Extension of a stochastic Gronwall lemma

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Summary. A stochastic Gronwall lemma is proved in Scheutzow (2013) in the case when the exponent \( p \) lies in the interval \( 0 < p < 1 \). In this paper, we extend the lemma to the entire interval \( 0 < p < \infty \). We construct simple examples to illustrate the present result.

1. Introduction. In [8], Scheutzow proved a stochastic version of the celebrated Gronwall lemma of the following type:

**Theorem 1.1.** Let \( Z \) and \( H \) be non-negative, adapted processes with continuous paths and assume that \( \psi \) is a non-negative and progressively measurable function. Let \( M \) be a continuous local martingale starting at zero. If

\[
Z(t) \leq H(t) + \int_0^t \psi(s)Z(s)\,ds + M(t)
\]

for all \( t \geq 0 \), then for \( p \in (0, 1) \), and \( \mu, \nu > 1 \) such that \( 1/\mu + 1/\nu = 1 \) and \( p\nu < 1 \), we have

\[
\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^p \leq \left(c_{p\nu} + 1\right)^{1/\nu} \left(\mathbb{E} \exp\left(p\mu \int_0^t \psi(s)\,ds\right)\right)^{1/\mu} \left(\mathbb{E}(H^*(t))^{p\nu}\right)^{1/\nu},
\]

where \( H^* \) is the maximal function of \( H \) and \( c_{p\nu} \) is a positive constant given by

\[
c_{p\nu} = \left(4 \wedge \frac{1}{p\nu}\right) \frac{\pi p\nu}{\sin(\pi p\nu)}.
\]
It is pointed out in [8] that the exact constant $c_{p\nu}$ in [2] is not optimal. We also remark that the above result in Theorem 4 of Scheutzow [8] holds true only for $0 < p < 1$. A key question naturally arises: can this result be extended to the entire interval $(0, \infty)$? The main aim of the present paper is to answer this question in the affirmative. For the rest of the paper, we shall assume that

\begin{equation}
Z(t)\alpha \leq H(t) + \left( \int_0^t \psi(s)Z(s)\beta \, ds \right)\alpha/\beta + M(t)
\end{equation}

for all $t \geq 0$, where $\alpha, \beta$ are positive real numbers. Throughout the paper, we impose similar assumptions on the real-valued processes $Z = (Z_t)_{t \geq 0}$, $H = (H_t)_{t \geq 0}$, $\psi = (\psi_t)_{t \geq 0}$ and $M = (M_t)_{t \geq 0}$ to those in Theorem 1.1. A general account of these processes is given in [6] and [7].

It is essential to note that in the special case when $\alpha = 1$ and $\beta = 1$ in (3), we have the linear stochastic integral inequality (1). The main point here is that the results of this paper contain and extend Theorem 4 of Scheutzow [8].

2. Main results. In this paper, we shall prove the following two theorems.

**Theorem 2.1.** Let $0 < \alpha \leq \beta < \infty$ and $1 < r < \infty$, and let $\theta, q > 1$ be such that $1/\theta + 1/q = 1$. Suppose that (3) holds. Then there exists a positive constant $A_{rq}$, depending only on $r$ and $q$, such that

\begin{equation}
\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^\alpha \leq 2^{1-1/q} \left( \mathbb{E} \left( 1 + \frac{(1 - e(t))^{\alpha/\beta}}{1 - (1 - e(t))^{\alpha/\beta}} \right)^\theta \right)^{1/\theta} \times ((\mathbb{E}(H^*(t))^q)^{1/q} + A_{rq} (\mathbb{E}\langle M \rangle_{r^q/2}^{rq/2})^{1/(rq)})
\end{equation}

for all $t \geq 0$, where $H^*(t) = \sup_{0 \leq s \leq t} H_s$, $\langle M \rangle$ is the quadratic variation of a continuous local martingale $M$ with $M(0) = 0$, and $e(t)$ is the process given by

\begin{equation}e(t) = \exp \left( - \int_0^t \psi(u) \, du \right).
\end{equation}

**Proof.** We argue similarly to [8], with minor modifications. The change of variable $Y(t) = Z(t)^\alpha$ in (3) yields

\begin{equation}
Y(t) \leq H(t) + M(t) + \left( \int_0^t \psi(s)Y(s)^{\beta/\alpha} \, ds \right)^{\alpha/\beta}
\end{equation}

for all $0 \leq t < \infty$. 

