

On the Cauchy transform vanishing outside a compact

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Summary. Motivated by a problem in holomorphic dynamics, we present a certain generalization of the celebrated F. and M. Riesz theorem.

1. Introduction. Given a finite complex measure ν with compact support on \mathbb{C} , let

$$\hat{\nu}(z) = \int \frac{d\nu(w)}{w - z}$$

be the Cauchy transform of ν . For the following facts, see e.g. [12]: $\hat{\nu}$ is locally in $L^1(dx dy)$, $\hat{\nu}$ exists almost everywhere on \mathbb{C} and is holomorphic outside $\text{supp}(\nu)$, and $\hat{\nu}(\infty) = 0$. Moreover, if for an open set U and an analytic function h on U , $h = \hat{\nu} dx dy$ -almost everywhere on U , then $|\nu|(U) = 0$.

The F. and M. Riesz theorem asserts that, given a measure ν on the unit circle S^1 , if $\int_{S^1} w^n d\nu(w) = 0$ for all $n = 1, 2, \dots$, in other words, if the Cauchy transform $\hat{\nu}$ of ν vanishes outside the closed unit disk, then ν is absolutely continuous with respect to the Lebesgue measure on S^1 . Theorem 1 below is a generalization of this theorem to unions of pairwise disjoint bounded finitely connected domains.

We use the following notations and terminology. Given a compact subset K of the plane, $A(K)$ is the algebra of all continuous functions on K which are analytic in the interior of K , and $R(K)$ is the algebra of uniform limits on K of rational functions with poles outside K (= uniform limits on K of

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functions holomorphic on K). If $R(K) = A(K)$, we call K an *A-compact*, and if an A-compact K is nowhere dense, then K is a *C-compact* ($C =$ continuous since in this case $A(K) = C(K)$), the set of all continuous functions on K). Given an open bounded set U , let $A(U)$ be the algebra of all analytic functions in U which extend continuously to \bar{U} .

Necessary and sufficient conditions for a compact in the plane to be an A- or a C-compact are given by Vitushkin [22]. Here are some sufficient conditions [10], [22]: K is a C-compact if the area of K is zero; K is a C-compact (respectively, an A-compact) if every point of K (respectively, every point of ∂K) belongs to the boundary of a component of the complement $\mathbb{C} \setminus K$. Moreover, the boundary of an A-compact is always a C-compact (see e.g. [10]), though not vice versa.

We would like to know about non-trivial measures ν supported on the boundary ∂K of K such that $\hat{\nu} = 0$ outside K . It is well known [10] that $\hat{\nu} = 0$ off K if and only if $\int f d\nu = 0$ for all $f \in R(K)$. This implies that if K is nowhere dense, such a non-trivial measure ν exists if and only if K is not a C-compact.

Let us formulate a question motivated by a problem in holomorphic dynamics. Given a nowhere dense compact E , let $V = \bigcup_k \Omega_k$ be the union of some non-empty collection $\{\Omega_k\}$ of bounded components of the complement $\mathbb{C} \setminus E$. The problem we are interested in is the following:

Describe the sets V and E for which any measure ν supported on E and such that $\hat{\nu} = 0$ in $\mathbb{C} \setminus (E \cup V)$ is in fact supported on $\bigcup_k \partial\Omega_k$ and absolutely continuous with respect to harmonic measures of Ω .

We prove in Theorem 1 that this is the case if each component Ω_k is finitely connected without isolated points in its boundary and the following three condition hold: (I) $\{\Omega_k\}$ is a so-called D-collection, (II) $A(\bar{V}) = A(V)$, (III) (i) \bar{V} is an A-compact, (ii) E is a C-compact.

Let us comment on (I)–(III) (see below for more details). In condition (I), $\{\Omega_k\}$ is said to be a D-collection if harmonic measures of different domains are mutually singular and for each domain Ω of the collection, a holomorphic homeomorphism from a bounded circular domain Δ onto Ω extends as a one-to-one map onto a subset of full (arc) measure to $\partial\Delta$. Condition (II) means that every continuous function on \bar{V} which is holomorphic in V is in fact holomorphic in the interior of \bar{V} . This is obviously the case if $\text{Int}(\bar{V}) = V$. In condition (III), (ii) along with $\hat{\nu} = 0$ off $E \cup V$ implies that $\nu = 0$ on $E \setminus \partial V$ (see Lemma 2.1), i.e., in fact ν is supported on ∂V . Note that (III) holds, for example, if $\mathbb{C} \setminus E$ has finitely many components.

