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PROBABILITY THEORY AND STOCHASTIC PROCESSES

Extension of a stochastic Gronwall lemma

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Summary. A stochastic Gronwall lemma is proved in Scheutzow (2013) in the case when the exponent p lies in the interval $0 < p < 1$. In this paper, we extend the lemma to the entire interval $0 < p < \infty$. We construct simple examples to illustrate the present result.

1. Introduction. In [8], Scheutzow proved a stochastic version of the celebrated Gronwall lemma of the following type:

THEOREM 1.1. *Let Z and H be non-negative, adapted processes with continuous paths and assume that ψ is a non-negative and progressively measurable function. Let M be a continuous local martingale starting at zero. If*

$$Z(t) \le H(t) + \int_0^t \psi(s) Z(s)\, ds + M(t) \tag{1}$$

for all $t \ge 0$, then for $p \in (0,1)$, and $\mu, \nu > 1$ such that $1/\mu + 1/\nu = 1$ and $p\nu < 1$, we have

$$\mathbf{E} \sup_{0 \le s \le t} Z(s)^p \le (c_{p\nu} + 1)^{1/\nu} \Big(\mathbf{E} \exp\Big\{p\mu \int_0^t \psi(s)\, ds\Big\}\Big)^{1/\mu} (\mathbf{E}(H^*(t))^{p\nu})^{1/\nu},$$

where H^* is the maximal function of H and $c_{p\nu}$ is a positive constant given by

$$c_{p\nu} = \Big(4 \wedge \frac{1}{p\nu}\Big) \frac{\pi p\nu}{\sin(\pi p\nu)}. \tag{2}$$

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It is pointed out in [8] that the exact constant $c_{p\nu}$ in (2) is not optimal. We also remark that the above result in Theorem 4 of Scheutzow [8] holds true only for $0 < p < 1$. A key question naturally arises: can this result be extended to the entire interval $(0, \infty)$? The main aim of the present paper is to answer this question in the affirmative. For the rest of the paper, we shall assume that

$$Z(t)^\alpha \le H(t) + \Big(\int_0^t \psi(s) Z(s)^\beta \, ds\Big)^{\alpha/\beta} + M(t) \tag{3}$$

for all $t \ge 0$, where α, β are positive real numbers. Throughout the paper, we impose similar assumptions on the real-valued processes $Z = (Z_t)_{t\ge0}$, $H = (H_t)_{t\ge0}$, $\psi = (\psi_t)_{t\ge0}$ and $M = (M_t)_{t\ge0}$ to those in Theorem 1.1. A general account of these processes is given in [6] and [7].

It is essential to note that in the special case when $\alpha = 1$ and $\beta = 1$ in (3), we have the linear stochastic integral inequality (1). The main point here is that the results of this paper contain and extend Theorem 4 of Scheutzow [8].

2. Main results. In this paper, we shall prove the following two theorems.

THEOREM 2.1. *Let $0 < \alpha \le \beta < \infty$ and $1 < r < \infty$, and let $\theta, q > 1$ be such that $1/\theta + 1/q = 1$. Suppose that (3) holds. Then there exists a positive constant A_{rq}, depending only on r and q, such that*

$$\mathbf{E} \sup_{0\le s\le t} Z(s)^\alpha \le 2^{1-1/q}\bigg(\mathbf{E}\Big(1 + \frac{(1-e(t))^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}}\Big)^\theta\bigg)^{1/\theta} \times \big((\mathbf{E}(H^*(t))^q)^{1/q} + A_{rq}(\mathbf{E}\langle M\rangle_t^{rq/2})^{1/(rq)}\big) \tag{4}$$

for all $t \ge 0$, where

$$H^*(t) = \sup_{0\le s\le t} H_s,$$

$\langle M\rangle$ is the quadratic variation of a continuous local martingale M with $M(0) = 0$, and $e(t)$ is the process given by

$$e(t) = \exp\Big(-\int_0^t \psi(u)\, du\Big). \tag{5}$$

Proof. We argue similarly to [8], with minor modifications. The change of variable $Y(t) = Z(t)^\alpha$ in (3) yields

$$Y(t) \le H(t) + M(t) + \Big(\int_0^t \psi(s) Y(s)^{\beta/\alpha}\, ds\Big)^{\alpha/\beta} \tag{6}$$

for all $0 \le t < \infty$.

