

The consistency of level by level equivalence with $V = \text{HOD}$, the Ground Axiom, and instances of square and diamond

by

Arthur W. APTER

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Summary. We construct via forcing a model for the level by level equivalence between strong compactness and supercompactness in which both $V = \text{HOD}$ and the Ground Axiom (GA) are true. In our model, various versions of the combinatorial principles \square and \diamond hold. In the model constructed, there are no restrictions on the class of supercompact cardinals.

1. Introduction and preliminaries. In [3], the following theorem was proven.

THEOREM 1. *Let $V \models \text{“ZFC} + \text{GCH} + \text{The class of supercompact cardinals is nonempty”}$. There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models \text{“ZFC} + \text{GCH} + \text{The class of supercompact cardinals is the same as in } V + \text{Level by level equivalence between strong compactness and supercompactness holds”}$. In $V^{\mathbb{P}}$, \square_{γ}^S holds for every infinite cardinal γ , where $S = \text{Safe}(\gamma)$. In addition, in $V^{\mathbb{P}}$, \diamond_{μ} holds for every μ which is inaccessible or the successor of a singular cardinal, and \diamond_{μ}^+ holds for every μ which is the successor of a regular cardinal.*

Quoting from [3], in terminology used by Woodin, this theorem can be classified as an “inner model theorem proven via forcing”. This is since the

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witnessing model satisfies pleasant properties one usually associates with an inner model, namely GCH and many instances of square and diamond, along with a property one might perhaps expect if a “nice” inner model containing supercompact cardinals ever were to be constructed (see [22] for a discussion of this topic), namely level by level equivalence between strong compactness and supercompactness. Theorem 1 can also be considered as a part of S.-D. Friedman’s “outer model program”, as first described in [10].

The purpose of this paper is to continue the outer model program by creating via forcing a model where the class of supercompact cardinals can be arbitrary which is significantly more “inner-model like” than any previously constructed model for level by level equivalence between strong compactness and supercompactness. We will prove a theorem in which we construct a model for the level by level equivalence between strong compactness and supercompactness satisfying $V = \text{HOD}$, the Ground Axiom (GA), and various versions of diamond and square. Specifically, we will establish the following.

THEOREM 2. *Let $V \models \text{“ZFC} + \text{GCH} + \text{The class of supercompact cardinals is nonempty”}$. There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models \text{“ZFC} + \text{GCH} + \text{The class of supercompact cardinals is the same as in } V + \text{Level by level equivalence between strong compactness and supercompactness holds} + V = \text{HOD} + \text{GA”}$. In $V^{\mathbb{P}}$, \square_{γ}^S holds for every infinite cardinal γ , where $S = \text{Safe}(\gamma)$. In addition, in $V^{\mathbb{P}}$, \diamond_{μ} holds for every regular cardinal μ , and \diamond_{μ}^+ holds for every μ such that $\mu = \lambda^+$, λ is regular, but λ is not the successor of a singular cardinal.*

We recall that the *Ground Axiom* (GA), introduced in [11, 20], is the assertion that the set-theoretic universe V is not a forcing extension of any inner model $W \subseteq V$ by nontrivial set forcing $\mathbb{P} \in W$. Certain (although not all) canonical inner models, e.g., L and $L_{\mathcal{U}}$ for κ a measurable cardinal and \mathcal{U} a normal measure over κ , satisfy GA.

We take this opportunity to make a few additional remarks concerning Theorem 2. Although the model witnessing the conclusions of Theorem 2 satisfies the same instances of the kind of square that Theorem 1 does, it does not satisfy exactly the same instances of the different versions of diamond. This is due to the coding partial ordering that we use to force $V = \text{HOD}$, which will decide generically where \diamond_{μ}^* holds or fails whenever μ is the double successor of a singular cardinal. We will be able to infer that \diamond_{μ} holds for every successor cardinal $\mu > \aleph_1$ by Shelah’s celebrated result of [21], since our forcing will preserve GCH. Also, we emphasize that Theorem 2 represents the first time a model for the level by level equivalence between strong compactness and supercompactness has been constructed in which either $V = \text{HOD}$ or GA is true.

