \in B$ we have the Lipschitz type condition

$$|g(x) - g(y)| \leq \sum_{j=1}^{n}\left\|\frac{\partial g}{\partial x_j}\right\|_{B,\infty}|x_j - y_j|$$

where

$$\left\|\frac{\partial g}{\partial x_j}\right\|_{B,\infty} := \sup_{x\in B}\left|\frac{\partial g(x)}{\partial x_j}\right| < \infty.$$

We can state the following result that is more convenient to apply:

Corollary 9. *Let B be a bounded closed convex subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a twice differentiable function defined in $\mathbb{R}^n$, or at least in an open neighbourhood of B, and with complex values and assume that*

$$\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} := \sup_{x\in B}\left|\frac{\partial^2 f(x)}{\partial x_k \partial x_j}\right| < \infty$$

for all $k, j \in \{1, \dots, n\}$. Then for all $u \in B$ we have

$$\begin{aligned}(5.14)\quad &\left|\frac{1}{V(B)}\int_B f(x)\,dx - \sum_{k=1}^n \alpha_k \frac{\partial f(u)}{\partial x_k}(u_k - \overline{x_{B,k}})\right.\\ &\qquad\left. - \sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B} \alpha_k (x_k - u_k) f(x) n_k(x)\,dA\right|\\ &\qquad\leq \sum_{k=1}^n\sum_{j=1}^n \left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} |\alpha_k| \frac{1}{V(B)}\int_B |u_k - x_k|\,|x_j - u_j|\,dx,\end{aligned}$$

and in particular

$$\begin{aligned}(5.15)\quad &\left|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^n \frac{\partial f(u)}{\partial x_k}(u_k - \overline{x_{B,k}})\right.\\ &\qquad\left. - \frac{1}{n}\sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B} (x_k - u_k) f(x) n_k(x)\,dA\right|\\ &\qquad\leq \frac{1}{n}\sum_{k=1}^n\sum_{j=1}^n \left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} \frac{1}{V(B)}\int_B |u_k - x_k|\,|x_j - u_j|\,dx.\end{aligned}$$

We also have the centre of gravity inequality

$$\begin{aligned}(5.16)\quad &\left|\frac{1}{V(B)}\int_B f(x)\,dx - \sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B} \alpha_k (x_k - \overline{x_{B,k}}) f(x) n_k(x)\,dA\right|\\ &\qquad\leq \sum_{k=1}^n\sum_{j=1}^n |\alpha_k| \left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} \frac{1}{V(B)}\int_B |\overline{x_{B,k}} - x_k|\,|x_j - \overline{x_{B,k}}|\,dx,\end{aligned}$$

and in particular

$$\begin{aligned}(5.17)\quad &\left|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{n}\sum_{k=1}^n \frac{1}{V(B)}\int_{\partial B} (x_k - \overline{x_{B,k}}) f(x) n_k(x)\,dA\right|\\ &\qquad\leq \frac{1}{n}\sum_{k=1}^n\sum_{j=1}^n \left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} \frac{1}{V(B)}\int_B |\overline{x_{B,k}} - x_k|\,|x_j - \overline{x_{B,k}}|\,dx.\end{aligned}$$

6. Example for 3-dimensional spaces. Let B be a bounded closed convex subset of $\mathbb{R}^3$ with smooth (or piecewise smooth) boundary ∂B. Let f be a twice differentiable function defined in $\mathbb{R}^3$, or at least in an open neighbourhood of B, and with complex values and assume that

$$\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty} := \sup_{x\in B}\left|\frac{\partial^2 f(x)}{\partial x_k \partial x_j}\right| < \infty \quad \text{for all } k,j \in \{1,2,3\}.$$