If $E = S^1$, the unit circle, and $V = \mathbb{D}$, the unit disk, then (I)–(III) are satisfied and we recover the F. and M. Riesz theorem.

All conditions of Theorem 1 turn out to be essential: the conclusion about the measure ν breaks down in general if one of the conditions (I), (II) or (III) does not hold; see Proposition 1 along with Examples 1–2.

There also exists an abstract F. and M. Riesz theorem: see [10, Theorem 7.6]. It would be interesting to derive Theorem 1 from that result. The reason we proved Theorem 1 was to apply it to a problem in holomorphic dynamics: see Corollary 2.1 and [18].

After completing this note (arXiv:1911.05336, Nov 13, 2019) we found Erret Bishop's paper [4] and its sequel [5]. The main Theorem 3 of [5] is a particular case of our Theorem 1 in the case when the boundary E of $E \cup V$ is equal to the boundary of the unbounded component of the complement to $E \cup V$ (it is fairly easy to see that this implies that all conditions (I)–(III) hold). In the concluding Remark III of [5] E. Bishop asks whether the analog of his Theorem 3 holds in a more general setting noting that “this seems to be a difficult question” and that it is clear that some extra hypotheses are necessary. Our Theorem 1 thus partially answers this question.

For another line of development of the F. and M. Riesz theorem and [4]–[5], see more recent papers [15], [16] by Dmitry Khavinson. Our Theorem 1 is close in spirit to [16, Theorem 1]. While we study measures on ∂X for a compact X by uniformizing components of the interior of X , in [15], [16] measures on ∂X are approximated from outside X . Theorem 1 of [16] essentially states that given a compact X for which the Dirichlet problem is always solvable, every measure μ on ∂X such that $\hat{\mu} = 0$ off X is a weak-* limit of $\{\mu_n\}$ with $\|\mu_n\| \leq \|\mu\|$ and $d\mu_n = f_n(z)dz|_{\partial X_n}$, for any sequence $X_n (\supset X)$ of compacts decreasing to X with analytic boundaries and some $f_n \in R(X_n)$.

Let us remark finally that we try to keep the proofs as self-contained and elementary as possible.

2. Statements

2.1. Main result and (counter-)examples. Unless stated otherwise, all measures considered are assumed to be complex and finite. Given two measures λ and μ where μ is positive we write $\lambda \ll \mu$ if λ is absolutely continuous with respect to μ , and $\lambda_1 \perp \lambda_2$ for two mutually singular measures λ_1, λ_2 . Given a bounded plane domain Ω (i.e., a connected open subset of the plane) let ω_Ω denote the harmonic measure on $\partial\Omega$ with respect to a fixed point in Ω .

Recall that a domain of the plane is *circular* if its boundary consists of a finite number of disjoint circles. An example is the unit disk \mathbb{D} . Given a finitely connected bounded domain Ω whose boundary contains no isolated points, it is a classical result that there exist a bounded circular domain Δ_Ω

and a conformal homeomorphism

$$\psi_\Omega : \Delta_\Omega \rightarrow \Omega.$$

By the Fatou theorem on radial limits, for (Lebesgue) almost every point $w \in \partial\Delta_\Omega$, the radial boundary value $\psi_\Omega(w)$ is defined.

DEFINITION 2.1 (cf. [13], [1]). Given a (finite or infinite) collection $\{\Omega_i\}$ of pairwise disjoint finitely connected domains without isolated points on their boundaries, we call $\{\Omega_i\}$ a *D-collection* ($D = \text{Davie}$, see [7]) if each ψ_{Ω_i} extends radially as a one-to-one map on a subset of full measure of the boundary of Δ_{Ω_i} , and $\omega_{\Omega_i} \perp \omega_{\Omega_j}$ for $i \neq j$.

