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Quantitative results on Diophantine equations in many variables

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1. Introduction. Consider a system $\underline{f}$ of polynomials $f_1, \ldots, f_r \in \mathbb{Z}[x_1, \ldots, x_n] = \mathbb{Z}[\underline{x}]$ of degree $d \geq 2$. It was shown by Birch [2] that if these polynomials are homogeneous, they satisfy the smooth Hasse principle provided

$$n - \dim V^* > r(r+1)(d-1)2^{d-1}, \tag{1}$$

where V^* is the so-called *Birch singular locus* of the the projective variety V corresponding to $\underline{f}$. Let V^{sm} be the *smooth locus* of V (which consists of the points where the the Jacobian matrix of $\underline{f}$ has rank r). Then the system $\underline{f}$ is said to satisfy the *smooth Hasse principle* if

$$\prod_\nu V^{\mathrm{sm}}(\mathbb{Q}_\nu) \neq \emptyset \quad \text{implies that} \quad V(\mathbb{Q}) \neq \emptyset.$$

Here, the product is over all places ν of $\mathbb{Q}$ and we set $\mathbb{Q}_\infty = \mathbb{R}$.

In this paper we are interested in the distribution and the size of the rational points on V (or integer points on V when the system is not assumed to be homogeneous). More specifically, let $\mathcal{V}_\mathbb{Z}$ be an integral model of V. Let $\mathbb{A}^\infty$ be the adele ring of $\mathbb{Q}$ outside the place ∞ and let $\mathcal{V}_{\mathbb{A}^\infty}$ be the base change of $\mathcal{V}_\mathbb{Z}$ to $\mathbb{A}^\infty$. We say that V satisfies *strong approximation outside* ∞ if the image of the diagonal map $\mathcal{V}_\mathbb{Z} \to \mathcal{V}_{\mathbb{A}^\infty}$ is dense. Note that strong approximation outside ∞ implies the smooth Hasse principle. For V as in Birch's theorem strong approximation outside ∞ holds. Theorem 3.11 below is a quantitative version of this statement, which is a first step in understanding the distribution of the integer zeros of arbitrary systems of integer polynomials. This result follows directly from our main theorem

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stated in Section 1.2. In order to obtain this result we generalise the work of Birch [2] to find quantitative asymptotics (in terms of the maximum of the absolute value of the coefficients of these polynomials) for the number of integer zeros of this system within a growing box. Using a quantitative version of the Nullstellensatz, we obtain an upper bound on the smallest non-trivial common zero of $\underline{f}$.

1.1. Related work. There are many improvements on Birch's result if we restrict to a single form. For example, Heath-Brown [13] showed that a cubic form has a non-trivial integer zero provided only that $n \geq 14$. Assuming that the variety V is non-singular, a form of degree 2, 3, or 4 satisfies the smooth Hasse principle provided that $n \geq 3$, $n \geq 9$ or $n \geq 40$ respectively [12, 14, 11]. Browning and Prendiville [5] slightly relaxed condition (1), by showing that for a form of degree $d \geq 3$ the smooth Hasse principle holds provided that $n - \dim V^* \geq \big(d - \frac{1}{2}\sqrt{d}\big)2^d$.

Recent results by Rydin Myerson [17, 18] improve on Birch's result for systems of forms when V is a complete intersection (which is implied by (1) in Birch's theorem). He shows that under this condition one can replace condition (1) by $n \geq 9r$ respectively $n \geq 25r$ for systems of degree 2 respectively 3.

Unconditional improvements include the observation that $\dim V^*$ can be replaced by a smaller quantity Δ, to be defined in (2), as shown independently by Dietmann and Schindler [9, 20]. Another improvement is the observation by Schmidt [23] that the assumption that the system of polynomials is homogeneous is not necessary. We make use of these improvements in this work.

There are known results on the smallest zero of a single form in many variables. Let $\Lambda(f)$ be the smallest integer zero of a form $f \in \mathbb{Z}[x_1, \ldots, x_n]$ with coefficients bounded in absolute value by C. For $d = 2$, Cassels [6, 7] showed that

$$\Lambda(f) \leq c_n C^{(n-1)/2},$$

where the constant c_n is explicit and depends only on n. This estimate has the best possible exponent, i.e. for all n and C, there exists an $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and a constant d_n only depending on n such that $\Lambda(f) \geq d_n C^{(n-1)/2}$. However, for generic quadratic forms one can do much better [3]. Sardari [19] proved an optimal strong approximation theorem for $f - N$, where f is a non-degenerate quadratic form and N a sufficiently large integer.

If $d = 3$, the best possible exponent for sufficiently large n is smaller than the exponent $(n-1)/2$ in Cassels' result. Browning, Dietmann and Elliott [4] showed that $\Lambda(f) \leq cC^{360000}$ for some absolute constant c provided $n \geq 17$, whereas by a result due to Pitman [16], for any $\varepsilon > 0$ and sufficiently large n one has $\Lambda(f) \leq c_{n,\varepsilon} C^{25/6+\varepsilon}$ for some constant $c_{n,\varepsilon}$. In case the hypersurface corresponding to f has at most isolated ordinary singularities, the former authors provide visibly better bounds, e.g. $\Lambda(f) \le cC^{1071}$ for $n = 17$. In fact, Browning, Dietmann and Elliott [4] wonder whether their ideas "could be adapted to handle non-singular forms of degree exceeding 3" analogous to "the extension of Birch [2] to higher degree of Davenport's treatment of cubic forms [8]". The main result of the present paper is that this is indeed possible, although their method to achieve effective lower bounds for the singular series and integral is completely different from ours.

1.2. Main result. Let $\underline{\tilde{f}}$ denote the top degree part of the system $\underline{f}$. Let V and $\widetilde{V}$ denote the affine and projective variety corresponding to $\underline{f}$ and $\underline{\tilde{f}}$ respectively. Let C and $\widetilde{C}$ be the (real) maximum of the absolute value of the coefficients of $\underline{f}$ and $\underline{\tilde{f}}$ respectively. For any $\underline{b} \in \mathbb{Z}^r$, we let $\tilde{f}_{\underline{b}} = b_1\tilde{f}_1 + \cdots + b_r\tilde{f}_r$. For a form g we let $\operatorname{Sing}(g)$ be the singular locus of g in affine space. Define the quantity Δ of Dietmann and Schindler and define K by

$$\Delta = \max_{\underline{b}\in\mathbb{Z}^r\setminus\{0\}} (\dim \operatorname{Sing}(\tilde{f}_{\underline{b}})) \quad \text{and} \quad K = \frac{n-\Delta}{2^{d-1}}. \tag{2}$$

Throughout this work we assume that

$$K > r(r+1)(d-1). \tag{3}$$

In particular, V is a complete intersection. Note that (3) corresponds to Birch's assumption on the number of variables (i.e. (1)) after replacing the dimension of the Birch singular locus by Δ.

Our main theorem, which is proven in Section 3.4, makes Birch's result, stated in the second sentence of this paper, quantitative in terms of C and $\widetilde{C}$:

Theorem 1.1. *Let $f_i \in \mathbb{Z}[\underline{x}] = \mathbb{Z}[x_1,\ldots,x_n]$ for $i = 1,\ldots,r$ be polynomials of degree d such that $K - r(r+1)(d-1) > 0$, $\underline{f}$ has a zero over $\mathbb{Z}_p$ for all primes p, and $\underline{\tilde{f}}$ has a real zero. Assume that the affine and projective varieties V and $\widetilde{V}$ corresponding to $\underline{f}$ and $\underline{\tilde{f}}$ respectively are non-singular. Then there exists an $\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$, polynomially bounded by C and $\widetilde{C}$, such that $\underline{f}(\underline{x}) = \underline{0}$ in fact*

$$\max_{1\le i\le n} |x_i| \le c(C^3\widetilde{C}^2)^{2n^2r^{n+1}(d-1)^n d\cdot\frac{K+r(r+1)(d-1)}{K-r(r+1)(d-1)}}, \tag{4}$$

where the constant c does not depend on C or $\widetilde{C}$.

The case that the system $\underline{f}$ is homogeneous is treated separately in Theorem 3.10.

Remark 1.2. The bound in the above theorem is in no sense believed to be optimal, and is far worse than the known bounds for small degrees discussed in Section 1.1. However, we provided an upper bound in a far more general setting; it has not been shown before that such an upper bound exists in our setting. Moreover, as pointed out in Remark 3.9, in contrast to the bounds for small degrees discussed before, the exponent should grow exponentially in n when $d \geq 4$ is even. The main contribution to this bound is due to our lower bound for the singular series and integral, which follows from a quantitative version of the Nullstellensatz by D'Andrea, Krick and Sombra [1] as discussed in Section 3.1. This theorem, although sharp in general, is not believed to be sharp in the present setting. It would be interesting to explore whether a stronger quantitative version of the Nullstellensatz can be applied in this setting, yielding a significant improvement on (4).

1.3. Structure of this paper. In Section 2 we generalise the work of Birch [2] to deduce an asymptotic formula (quantitative in C and $\widetilde{C}$) for the number of integer points on V within a box $P\mathcal{B}$ for $P \to \infty$, which is the content of Theorem 2.15. Familiarity with Birch's work is not necessary: we prove all results which are direct generalisations of his work. We obtain lower bounds for the singular series and integral (introduced in Sections 2.4 and 2.5 respectively) in Sections 3.2 and 3.3. We end with the proofs of our main theorems in Section 3.4.