Assume that ∂B is described by the vector equation

$$r(u,v) = x_1(u,v)\vec{i} + x_2(u,v)\vec{j} + x_3(u,v)\vec{k}$$

where $(u,v) \in [a,b]\times[c,d]$. Then, by using the notations from the second section, we have

$$\begin{aligned}
(6.1)\quad & \Bigg|\frac{1}{V(B)}\int_B f(x)\,dx - \sum_{k=1}^{3}\alpha_k \frac{\partial f(y_1,y_2,y_3)}{\partial x_k}(y_k - \overline{x_{B,k}}) \\
&- \frac{1}{V(B)}\int_a^b\int_c^d \alpha_1(x_1(u,v)-y_1) f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}\,du\,dv \\
&- \frac{1}{V(B)}\int_a^b\int_c^d \alpha_2(x_2(u,v)-y_2) f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}\,du\,dv \\
&- \frac{1}{V(B)}\int_a^b\int_c^d \alpha_3(x_3(u,v)-y_3) f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}\,du\,dv\Bigg| \\
&\le \sum_{k=1}^{3}\sum_{j=1}^{3}|\alpha_k|\left\|\frac{\partial^2 f}{\partial x_k\partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B |y_k - x_k|\,|x_j - y_j|\,dx
\end{aligned}$$

for all $(y_1,y_2,y_3)\in B$ and $\alpha_k \in \mathbb{C}$, $k\in\{1,2,3\}$, with $\alpha_1+\alpha_2+\alpha_3 = 1$.

In particular, we have

$$\begin{aligned}
(6.2)\quad & \Bigg|\frac{1}{V(B)}\int_B f(x)\,dx - \frac{1}{3}\sum_{k=1}^{3}\frac{\partial f(y_1,y_2,y_3)}{\partial x_k}(y_k - \overline{x_{B,k}}) \\
&- \frac{1}{3V(B)}\int_a^b\int_c^d (x_1(u,v)-y_1) f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}\,du\,dv \\
&- \frac{1}{3V(B)}\int_a^b\int_c^d (x_2(u,v)-y_2) f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}\,du\,dv \\
&- \frac{1}{3V(B)}\int_a^b\int_c^d (x_3(u,v)-y_3) f(x_1(u,v),x_2(u,v),x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}\,du\,dv\Bigg|
\end{aligned}$$

$$\le \frac{1}{3}\sum_{k=1}^{3}\sum_{j=1}^{3}\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B |y_k - x_k|\,|x_j - y_j|\,dx$$

for all $(y_1, y_2, y_3) \in B$.

We also have the *centre of gravity inequalities*

$$\begin{aligned}(6.3)\quad &\left|\frac{1}{V(B)}\int_B f(x)\,dx\right. \\ &- \frac{1}{V(B)}\int_a^b\int_c^d \alpha_1(x_1(u,v) - \overline{x_{B,1}})f(x_1(u,v), x_2(u,v), x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}\,du\,dv \\ &- \frac{1}{V(B)}\int_a^b\int_c^d \alpha_2(x_2(u,v) - \overline{x_{B,2}})f(x_1(u,v), x_2(u,v), x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}\,du\,dv \\ &\left.- \frac{1}{V(B)}\int_a^b\int_c^d \alpha_3(x_3(u,v) - \overline{x_{B,3}})f(x_1(u,v), x_2(u,v), x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}\,du\,dv\right| \\ &\le \sum_{k=1}^{3}\sum_{j=1}^{3}|\alpha_k|\left\|\frac{\partial^2 f}{\partial x_k \partial x_j}\right\|_{B,\infty}\frac{1}{V(B)}\int_B |\overline{x_{B,k}} - x_k|\,|x_j - \overline{x_{B,j}}|\,dx\end{aligned}$$

for all $\alpha_k \in \mathbb{C}$, $k \in \{1,2,3\}$, with $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