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For any real-valued continuous local martingale \( M = (M_t)_{t \geq 0} \), it follows from (6) that

\[
Y(t) \leq H(t) + |M(t)| + \left( \int_0^t \psi(s)Y(s)^{\beta/\alpha} \, ds \right)^{\alpha/\beta}.
\]

We now proceed to estimate \( Y(t) \) from above. Let \( e(t) \) be defined by (5). Using a nonlinear version of the Gronwall lemma (see [9, Theorem 1]) in (7), we get the estimate

\[
Y(t) \leq H(t) + |M(t)| + \frac{\left( \int_0^t \psi(s)e(s) \, ds \right)^{\alpha/\beta}}{1 - (1 - e(t))^{\alpha/\beta}}(H^*(t) + M^*(t))
\]

\[
= H(t) + |M(t)| + \frac{(1 - e(t))^{\alpha/\beta}}{1 - (1 - e(t))^{\alpha/\beta}}(H^*(t) + M^*(t)),
\]

where \( H^*(t) = \sup_{0 \leq s \leq t} H_s \) and \( M^*(t) = \sup_{0 \leq s \leq t} |M_s| \). Therefore,

\[
Z(t)^{\alpha} \leq H(t) + |M(t)| + \frac{(1 - e(t))^{\alpha/\beta}}{1 - (1 - e(t))^{\alpha/\beta}}(H^*(t) + M^*(t)).
\]

Consequently, assuming that \( \psi(s) \) is non-deterministic in (9), applying the Hölder inequality we obtain

\[
\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^{\alpha} \leq \mathbb{E} \left( 1 + \frac{(1 - e(t))^{\alpha/\beta}}{1 - (1 - e(t))^{\alpha/\beta}} \right)^{\theta} (H^*(t) + M^*(t))^{1/q} \leq \frac{(1 - e(t))^{\alpha/\beta}}{1 - (1 - e(t))^{\alpha/\beta}}(H^*(t) + M^*(t))
\]

for \( \theta, q > 1 \) and \( 1/\theta + 1/q = 1 \).

Let \( \langle M \rangle \) be the quadratic variation of a continuous local martingale \( M \) with \( M(0) = 0 \). Then, by a continuous martingale inequality (see [2, Theorem 1]), there exists a positive constant \( C_{rq} \) for \( 1 < r < \infty \) such that

\[
\mathbb{E}(M^*(t))^q \leq C_{rq}(\mathbb{E}\langle M \rangle_t)^{rq/2})^{1/r}.
\]

The desired result (4) now follows by combining (10) and (11).
Remark. In the case when $\psi(s)$ is deterministic in (9), it follows immediately using the continuous martingale inequality [2] that
\[
E \sup_{0 \leq s \leq t} Z(s)^\alpha \leq \left(1 + \frac{(1 - e(t))^{\alpha/\beta}}{1 - (1 - e(t))^{\alpha/\beta}}\right) \left(\mathbb{E}(H^*(t)) + B_{r/2}(\mathbb{E}(M)_{r}^{r/2})^{1/r}\right),
\]
where $B_{r}$ is some constant with $1 < r < \infty$, and $0 < \alpha \leq \beta$.

A remarkable feature of Scheutzow’s result [8] is that the upper estimate is independent of the quadratic variation $\langle M \rangle$ of the continuous local martingale $M$. This is a consequence of the Burkholder martingale inequality [3]. Indeed, our result in Theorem 2.1 is given in terms of $\langle M \rangle$. In what follows, assuming that (3) holds, we shall prove a result similar to that in [8] with the upper bound independent of the quadratic variation $\langle M \rangle$. The result extends [8, Theorem 4] to the case when $0 < p < \infty$. We shall construct simple examples to illustrate this result.

**Theorem 2.2.** Let $0 < p < \beta \leq \alpha < \infty$, and let $\theta, q > 1$ be such that $1/\theta + 1/q = 1$ and $pq/\alpha < 1$. Assume that (3) holds. Then there exists a positive constant $B_{pq/\beta}$, depending only on $p, q$ and $\beta$, such that
\[
E \sup_{0 \leq s \leq t} Z(s)^p \leq (1 + B_{pq/\beta})^{1/q} \left(\mathbb{E}\exp\left(p\theta - \frac{t}{\beta} \int_0^\beta \psi(s) ds\right)\right)^{1/\theta} \left(\mathbb{E}(H^*(t))^{pq/\alpha}\right)^{1/q}
\]
for all $t \geq 0$, where $H^*(t) = \sup_{0 \leq s \leq t} H_s$.