Our main result is

THEOREM 1. *Let V be a bounded open set such that each component of V is finitely connected without isolated boundary points. Let $\{\Omega_i, \kappa_i\}_{i=1}^N$, $1 \leq N \leq \infty$, be a set of couples where $\{\Omega_i\}_{i=1}^N$ is the collection of all components of V and each κ_i is a holomorphic function on Ω_i . For each i , fix a uniformization $\psi_{\Omega_i} : \Delta_{\Omega_i} \rightarrow \Omega_i$. Let $E \subset \mathbb{C}$ be a compact set without interior points such that $E \subset \mathbb{C} \setminus V$ and $E \supset \partial V$.*

(P1) *Assume that*

- (I) $\{\Omega_i\}_{i=1}^N$ is a *D-collection*,
- (II) $A(V) = A(\bar{V})$,
- (III) (i) \bar{V} is an *A-compact*, (ii) E is a *C-compact*.

Then (a) implies (b) where

(a) *there exists a measure ν supported in E such that*

$$(1) \quad \hat{\nu}(z) = \begin{cases} \kappa_i(z) & \text{if } z \in \Omega_i, \text{ for each } i, \\ 0 & \text{if } z \in \mathbb{C} \setminus (E \cup V), \end{cases}$$

(b) *for every i ,*

$$(2) \quad \|\kappa_i\| := \limsup_{\epsilon \rightarrow 0} \int_{\partial\Omega_{i,\epsilon}} |\kappa_i(z)| |dz| < \infty$$

where $\Omega_{i,\epsilon} = \psi_{\Omega_i}(\{w : \text{dist}(w, \partial\Delta_{\Omega_i}) > \epsilon\})$. Moreover,

(b1)

$$(3) \quad \sum_{i=1}^N \|\kappa_i\| < \infty,$$

(b2) *the following representation holds:*

$$(4) \quad \nu = \sum_{i=1}^N \nu_i$$

where $\{\nu_i\}_{i=1}^N$ are pairwise mutually singular measures, ν_i is a measure on $\partial\Omega_i$ such that $\nu_i \ll \omega_{\Omega_i}$, and $\|\kappa_i\| = \|\nu_i\|$, the total variation of ν_i . In particular, ν has no atoms.

(P2) Conversely, assume that (2)–(3) hold. Then there exists a measure ν supported on $\bigcup_{k=1}^N \partial\Omega_k$ such that

$$(5) \quad \hat{\nu}(z) = \begin{cases} \kappa_i(z) & \text{if } z \in \Omega_i \text{ for each } i, \\ 0 & \text{if } z \in \mathbb{C} \setminus \bar{V}. \end{cases}$$

Moreover, if harmonic measures on different Ω_k are mutually singular, then ν admits a representation (4) as in (b2).

Notice that if $E = \partial V$ then (III)(i) implies (III)(ii).

Applying Theorem 1 with $E = \partial V$ and $\kappa_i = 1$ for all i we get an answer to [12, Problem 4.2, p. 55] for sets V that satisfy conditions (I)–(III): Let $V = \bigcup_k \Omega_k$ be a bounded open set such that its components $\{\Omega_k\}$ form a D-collection and $A(V) = A(\bar{V}) = R(\bar{V})$. Then there is a measure μ on ∂V such that $\hat{\mu} = 1$ in V and $\hat{\mu} = 0$ off \bar{V} if and only if for each k the linear measure $\Lambda(\partial\Omega_k)$ of $\partial\Omega_k$ is finite and $\sum_k \Lambda(\partial\Omega_k) < \infty$.