1.4. Notation. On the vector space $\mathbb{Q}_p^n$ (p prime or $p = \infty$) we introduce the sup norm $|\underline{\alpha}|_p = \max_{1\leq i\leq n} |\alpha_i|_p$, where $|\cdot|_p$ is the absolute value on $\mathbb{Q}_p$. We write $|\cdot|$ for $|\cdot|_\infty$. For $\beta \in \mathbb{R}$, we let $\|\beta\| = \min_{i\in\mathbb{Z}} |i - \beta|$ be the least distance from β to an integer, and for a point $\underline{\alpha} \in \mathbb{R}^n$ we write $\|\underline{\alpha}\| = \max_{1\leq i\leq n} \|\alpha_i\|$. If $\underline{a} \in \mathbb{Z}^m$ and $q \in \mathbb{Z}$, we abbreviate $\gcd(a_1, \ldots, a_m, q)$ to $\gcd(\underline{a}, q)$. For $x \in \mathbb{R}$ and $q \in \mathbb{Z}$ we write $e(x)$ for $e^{2\pi ix}$ and $e_q(x)$ for $e^{2\pi ix/q}$. For functions f, g defined on a subset of the real numbers we use Vinogradov's notation $f \ll g$ to mean $f = O(g)$. Without a specific indication, the implied constant may depend on n, r and d, but not on C or $\widetilde{C}$.

Let $\mathcal{E}$ denote the box $[-1, 1]^n$ and let $\mathcal{B}$ be an n-dimensional box contained in $\mathcal{E}$ of side length at most 1, i.e. there are $a_j, b_j \in \mathbb{R}$ with $-1 \leq a_j \leq b_j \leq 1$ and $0 < b_j - a_j < 1$ such that $\mathcal{B}$ is given by $\prod_{j=1}^n [a_j, b_j]$. We write sums of the form $\sum_{\underline{x}\in P\mathcal{B}\cap\mathbb{Z}^n}$ as $\sum_{\underline{x}\in P\mathcal{B}}$.

2. A quantitative asymptotic formula for the number of integer zeros

2.1. Estimates of exponential sums. Let $\underline{\alpha} \in [0, 1)^r$ and $\underline{\nu} \in \mathbb{Z}^r$. We obtain estimates for the exponential sums

$$S(\underline{\alpha}) = \sum_{\underline{x}\in P\mathcal{B}} e(\underline{\alpha} \cdot \underline{f}(\underline{x})) \quad \text{and} \quad S(\underline{\alpha}, \underline{\nu}) = S(\underline{\alpha})e(-\underline{\alpha} \cdot \underline{\nu})$$

depending on $\alpha_1, \ldots, \alpha_r$ not being too well approximable by rational numbers with small denominators.

Let $M(P,\underline{\nu})$ denote the number of integer points in the box $P\mathcal{B}$ satisfying $\underline{f}(\underline{x}) = \underline{\nu}$. This counting function is the principal object of study in the first part of this work. Observe that

$$M(P,\underline{\nu}) = \int\limits_{\underline{\alpha}\in[0,1)^r} S(\underline{\alpha},\underline{\nu})\,\mathrm{d}\underline{\alpha}. \tag{5}$$

The following lemma, generalising [20, Lemma 2], enables us to split the right-hand side of (5) into a main term and an error term.

LEMMA 2.1. *Let $\varepsilon > 0$ and $0 < \theta < 1$. One of the following holds:*

(i) $|S(\underline{\alpha})| \ll P^{n-K\theta+\varepsilon}$;
(ii) (rational approximation to $\underline{\alpha}$ with respect to the parameter θ) *there are $\underline{a} \in \mathbb{Z}^r$ and $q \in \mathbb{Z}_{>0}$ such that* $\gcd(\underline{a},q) = 1$,

$$2|q\underline{\alpha} - \underline{a}| < \widetilde{C}^{r-1}P^{-d+r(d-1)\theta} \quad \textit{and} \quad q < \widetilde{C}^r P^{r(d-1)\theta}.$$

Proof. Following [20, proof of Lemma 2], let $\Gamma_i(\underline{x}^{(1)},\dots,\underline{x}^{(d)})$ for $1 \le i \le r$ be the multilinear form associated to $\tilde{f}_i$, satisfying $\Gamma_i(\underline{x},\dots,\underline{x}) = d!\tilde{f}_i(\underline{x})$. Let $N(P^\xi, P^{-\eta};\underline{\alpha})$ be the number of integer vectors $\underline{x}^{(i)} \in \mathbb{Z}^n$ for $2 \le i \le d$ such that $|\underline{x}^{(i)}| \le P^\xi$ and

$$\Big\|\sum_{i=1}^r \alpha_i\Gamma_i(\underline{e}_j,\underline{x}^{(2)},\dots,\underline{x}^{(d)})\Big\| < P^{-\eta} \quad \text{for all } 1 \le j \le n.$$

We introduce an $r \times nN(P^\theta, P^{-d+(d-1)\theta};\underline{\alpha})$ matrix Ψ, the rank of which enables us to distinguish between two cases. The entries of Ψ are given by $\Gamma_i(\underline{e}_j,\underline{x}^{(2)},\dots,\underline{x}^{(d)})$ where the rows of Ψ are indexed by i and the columns by $(j,\underline{x}^{(2)},\dots,\underline{x}^{(d)})$ for $1 \le j \le n$ and $\underline{x}^{(2)},\dots,\underline{x}^{(d)}$ running over all vectors counted by $N(P^\theta, P^{-d+(d-1)\theta};\underline{\alpha})$.

CASE 1: $\operatorname{rank}\Psi < r$. In this case, there exists a $\underline{b} \in \mathbb{Z}^r \setminus \{\underline{0}\}$ such that $\sum_{i=1}^r b_i\Psi_{i,l} = 0$ for all l. In particular, the system of equations

$$\sum_{i=1}^r b_i\Gamma_i(\underline{e}_j,\underline{x}^{(2)},\dots,\underline{x}^{(d)}) = 0 \quad (1 \le j \le n)$$

has at least $N(P^\theta, P^{-d+(d-1)\theta};\underline{\alpha})$ integer solutions $\underline{x}^{(i)} \in \mathbb{Z}^n$ for $2 \le i \le d$ with $|\underline{x}^{(i)}| \le P^\theta$. On the other hand, the number of solutions is $\ll P^{(d-2)n\theta+\Delta\theta+\varepsilon}$ by [20, the last equation on p. 212 and the third equation on p. 216].

Now, suppose k is such that $|S(\underline{\alpha})| > P^{n-k}$. Then [2, Lemma 2.4] implies

$$N(P^\theta, P^{-d+(d-1)\theta};\underline{\alpha}) \gg P^{(d-1)n\theta-2^{d-1}k-\varepsilon},$$

which follows from Weyl's inequality and Davenport's application of the geometry of numbers. This estimate is independent of the coefficients of $\underline{f}$, i.e. the implied constant does not depend on C or $\widetilde{C}$. Hence,

$$P^{(d-2)n\theta+\Delta\theta+\varepsilon} \gg P^{(d-1)n\theta-2^{d-1}k-\varepsilon},$$

so that $k \geq K\theta$. Therefore,

$$|S(\underline{\alpha})| < P^{n-K\theta+\varepsilon}.$$

CASE 2: $\operatorname{rank}\Psi = r$. In this case there is an $r \times r$ submatrix of Ψ of full rank, which we denote by $\hat{\Psi}$. Let $\underline{p_l}$ denote the lth column of $\hat{\Psi}^{-1}$. We now define

$$q = \det \hat{\Psi}, \qquad \underline{a} = q \sum_{l=1}^{r} \underline{p_l}\big((\hat{\Psi}\underline{\alpha})_l - \|(\hat{\Psi}\underline{\alpha})_l\|\big).$$

As $\hat{\Psi}$ is an integer matrix and $(q\underline{p_l})_{l=1}^r$ is the adjoint of $\hat{\Psi}$, it follows that $\underline{a} \in (\mathbb{Z}_{\geq 0})^r$ and $q \in \mathbb{Z}$. Moreover, q is non-zero as $\hat{\Psi}$ has full rank. After removing a common factor of q and the a_i, the conditions $\gcd(q, \underline{a}) = 1$ and $q > 0$ are satisfied. We check that $\underline{a}$ and q satisfy the remaining properties of clause (ii). As every entry of Ψ can be estimated by $\widetilde{C}P^{(d-1)\theta}$, we find

$$q \ll \widetilde{C}^r P^{r(d-1)\theta}. \tag{6}$$

Moreover,

$$|q\alpha_i - a_i| = |q|\Big|\sum_{l=1}^{r} (\hat{\Psi}^{-1})_{i,l}\|(\hat{\Psi}\underline{\alpha})_l\|\Big|.$$

By Cramer's rule, $q(\hat{\Psi}^{-1})_{i,l} \ll \widetilde{C}^{r-1}P^{(r-1)(d-1)\theta}$ and one has $\|(\hat{\Psi}\underline{\alpha})_l\| \ll P^{-d+(d-1)\theta}$ by construction of Ψ. Hence,

$$|q\alpha_i - a_i| \ll \widetilde{C}^{r-1}P^{(r-1)(d-1)\theta}P^{-d+(d-1)\theta}. \tag{7}$$

Finally, note that by scaling θ, the implied constants in (6) and (7) can be transferred to the implied constant in (i). ∎

For $\underline{a} \in \mathbb{Z}^r$ and $q \in \mathbb{Z}_{>0}$, let

$$S_{\underline{a},q} = \sum_{\underline{x}\,(q)} e_q(\underline{a}\cdot \underline{f}(\underline{x})), \qquad S_{\underline{a},q}(\underline{\nu}) = S_{\underline{a},q}e_q(-\underline{a}\cdot\underline{\nu}).$$

Here, the summation is over a complete set of residues modulo q for every vector component of $\underline{x}$. The following lemma, which is a corollary of Lemma 2.1 and generalises [2, Lemma 5.4], will be useful when we define the singular series in terms of $S_{\underline{a},q}$ in Section 2.4.