**Proof.** The proof is similar to that in [8] with appropriate modifications. Taking the $\beta/\alpha$th power on both sides of (7), we get
\[
Y(t)^{\beta/\alpha} \leq H(t)^{\beta/\alpha} + |M(t)|^{\beta/\alpha} + \int_0^t \psi(s)Y(s)^{\beta/\alpha} ds.
\]
Let $N(t) = Y(t)^{\beta/\alpha}$. Then
\[
N(t) \leq H(t)^{\beta/\alpha} + |M(t)|^{\beta/\alpha} + \int_0^t \psi(s)N(s) ds.
\]
Now applying the usual Gronwall lemma in (14) and integrating by parts, we have
\[
N(t) \leq \exp\left(\int_0^t \psi(s) ds\right)\left(\int_0^t e^{-\frac{t}{\beta}} \psi(s) ds d|M(r)|^{\beta/\alpha} + H^*(t)^{\beta/\alpha}\right).
\]
It is clear that the stochastic integrator $|M|^{\beta/\alpha}$ in (15) is not a continuous local martingale. Let $\delta > 0$ and define a process $P(r)$ by $P(r) = \int_0^r \psi(s) ds$...
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\((\delta + |M(r)|^2)^{\beta/(2\alpha)}\). Applying Ito’s lemma to \(P(r)\), we obtain

\[
dP(r) = \frac{\beta}{\alpha}(\delta + |M(r)|^2)^{\frac{\beta}{2\alpha} - 1}|M(r)|\text{sgn}(M(r))dM(r)
+ \frac{\beta}{2\alpha} \left( (\delta + |M(r)|^2)^{\frac{\beta}{2\alpha} - 1} + \left( \frac{\beta}{\alpha} - 2 \right)(\delta + |M(r)|^2)^{\frac{\beta}{2\alpha} - 2}|M(r)|^2 \right)d\langle M \rangle_r,
\]

where \(\text{sgn}(x)\) is 1 if \(x \geq 0\), and \(-1\) if \(x < 0\).

Letting \(\delta \downarrow 0\) and using the fact that \(\beta/\alpha - 2 \leq -1\), it now follows that

\[
d|M(r)|^{\frac{\beta}{\alpha}} \leq \frac{\beta}{\alpha} |M(r)|^{\frac{\beta}{\alpha} - 1}\text{sgn}(M(r))dM(r).
\]

Hence,

\[
\int_0^t e^{-\int_0^s \psi(r) dr} d|M(r)|^{\frac{\beta}{\alpha}} \leq \frac{\beta}{\alpha} \int_0^t e^{-\int_0^s \psi(r) dr} |M(r)|^{\frac{\beta}{\alpha} - 1}\text{sgn}(M(r))dM(r)
\]

with the stochastic integrator on the right-hand side being a continuous local martingale.

Define a local martingale \(A_t = (A_t)_{t \geq 0}\) by

\[
A(t) = \frac{\beta}{\alpha} \int_0^t e^{-\int_0^s \psi(r) dr} |M(r)|^{\frac{\beta}{\alpha} - 1}\text{sgn}(M(r))dM(r).
\]

We shall prove the existence of a continuous version of the local martingale \(A_t\). Using [5, Lemma 2.20], it suffices to show that \(\langle A \rangle_t < \infty\) for all \(t \geq 0\). Let \(\kappa \geq 1\) be fixed, \(A_t = (A_t)_{t \geq 0}\) be defined by (18) and let \(M^*(t) = \sup_{0 \leq r \leq t} |M_r|\). Then

\[
\langle A \rangle_t = \frac{\beta^2}{\alpha^2} \int_0^t e^{-\int_0^s \psi(r) dr} |M(r)|^{2(\frac{\beta}{\alpha} - 1)} d\langle M \rangle_r
= \frac{\beta^2}{\alpha^2} \int_0^t e^{-\int_0^s \psi(r) dr} \frac{|M(r)|^{2(\frac{\beta}{\alpha} - 1) + \kappa}}{|M(r)|^\kappa} d\langle M \rangle_r
\leq \frac{\beta^2}{\alpha^2} M^*(t)^{2\frac{\beta}{\alpha} + \kappa} \int_0^t e^{-\int_0^s \psi(r) dr} \frac{d\langle M \rangle_r}{|M(r)|^{2+\kappa}}
\leq \frac{\beta^2}{\alpha^2} M^*(t)^{2\frac{\beta}{\alpha} + \kappa} \int_0^t \frac{d\langle M \rangle_r}{|M(r)|^{2+\kappa}} < \infty
\]

provided that \(M^*\) and \(\int_0^t \frac{d\langle M \rangle_r}{|M(r)|^{2+\kappa}}\) are both finite.
This proves that there exists a continuous version (local martingale) \( \mathcal{L}(t) \) of the local martingale \( A(t) \). It now follows from (15), (17) and (18) that

\[
N(t) \leq \exp \left( \int_0^t \psi(s) \, ds \right) \left( L(t) + H^*(t)^{\beta/\alpha} \right)
\]

using the continuous version of \( A(t) \).