We conclude Section 1 with a discussion of some preliminary material. Suppose V is a model of ZFC in which for all regular cardinals $\kappa < \lambda$, κ is λ strongly compact iff κ is λ supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ supercompact. Such a universe will be said to witness *level by level equivalence between strong compactness and supercompactness*. For brevity, we will henceforth abbreviate this as just *level by level equivalence*. The exception is provided by a theorem of Menas [19], who showed that if κ is a measurable limit of cardinals δ which are λ strongly compact, then κ is λ strongly compact but need not be λ supercompact. Any model of ZFC with this property also witnesses the Kimchi–Magidor property [17] that the classes of strongly compact and supercompact cardinals coincide precisely, except at measurable limit points. Models in which GCH and level by level equivalence between strong compactness and supercompactness hold nontrivially were first constructed by Shelah and the author in [5].

We presume a basic knowledge of large cardinals and forcing. A good reference in this regard is [16]. When forcing, $q \geq p$ means that q is *stronger than* p . We will have some slight abuses of notation. In particular, when G is V -generic over \mathbb{P} , we take both $V[G]$ and $V^{\mathbb{P}}$ as being the generic extension of V by \mathbb{P} . We will also, from time to time, confuse terms with the sets they denote and write x when we actually mean \dot{x} or \check{x} . For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. For $\kappa < \lambda$ regular cardinals, $\text{Coll}(\kappa, \lambda)$ is the usual Lévy collapse of all cardinals in the half-open interval $(\kappa, \lambda]$ to κ . For κ a regular cardinal and λ an ordinal, $\text{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ many Cohen subsets of κ . The partial ordering \mathbb{P} is κ -directed closed if every directed set of conditions of size less than κ has an upper bound.

We recall for the benefit of readers the definition given by Hamkins in [15, Section 3] of the lottery sum of a collection of partial orderings. If \mathfrak{A} is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus \mathfrak{A} = \{ \langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A} \text{ and } p \in \mathbb{P} \} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p \rangle \leq \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \leq_{\mathbb{P}} p'$. Intuitively, if G is V -generic over $\oplus \mathfrak{A}$, then G first selects an element of \mathfrak{A} (or as Hamkins says in [15], “holds a lottery among the posets in \mathfrak{A} ”) and then forces with it ⁽¹⁾.

A corollary of Hamkins’ work of [12] on the approximation and cover properties (which is a generalization of his gap forcing results found in [13, 14]) will be employed in the proof of Theorem 2. This corollary follows from [12, Theorems 3, 31, and Corollary 14]. We therefore state as

⁽¹⁾ The terminology “lottery sum” is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names “disjoint sum of partial orderings”, “side-by-side forcing”, and “choosing which partial ordering to force with generically”.

a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [13, 14, 12] when appropriate. Suppose \mathbb{P} is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| \leq \delta$, \mathbb{Q} is nontrivial, and $\Vdash_{\mathbb{Q}} \dot{\mathbb{R}}$ is δ^+ -directed closed". In Hamkins' terminology of [12], \mathbb{P} *admits a closure point at δ* . In Hamkins' terminology of [13, 14, 12], \mathbb{P} is *mild with respect to a cardinal κ* iff every set of ordinals x in $V^{\mathbb{P}}$ of size below κ has a "nice" name τ in V of size below κ , i.e., there is a set y in V , $|y| < \kappa$, such that any ordinal forced by a condition in \mathbb{P} to be in τ is an element of y . Also, as in the terminology of [13, 14, 12] and elsewhere, an embedding $j : V \rightarrow \bar{M}$ is *amenable to V* when $j \restriction A \in V$ for any $A \in V$. The specific corollary of Hamkins' work from [12] we will be using is then the following.

THEOREM 3 (Hamkins). *Suppose that $V[G]$ is a generic extension obtained by forcing with \mathbb{P} that admits a closure point at some regular $\delta < \kappa$. Suppose further that $j : V[G] \rightarrow M[j(G)]$ is an elementary embedding with critical point κ for which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^{\delta} \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \rightarrow M$ is amenable to V . If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V . Finally, if \mathbb{P} is mild with respect to κ and κ is λ strongly compact in $V[G]$ for any $\lambda \geq \kappa$, then κ is λ strongly compact in V .*

It immediately follows from Theorem 3 that any cardinal κ which is λ supercompact in a generic extension obtained by forcing that admits a closure point below κ (such as at ω) must also be λ supercompact in the ground model. In particular, if \bar{V} is a generic extension of V by a partial ordering admitting a closure point at ω in which each supercompact cardinal is preserved, the class of supercompact cardinals in \bar{V} remains the same as in V . In addition, it follows from Theorem 3 that if \mathbb{P} admits a closure point at ω , \mathbb{P} is mild with respect to κ , and $V^{\mathbb{P}} \models \text{"}\kappa \text{ is } \lambda \text{ strongly compact"}$, then it is also true that $V \models \text{"}\kappa \text{ is } \lambda \text{ strongly compact"}$.