Given a measure supported on a compact in the closed unit disk $\bar{\mathbb{D}}$, the unit circle can be used as a “screen” to kill the Cauchy transform of this measure off $\bar{\mathbb{D}}$:

EXAMPLE 1 (A. Volberg). Let $V = \mathbb{D} \setminus K$ where $\mathbb{D} = \{|z| < 1\}$ and $K = [0, 1]$. Let ν_K be a measure supported in K . Assume that $\hat{\nu}_K(e^{it}) \in L^1(0, 2\pi)$. On the unit circle $S^1 = \{|z| = 1\}$ with the one-dimensional Lebesgue measure (i.e., the arc length measure dt , $0 \leq t \leq 2\pi$), we define a new measure ν_c so that $d\nu_c(e^{it}) = h_c dt$ with density

$$h_c(e^{it}) = \frac{1}{2\pi} e^{it} \hat{\nu}_K(e^{it}).$$

Let ν be a (finite) measure on ∂V which is defined as follows: $\nu = \nu_K$ on K and $\nu = \nu_c$ on S^1 . Then

$$(6) \quad \hat{\nu}(z) = \begin{cases} \hat{\nu}_K(z) & \text{if } z \in V, \\ 0 & \text{if } z \in \mathbb{C} \setminus \bar{V}. \end{cases}$$

Indeed, for $z \in \mathbb{C} \setminus (K \cup S^1)$,

$$\hat{\nu}_c(z) = \int_{S^1} \frac{d\nu_c(w)}{w - z} = \int_K \frac{d\nu_K(u)}{u - z} \int_{S^1} \frac{dw}{2\pi i (w - z)(u - w)}$$

where the inner integral is equal to -1 for $|z| > 1$ and 0 for $z \in \mathbb{D} \setminus K$. The same calculation, hence (6) as well, hold if $K \subset \bar{\mathbb{D}}$ is any compact such that the length of $K \cap \partial\mathbb{D}$ is zero and ν_K is any measure on K such that $\hat{\nu}_K(e^{it}) \in L^1(0, 2\pi)$.

PROPOSITION 1. *Conditions (I)–(III) of Theorem 1 are necessary for its conclusion: there exist open sets V and measures ν supported on $E := \partial V$ such that V is simply connected, $\hat{\nu} = 0$ off \bar{V} and ν has atoms (so is not absolutely continuous with respect to harmonic measure on ∂V) while, in the notations of Theorem 1, for the sets V and E one and only one of the conditions (I), (II), (III) breaks down.*

Proof. Let K be any nowhere dense compact as in Example 1 such that $V = \mathbb{D} \setminus K$ is simply connected. Let $E = K \cup S^1 = \partial V$. Since $\bar{V} = \bar{\mathbb{D}}$, \bar{V} is an A-compact and $E = \partial V$ is a C-compact, so condition (III) holds. On the other hand, taking ν_K to be a discrete measure with $\text{supp}(\nu_K) = K$ we see that the conclusion of Theorem 1 does not hold. This means that (I) or (II) has to break down. In particular, for $K = [0, 1]$, condition (I) does not hold while (II) does, so condition (I) is necessary indeed. As for the necessity of (II), we choose $K = J$ where J is a Jordan arc such that $V = \mathbb{D} \setminus J$ is a simply connected domain which satisfies (I), i.e., the Riemann map $\psi_V : \mathbb{D} \rightarrow V$ extends to a one-to-one map on a set of full (arc) measure on S^1 . Hence, (II) cannot hold in this case (this also follows directly from [6, Theorem 1']; see moreover [2], [3]). The existence of such a Jordan arc J follows from Browder and Wermer [6]. Indeed, in [6] an example of a Jordan arc J is constructed such that the Riemann map $h : \mathbb{C} \setminus \bar{\mathbb{D}} \rightarrow \mathbb{C} \setminus J$ with $h(\infty) = \infty$ extends one-to-one on a set of full Lebesgue measure on S^1 . Then it is easy to see that if we take $J \subset \bar{\mathbb{D}}$ with $J \cap S^1 = \{1\}$, then $V = \mathbb{D} \setminus J$ satisfies (I). [Proof: since $V \subset \mathbb{C} \setminus J$, $h^{-1}(V)$ is a well defined simply connected bounded domain with pairwise analytic boundary; hence, if $\beta : \mathbb{D} \rightarrow h^{-1}(V)$ is a Riemann map then $\psi_V := h \circ \beta : \mathbb{D} \rightarrow V$ extends to a one-to-one-map on a set of full measure on S^1 .]