LEMMA 2.2. *For every $\varepsilon > 0$ we have*

$$|S_{\underline{a},q}| \ll \widetilde{C}^{K/(d-1)}q^{n-K/(r(d-1))+\varepsilon}. \tag{8}$$

Proof. Observe that $S_{\underline{a}',q'}$ is a particular case of $S(\underline{\alpha})$ with $P = q'$ and $\underline{\alpha} = \underline{a}'/q'$. Take θ such that $r(d-1)\theta < 1 - \log_q(\widetilde{C}^r)$. Then the inequalities in Lemma 2.1(ii) read

$$|q\underline{a}' - q'\underline{a}| < q'^{2-d}, \quad 1 \le q < q'.$$

As $d \ge 2$, this system has no solutions, hence (i) is satisfied. ∎

2.2. Minor arcs. Let

$$\mathcal{M}(\theta) = \{(\underline{a}, q) \in \mathbb{Z}^r_{\ge 0} \times \mathbb{Z}_{>0} : \gcd(\underline{a}, q) = 1,\ |\underline{a}| \le q < \widetilde{C}^r P^{r(d-1)\theta}\}$$

be the set of all pairs $(\underline{a}, q)$ occurring in Lemma 2.1(ii). Given $(\underline{a}, q) \in \mathbb{Z}^r \times \mathbb{Z}_{>0}$ and $0 < \theta \le 1$, define a *major arc* by

$$\mathfrak{M}_{\underline{a},q}(\theta) = \{\alpha \in [0,1)^r : 2|q\underline{\alpha} - \underline{a}| < \widetilde{C}^{r-1} P^{-d+r(d-1)\theta}\}.$$

Then, define *the major arcs* to be

$$\mathfrak{M}(\theta) = \bigcup_{(\underline{a},q) \in \mathcal{M}(\theta)} \mathfrak{M}_{\underline{a},q}(\theta). \tag{9}$$

Observe that $\mathfrak{M}(\theta)$ consists of all $\underline{\alpha}$ satisfying (ii) of Lemma 2.1. Define *the minor arcs* by $\mathfrak{m} = [0,1)^r \setminus \mathfrak{M}$. The contribution of $\underline{\alpha} \in \mathfrak{m}$ to the integral in (5) will be considered as an error term. In order to estimate this error term, we first estimate the volume of $\mathfrak{M}(\theta)$, generalising [2, Lemma 4.2]:

Lemma 2.3. *The major arcs $\mathfrak{M}(\theta)$ have volume at most*

$$\widetilde{C}^{r^2} P^{-rd+r(r+1)(d-1)\theta}.$$

Proof. Each major arc $\mathfrak{M}_{\underline{a},q}(\theta)$ has volume $(q^{-1}\widetilde{C}^{r-1}P^{-d+r(d-1)\theta})^r$. As $\mathfrak{M}(\theta)$ is the (not necessarily disjoint) union of major arcs, an upper bound for the volume of $\mathfrak{M}(\theta)$ is given by

$$\sum_{(\underline{a},q) \in \mathcal{M}(\theta)} (q^{-1}\widetilde{C}^{r-1}P^{-d+r(d-1)\theta})^r. \ ∎$$

If the major arcs are disjoint, we can write

$$\int_{\mathfrak{M}(\theta)} S(\underline{\alpha}, \underline{\nu})\, \mathrm{d}\alpha = \sum_{(\underline{a},q) \in \mathcal{M}(\theta)} \int_{\mathfrak{M}_{\underline{a},q}(\theta)} S(\underline{\alpha}, \underline{\nu})\, \mathrm{d}\alpha.$$

This is the case for θ small enough, which generalises [2, Lemma 4.1]:

Lemma 2.4. *Suppose θ is such that $d > 2r(d-1)\theta + (2r-1)\log_P(\widetilde{C})$. Then $\mathfrak{M}(\theta)$ in (9) is a disjoint union of $\mathfrak{M}_{\underline{a},q}(\theta)$.*

Proof. Suppose that α lies in distinct sets $\mathfrak{M}_{\underline{b},q}(\theta)$ and $\mathfrak{M}_{\underline{b}',q'}(\theta)$. It follows that there is an i such that $b_i/q \ne b_i'/q'$. Then

$$1 \le |b_i'q - q'b_i| \le q|q'\alpha_i - b_i'| + q'|q\alpha_i - b_i| < \widetilde{C}^{2r-1}P^{-d+2r(d-1)\theta},$$

which contradicts our assumption on θ. ∎

Now, take major arcs $\mathfrak{M}(\theta_0)$, where $0 < \theta_0 < 1$, $0 < \delta < 1$ and η are such that

$$\eta = r(d-1)\theta_0, \tag{10}$$

$$d > 2\eta + (2r-1)\log_P(\widetilde{C}), \tag{11}$$

$$\frac{K}{r(d-1)} - (r+1) > \delta\eta^{-1}. \tag{12}$$

Observe that assumption (12) is a quantitative version of our main assumption (3). By (11) the major arcs $\mathfrak{M}_{a,q}(\theta_0)$ are disjoint. Later, we choose η and δ satisfying (11) and (12). From now on, write $\mathfrak{M}_{\underline{a},q}$ for $\mathfrak{M}_{\underline{a},q}(\theta_0)$.

We use Birch's idea of a sliding scale to bound $S(\underline{\alpha},\underline{\nu})$ on the minor arcs. Note that the estimate

$$S(\underline{\alpha}) \ll P^{n-K\theta+\varepsilon}$$

for $\alpha \in \mathfrak{m}$ is the stronger the larger θ is. Therefore, in order to show that $\int_{\mathfrak{m}} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha}$ is negligible, we let θ depend on $\underline{\alpha}$. For most $\underline{\alpha}$, we take θ large and have a strong estimate for $|S(\underline{\alpha})|$. When this estimate is invalid, we have to use a smaller value of θ, but we have the compensation that this only happens for a set of α of small measure by the previous lemma. So, the worse the estimate for $|S(\underline{\alpha})|$, the smaller the set of $\underline{\alpha}$ for which it is necessary to use this estimate. Hence, we find the following generalisation of [2, Lemma 4.4]:

LEMMA 2.5.

$$\int_{\mathfrak{m}} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} = O(\widetilde{C}^{r^2}P^{n-rd-\delta}),$$

where O does not depend on C or $\widetilde{C}$.

Proof. First, observe that $|S(\underline{\alpha},\underline{\nu})| = |S(\underline{\alpha})|$. Let $\varepsilon > 0$ be small. Now, define a sequence

$$\theta_T > \theta_{T-1} > \cdots > \theta_1 > \theta = \theta_0 > 0 \tag{13}$$

such that

$$(r+1)(d-1)\theta_T = 2d,$$

$$r(r+1)(d-1)(\theta_{t+1}-\theta_t) < \delta\varepsilon \quad \text{for } 0 \le t \le T-1. \tag{14}$$

This can be done with $T \ll P^{\delta\varepsilon}$ (independent of $\widetilde{C}$).

By Lemma 2.1 and as $-K\theta_T + \varepsilon < -2rd$ by (3), we find

$$\int_{\underline{\alpha}\notin\mathfrak{M}(\theta_T)} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} \ll P^{n-2rd}.$$

By Lemmata 2.3 and 2.1,

$$\int\limits_{\mathfrak{M}(\theta_{t+1})-\mathfrak{M}(\theta_t)} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} \le \int\limits_{\mathfrak{M}(\theta_{t+1})} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} \ll \widetilde{C}^{r^2} P^{-rd+r(r+1)(d-1)\theta_{t+1}} P^{n-K\theta_t+\varepsilon}.$$

By (14), (12), (10) and (13) we have

$$r(r+1)(d-1)\theta_{t+1} - K\theta_t + \varepsilon < -\delta - \delta\varepsilon,$$

so that

$$\int\limits_{\mathfrak{M}(\theta_{t+1})-\mathfrak{M}(\theta_t)} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} \ll \widetilde{C}^{r^2} P^{n-rd-\delta-\delta\varepsilon}.$$

Therefore,

$$\int\limits_{\alpha\notin\mathfrak{M}(\theta_0)} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} = \int\limits_{\alpha\notin\mathfrak{M}(\theta_T)} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} + \sum_{t=0}^{T-1} \int\limits_{\mathfrak{M}(\theta_{t+1})-\mathfrak{M}(\theta_t)} |S(\underline{\alpha},\underline{\nu})|\,\mathrm{d}\underline{\alpha} \ll P^{n-2rd} + P^{\delta\varepsilon}\widetilde{C}^{r^2} P^{n-rd-\delta-\delta\varepsilon}. \qquad \blacksquare$$

Since for our choice of δ and θ_0 the major arcs are disjoint, we obtain the following generalisation of [2, Lemma 4.5].