We end Section 1 by stating the definitions of the combinatorial notions we will be using. Readers may consult [3] for additional details, from which we will feel free to quote verbatim when appropriate. If κ is a regular uncountable cardinal, \diamond_{κ} is the principle stating that there exists a sequence $\langle S_{\alpha} \mid \alpha < \kappa \rangle$ of sets such that $S_{\alpha} \subseteq \alpha$, with the additional property that for every $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha = S_{\alpha}\}$ is a stationary subset of κ . We recall that if λ is an infinite cardinal then:

1. \diamond_{λ}^* is the assertion that there exists a sequence $\langle S_{\alpha} \mid \alpha < \lambda \rangle$ such that:
 - (a) For every α , S_{α} is a family of subsets of α with $|S_{\alpha}| < \lambda$.
 - (b) For every $X \subseteq \lambda$, there is $C \subseteq \lambda$ a club set such that $\forall \alpha \in C \ X \cap \alpha \in S_{\alpha}$.

2. \diamond_λ^+ is the assertion that there exists a sequence $\langle \mathcal{S}_\alpha \mid \alpha < \lambda \rangle$ such that:
- (a) For every α , \mathcal{S}_α is a family of subsets of α with $|\mathcal{S}_\alpha| < \lambda$.
 - (b) For every $X \subseteq \lambda$, there is $C \subseteq \lambda$ a club set such that $\forall \alpha \in C$, $X \cap \alpha, C \cap \alpha \in \mathcal{S}_\alpha$.

We explicitly make the observation that trivially, by their definitions, $\diamond_\lambda^+ \Rightarrow \diamond_\lambda^*$. This of course means that if \diamond_λ^* fails, then so does \diamond_λ^+ . We will use this explicitly in our coding and in our proof of Lemma 2.3 showing that in the desired generic extension, $V = \text{HOD}$.

We now give a partial version of square, \square_κ^S for κ an uncountable cardinal and $S \subseteq \kappa$ a set of regular cardinals, compatible with supercompact cardinals. As was mentioned in [3, Section 2.1], square sequences of this kind were first shown to be consistent with supercompactness by Foreman and Magidor [9, page 191], using techniques of Baumgartner. In the notation of Definition 1.1 found immediately below, they showed that $\square_{\kappa^+}^{\{\kappa^{++n} \mid n < \omega\}}$ is consistent with κ being supercompact.

Given a set S of regular cardinals, we denote by $\text{cof}(S)$ the class of ordinals α such that $\text{cf}(\alpha) \in S$. We also let REG stand for the class of regular cardinals.

DEFINITION 1.1. Let κ be an infinite cardinal and let S be a set of regular cardinals which are less than or equal to κ . Then a \square_κ^S -sequence is a sequence $\langle C_\alpha \mid \alpha \in \kappa^+ \cap \text{cof}(S) \rangle$ such that:

- C_α is club in α and $\text{ot}(C_\alpha) \leq \kappa$.
- If $\beta \in \lim(C_\alpha) \cap \lim(C_{\alpha'})$ then $C_\alpha \cap \beta = C_{\alpha'} \cap \beta$.

\square_κ^S holds if and only if there is a \square_κ^S -sequence.

DEFINITION 1.2. For each infinite cardinal κ , a regular cardinal μ is *safe* for κ if and only if:

- $\mu \leq \kappa$.
- For every cardinal $\lambda \leq \kappa$, if λ is κ^+ supercompact then $\lambda \leq \mu$.

$\text{Safe}(\kappa)$ is the set of safe regular cardinals for κ .

We note that the safe set is a final segment of $\text{REG} \cap (\kappa + 1)$, and that the safe set can only be empty when κ is a singular limit of cardinals which are κ^+ supercompact. In addition, by the remarks immediately following the statement of Theorem 3, $\text{Safe}(\gamma)$ remains the same in any cardinal and cofinality preserving generic extension by a partial ordering admitting a closure point at ω and preserving all regular instances of supercompactness.