That (III) is necessary as well is shown by the following example. ■

EXAMPLE 2. Let J be the Jordan arc as in [6] (see the proof of Proposition 1). One can assume that $J \subset \mathbb{D} \cup \{1\}$ and the endpoints of J are 0 and 1. Let $\{D_j\}_{j=1}^\infty$ be a collection of open disks in $\mathbb{D} \setminus J$ with pairwise disjoint closures such that each D_j touches J at precisely one point which is neither 0 nor 1, and the set of all such points is dense in J and

$$\sum_{j=1}^{\infty} \frac{r_j}{d_j} < \infty$$

where r_j is the radius of D_j and d_j is the distance between 0 and D_j . (It is not difficult to show that such a choice of disks is possible.) Define $V = \mathbb{D} \setminus (J \cup \bigcup_{j=1}^\infty \bar{D}_j)$ and $E = \partial V = S^1 \cup J \cup \bigcup_{j=1}^\infty S_j$ where $S_j = \partial D_j$. Since there are non-constant continuous functions on the Riemann sphere which are holomorphic in $\mathbb{C} \setminus J$, by [9], [10] the compact $\bar{V} = \bar{\mathbb{D}} \setminus \bigcup_{j=1}^\infty \bar{D}_j$ is not an A-compact, i.e., $R(\bar{V}) \neq A(\bar{V})$.

Now we define a measure ν on E as follows. Let $\nu = \delta_0 + \sum_{j=0}^{\infty} \nu_j$ where δ_0 is the Dirac measure at 0, $d\nu_0(w) = \hat{\delta}_0(w)dw$, $w \in S^1$, is a measure on S^1 and, for each j , $d\nu_j(w) = -\hat{\delta}_0(w)dw$, $w \in S_j$, is a measure on S_j . Since for every $j \geq 0$, $\|\nu_j\| \leq 2\pi r_j/d_j$, we have

$$\|\nu\| \leq 1 + 2\pi \left(1 + \sum_{j=1}^{\infty} \frac{r_j}{d_j} \right) < \infty.$$

Similar to Example 1, we get $\hat{\nu} = 0$ off \tilde{E} and $\hat{\nu} = \hat{\delta}_0$ in V . On the other hand, ν is not absolutely continuous with respect to harmonic measure of V because ν has an atom at 0. Notice that V is simply connected and, for the sets V , E as above, conditions (I), (II) of Theorem 1 hold but (III) does not: \bar{V} is not an A-compact. Note that at the same time, $E = \partial V$ is a C-compact as every point of E is at the boundary of the component V of the complement to E .

2.2. Local removability of C-compacts. We need the following

LEMMA 2.1.

- (a) *Every closed subset of a C-compact is a C-compact.*
- (b) *Let K be a nowhere dense compact in \mathbb{C} and μ a measure on K . Suppose that for a neighborhood W of a point $x \in K$, $K \cap \bar{W}$ is a C-compact and $\hat{\mu} = 0$ on $W \setminus K$. Then μ vanishes on $K \cap W$, i.e., $|\mu|(W) = 0$.*

Proof. (a) follows from the fact that any continuous function on a closed subset of a compact extends to a continuous function on the whole compact.

Let us prove (b). Let U be the union of all components of $\mathbb{C} \setminus K$ that intersect W . Then $W \setminus K \subset U$ and $\hat{\mu} = 0$ on U . One can assume that $\infty \in U$ as otherwise, for $M(z) = 1/(x_0 - z)$ with some $x_0 \in U$, we replace K by $\tilde{K} = M(K)$ and μ by a measure $\tilde{\mu}$ on \tilde{K} such that $d\tilde{\mu}(w) = wd\mu(x_0 - 1/w)$. Thus $\infty \in U$.

Now, to get $|\mu|(W) = 0$ it is enough to prove that for each $x \in K \cap W$ there is a neighborhood $W_x \subset W$ such that for all continuous functions g with compact support in W_x , $\int g d\mu = 0$. So fix $x \in K \cap W$ and choose $W_x = B(x, r)$ where $r > 0$ is such that $B(x, 2r) \subset W$. Let g be a continuous function on \mathbb{C} which is compactly supported in $B(x, r)$. Let $\hat{K} = \mathbb{C} \setminus U$. It is enough to prove that $g \in R(\hat{K})$.