The rest of the proof now follows as in [8] with minor modifications. Using the change-of-variables \( N(t) = Y(t)^{\alpha/\beta} \) and \( Y(t) = Z(t)^{\alpha/pq/\beta} \), we obtain

\[
N(t) = Z(t)^{\beta/\alpha}.
\]

Then (20) implies that

\[
Z(t) \leq \exp \left( \frac{1}{\beta} \int_0^t \psi(s) \, ds \right) \left( L(t) + H^*(t)^{\beta/\alpha} \right)^{1/\beta}.
\]

By the non-negativity of the process \( Z \), we have \( -L(t) \leq H^*(t)^{\beta/\alpha} \) for all \( t \geq 0 \). On the other hand, by the Hölder inequality with exponents \( \theta, q > 1 \) such that \( 1/\theta + 1/q = 1 \), we have

\[
\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^p \leq \mathbb{E} \exp \left( \frac{p}{\beta} \int_0^t \psi(s) \, ds \right) (L^*(t) + H^*(t)^{\beta/\alpha})^{p/\beta}
\]

\[
\leq \left( \mathbb{E} \exp \left( \frac{p \theta}{\beta} \int_0^t \psi(s) \, ds \right) \right)^{1/\theta} \left( \mathbb{E} (L^*(t))^{pq/\beta} + \mathbb{E} (H^*(t))^{pq/\alpha} \right)^{1/q},
\]

where \( L^*(t) = \sup_{0 \leq s \leq t} |L_s| \).

The proof will be complete once we have an estimate for \( \mathbb{E} (L^*(t))^{pq/\beta} \). This follows by using the Burkholder martingale inequality (see [3, p. 432]). For \( 0 < pq/\beta < 1 \) and \( L(t) \) being a local martingale with continuous sample paths and \( L(0) = 0 \), there exists a positive constant \( B_{pq/\beta} \) such that

\[
\mathbb{E} (L^*(t))^{pq/\beta} \leq B_{pq/\beta} \mathbb{E} (L^-(t))^{pq/\beta},
\]

where \( L^-(t) := -\inf_{0 \leq s \leq t} L_s \vee 0 \).

Now from \( -L(t) \leq H^*(t)^{\beta/\alpha} \), we have \( L^-(t) \leq H^*(t)^{\beta/\alpha} \). Hence,

\[
\mathbb{E} (L^-(t))^{pq/\beta} \leq \mathbb{E} (H^*(t))^{pq/\alpha}.
\]

Then

\[
\mathbb{E} (L^*(t))^{pq/\beta} \leq B_{pq/\beta} \mathbb{E} (H^*(t))^{pq/\alpha},
\]

which follows immediately from (23) and (24).

Now combining (22) and (25), we obtain (12), which completes the proof of the theorem. ■
REMARK. It should be noted that if $\alpha = \beta = 1$, then Theorem 2.2 is proved in Scheutzow [8] with an exact constant $B_{pq}$ and for $0 < p < 1$. In this particular case, the optimal constant $B_{pq}$ follows from Theorem 1.4 in Bañuelos and Osekiowski [1].

REMARK. For a class of continuous local martingales $M$ such that
\[ \int_0^t \frac{d\langle M \rangle_r}{|M(r)|^{2+\kappa}} < \infty, \]

We conclude with two simple examples of independent interest. These fall under our Theorem 2.2 but not under [8] Theorem 4 where the upper estimates for $\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^p$ are given in the case $0 < p < 1$.

**Example 2.3.** Fix $1 < \beta \leq \alpha$. Let $\theta$ and $q$ satisfy the conditions in Theorem 2.2. Hence, assuming (3) holds, $\mathbb{E} \sup_{0 \leq s \leq t} Z(s)$ is majorized by the upper estimate in (12) with $p = 1$.

**Example 2.4.** Let $2 < \beta \leq \alpha$, and let $\theta, q > 1$ be such that $1/\theta + 1/q = 1$ and $2q/\beta < 1$. Suppose that (3) holds. Then an upper bound for $\mathbb{E} \sup_{0 \leq s \leq t} Z(s)^2$ follows from Theorem 2.2 for $p = 2$.

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**References**


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