2. The proof of Theorem 2. We turn now to the proof of Theorem 2. Suppose $V \models \text{“ZFC} + \text{GCH} + \text{The class of supercompact cardinals is nonempty”}$. By Theorem 1, we may assume in addition that in V , level

by level equivalence holds, and that \square_κ^S holds for every infinite cardinal κ , where $S = \text{Safe}(\kappa)$.

We are in a position to define the partial ordering \mathbb{P} used in the proof of Theorem 2. We pattern our definition after the iteration found in [3, Section 3.2]. First, for λ a cardinal, let $\mathbb{Q}^\diamond(\lambda^+)$ be the partial ordering of [3, p. 71] which adds $\diamond_{\lambda^+}^+$. For the exact definition, we refer readers to [3]. We note only that $\mathbb{Q}^\diamond(\lambda^+)$ is λ^+ -directed closed, and λ^{++} -c.c. $\mathbb{P} = \langle \langle \mathbb{P}_\delta, \dot{Q}_\delta \rangle \mid \delta \in \text{ORD} \rangle$ may now be defined as the proper class Easton support iteration which begins by forcing with $\text{Add}(\omega, 1)$ (i.e., $\mathbb{P}_0 = \{\emptyset\}$ and $\mathbb{Q}_0 = \text{Add}(\omega, 1)$) so as to ensure a closure point at ω . We then let \dot{Q}_δ be a term for:

- $\text{Add}(\delta, 1)$ for δ inaccessible.
- The lottery sum of $\text{Add}(\delta, \delta^+)$ and $\mathbb{Q}^\diamond(\delta)$ at cardinals δ where $\delta = \lambda^+$ for λ the successor of a singular cardinal (so in particular, $\lambda = \rho^+$ and $\delta = \rho^{++}$ where ρ is a singular cardinal).
- The partial ordering $\mathbb{Q}^\diamond(\delta)$ at cardinals δ where $\delta = \lambda^+$ and λ is a regular cardinal which is not the successor of a singular cardinal (so in particular, either λ is inaccessible, or $\lambda = \rho^+$ where ρ is a regular cardinal).
- Trivial forcing if δ does not fall into any of the above three categories (so in particular, trivial forcing occurs if either δ is a singular cardinal or $\delta = \lambda^+$ and λ is a singular cardinal).

As in [3], we need to show that forcing with \mathbb{P} preserves all regular instances of supercompactness. Standard arguments show that $V^\mathbb{P} \models \text{ZFC}$ and that forcing with \mathbb{P} preserves all cardinals, cofinalities, and GCH. In addition, in analogy to what was done in [3], using the arguments of [1, Lemma 1.1] and [8, Theorem 12.2], it is easily verified that in $V^\mathbb{P}$, \diamond_μ^+ holds for every μ which is inaccessible, and \diamond_μ^+ holds at cardinals μ where $\mu = \lambda^+$ for λ a regular cardinal which is not the successor of a singular cardinal. Since GCH holds in $V^\mathbb{P}$, by Shelah's work of [21], \diamond_μ^+ holds in $V^\mathbb{P}$ for every successor cardinal $\mu > \aleph_1$ and hence holds in $V^\mathbb{P}$ for every regular cardinal μ .

LEMMA 2.1. *If $V \models \text{"}\kappa < \lambda \text{ are such that } \kappa \text{ is } \lambda \text{ supercompact and } \lambda \text{ is regular"}$, then $V^\mathbb{P} \models \text{"}\kappa \text{ is } \lambda \text{ supercompact"}$.*