COROLLARY 2.6.

$$M(P,\underline{\nu}) = \sum_{(\underline{a},q)\in\mathcal{M}(\theta_0)} \int\limits_{\mathfrak{M}_{\underline{a},q}} S(\underline{\alpha},\underline{\nu})\,\mathrm{d}\alpha + O(\widetilde{C}^{r^2} P^{n-rd-\delta}).$$

2.3. Approximating exponential sums by integrals. Given $\underline{\alpha}$ in $\mathfrak{M}_{\underline{a},q}$, we let $\underline{\beta} = \underline{\alpha} - \underline{a}/q$. Similarly, given $\underline{x} \in P\mathcal{B} \cap \mathbb{Z}^n$, we let $\underline{z} = \underline{x} - q\underline{y}$ for $\underline{y} \in \mathbb{Z}^n$ such that $0 \le z_i < q$ for all i. Then

$$S(\underline{\alpha},\underline{\nu}) = \sum_{\underline{z}(q)} \sum_{\underline{z}+q\underline{y}\in P\mathcal{B}} e\big(\underline{\alpha}\cdot(\underline{f}(\underline{z}+q\underline{y}) - \underline{\nu})\big) = \sum_{\underline{z}(q)} e_q\big(\underline{a}\cdot(\underline{f}(\underline{z}) - \underline{\nu})\big) \sum_{\underline{z}+q\underline{y}\in P\mathcal{B}} e\big(\underline{\beta}\cdot(\underline{f}(\underline{z}+q\underline{y}) - \underline{\nu})\big). \tag{15}$$

The following lemma replaces the sum

$$\sum_{\underline{z}+q\underline{y}\in P\mathcal{B}} e(\underline{\beta}\cdot\underline{f}(\underline{z}+q\underline{y}))$$

by an integral. For a measurable subset $\mathcal{C}$ of $\mathcal{E}$ and $\underline{\gamma} \in \mathbb{R}^r$, we write

$$I(\mathcal{C},\underline{\gamma}) = \int\limits_{\underline{\zeta}\in\mathcal{C}} e(\underline{\gamma}\cdot\underline{\tilde{f}}(\underline{\zeta}))\,\mathrm{d}\underline{\zeta}. \tag{16}$$

LEMMA 2.7. *Given $\underline{z} \in \mathbb{Z}^n$, $\underline{\beta} \in \mathbb{Z}^r$ and $q \in \mathbb{Z}_{>0}$, we have*

$$\sum_{\underline{z}+q\underline{y}\in P\mathcal{B}} e(\underline{\beta}\cdot\underline{f}(\underline{z}+q\underline{y})) = I(\mathcal{B},P^d\underline{\beta})\frac{P^n}{q^n} + O\biggl((C|P^d\underline{\beta}|+1)\frac{P^{n-1}}{q^{n-1}}\biggr). \tag{17}$$

Proof. For the system of polynomials $\underline{r} = \underline{f} - \tilde{\underline{f}}$ of degree at most $d-1$ we have

$$|e(\underline{\beta}\cdot\underline{r}(\underline{z}+q\underline{y})) - 1| \ll |\underline{\beta}|\,|\underline{r}(\underline{z}+q\underline{y}))| \ll |\underline{\beta}|CP^{d-1},$$

where we assume that $\underline{z}+q\underline{y} \in P\mathcal{B}$. There are $O((P/q)^n)$ values of $\underline{y}$ in the sum, hence

$$\sum_{\underline{z}+q\underline{y}\in P\mathcal{B}} e(\underline{\beta}\cdot\underline{f}(\underline{z}+q\underline{y})) = \sum_{\underline{z}+q\underline{y}\in P\mathcal{B}} e(\underline{\beta}\cdot\tilde{\underline{f}}(\underline{z}+q\underline{y})) + O\biggl(|\underline{\beta}|C\frac{P^{n+d-1}}{q^n}\biggr).$$

Next, we replace the sum on the right-hand side by the integral

$$\int_{\underline{z}+q\underline{\omega}\in P\mathcal{B}} e(\underline{\beta}\cdot\tilde{\underline{f}}(\underline{z}+q\underline{\omega}))\,\mathrm{d}\underline{\omega}. \tag{18}$$

The edges of the cube of summation and integration have length P/q. In the replacement of the sum by the integral, we have an error of at most $\ll (P/q)^{n-1}$ coming from the boundaries. The variation in $e(\underline{\beta}\cdot\tilde{\underline{f}}(\underline{z}+q\underline{y}))$ results in an error of at most $O(|\underline{\beta}|q\widetilde{C}P^{d-1}(P/q)^n)$. Hence, the total error in (17) is

$$\ll |\underline{\beta}|C\frac{P^{n+d-1}}{q^n} + |\underline{\beta}|\widetilde{C}\frac{P^{n+d-1}}{q^{n-1}} + \frac{P^{n-1}}{q^{n-1}} \ll (C|P^d\underline{\beta}|+1)\frac{P^{n-1}}{q^{n-1}}.$$

Applying the substitution $\underline{z}+q\underline{\omega} = P\underline{\zeta}$ to (18) gives the desired result. ∎

COROLLARY 2.8. *Given $\underline{z} \in \mathbb{Z}^n$ and $\underline{\alpha} \in \mathfrak{M}_{\underline{a},q}$ with $\underline{\beta} = \underline{\alpha} - \underline{a}/q$, we have*

$$\sum_{\underline{z}+q\underline{y}\in P\mathcal{B}} e(\underline{\beta}\cdot\underline{f}(\underline{z}+q\underline{y})) = I(\mathcal{B},P^d\underline{\beta})\frac{P^n}{q^n} + O\biggl(C\widetilde{C}^{r-1}\frac{P^{n+\eta-1}}{q^n}\biggr).$$

Proof. Estimate the error term in Lemma 2.7 by observing that for $\underline{\alpha}$ in $\mathfrak{M}_{\underline{a},q}$ we have $|P^d\underline{\beta}| \le \widetilde{C}^{r-1}q^{-1}P^\eta$ and $1 \le \widetilde{C}^r q^{-1}P^\eta$. ∎

Recall $S_{\underline{a},q}(\underline{\nu})$ is defined by (2.1). Applying the corollary to (15) we obtain (compare with [2, Lemma 5.1]):

COROLLARY 2.9. *Let $\underline{\alpha} = \underline{a}/q + \underline{\beta} \in \mathfrak{M}_{\underline{a},q}$. Then*

$$S(\underline{\alpha},\underline{\nu}) = q^{-n}S_{\underline{a},q}(\underline{\nu})\cdot I(\mathcal{B},P^d\underline{\beta})\cdot e(-\underline{\beta}\cdot\underline{\nu})\cdot P^n + O(C\widetilde{C}^{r-1}P^{n+\eta-1}).$$

In the next two sections we study the singular series and singular integral which will be obtained by putting together $q^{-n}S_{\underline{a},q}(\underline{\nu})$ and $I(\mathcal{B},P^d\underline{\beta})$ respectively for all $\underline{\alpha} \in \mathfrak{M}$.

2.4. Singular series. Define the *singular series* by

$$\mathfrak{S}(\underline{\nu}) = \sum_{q=1}^{\infty} q^{-n} \sum_{\substack{\underline{a}(q) \\ \gcd(\underline{a},q)=1}} S_{\underline{a},q}(\underline{\nu}).$$

It converges absolutely under assumption (12) on K. This is made quantitative in the following lemma, generalising [2, p. 256]:

LEMMA 2.10. *For all $\tau \geq 0$ we have*

$$\sum_{q>P^{\tau\eta}} q^{-n} \sum_{\substack{\underline{a}(q) \\ \gcd(\underline{a},q)=1}} |S_{\underline{a},q}(\underline{\nu})| \ll \widetilde{C}^{K/(d-1)} P^{-\tau\delta}.$$

Proof. Observe that $|S_{\underline{a},q}(\underline{\nu})| = |S_{\underline{a},q}|$. By Lemma 2.2 and (12), we have

$$\sum_{q>P^{\tau\eta}} q^{-n} \sum_{\substack{\underline{a}(q) \\ \gcd(\underline{a},q)=1}} |S_{\underline{a},q}(\underline{\nu})| \ll \sum_{q>P^{\tau\eta}} q^{-n} \sum_{\substack{\underline{a}(q) \\ \gcd(\underline{a},q)=1}} \widetilde{C}^{K/(d-1)} q^{n-K/(r(d-1))+\varepsilon}$$
$$\ll \widetilde{C}^{K/(d-1)} \sum_{q>P^{\tau\eta}} q^{-1-\delta\eta^{-1}} \ll \widetilde{C}^{K/(d-1)} P^{-\tau\delta}. \quad \blacksquare$$

2.5. Singular integral. For $T \in \mathbb{R}$, define a continuous function $J_T : \mathbb{R}^r \to \mathbb{R}$ by

$$J_T(\underline{\mu}) = \int_{|\underline{\gamma}|\leq T} I(\mathcal{B}, \underline{\gamma}) e(-\underline{\gamma} \cdot \underline{\mu}) \, \mathrm{d}\underline{\gamma},$$

where $I(\mathcal{B}, \underline{\gamma})$ is defined by (16). The next two lemmata, generalising [2, Lemmata 5.2 and 5.3], show that the sequence $(J_T)_{T\in\mathbb{N}}$ converges uniformly in $\underline{\mu}$ to a function $J(\underline{\mu})$ which we call the *singular integral.*

LEMMA 2.11. *For all $\varepsilon > 0$ one has*

$$|I(\mathcal{B}, \underline{\gamma})| \ll \min\bigl(1, (\widetilde{C}^{1-r}|\underline{\gamma}|)^{-r-1-\delta\eta^{-1}} (\widetilde{C}|\underline{\gamma}|)^{\varepsilon}\bigr).$$

Proof. $|I(\mathcal{B}, \underline{\gamma})| \ll 1$ follows directly. Therefore, in proving the second part of the inequality we may assume that

$$\widetilde{C}^{1-r}|\gamma| > 1. \tag{19}$$

Take $P = \widetilde{C}|\underline{\gamma}|(\widetilde{C}^{1-r}|\underline{\gamma}|)^{K/(r(d-1))}$. By (19) and $d \geq 2$ we find that $P > (\widetilde{C}|\underline{\gamma}|)^{2/d}$. Hence, for $\underline{\alpha} = P^{-d}\underline{\gamma}$ we have $|\underline{\alpha}| < (\widetilde{C}P^d)^{-1/2}$. Let φ satisfy

$$|\underline{\alpha}| = \widetilde{C}^{r-1} P^{-d+r(d-1)\varphi}.$$

Then by Lemma 2.4 we find that $\mathfrak{M}(\varphi)$ is given as a disjoint union of $\mathfrak{M}_{\underline{a},q}(\varphi)$. Observe that $\underline{\alpha}$ lies on the boundary of the open set $\mathfrak{M}_{\underline{0},1}(\varphi)$. Hence, $\underline{\alpha}$ is not in $\mathfrak{M}(\varphi)$. Lemma 2.1 now implies that

$$|S(\underline{\alpha})| \ll P^{n+\varepsilon} (\widetilde{C}^{1-r} P^d |\underline{\alpha}|)^{-K/(r(d-1))}. \tag{20}$$