For any real-valued continuous local martingale $M = (M_t)_{t\geq 0}$, it follows from (6) that

$$Y(t) \leq H(t) + |M(t)| + \Big(\int_0^t \psi(s) Y(s)^{\beta/\alpha}\, ds\Big)^{\alpha/\beta}. \tag{7}$$

We now proceed to estimate $Y(t)$ from above. Let $e(t)$ be defined by (5). Using a nonlinear version of the Gronwall lemma (see [9, Theorem 1]) in (7), we get the estimate

$$\begin{aligned} Y(t) &\leq H(t) + |M(t)| + \frac{(\int_0^t \psi(s)(H(s) + |M(s)|)^{\beta/\alpha} e(s)\, ds)^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}} \\ &\leq H(t) + |M(t)| + \frac{(\int_0^t \psi(s) e(s)\, ds)^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}} (H^*(t) + M^*(t)) \\ &= H(t) + |M(t)| + \frac{(1-e(t))^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}} (H^*(t) + M^*(t)), \end{aligned} \tag{8}$$

where $H^*(t) = \sup_{0\leq s\leq t} H_s$ and $M^*(t) = \sup_{0\leq s\leq t} |M_s|$. Therefore,

$$Z(t)^\alpha \leq H(t) + |M(t)| + \frac{(1-e(t))^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}} (H^*(t) + M^*(t)). \tag{9}$$

Consequently, assuming that $\psi(s)$ is non-deterministic in (9), applying the Hölder inequality we obtain

$$\begin{aligned} \mathbf{E} \sup_{0\leq s\leq t} Z(s)^\alpha &\leq \mathbf{E}\Big(1 + \frac{(1-e(t))^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}}\Big)(H^*(t) + M^*(t)) \\ &\leq \Big(\mathbf{E}\Big(1 + \frac{(1-e(t))^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}}\Big)^\theta\Big)^{1/\theta} \big(\mathbf{E}(H^*(t) + M^*(t))^q\big)^{1/q} \\ &\leq 2^{1-1/q}\Big(\mathbf{E}\Big(1 + \frac{(1-e(t))^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}}\Big)^\theta\Big)^{1/\theta} \\ &\quad \times \big((\mathbf{E}(H^*(t))^q)^{1/q} + (\mathbf{E}(M^*(t))^q)^{1/q}\big) \end{aligned} \tag{10}$$

for $\theta, q > 1$ and $1/\theta + 1/q = 1$.

Let $\langle M\rangle$ be the quadratic variation of a continuous local martingale M with $M(0) = 0$. Then, by a continuous martingale inequality (see [2, Theorem 1]), there exists a positive constant C_{rq} for $1 < r < \infty$ such that

$$\mathbf{E}(M^*(t))^q \leq C_{rq}(\mathbf{E}\langle M\rangle_t^{rq/2})^{1/r}. \tag{11}$$

The desired result (4) now follows by combining (10) and (11). ∎

Remark. In the case when $\psi(s)$ is deterministic in (9), it follows immediately using the continuous martingale inequality [2] that

$$\mathbf{E}\sup_{0\le s\le t} Z(s)^{\alpha} \le \left(1+\frac{(1-e(t))^{\alpha/\beta}}{1-(1-e(t))^{\alpha/\beta}}\right)\big(\mathbf{E}(H^*(t))+B_r(\mathbf{E}\langle M\rangle_t^{r/2})^{1/r}\big),$$

where B_r is some constant with $1<r<\infty$, and $0<\alpha\le\beta$.

A remarkable feature of Scheutzow's result [8] is that the upper estimate is independent of the quadratic variation $\langle M\rangle$ of the continuous local martingale M. This is a consequence of the Burkholder martingale inequality [3]. Indeed, our result in Theorem 2.1 is given in terms of $\langle M\rangle$. In what follows, assuming that (3) holds, we shall prove a result similar to that in [8] with the upper bound independent of the quadratic variation $\langle M\rangle$. The result extends [8, Theorem 4] to the case when $0<p<\infty$. We shall construct simple examples to illustrate this result.

Theorem 2.2. *Let $0<p<\beta\le\alpha<\infty$, and let $\theta,q>1$ be such that $1/\theta+1/q=1$ and $pq/\beta<1$. Assume that* (3) *holds. Then there exists a positive constant $B_{pq/\beta}$, depending only on p,q and β, such that*

$$\begin{aligned}(12)\qquad &\mathbf{E}\sup_{0\le s\le t} Z(s)^{p}\\ &\le (1+B_{pq/\beta})^{1/q}\left(\mathbf{E}\exp\left(\frac{p\theta}{\beta}\int_0^t\psi(s)\,ds\right)\right)^{1/\theta}\big(\mathbf{E}(H^*(t))^{pq/\alpha}\big)^{1/q}\end{aligned}$$

for all $t\ge 0$, where $H^*(t)=\sup_{0\le s\le t}H_s$.