Proof. Let $\kappa < \lambda$ be as in the hypotheses of Lemma 2.1. If λ is either inaccessible or $\lambda = \delta^+$ where δ is a singular cardinal, then the argument that $V^\mathbb{P} \models \text{"}\kappa \text{ is } \lambda \text{ supercompact"}$ is exactly the same as in [3, Theorem 5, bottom of page 73 up to and including the first three paragraphs on page 74]. If $\lambda = \delta^+$ where δ is a regular cardinal which is not the successor of a singular cardinal, then the argument that $V^\mathbb{P} \models \text{"}\kappa \text{ is } \lambda \text{ supercompact"}$ is exactly the same as in [3, Theorem 5, fourth paragraph on page 74 through the end of the proof on page 76]. Suppose now that $\lambda = \delta^+$ where δ is the successor of a singular cardinal. Assume that $\mathbb{Q}_\lambda^\diamond$ is chosen in the stage λ lottery held in

the definition of \mathbb{P} , and that we are forcing above $p_0 \in \mathbb{P}$ which forces that this is indeed the case. The argument that $V^{\mathbb{P}} \models “\kappa \text{ is } \lambda \text{ supercompact}”$ is then once again exactly the same as in [3, Theorem 5, fourth paragraph on page 74 through the end of the proof on page 76]. If $\text{Add}(\lambda, \lambda^+)$ is chosen in the stage λ lottery held in the definition of \mathbb{P} , then the argument that $V^{\mathbb{P}} \models “\kappa \text{ is } \lambda \text{ supercompact}”$ combines standard techniques with ideas originally due to Magidor [18]. It is sketched in [3, Theorem 5, ninth paragraph on page 74 through line 6 on page 75] and given completely in both [5, pages 119–120] and [2, Lemma 2.4], to which we refer readers for additional details. This completes the proof of Lemma 2.1. ■

Lemma 2.1 immediately implies that all V -supercompact cardinals are preserved to $V^{\mathbb{P}}$. By its definition, we may write $\mathbb{P} = \mathbb{P}' * \dot{\mathbb{P}}''$, where $|\mathbb{P}'| = \omega$, $\mathbb{P}' = \text{Add}(\omega, 1)$ is nontrivial, and $\Vdash_{\mathbb{P}'} “\dot{\mathbb{P}}'' \text{ is } \aleph_1\text{-directed closed}”$. Therefore, by our remarks in the paragraph immediately following Theorem 3, the class of supercompact cardinals remains the same in both V and $V^{\mathbb{P}}$. Further, by the upwards absoluteness of any form of square in a cardinal preserving generic extension (see the discussion given in the proof of [1, Theorem 1]) and our remarks in the last paragraph of Section 1, in $V^{\mathbb{P}}$, \square_{γ}^S holds for every infinite cardinal γ , where $S = \text{Safe}(\gamma)$.

LEMMA 2.2. $V^{\mathbb{P}} \models “\text{Level by level equivalence holds}”$.

Proof. We mimic the proofs of [1, Lemma 1.3] and [3, Lemma 4.1]. Suppose $V^{\mathbb{P}} \models “\kappa < \lambda \text{ are regular cardinals such that } \kappa \text{ is } \lambda \text{ strongly compact and } \kappa \text{ is not a measurable limit of cardinals } \delta \text{ which are } \lambda \text{ supercompact}”$. By Lemma 2.1, any cardinal δ such that δ is λ supercompact in V remains λ supercompact in $V^{\mathbb{P}}$. We may therefore infer that $V \models “\kappa < \lambda \text{ are regular cardinals such that } \kappa \text{ is not a measurable limit of cardinals } \delta \text{ which are } \lambda \text{ supercompact}”$.

By the definition of \mathbb{P} , it is easily seen that \mathbb{P} is mild with respect to κ . Hence, by the factorization of \mathbb{P} given above and Theorem 3, $V \models “\kappa \text{ is } \lambda \text{ strongly compact}”$. Consequently, by level by level equivalence between strong compactness and supercompactness in V , $V \models “\kappa \text{ is } \lambda \text{ supercompact}”$, so another application of Lemma 2.1 shows that $V^{\mathbb{P}} \models “\kappa \text{ is } \lambda \text{ supercompact}”$. This completes the proof of Lemma 2.2. ■

LEMMA 2.3. $V^{\mathbb{P}} \models “V = \text{HOD}”$.

Proof. We use ideas given by Brooke-Taylor in [6, Theorem 9] (see also the proof of [4, Lemma 4.3]). In particular, to show that $V^{\mathbb{P}} \models “V = \text{HOD}”$, it will suffice to show that every set of ordinals in $V^{\mathbb{P}}$ is ordinal definable using a coding oracle given by where $\diamond_{\kappa^{++}}^*$ holds or fails for κ a singular cardinal.