Indeed, assume that there is a sequence of rational functions R_n with poles outside \hat{K} converging uniformly on \hat{K} to g . Perturbing some of R_n if necessary one can further assume that all R_n have simple poles. If say $R_n(z) = P(z) + \sum_{j=1}^m \alpha_j/(z - b_j)$ where P is a polynomial and all b_j are in U , then $\int R_n d\mu = \int P d\mu + \sum_{j=1}^m \alpha_j \hat{\mu}(b_j) = 0$ because $\hat{\mu} = 0$ in U and $\int z^n d\mu(z) = 0$ for all $n > 0$ (this is because $\hat{\mu} = 0$ in a neighborhood of ∞).

Then $\int g d\mu = \lim_n \int R_n d\mu = 0$. It remains to show that $g \in R(\hat{K})$. We use Bishop's theorem [23, 11.8]: *Given a compact $X \subset \mathbb{C}$ and a continuous function f on \mathbb{C} , assume that for each $z \in X$ there is a closed neighborhood $B_z = \{|w - z| \leq \delta_z\}$ with $\delta_z > 0$ such that $f|_{X \cap B_z} \in R(X \cap B_z)$. Then $f \in R(X)$.* Applying this to the compact \hat{K} and the function g shows that if $z \in \hat{K} \setminus \overline{B}(x, r)$, then, for $\delta_z = |z - x| - r > 0$, $g|_{\hat{K} \cap B_z} = 0 \in R(\hat{K} \cap B_z)$. On the other hand, for $z \in \overline{B}(x, r)$ and $\delta_z = r$, $g|_{\hat{K} \cap B_z} \in R(\hat{K} \cap B_z)$ because $\hat{K} \cap B_z \subset \hat{K} \cap \overline{B}(x, 2r) \subset K \cap \overline{W}$ while $K \cap \overline{W}$ is a \mathbb{C} -compact. Notice that $g \in R(\hat{K})$ also follows from Vitushkin's necessary and sufficient condition for a function to be in $R(X)$ (see [22]). ■

2.3. A particular case: rotation domains. Let $\{\Omega_i\}$ be a collection of pairwise disjoint Jordan domains such that $\overline{\Omega}_i \cap \overline{\Omega}_j$ is at most a single point for all $i \neq j$, and $V = \bigcup_i \Omega_i$ is bounded. Then obviously conditions (I)–(II) of Theorem 1 hold.

Here is a more interesting case which originates from holomorphic dynamics. Recall that a simply or doubly connected domain A is called a *rotation domain* for a rational function f of degree at least 2 if $f^Q : A \rightarrow A$ is a homeomorphism for some $Q \geq 1$ and is conjugate to an irrational rotation: for a conformal homeomorphism $\psi_A : \Delta_A \rightarrow A$ where Δ_A is either a round disk or a round annulus, the conjugate map $R := \psi_A^{-1} \circ f^Q \circ \psi_A : \Delta_A \rightarrow \Delta_A$ is an irrational rotation of Δ_A . (A is called a *Siegel disk* or a *Herman ring* depending on whether Δ_A is a disk or an annulus.)

PROPOSITION 2. *Let $\{A\}$ be a collection of different rotation domains of a rational function f , i.e., each A is either a Siegel disk or a Herman ring. Then $\{A\}$ satisfies conditions (I)–(II) of Theorem 1, i.e.,*

- (1) *for every A , there is a subset \tilde{X} of $\partial\Delta_A$ of full Lebesgue (length) measure on which ψ_A is one-to-one,*
- (2) *for any two different rotation domains A_1, A_2 , their harmonic measures ω_{A_1} and ω_{A_2} are mutually singular,*
- (3) *if $V_f := \bigcup_{A \in \{A\}} A$, then the interior of \overline{V}_f is equal to V_f , and therefore $A(V_f) = A(\overline{V}_f)$.*