On the other hand, by Lemma 2.7 with $\underline{z}=\underline{0}$, $q=1$, $\underline{a}=\underline{0}$, we obtain

$$S(\underline{\alpha})=\sum_{\underline{y}\in P\mathcal{B}} e(\underline{\alpha}\cdot\underline{f}(\underline{y}))=I(\mathcal{B},P^d\underline{\alpha})P^n+O((\widetilde{C}|P^d\underline{\alpha}|+1)P^{n-1}). \tag{21}$$

Hence, combining (20) and (21) yields

$$|I(\mathcal{B},\underline{\gamma})|\ll(\widetilde{C}^{1-r}|\underline{\gamma}|)^{-K/(r(d-1))}(\widetilde{C}|\underline{\gamma}|)^{\varepsilon}.$$

Estimating $K/(r(d-1))$ by $r+1+\delta\eta^{-1}$ using (12) completes the proof. ∎

LEMMA 2.12. *If $T_2>T_1$, then for all $\varepsilon>0$ one has*

$$|J_{T_1}(\underline{\mu})-J_{T_2}(\underline{\mu})|\ll\widetilde{C}^{r^2-1+(r-1)\delta\eta^{-1}+\varepsilon}T_1^{-1-\delta\eta^{-1}+\varepsilon}.$$

Proof. Using Lemma 2.11 we find

$$\begin{aligned}
J_{T_2}(\underline{\mu})-J_{T_1}(\underline{\mu}) &= \int\limits_{T_1\le|\underline{\gamma}|\le T_2} I(\mathcal{B},\underline{\gamma})e(-\underline{\gamma}\cdot\underline{\mu})\,\mathrm{d}\underline{\gamma}\\
&\ll \int\limits_{T_1\le|\underline{\gamma}|\le T_2} (\widetilde{C}^{1-r}|\underline{\gamma}|)^{-r-1-\delta\eta^{-1}}(\widetilde{C}|\underline{\gamma}|)^{\varepsilon}\,\mathrm{d}\underline{\gamma}\\
&\ll \int_{T_1}^{T_2}\widetilde{C}^{r^2-1+(r-1)\delta\eta^{-1}+\varepsilon}\Gamma^{-r-1-\delta\eta^{-1}+\varepsilon}\Gamma^{r-1}\,\mathrm{d}\Gamma\\
&\ll \widetilde{C}^{r^2-1+(r-1)\delta\eta^{-1}+\varepsilon}T_1^{-1-\delta\eta^{-1}+\varepsilon}. \quad ∎
\end{aligned}$$

Taking the limit as $T_2\to\infty$ we obtain

$$J_{T_1}(\underline{\mu})-J(\underline{\mu})\ll\widetilde{C}^{r^2-1+(r-1)\delta\eta^{-1}+\varepsilon}T_1^{-1-\delta\eta^{-1}+\varepsilon}. \tag{22}$$

This implies the following upper bound for $J(\underline{\mu})$:

COROLLARY 2.13. *For all $\underline{\mu}\in\mathbb{Z}^r$ and $\varepsilon>0$ we have*

$$J(\underline{\mu})\ll\widetilde{C}^{r(r-1)+\varepsilon}.$$

Proof. By the trivial bound in Lemma 2.11 we have $J_{\widetilde{C}^{r-1}}(\underline{\mu})\ll\widetilde{C}^{r(r-1)}$. By (22) we see that $J(\underline{\mu})-J_{\widetilde{C}^{r-1}}(\underline{\mu})\ll\widetilde{C}^{r(r-1)+\varepsilon}$. The result follows by the triangle inequality. ∎

2.6. Major arcs. Combining the results in the previous sections, we obtain a quantitative asymptotic theorem for the number of integer zeros of $\underline{f}-\underline{\nu}$ in a box $P\mathcal{B}$, generalising [2, Lemma 5.5]:

LEMMA 2.14.

$$\frac{M(P,\underline{\nu})}{P^{n-rd}}=\mathfrak{S}(\underline{\nu})J(P^{-d}\underline{\nu})+\mathfrak{O}_1+\mathfrak{O}_2,$$

where

$$\mathfrak{O}_1=O(C\widetilde{C}^{2r^2+r-1}P^{-1+2(r+1)\eta})\quad\text{and}\quad\mathfrak{O}_2=O(\widetilde{C}^{K/(d-1)+r^2-r+\varepsilon}P^{-\delta}).$$

Proof. By Corollary 2.6 we have

$$M(P,\underline{\nu}) = \sum_{(\underline{a},q)\in\mathcal{M}(\theta_0)} \int_{\mathfrak{M}_{\underline{a},q}} S(\underline{\alpha},\underline{\nu})\,\mathrm{d}\underline{\alpha} + O(\widetilde{C}^{r^2}P^{n-rd-\delta}).$$

Note that $K/(d-1)-r+\varepsilon>0$. Let $\underline{\beta}=\underline{\alpha}-\underline{a}/q$. Then

$$M(P,\underline{\nu}) = \sum_{(\underline{a},q)\in\mathcal{M}(\theta_0)} \int_{\mathfrak{M}_{\underline{0},q}} S(\underline{a}/q+\underline{\beta},\underline{\nu})\,\mathrm{d}\underline{\beta} + P^{n-rd}\mathfrak{O}_2.$$

As $|S_{\underline{a},q}(\underline{\nu})|\le q^n$ and $|\mathcal{M}(\theta_0)|\le(\widetilde{C}^rP^\eta)^{r+1}$,

$$\sum_{(\underline{a},q)\in\mathcal{M}(\theta_0)} \frac{|S_{\underline{a},q}(\underline{\nu})|}{q^n}C\widetilde{C}^{r-1}P^{n+\eta-1}\int_{\mathfrak{M}_{\underline{0},q}}\mathrm{d}\underline{\beta} = P^{n-rd}\mathfrak{O}_1.$$

Hence, Corollary 2.9 implies that

$$\frac{M(P,\underline{\nu})}{P^{n-rd}} = P^{rd}\sum_{(\underline{a},q)\in\mathcal{M}(\theta_0)} \frac{S_{\underline{a},q}(\underline{\nu})}{q^n}J_{\widetilde{C}^{r-1}P^\eta}(P^{-d}\underline{\nu}) + \mathfrak{O}_1+\mathfrak{O}_2. \tag{23}$$

By Lemma 2.10 for $\tau=0$ we find

$$\sum_{(\underline{a},q)\in\mathcal{M}(\theta_0)} \frac{|S_{\underline{a},q}(\underline{\nu})|}{q^n}\widetilde{C}^{r^2-1+(r-1)\delta\eta^{-1}+\varepsilon}(\widetilde{C}^{r-1}P^\eta)^{-1-\delta\eta^{-1}+\varepsilon} = \mathfrak{O}_2.$$

Hence, using (22) for $T_1=\widetilde{C}^{r-1}P^\eta$ to rewrite (23) we obtain

$$\frac{M(P,\underline{\nu})}{P^{n-rd}} = \sum_{(\underline{a},q)\in\mathcal{M}(\theta_0)} \frac{S_{\underline{a},q}(\underline{\nu})}{q^n}J(P^{-d}\underline{\nu}) + \mathfrak{O}_1+\mathfrak{O}_2.$$

By Lemma 2.10 for $\tau=1$ and Corollary 2.13 we can plug in the singular series and obtain

$$\begin{aligned}\frac{M(P,\underline{\nu})}{P^{n-rd}} &= (\mathfrak{S}(\underline{\nu})+O(\widetilde{C}^{K/(d-1)}P^{-\delta}))J(P^{-d}\underline{\nu})+\mathfrak{O}_1+\mathfrak{O}_2\\ &= \mathfrak{S}(\underline{\nu})J(P^{-d}\underline{\nu})+\mathfrak{O}_1+\mathfrak{O}_2. \ \blacksquare\end{aligned}$$

THEOREM 2.15.

$$M(P,\underline{\nu}) = P^{n-rd}\mathfrak{S}(\underline{\nu})J(P^{-d}\underline{\nu}) + O(C\widetilde{C}^{K/(d-1)+r^2-1}P^{n-rd-\delta}),$$

where

$$\delta < \frac{K-r(r+1)(d-1)}{K+r(r+1)(d-1)}. \tag{24}$$

Proof. Let δ be as in (24) and let

$$\eta = \frac{r(d-1)}{K+r(r+1)(d-1)}.$$

As $\eta<\frac{1}{r+1}$, condition (11) is satisfied for P large enough (e.g. $P>\widetilde{C}^{4r/d}$). Moreover, (12) is satisfied. As $-1+2(r+1)\eta<-\delta$ and $r^2+r<K/(d-1)$,

both error terms $\mathfrak{O}_1$ and $\mathfrak{O}_2$ in Lemma 2.14 are $O(C\widetilde{C}^{K/(d-1)+r^2-1}P^{-\delta})$. The statement now follows from that result. ∎

3. Quantitative strong approximation

3.1. The Nullstellensatz for non-singular varieties. From now on assume that V and $\widetilde{V}$ are non-singular over $\overline{\mathbb{Q}}$ as affine and projective varieties respectively (see Section 1.2 for both our main assumption (3) ([1]) and the definition of V and $\widetilde{V}$). Then, if $\underline{f}$ has a zero modulo a prime power, we can invoke Hensel's lemma to find more zeros modulo higher powers of the same prime. The real analogue of this statement is the implicit function theorem around a zero of $\underline{\tilde{f}}$. These ideas can be used to deduce known results on the non-vanishing of the singular series and integral provided the existence of local zeros. In Proposition 3.4 and Corollary 3.8 we prove this in a quantitative manner. In order to do so, we need to control the minors of the Jacobian matrices of $\underline{f}$ and $\underline{\tilde{f}}$, for which we use a quantitative version of the Nullstellensatz, as explained below.