To do this, suppose $p \in \mathbb{P}$, \dot{x} , and α are such that $p \Vdash “\dot{x} \subseteq \alpha”$ and that α is a limit ordinal. By the definition of \mathbb{P} , there must exist some ordinal β

such that $p \Vdash \dot{x} \in V^{\mathbb{P}\beta}$. Also, since \mathbb{P} is an Easton support proper class length iteration, there must be some ordinal γ such that $\text{support}(p) \subseteq \gamma$. This allows us to write $p = \langle p_\sigma \mid \sigma < \gamma \rangle$. Let $\delta > \max(\alpha, \beta, \gamma)$ be fixed but arbitrary, with $\langle \delta_\sigma \mid \sigma < \alpha \rangle$ the first α many singular cardinals greater than δ . Define $\rho = \sup(\{\delta_\sigma \mid \sigma < \alpha\})$. Take $q > p$, $q = \langle q_\sigma \mid \sigma < \rho \rangle$ as the condition such that $q_\sigma = p_\sigma$ for $\sigma < \gamma$. For $\sigma \geq \gamma$, $\sigma < \rho$, q_σ is the trivial condition, except if $\zeta < \alpha$ and $\sigma = \delta_\zeta^{++}$. At such a σ , q_σ is defined as the term such that $\Vdash_{\mathbb{P}_\sigma} "q_\sigma \in \dot{\mathbb{Q}}_\sigma \text{ forces that } \dot{\mathbb{Q}}^\diamond(\sigma) \text{ is chosen in the stage } \sigma \text{ lottery if } \zeta \in \dot{x}, \text{ but } \text{Add}(\sigma, \sigma^+) \text{ is chosen in the stage } \sigma \text{ lottery if } \zeta \notin \dot{x}"$. The definition of \mathbb{P} tells us that in $V^{\mathbb{P}}$, \diamond_σ^+ , and hence also \diamond_σ^* , both hold if $\mathbb{Q}^\diamond(\sigma)$ is chosen in the stage σ lottery. On the other hand, the arguments found in [6, paragraph immediately preceding Section 3, pages 644–645] combined with the closure properties of \mathbb{P} tell us that \diamond_σ^* fails if $\text{Add}(\sigma, \sigma^+)$ is chosen in the stage σ lottery. This means that as in the proofs of [6, Theorem 9] and [4, Lemma 4.3], the proper class of conditions forcing that the set x is coded using the coding oracle mentioned in the first paragraph of the proof of this lemma is dense in \mathbb{P} . Since x is an arbitrary set of ordinals, again as in the proofs of [6, Theorem 9] and [4, Lemma 4.3], $V^{\mathbb{P}} \models "V = \text{HOD}"$. This completes the proof of Lemma 2.3. ■

LEMMA 2.4. $V^{\mathbb{P}} \models \text{GA}$.

Proof. We follow the proofs of [4, Theorem 4.7] and [20, Theorem 10]. Let $V^{\mathbb{P}} = \bar{V}$. Towards a contradiction, suppose that \bar{V} is a set forcing extension of an inner model W of ZFC. In particular, we assume that $W \subseteq \bar{V}$ is such that $\bar{V} = W[h]$, where h is W -generic for some set partial ordering $\mathbb{Q} \in W$. By [7, Lemma 19], for $\kappa > |\mathbb{Q}|$ a singular cardinal, the models W and \bar{V} will agree on the properties " $\diamond_{\kappa^{++}}^*$ holds" and " $\diamond_{\kappa^{++}}^*$ fails". As the proof of Lemma 2.3 shows, every set of ordinals $x \in \bar{V}$ is coded using the oracle "Either $\diamond_{\kappa^{++}}^*$ holds or fails for κ a singular cardinal". The claim is that one such code for x must appear above $|\mathbb{Q}|$. If $p \Vdash \dot{x} \subseteq \alpha$ and $|\dot{\mathbb{Q}}| = \epsilon$, let $\epsilon^* = \max(\alpha, \epsilon)$. The proof of Lemma 2.3, with ϵ^* replacing α in the definition of δ , then shows that there is a dense set of conditions forcing that x is coded above $|\mathbb{Q}|$. This means that the code also appears in W . Consequently, $x \in W$, and so every set of ordinals of \bar{V} is also in W . This shows that $\bar{V} = W$, which means that the forcing \mathbb{Q} was trivial. Thus, $V^{\mathbb{P}} \models \text{GA}$. This completes the proof of Lemma 2.4. ■

Lemmas 2.1–2.4, the paragraph immediately preceding the proof of Lemma 2.1, and the intervening remarks complete the proof of Theorem 2. ■

We conclude this paper by asking what other combinatorial properties can consistently hold in a model containing supercompact cardinals which satisfies level by level equivalence, $V = \text{HOD}$, and GA. In particular, can

such a model satisfy the same combinatorial properties as the model of Theorem 1? If this sort of model were to be constructed via forcing, one would most likely have to employ different kinds of coding oracles from the ones used above.