Proof. The proof is similar to that in [8] with appropriate modifications. Taking the $\frac{\beta}{\alpha}$th power on both sides of (7), we get

$$Y(t)^{\beta/\alpha}\le H(t)^{\beta/\alpha}+|M(t)|^{\beta/\alpha}+\int_0^t\psi(s)Y(s)^{\beta/\alpha}\,ds. \tag{13}$$

Let $N(t)=Y(t)^{\beta/\alpha}$. Then

$$N(t)\le H(t)^{\beta/\alpha}+|M(t)|^{\beta/\alpha}+\int_0^t\psi(s)N(s)\,ds. \tag{14}$$

Now applying the usual Gronwall lemma in (14) and integrating by parts, we have

$$N(t)\le\exp\Big(\int_0^t\psi(s)\,ds\Big)\Big(\int_0^t e^{-\int_0^r\psi(s)\,ds}\,d|M(r)|^{\beta/\alpha}+H^*(t)^{\beta/\alpha}\Big). \tag{15}$$

It is clear that the stochastic integrator $|M|^{\beta/\alpha}$ in (15) is not a continuous local martingale. Let $\delta>0$ and define a process $P(r)$ by $P(r)=$

$(\delta+|M(r)|^2)^{\beta/(2\alpha)}$. Applying Ito's lemma to $P(r)$, we obtain

$$dP(r) = \frac{\beta}{\alpha}(\delta+|M(r)|^2)^{\frac{\beta}{2\alpha}-1}|M(r)|\,\mathrm{sgn}(M(r))\,dM(r) + \frac{\beta}{2\alpha}\bigg((\delta+|M(r)|^2)^{\frac{\beta}{2\alpha}-1} + \Big(\frac{\beta}{\alpha}-2\Big)(\delta+|M(r)|^2)^{\frac{\beta}{2\alpha}-2}|M(r)|^2\bigg)d\langle M\rangle_r,$$

where $\mathrm{sgn}(x)$ is 1 if $x \geq 0$, and -1 if $x < 0$.

Letting $\delta \downarrow 0$ and using the fact that $\beta/\alpha - 2 \leq -1$, it now follows that

$$d|M(r)|^{\beta/\alpha} \leq \frac{\beta}{\alpha}|M(r)|^{\beta/\alpha-1}\,\mathrm{sgn}(M(r))dM(r). \tag{16}$$

Hence,

$$\int_0^t e^{-\int_0^r \psi(s)\,ds}\,d|M(r)|^{\beta/\alpha} \leq \frac{\beta}{\alpha}\int_0^t e^{-\int_0^r \psi(s)\,ds}|M(r)|^{\beta/\alpha-1}\,\mathrm{sgn}(M(r))\,dM(r) \tag{17}$$

with the stochastic integrator on the right-hand side being a continuous local martingale.

Define a local martingale $A=(A_t)_{t\geq 0}$ by

$$A(t) = \frac{\beta}{\alpha}\int_0^t e^{-\int_0^r \psi(s)\,ds}|M(r)|^{\beta/\alpha-1}\,\mathrm{sgn}(M(r))\,dM(r). \tag{18}$$

We shall prove the existence of a continuous version of the local martingale $A(t)$. Using [5, Lemma 2.20], it suffices to show that $\langle A\rangle_t < \infty$ for all $t \geq 0$. Let $\kappa \geq 1$ be fixed, $A=(A_t)_{t\geq 0}$ be defined by (18) and let $M^*(t) = \sup_{0\leq r\leq t}|M_r|$. Then

$$\begin{aligned}\langle A\rangle_t &= \frac{\beta^2}{\alpha^2}\int_0^t e^{-2\int_0^r \psi(s)\,ds}|M(r)|^{2(\beta/\alpha-1)}\,d\langle M\rangle_r \\ &= \frac{\beta^2}{\alpha^2}\int_0^t e^{-2\int_0^r \psi(s)\,ds}\frac{|M(r)|^{2(\beta/\alpha-1)+\kappa}}{|M(r)|^{\kappa}}\,d\langle M\rangle_r \\ &\leq \frac{\beta^2}{\alpha^2}M^*(t)^{2\beta/\alpha+\kappa}\int_0^t e^{-2\int_0^r \psi(s)\,ds}\,\frac{d\langle M\rangle_r}{|M(r)|^{2+\kappa}} \\ &\leq \frac{\beta^2}{\alpha^2}M^*(t)^{2\beta/\alpha+\kappa}\int_0^t \frac{d\langle M\rangle_r}{|M(r)|^{2+\kappa}} < \infty\end{aligned} \tag{19}$$

provided that M^* and $\int_0^t \frac{d\langle M\rangle_r}{|M(r)|^{2+\kappa}}$ are both finite.