In particular,

$$\begin{aligned}(6.4)\quad &\left|\frac{1}{V(B)}\int_B f(x)\,dx\right. \\ &- \frac{1}{3V(B)}\int_a^b\int_c^d (x_1(u,v) - \overline{x_{B,1}})f(x_1(u,v), x_2(u,v), x_3(u,v))\frac{\partial(x_2,x_3)}{\partial(u,v)}\,du\,dv \\ &- \frac{1}{3V(B)}\int_a^b\int_c^d (x_2(u,v) - \overline{x_{B,2}})f(x_1(u,v), x_2(u,v), x_3(u,v))\frac{\partial(x_3,x_1)}{\partial(u,v)}\,du\,dv \\ &\left.- \frac{1}{3V(B)}\int_a^b\int_c^d (x_3(u,v) - \overline{x_{B,3}})f(x_1(u,v), x_2(u,v), x_3(u,v))\frac{\partial(x_1,x_2)}{\partial(u,v)}\,du\,dv\right| \\ &\le \frac{1}{3}\,\frac{1}{V(B)}\left[\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{B,\infty}\int_B (\overline{x_{B,1}} - x_1)^2\,dx\right. \\ &\left.+ \left\|\frac{\partial^2 f}{\partial x_2^2}\right\|_{B,\infty}\int_B (\overline{x_{B,2}} - x_2)^2\,dx + \left\|\frac{\partial^2 f}{\partial x_3^2}\right\|_{B,\infty}\int_B (\overline{x_{B,3}} - x_3)^2\,dx\right] \\ &+ \frac{2}{3}\,\frac{1}{V(B)}\left[\left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|_{B,\infty}\int_B |\overline{x_{B,1}} - x_1|\,|x_2 - \overline{x_{B,2}}|\,dx\right.\end{aligned}$$

$$+ \left\| \frac{\partial^2 f}{\partial x_2 \partial x_3} \right\|_{B,\infty} \int_B |\overline{x_{B,2}} - x_2| \, |x_3 - \overline{x_{B,3}}| \, dx$$

$$+ \left\| \frac{\partial^2 f}{\partial x_1 \partial x_3} \right\|_{B,\infty} \int_B |\overline{x_{B,1}} - x_1| \, |x_3 - \overline{x_{B,3}}| \, dx \Bigg].$$

7. Example for 3-dimensional balls. Consider the 3-dimensional ball centred at $C = (a, b, c)$ and having radius $R > 0$,

$$B(C, R) := \{(x, y, z) \in \mathbb{R}^3 \mid (x - a)^2 + (y - b)^2 + (z - c)^2 \leq R^2\}$$

and the sphere

$$S(C, R) := \{(x, y, z) \in \mathbb{R}^3 \mid (x - a)^2 + (y - b)^2 + (z - c)^2 = R^2\}.$$

Consider the parametrisation of $B(C, R)$ and $S(C, R)$ given by

$$B(C,R): \begin{cases} x = r \cos\psi \cos\varphi + a, \\ y = r \cos\psi \sin\varphi + b, \\ z = r \sin\psi + c, \end{cases} \qquad (r, \psi, \varphi) \in [0, R] \times [-\pi/2, \pi/2] \times [0, 2\pi],$$

and

$$S(C,R): \begin{cases} x = R \cos\psi \cos\varphi + a, \\ y = R \cos\psi \sin\varphi + b, \\ z = R \sin\psi + c, \end{cases} \qquad (\psi, \varphi) \in [-\pi/2, \pi/2] \times [0, 2\pi].$$

Observe that

$$\begin{vmatrix} \frac{\partial y}{\partial \psi} & \frac{\partial z}{\partial \psi} \\ \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = -R^2 \cos^2\psi \cos\varphi,$$

$$\begin{vmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial z}{\partial \psi} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = R^2 \cos^2\psi \sin\varphi,$$

$$\begin{vmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial y}{\partial \psi} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = -R^2 \sin\psi \cos\psi.$$

Let us consider the transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T_2(r, \psi, \varphi) := (r \cos\psi \cos\varphi + a, r \cos\psi \sin\varphi + b, r \sin\psi + c).$$