Let I be a subset of $[n] := \{1, \dots, n\}$ of size r and let $\Delta_I(\underline{x})$ be the $r \times r$ minor of the Jacobian matrix of $\underline{f}$ (of dimensions $r \times n$) with columns given by the elements of I. Similarly, let $\tilde{\Delta}_I(\underline{x})$ be the $r \times r$ minor of the Jacobian matrix of $\underline{\tilde{f}}$ with columns given by the elements of I.

Consider the polynomials $\underline{f}$ and all $r \times r$ minors Δ_I. As V is non-singular, these polynomials have no common zero over $\overline{\mathbb{Q}}$. Hence, by the Nullstellensatz, the ideal generated by these polynomials equals $\overline{\mathbb{Q}}[\underline{x}]$. This is made quantitative in [1, Theorem 2] (we take $V = \mathbb{A}^n(\overline{\mathbb{Q}}), g = 1$ and $s = r + \binom{n}{r}$ in that result): there exists an $N \in \mathbb{Z}_{>0}$ and polynomials $g_1, \dots, g_r$ and g_I in $\mathbb{Z}[\underline{x}]$ for all $I \subset [n]$ with $|I| = r$ such that

$$\sum_{i=1}^{r} f_i(\underline{x})g_i(\underline{x}) + \sum_I \Delta_I(\underline{x})g_I(\underline{x}) = N, \tag{25}$$

satisfying the estimate $\log(N) - 2(n+1)r^n(d-1)^{n-1}d\log(C^r) \ll 1$. Hence,

$$N \ll C^{2(n+1)r^{n+1}(d-1)^{n-1}d} = \mathfrak{C}, \tag{26}$$

where the above equality defines $\mathfrak{C}$.

For the projective variety $\widetilde{V}$ we have a similar reasoning for every affine patch of $\widetilde{V}$ obtained by setting one of the coordinates x_j equal to 1. Let $1 \le j \le n$ be given. Because $\widetilde{V}$ is non-singular over $\overline{\mathbb{Q}}$, we find $\tilde{N}_j \in \mathbb{Z}_{>0}$ and polynomials $\tilde{g}_{1,j}, \dots, \tilde{g}_{r,j}$ and $\tilde{g}_{I,j}$ in $\mathbb{Z}[\underline{x}]$ for all $I \subset [n]$ with $|I| = r$ such

([1]) In fact, for the lower bounds for the singular series and singular integral it suffices to assume $n \ge r$, but we need (3) again in Section 3.4.

that

$$\sum_{i=1}^{r} \tilde{f}_i(\underline{x})\tilde{g}_{i,j}(\underline{x}) + \sum_{I} \tilde{\Delta}_I(\underline{x})\tilde{g}_{I,j}(\underline{x}) = \tilde{N}_j \tag{27}$$

for all $\underline{x}$ with $x_j = 1$. Let $\|g\|_\infty$ denote the *height* of a polynomial g, that is, the maximum of the absolute values of the coefficients of g. Then, by the same result [1, Theorem 2], we have

$$\log(\|\tilde{g}_{I,j}\|_\infty) - 2nr^{n-1}(d-1)^{n-2}d\log(\widetilde{C}^r) \ll 1$$

for all $I \subset [n]$ with $|I| = r$. Hence,

$$\|\tilde{g}_{I,j}\|_\infty \ll \widetilde{C}^{2nr^n(d-1)^{n-2}d} = \tilde{\mathfrak{C}}, \tag{28}$$

where the above equation defines $\tilde{\mathfrak{C}}$. Let $\tilde{N} = \min_{j=1}^n \tilde{N}_j$.

3.2. Lower bound for the singular series. As usual, for each prime p define the *local density* at p to be

$$\sigma_p(\underline{\nu}) = \lim_{m\to\infty} \frac{\#\{\underline{x} \bmod p^m : \underline{f}(\underline{x}) \equiv \underline{\nu} \bmod p^m\}}{p^{m(n-r)}}. \tag{29}$$

Observe that

$$p^{m(n-r)} \sum_{k=0}^{m} \sum_{\substack{\underline{a}(p^k) \\ \gcd(\underline{a},p)=1}} p^{-kn} S_{\underline{a},p^k}(\underline{\nu})$$

is the number of points satisfying $\underline{f}(\underline{x}) = \underline{\nu} \bmod p^m$. So, equivalently we could have defined

$$\sigma_p(\underline{\nu}) = \sum_{k=0}^{\infty} \sum_{\substack{\underline{a}(p^k) \\ \gcd(\underline{a},p)=1}} p^{-kn} S_{\underline{a},p^k}(\underline{\nu}).$$

Then, by multiplicativity of $S_{\underline{a},q}$, we can factorise the singular series as a product over the local densities, i.e. $\mathfrak{S}(\underline{\nu}) = \prod_{p \text{ prime}} \sigma_p(\underline{\nu})$. The rest of this section is devoted to providing quantitative lower bounds for the local densities and singular series, using the ideas described in the previous section.

LEMMA 3.1. *If there exists a non-singular solution $\underline{x}_0 \in \mathbb{Z}_p^n$ to $\underline{f}(\underline{x}_0) = \underline{\nu}$, then*

$$\sigma_p(\underline{\nu}) \ge \Big(p^{-1} \max_I |\Delta_I(\underline{x}_0)|_p^2\Big)^{n-r}.$$

Proof. Take $e \in \mathbb{Z}$ such that $\max_I |\Delta_I(\underline{x}_0)|_p = p^{-e}$ and assume that $m > 2e+1$. The non-singular solution $\underline{x}_0 \in \mathbb{Z}_p^n$ gives a non-singular solution $\underline{a}$ modulo p^{2e+1}. Using Hensel's lemma (see, for example, [10, Proposition 5.21 and Note 5.22]), we can lift this solution to at least $p^{(n-r)(m-2e-1)}$ non-singular solutions of $\underline{f}(\underline{x}) \equiv \underline{\nu} \bmod p^m$. Hence, by (29) the result follows. ∎

Lemma 3.2. *For all primes p for which there exists a solution $\underline{x}_0 \in \mathbb{Z}_p^n$ of $\underline{f}(\underline{x}_0) = \underline{0}$ we have*

$$\max_I |\Delta_I(\underline{x}_0)|_p \geq |N|_p.$$

Proof. Let p be a prime such that there is an $\underline{x}_0 \in \mathbb{Z}_p^n$ with $\underline{f}(\underline{x}_0) = \underline{0}$, so that the first set of terms on the left-hand side of (25) vanish for $\underline{x} = \underline{x}_0$. Then taking p-adic absolute values in (25) shows that

$$\max_I |\Delta_I(\underline{x}_0)|_p \max_I |g_I(\underline{x}_0)|_p \geq |N|_p.$$

As $g_I \in \mathbb{Z}[\underline{x}]$, we obtain $\max_I |\Delta_I(\underline{x}_0)|_p \geq |N|_p$. ∎

Lemma 3.3. *If p is prime such that $p \nmid d$ and $p \nmid N$, then*

$$\sigma_p(\underline{0}) - 1 \ll p^{-n/2+r+\varepsilon}. \tag{30}$$

Proof. Suppose V is singular over $\mathbb{F}_p$. Then there exists an $\underline{x} \in \mathbb{F}_p^n$ such that $\underline{f}(\underline{x}) = \underline{0}$ and $\Delta_I(\underline{x}) = 0$ over $\mathbb{F}_p$ for all $I \subset [n]$ with $|I| = r$. Considering (25) over $\mathbb{F}_p$, we deduce that $N \equiv 0 \bmod p$. This contradicts our assumption, so V is non-singular over $\mathbb{F}_p$.

As pointed out by Schmidt [22], a result of Deligne, worked out in [24, Appendice], then shows that

$$\#V_{\mathbb{F}_p}(\underline{0}) = p^{n-r} + O(p^{n/2+\varepsilon})$$

provided $p \nmid d$, where the implied constant depends at most on n and d. Observe that if $\underline{x} \in \mathbb{Z}^n$ is a solution of $\underline{f}(\underline{x}) = \underline{0} \bmod p^e$ for some $e \in \mathbb{Z}_{>0}$, then $\underline{x}$ reduces to a non-singular point on $V_{\mathbb{F}_p}$. Hence, $\underline{x} \bmod p^e$ can be obtained by lifting a point of $V_{\mathbb{F}_p}$. We conclude that

$$\#\{\underline{x} \bmod p^m : \underline{f}(\underline{x}) \equiv \underline{\nu} \bmod p^m\} = p^{m(n-r)} + O(p^{(m-1)(n-r)+n/2+\varepsilon}).$$

Using (29), we obtain (30). ∎

Proposition 3.4. *Suppose that for each prime p there exists a solution $\underline{x}_0 \in \mathbb{Z}_p^n$ to $\underline{f}(\underline{x}_0) = \underline{0}$. Then*

$$\mathfrak{S}(\underline{0}) \gg N^{-3(n-r)}.$$

Proof. Let S be the finite set of primes for which $p \mid dN$. Applying Lemmata 3.1 and 3.2 and using the product formula for $|\cdot|_p$ we obtain

$$\prod_{p\in S} \sigma_p(\underline{0}) \geq \prod_{p\in S} (p^{-1}|N|_p^2)^{n-r} \gg (N^{-1}N^{-2})^{n-r} = N^{3(r-n)}.$$

It follows from Lemma 3.3 that

$$\prod_{p\notin S} \sigma_p(\underline{0}) = \prod_{p\notin S} 1 + O(p^{-n/2+r+\varepsilon}) \gg 1,$$

where the implied constant does not depend on C. We conclude that

$$\mathfrak{S}(\underline{0}) = \prod_{p\in S} \sigma_p(\underline{0}) \prod_{p\notin S} \sigma_p(\underline{0}) \gg N^{-3(n-r)}. \blacksquare$$

3.3. Lower bound for the singular integral. The following lemma is a quantitative version of the implicit function theorem. We use this result to prove Lemma 3.6, which is the real analogue of Lemma 3.1. Recall that $\tilde{\Delta}_I(\underline{x})$ is the $r\times r$ minor of the Jacobian of $\tilde{f}$ with columns determined by I. Abbreviate $\tilde{\Delta}_{\{1,\dots,r\}}(\underline{x})$ to $\tilde{\Delta}(\underline{x})$.