This proves that there exists a continuous version (local martingale) $L(t)$ of the local martingale $A(t)$. It now follows from (15), (17) and (18) that

$$N(t) \le \exp\Big(\int_0^t \psi(s)\,ds\Big)\big(L(t) + H^*(t)^{\beta/\alpha}\big) \tag{20}$$

using the continuous version of $A(t)$.

The rest of the proof now follows as in [8] with minor modifications. Using the change-of-variables $N(t) = Y(t)^{\beta/\alpha}$ and $Y(t) = Z(t)^{\alpha}$, we obtain $N(t) = Z(t)^{\beta}$. Then (20) implies that

$$Z(t) \le \exp\bigg(\frac{1}{\beta}\int_0^t \psi(s)\,ds\bigg)\big(L(t) + H^*(t)^{\beta/\alpha}\big)^{1/\beta}. \tag{21}$$

By the non-negativity of the process Z, we have $-L(t) \le H^*(t)^{\beta/\alpha}$ for all $t \ge 0$. On the other hand, by the Hölder inequality with exponents $\theta, q > 1$ such that $1/\theta + 1/q = 1$, we have

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le t} Z(s)^p &\le \mathbf{E}\exp\bigg(\frac{p}{\beta}\int_0^t \psi(s)\,ds\bigg)\big(L^*(t) + H^*(t)^{\beta/\alpha}\big)^{p/\beta} \\ &\le \bigg(\mathbf{E}\exp\bigg(\frac{p\theta}{\beta}\int_0^t \psi(s)\,ds\bigg)\bigg)^{1/\theta}\big(\mathbf{E}(L^*(t))^{pq/\beta} + \mathbf{E}(H^*(t))^{pq/\alpha}\big)^{1/q},\end{aligned} \tag{22}$$

where $L^*(t) = \sup_{0\le s\le t}|L_s|$.

The proof will be complete once we have an estimate for $\mathbf{E}(L^*(t))^{pq/\beta}$. This follows by using the Burkholder martingale inequality (see [3, p. 432]). For $0 < pq/\beta < 1$ and $L(t)$ being a local martingale with continuous sample paths and $L(0) = 0$, there exists a positive constant $B_{pq/\beta}$ such that

$$\mathbf{E}(L^*(t))^{pq/\beta} \le B_{pq/\beta}\mathbf{E}(L^-(t))^{pq/\beta}, \tag{23}$$

where $L^-(t) := -\inf_{0\le s\le t} L_s \vee 0$.

Now from $-L(t) \le H^*(t)^{\beta/\alpha}$, we have $L^-(t) \le H^*(t)^{\beta/\alpha}$. Hence,

$$\mathbf{E}(L^-(t))^{pq/\beta} \le \mathbf{E}(H^*(t))^{pq/\alpha}. \tag{24}$$

Then

$$\mathbf{E}(L^*(t))^{pq/\beta} \le B_{pq/\beta}\mathbf{E}(H^*(t))^{pq/\alpha}, \tag{25}$$

which follows immediately from (23) and (24).

Now combining (22) and (25), we obtain (12), which completes the proof of the theorem. ∎

Remark. It should be noted that if $\alpha = \beta = 1$, then Theorem 2.2 is proved in Scheutzow [8] with an exact constant B_{pq} and for $0 < p < 1$. In this particular case, the optimal constant B_{pq} follows from Theorem 1.4 in Bañuelos and Osękowski [1].

Remark. For a class of continuous local martingales M such that

$$\int_0^t \frac{d\langle M\rangle_r}{|M(r)|^{2+\kappa}} < \infty,$$

see for instance [4, Theorem II.4 and remark on p. 171].

We conclude with two simple examples of independent interest. These fall under our Theorem 2.2, but not under [8, Theorem 4] where the upper estimates for $\mathbf{E}\sup_{0\le s\le t} Z(s)^p$ are given in the case $0 < p < 1$.

Example 2.3. Fix $1 < \beta \le \alpha$. Let θ and q satisfy the conditions in Theorem 2.2. Hence, assuming (3) holds, $\mathbf{E}\sup_{0\le s\le t} Z(s)$ is majorized by the upper estimate in (12) with $p = 1$.

Example 2.4. Let $2 < \beta \le \alpha$, and let $\theta, q > 1$ be such that $1/\theta + 1/q = 1$ and $2q/\beta < 1$. Suppose that (3) holds. Then an upper bound for $\mathbf{E}\sup_{0\le s\le t} Z(s)^2$ follows from Theorem 2.2 for $p = 2$.

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