It is well known that the Jacobian of T_2 is

$$J(T_2) = r^2 \cos\psi$$

and T_2 is a one-to-one mapping defined on $[0, R] \times [-\pi/2, \pi/2] \times [0, 2\pi]$, with values in $B(C, R)$. Thus we have the change of variables

$$\iiint_{B(C,R)} f(x,y,z)\,dx\,dy\,dz \tag{7.1}$$
$$= \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c) r^2 \cos\psi\,dr\,d\psi\,d\varphi.$$

We also have

$$\iiint_{B(C,R)} |z - \overline{z_{B(C,R)}}|^2\,dx\,dy\,dz = \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2 \sin^2\psi r^2 \cos\psi\,dr\,d\psi\,d\varphi$$
$$= \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^4 \sin^2\psi \cos\psi\,dr\,d\psi\,d\varphi = \frac{4}{15}\pi R^5,$$

and similarly

$$\iiint_{B(C,R)} |x - \overline{x_{B(C,R)}}|^2\,dx\,dy\,dz = \iiint_{B(C,R)} \left|y - \overline{y_{B(C,R)}}\right|^2\,dx\,dy\,dz = \frac{4}{15}\pi R^5.$$

Also

$$\iiint_{B(C,R)} |x - \overline{x_{B(C,R)}}|\,|y - \overline{y_{B(C,R)}}|\,dx\,dy\,dz$$
$$= \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} |r\cos\psi\cos\varphi|\;|r\cos\psi\sin\varphi| r^2 \cos\psi\,dr\,d\psi\,d\varphi$$
$$= \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^4 \cos^3\psi |\sin\varphi\cos\varphi|\,dr\,d\psi\,d\varphi = \frac{8}{15}R^5,$$

and similarly

$$\iiint_{B(C,R)} |x - \overline{x_{B(C,R)}}|\,|z - \overline{z_{B(C,R)}}|\,dx\,dy\,dz$$
$$= \iiint_{B(C,R)} |y - \overline{y_{B(C,R)}}|\,|z - \overline{z_{B(C,R)}}|\,dx\,dy\,dz = \frac{8}{15}R^5.$$

Since $V(B(C,R)) = \frac{4\pi R^3}{3}$, by (6.4) we get

$$\Bigg| \frac{1}{\frac{4\pi R^3}{3}} \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c) \tag{7.2}$$
$$\times r^2 \cos\psi\,dr\,d\psi\,d\varphi$$

$$+\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}\int_{0}^{2\pi} f(R\cos\psi\cos\varphi+a, R\cos\psi\sin\varphi+b, R\sin\psi+c)\cos^{3}\psi\cos^{2}\varphi\, d\psi\, d\varphi$$

$$-\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}\int_{0}^{2\pi} f(R\cos\psi\cos\varphi+a, R\cos\psi\sin\varphi+b, R\sin\psi+c)\cos^{3}\psi\sin^{2}\varphi\, d\psi\, d\varphi$$

$$+\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}\int_{0}^{2\pi} f(R\cos\psi\cos\varphi+a, R\cos\psi\sin\varphi+b, R\sin\psi+c)\sin^{2}\psi\cos\psi\, d\psi\, d\varphi\Bigg|$$

$$\le \frac{1}{15}R^2\left[\left\|\frac{\partial^2 f}{\partial x^2}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial y^2}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial z^2}\right\|_{B(C,R),\infty}\right]$$
$$+\frac{4}{15\pi}R^2\left[\left\|\frac{\partial^2 f}{\partial x\partial y}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial y\partial z}\right\|_{B(C,R),\infty}+\left\|\frac{\partial^2 f}{\partial z\partial x}\right\|_{B(C,R),\infty}\right].$$

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Silvestru Sever Dragomir
Mathematics, College of Engineering & Science
Victoria University
PO Box 14428
Melbourne, MC 8001, Australia
ORCID: 0000-0003-2902-6805
E-mail: sever.dragomir@vu.edu.au
http://rgmia.org/dragomir
and
DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences
School of Computer Science & Applied Mathematics
University of the Witwatersrand
Private Bag 3
Johannesburg 2050, South Africa