LEMMA 3.5. *Given $\underline{x}_0 \in \mathbb{R}^n$ with $|\underline{x}_0| \le 1$, assume that*

$$M := \max_{I\subset[n],\, |I|=r} |\tilde{\Delta}_I(\underline{x}_0)| = |\tilde{\Delta}(\underline{x}_0)| > 0.$$

Let $\underline{g}:\mathbb{R}^n\to\mathbb{R}^n$ be given by

$$\underline{g} : \underline{x} \mapsto (\tilde{f}_1(\underline{x}),\dots,\tilde{f}_r(\underline{x}), x_{r+1},\dots,x_n).$$

Then there are open subsets $U\subset\mathbb{R}^n$ and $W\subset\mathbb{R}^n$ with $\underline{x}_0\in U$ and $\underline{g}(\underline{x}_0) \in W$ such that $\underline{g}$ is a bijection from U to W and has differentiable inverse $\underline{g}^{-1}: W\to U$ satisfying $\det((\underline{g}^{-1})') \ge M^{-1}$. *Furthermore, one may choose*

$$W = \{\underline{y}\in\mathbb{R}^n : |\underline{g}(\underline{x}_0) - \underline{y}| < M^2/\widetilde{C}^{2r-1}\}.$$

Proof. We explicitly find a small open neighbourhood of $\underline{x}_0$ in which the implicit function theorem is applicable, following the proof of [25, Theorem 2.11] or [15, Lemma 9.3]. Note that $M \ll \widetilde{C}^r$. Let U be the closed rectangle given by

$$U = \{\underline{x}\in\mathbb{R}^n : |\underline{x}-\underline{x}_0| \le aM/\widetilde{C}^r\},$$

for a sufficiently small constant $a\in\mathbb{R}$ depending only on d, n and r. Then for $\underline{x}\in U$ we have $|\underline{x}| \le |\underline{x}-\underline{x}_0| + |\underline{x}_0| \ll 1$. Hence, for $\underline{x}\in U$ one finds that $\frac{\partial g_i}{\partial x_j x_k}(\underline{x}) \ll \widetilde{C}$ for all $1\le i,j,k\le n$. It follows that

$$\left|\frac{\partial g_i}{\partial x_j}(\underline{x}) - \frac{\partial g_i}{\partial x_j}(\underline{x}_0)\right| \ll \widetilde{C}|\underline{x}-\underline{x}_0| \ll a\widetilde{C}^{1-r}M.$$

Let $D\underline{g}$ be the Jacobian matrix of $\underline{g}$ and write $\underline{g}'(\underline{x}) = \underline{g}(\underline{x}) - D\underline{g}(\underline{x}_0)\cdot\underline{x}$. As

$$\frac{\partial(\underline{g}'(\underline{x}))_i}{\partial x_j} = \frac{\partial g_i}{\partial x_j}(\underline{x}) - \frac{\partial g_i}{\partial x_j}(\underline{x}_0),$$

for $\underline{x}_1, \underline{x}_2 \in U$ we have

$$|\underline{g}'(\underline{x}_1) - \underline{g}'(\underline{x}_2)| \ll a\widetilde{C}^{1-r}M|\underline{x}_1-\underline{x}_2|. \tag{31}$$

Given an invertible $n\times n$ matrix A, let $|A| = \max_{i,j}|A_{i,j}|$ be the max norm. For all $\underline{h}\in\mathbb{R}^n$ one has $|\underline{h}| \le \frac{|\mathrm{adj}(A)|}{\det A}|A\underline{h}|$ with $\mathrm{adj}(A)$ the adjugate of A. Now take $A = D\underline{g}(\underline{x}_0)$. Then $|\mathrm{adj}(A)| \ll \widetilde{C}^{r-1}$. Since $M = |\tilde{\Delta}(\underline{x}_0)| = |\det(D\underline{g}(\underline{x}_0))|$, for $\underline{x}_1, \underline{x}_2 \in U$ we have

$$|D\underline{g}(\underline{x}_0)(\underline{x}_1 - \underline{x}_2)| \gg \widetilde{C}^{1-r} M |\underline{x}_1 - \underline{x}_2|.$$

Hence,

$$|\underline{g}'(\underline{x}_1) - \underline{g}'(\underline{x}_2)| + |\underline{g}(\underline{x}_1) - \underline{g}(\underline{x}_2)| \geq |D\underline{g}(\underline{x}_0)(\underline{x}_1 - \underline{x}_2)| \gg \widetilde{C}^{1-r} M |\underline{x}_1 - \underline{x}_2|.$$

Therefore, using (31) for a small enough, we find for all $\underline{x}_1, \underline{x}_2 \in U$ that

$$|\underline{g}(\underline{x}_1) - \underline{g}(\underline{x}_2)| \gg \widetilde{C}^{1-r} M |\underline{x}_1 - \underline{x}_2|.$$

This implies that if $\underline{x}$ is on the boundary of U then

$$|\underline{g}(\underline{x}) - \underline{g}(\underline{x}_0)| \gg \widetilde{C}^{1-r} M |\underline{x} - \underline{x}_0| = aM^2/\widetilde{C}^{2r-1}.$$

Now set $b = aM^2/\widetilde{C}^{2r-1}$ so that for $\underline{x}$ on the boundary of U we have $|\underline{g}(\underline{x}) - \underline{g}(\underline{x}_0)| \gg b$, and define

$$W = \{\underline{y} \in \mathbb{R}^n : |\underline{y} - \underline{g}(\underline{x}_0)| < \tfrac{1}{2} b\}.$$

The proof of [15, Lemma 9.3] ensures that W has the required properties (after shrinking U). ∎

LEMMA 3.6. *Suppose that* $\underline{x}_0 \in \mathbb{R}^n$ *with* $|\underline{x}_0| \leq 1$ *satisfies* $\underline{\tilde{f}}(\underline{x}_0) = \underline{0}$ *and that* $M = \max_{I \subset [n],\, |I| = r} |\tilde{\Delta}_I(\underline{x}_0)| > 0$. *Then*

$$J(\underline{0}) \gg M^{-1} (M^2/\widetilde{C}^{2r-1})^{n-r}.$$

Proof. In [21, Paragraph 11], Schmidt shows that for $\underline{\mu} \in \mathbb{R}^r$ we have

$$J(\underline{\mu}) = \lim_{t \to \infty} t^r \int\limits_{|\underline{\tilde{f}}(\underline{x}) - \underline{\mu}| \leq t^{-1}} \prod_{i=1}^{r} (1 - t|\tilde{f}_i(\underline{x}) - \mu_i|) \, \mathrm{d}\underline{x}.$$

Let $\mathbb{1}_{1/(2t)} : \mathbb{R} \to \{0, 1\}$ be the characteristic function of the interval $\left[-\frac{1}{2t}, \frac{1}{2t}\right]$. Let U, W, g be as in Lemma 3.5. Then

$$J(\underline{0}) \geq \lim_{t \to \infty} \left(\frac{t}{2}\right)^r \int\limits_U \prod_{i=1}^{r} \mathbb{1}_{1/(2t)} \circ \tilde{f}_i(\underline{x}) \, \mathrm{d}\underline{x}.$$

Applying the change of variables as in Lemma 3.5 we obtain

$$\begin{aligned}
\int\limits_U \prod_{i=1}^{r} \mathbb{1}_{1/(2t)} \circ \tilde{f}_i(\underline{x}) \, \mathrm{d}\underline{x} &= \int\limits_W |\det((g^{-1})')| \prod_{i=1}^{r} \mathbb{1}_{1/(2t)}(y_i) \, \mathrm{d}\underline{y} \\
&\geq \int\limits_W M^{-1} \prod_{i=1}^{r} \mathbb{1}_{1/(2t)}(y_i) \, \mathrm{d}\underline{y}.
\end{aligned}$$

As for t sufficiently large, $\mathbb{1}_{1/(2t)} \equiv 0$ outside W, the theorem follows. ∎

LEMMA 3.7. *Let $\underline{x}_0 \in \mathbb{R}^n$ be such that $|\underline{x}_0| = 1$ and $\underline{\tilde{f}}(\underline{x}_0) = \underline{0}$. Then for some $1 \le j \le n$ one has*

$$\max_I |\tilde{\Delta}_I(\underline{x}_0)| \gg \tilde{\mathfrak{C}}^{-1}\tilde{N}_j.$$