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References

- [1] A. Apter, *Diamond, square, and level by level equivalence*, Arch. Math. Logic 44 (2005), 387–395.
- [2] A. Apter, *Expanding κ 's power set in its ultrapowers*, Radovi Mat. 10 (2001), 149–156.
- [3] A. Apter and J. Cummings, *An L -like model containing very large cardinals*, Arch. Math. Logic 47 (2008), 65–78.
- [4] A. Apter and Sh. Friedman, *Coding into HOD via normal measures with some applications*, Math. Logic Quart. 57 (2011), 366–372.
- [5] A. Apter and S. Shelah, *On the strong equality between supercompactness and strong compactness*, Trans. Amer. Math. Soc. 349 (1997), 103–128.
- [6] A. Brooke-Taylor, *Large cardinals and definable well-orders on the universe*, J. Symbolic Logic 74 (2009), 641–654.
- [7] Y. Cheng, S.-D. Friedman and J. D. Hamkins, *Large cardinals need not be large in HOD* , Ann. Pure Appl. Logic 166 (2015), 1186–1198.
- [8] J. Cummings, M. Foreman and M. Magidor, *Squares, scales, and stationary reflection*, J. Math. Logic 1 (2001), 35–98.
- [9] M. Foreman and M. Magidor, *A very weak square principle*, J. Symbolic Logic 62 (1997), 175–196.
- [10] S.-D. Friedman, *Large cardinals and L -like universes*, in: Set Theory: Recent Trends and Applications, A. Andretta (ed.), Quaderni Mat. 17, Seconda Univ. Napoli, 2006, 93–110.
- [11] G. Fuchs, J. D. Hamkins and J. Reitz, *Set-theoretic geology*, Ann. Pure Appl. Logic 166 (2015), 464–501.
- [12] J. D. Hamkins, *Extensions with the approximation and cover properties have no new large cardinals*, Fund. Math. 180 (2003), 257–277.
- [13] J. D. Hamkins, *Gap forcing*, Israel J. Math. 125 (2001), 237–252.
- [14] J. D. Hamkins, *Gap forcing: generalizing the Lévy–Solovay theorem*, Bull. Symbolic Logic 5 (1999), 264–272.
- [15] J. D. Hamkins, *The lottery preparation*, Ann. Pure Appl. Logic 101 (2000), 103–146.
- [16] T. Jech, *Set Theory: The Third Millennium Edition, Revised and Expanded*, Springer, Berlin, 2003.
- [17] Y. Kimchi and M. Magidor, *The independence between the concepts of compactness and supercompactness*, circulated manuscript.
- [18] M. Magidor, *On the existence of nonregular ultrafilters and the cardinality of ultrapowers*, Trans. Amer. Math. Soc. 249 (1979), 97–111.
- [19] T. Menas, *On strong compactness and supercompactness*, Ann. Math. Logic 7 (1974), 327–359.

- [20] J. Reitz, *The Ground Axiom*, J. Symbolic Logic 72 (2007), 1299–1317.
- [21] S. Shelah, *Diamonds*, Proc. Amer. Math. Soc. 138 (2010), 2151–2161.
- [22] W. H. Woodin, *In search of ultimate-L: the 19th Midrasha Mathematicae Lectures*, Bull. Symbolic Logic 23 (2017), 1–109.

Arthur W. Apter
Department of Mathematics
Baruch College of CUNY
New York, NY 10010, U.S.A.
and
The CUNY Graduate Center, Mathematics
365 Fifth Avenue
New York, NY 10016, U.S.A.
ORCID: 0000-0002-7091-3628
E-mail: awapter@alum.mit.edu
<http://faculty.baruch.cuny.edu/aapter>