Proof. This is essentially the same proof as that of Lemma 3.2. Substitute $\underline{x} = \underline{x}_0$ in (27) for a choice of j such that $(x_0)_j = |\underline{x}_0| = 1$. Then the first sum vanishes and we find that

$$\max_I |\tilde{\Delta}_I(\underline{x}_0)| \max_I |\tilde{g}_{I,j}(\underline{x}_0)| \gg |\tilde{N}_j|.$$

Note that $\tilde{g}_{I,j}(\underline{x}_0) \ll \|\tilde{g}_{I,j}\|_\infty \ll \widetilde{\mathfrak{C}}$. This implies that

$$\max_I |\tilde{\Delta}_I(\underline{x}_0)| \gg \widetilde{\mathfrak{C}}^{-1}\tilde{N}_j. \blacksquare$$

COROLLARY 3.8. *Suppose $\underline{\tilde{f}}$ has a real zero. Then*

$$J(\underline{0}) \gg \frac{\tilde{N}^{2(n-r)-1}}{\tilde{\mathfrak{C}}^{2(n-r)-1}\widetilde{C}^{(2r-1)(n-r)}}.$$

Proof. Observe that by homogeneity of $\underline{\tilde{f}}$ we can assume that the non-singular real zero $\underline{x}_0$ satisfies $|\underline{x}_0| = 1$. The corollary then follows directly from Lemmata 3.6 and 3.7. $\blacksquare$

3.4. Main theorems

Proof of Theorem 1.1. From Theorem 2.15, it follows that for P satisfying

$$P \gg \left(\frac{C\widetilde{C}^{K/(d-1)+r^2-1}}{\mathfrak{S}(\underline{0})J(\underline{0})}\right)^{1/\delta}$$

we have $M(P, \underline{0}) > 1$ (if the implied constant is large enough). Hence, there exists a non-trivial integer zero $\underline{x}$ of $\underline{f}$ with $|\underline{x}| \le P$.

By Proposition 3.4, Corollary 3.8, (26) and $\tilde{N} \ge 1$ it follows that

$$\begin{aligned}\mathfrak{S}(\underline{0})J(\underline{0}) &\gg \tilde{\mathfrak{C}}^{-(2(n-r)-1)}\widetilde{C}^{-2r(n-r)}\left(\frac{\tilde{N}^2}{N^3}\right)^{n-r}\tilde{N}^{-1} \\ &\gg \mathfrak{C}^{-3(n-r)}\tilde{\mathfrak{C}}^{-(2(n-r)-1)}\widetilde{C}^{-(2r-1)(n-r)}.\end{aligned}$$

Using $(n+1)(n-r) < n^2 - r$ one finds

$$\frac{C\widetilde{C}^{K/(d-1)+r^2-1}}{\mathfrak{S}(\underline{0})J(\underline{0})} \ll \mathfrak{C}^{\frac{3n^2}{n+1}}\tilde{\mathfrak{C}}^{\frac{2n^2}{n+1}}\frac{C}{\mathfrak{C}^{\frac{r}{n+1}}}\frac{\widetilde{C}^{K/(d-1)+(2r-1)(n-r)+r^2-1}}{\tilde{\mathfrak{C}}^{1+\frac{r}{n+1}}}$$

As $K \le n/2^{d-1}$, the two fractions on the right-hand side are bounded by 1, so one can take

$$P \gg (C^3\widetilde{C}^2)^{2n^2r^{n+1}(d-1)^n d\cdot\frac{K+r(r+1)(d-1)}{K-r(r+1)(d-1)}}. \blacksquare$$

Remark 3.9. The upper bound (4) should be compared with the following example generalising Kneser's example in [7]. Suppose $d \geq 4$ is even and let

$$f(\underline{x}) = x_1^d - \sum_{i=1}^{n-1} (x_{i+1} - c x_i^{d/2})^2 - 1$$

for some $c \in \mathbb{N}$. Then $C = \widetilde{C} = c^2$, f is non-singular, and $\underline{x}$ given by $x_i = c^{\sum_{j=1}^{i-1} (d/2)^{i-1}}$ is a zero of f. If $\underline{a}$ is a non-trivial integer zero of $\underline{f}$, then clearly $a_1 \neq 0$. Moreover,

$$|a_{i+1} - c a_i^{d/2}| \leq |a_1|^{d/2}$$

for all $i = 1, \ldots, n-1$. Inductively one can show that

$$|a_i| \gg c^{(d/2)^{i-2}} |a_1| \qquad (2 \leq i \leq n),$$

where the implied constant only depends on i. Hence, in case $r = 1$ and $d \geq 4$ is even, the right-hand side of (4) is at least $C^{(d/2)^{n-2}/2}$. Note that—in contrast to the cases $d = 2$ and $d = 3$—the exponent of C grows exponentially in n.

We can do slightly better than Theorem 1.1 in case we add the assumption that the polynomials $\underline{f}$ are homogeneous:

Theorem 3.10. *Suppose $\tilde{f}_i \in \mathbb{Z}[\underline{x}]$ for $i = 1, \ldots, r$ are homogeneous polynomials of degree d such that $K - r(r+1)(d-1) > 0$, $\underline{\tilde{f}}$ has a zero over $\mathbb{Z}_p$ for all primes p and a real zero. Assume that the corresponding projective variety $\widetilde{V}$ is non-singular. Then there exists an $\underline{x} \in \mathbb{Z}^n \setminus \{0\}$, polynomially bounded by C and $\widetilde{C}$, such that $\tilde{f}(\underline{x}) = \underline{0}$, more precisely*

$$|\underline{x}| \ll \widetilde{C}^{6n^2 r^n (d-1)^{n-2} d \cdot \frac{K + r(r+1)(d-1)}{K - r(r+1)(d-1)}}.$$

Proof. As in the proof of Theorem 1.1 (with $C = \widetilde{C}$) we use the fact that for P satisfying

$$P \gg \left(\frac{\widetilde{C}^{K/(d-1) + r^2}}{\mathfrak{S}(\underline{0}) J(\underline{0})} \right)^{1/\delta}$$

we have $M(P, \underline{0}) > 1$ (if the implied constant is large enough). Moreover, the quantitative version of the Nullstellensatz for $\underline{\tilde{f}}$ given in (28) does still hold. Hence, mutatis mutandis, the proof of Proposition 3.4 applies and we find that $\mathfrak{S}(\underline{0}) \geq \tilde{N}^{-3(n-r)}$. Together with Corollary 3.8 and (26) it follows that

$$\mathfrak{S}(\underline{0}) J(\underline{0}) \gg \tilde{\mathfrak{C}}^{-(2(n-r)-1)} \widetilde{C}^{-2r(n-r)} \tilde{N}^{-n+r-1} \gg \tilde{\mathfrak{C}}^{-(3(n-r)-2)} \widetilde{C}^{-2r(n-r)}.$$

One finds that one can take

$$P \gg \widetilde{C}^{6n(n-r) r^n (d-1)^{n-2} d \cdot \frac{K + r(r+1)(d-1)}{K - r(r+1)(d-1)}}. \blacksquare$$

As already indicated in the introduction, we provide a quantitative strong approximation theorem for systems $\underline{f}$ satisfying the same conditions as in Theorem 1.1. Call $\underline{x} \in \mathbb{R}^n$ *totally positive* if $x_i > 0$ for all i.

THEOREM 3.11. *Let $\underline{m}, \underline{M} \in \mathbb{Z}^n$. Suppose $f_i \in \mathbb{Z}[\underline{x}]$ for $i = 1, \dots, r$ are polynomials of degree d such that $K - r(r+1)(d-1) > 0$ and the corresponding varieties V and $\widetilde{V}$ are non-singular affine respectively projective. Suppose that for every prime p there exists a zero $\underline{y} \in \mathbb{Z}_p$ of $\underline{f}$ satisfying $y_i \equiv m_i \bmod M_i$ and suppose $\underline{\tilde{f}}$ has a totally positive real zero. Then there exists an $\underline{x} \in \mathbb{Z}^n_{>0}$, polynomially bounded by C and $\widetilde{C}$, such that*

$$\underline{f}(\underline{x}) = \underline{0} \quad \textit{and} \quad x_i \equiv m_i \bmod M_i$$

and

$$|\underline{x}| \ll (|\underline{M}|^{5d} C^3 \widetilde{C}^2)^{2n^2 r^{n+1}(d-1)^{n-1} d \cdot \frac{K + r(r+1)(d-1)}{K - r(r+1)(d-1)}},$$

where the implied constant does not depend on C, $\widetilde{C}$, $\underline{m}$ or $\underline{M}$.

Proof. Let

$$\underline{g}(\underline{y}) = \underline{f}(\underline{My} + \underline{m}) \quad \text{and} \quad \underline{\tilde{g}}(\underline{y}) = [\underline{f}(\underline{My} + \underline{m})]^{\sim} = \underline{\tilde{f}}(\underline{My}),$$

where $(\underline{My})_i = M_i y_i$. Observe that over $\overline{\mathbb{Q}}$ the system $\underline{f}$ is non-singular if and only if $\underline{g}$ is non-singular and similarly for $\widetilde{\underline{f}}$. Moreover, the condition on the existence of zeros of $\underline{f}$ ensures that $\underline{g}$ has zeros over $\mathbb{Z}_p$ for all primes p and that $\underline{\tilde{g}}$ has a totally positive zero over $\mathbb{R}$. After scaling, this zero lies in $\mathcal{B} \subset (0,1]^n$. Now, apply Theorem 1.1. The statement follows by noting that the maximal coefficient of $\underline{g}$ and $\underline{\tilde{g}}$ is $\ll |\underline{M}|^d C$ and $\ll |\underline{M}|^d \widetilde{C}$ respectively as we can assume without loss of generality that $|m_i| \le |M_i|$ for all $i = 1, \dots, n$. ∎

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