Part (1) is an easy corollary of the following claim which is proved in [19, Theorem 2] for the Siegel disk and the proof holds with obvious modifications for the Herman ring:

LEMMA 2.2. *Let A be a rotation domain of a rational function f such that $f(A) = A$ and $\psi_A : \Delta_A \rightarrow A$ as above. There are only two cases possible:*

- (i) *All radial limits $\psi_A(w)$ are different.*
- (ii) *There is a point $a \in \partial A$ such that $f(a) = a$, and if $\psi_A(w_1) = \psi_A(w_2)$ for some $w_1, w_2 \in \partial\Delta_A$ then $\psi_A(w_1) = \psi_A(w_2) = a$.*

Proof of Proposition 2. (1) follows at once from Lemma 2.2 and Riesz's uniqueness theorem for bounded analytic functions.

Let us prove (2). It is enough to show that the following is impossible: $\omega_{A_1}(Y) > 0$ and $\omega_{A_2}(Y) > 0$ for $Y = \overline{A_1} \cap \overline{A_2}$ where Y is a subset of a component L_1 of ∂A_1 as well as a component L_2 of ∂A_2 . So assume this is the case. First, since $f^Q(L_i) = L_i$ for some $Q > 0$ and $i = 1, 2$, we have $f^Q(Y) \subset Y$. Secondly, by the Fatou theorem on radial limits, there is a set $\tilde{Y} \subset \partial \Delta_{A_1}$ of length $|\tilde{Y}| > 0$ such that for all $w \in \tilde{Y}$, the radial limit $\psi_{A_1}(w)$ exists and is in Y . Moreover, since $f^Q(Y) \subset Y$, one can assume that $R(\tilde{Y}) \subset \tilde{Y}$ where $R = \psi_{A_1}^{-1} \circ f^Q \circ \psi_{A_1} : \Delta_{A_1} \rightarrow \Delta_{A_1}$ is an irrational rotation of Δ_{A_1} . Since $|\tilde{Y}| > 0$, we find that \tilde{Y} has full measure. Then Y is a closed subset of L_1 of full harmonic measure ω_{A_1} , and hence $Y = L_1$. Since $\omega_{A_2}(Y) > 0$, we have $Y = L_2$ as well, i.e., $L_1 = L_2$. Taking now a small disk B around some $x \in L_1 = L_2$, we see that all iterates $f^{jQ}B$, $j \geq 0$, stay in $\overline{A_1} \cup \overline{A_2}$, which is possible only if x is in the Fatou set, a contradiction. Moreover, that the interior of $\overline{V_f}$ coincides with V_f is also proved by a very similar argument. ■

Theorem 1 and Proposition 2 immediately imply

COROLLARY 2.1. *Suppose \mathcal{H} is a non-empty collection of bounded rotation domains of a rational function f . Let $V = \bigcup\{A : A \in \mathcal{H}\}$, $E \subset \mathbb{C} \setminus V$ a nowhere dense compact set such that $\partial V \subset E$, and ν a measure supported on E such that $\hat{\nu} = 0$ off $E \cup V$. If E is a C -compact and \overline{V} is an A -compact, then ν is, in fact, supported on $\partial V = \bigcup_{A \in \mathcal{H}} \partial A$ and, for each A , $\nu|_{\partial A} \ll \omega_A$. In particular, ν is non-atomic. Moreover, the function $\hat{\nu} \circ \psi'_A$ is in the H^1 -Hardy space, i.e.,*

$$\limsup_{\epsilon \rightarrow 0} \int_{\{z \in \Delta_A : \text{dist}(z, \partial A) = \epsilon\}} |\hat{\nu} \circ \psi'_A(z)| |dz| < \infty.$$

Conjecturally, $\overline{V} = \bigcup\{\overline{A} : A \in \mathcal{H}\}$ is *always* an A -compact.

3. Proof of Theorem 1

3.1. Preparatory statements. The proof is heavily based on some general results, mainly from the 1960s (see Theorems 2–4). The first one is a consequence of the Hahn–Banach theorem and the Riesz representation theorem (see [12]):

THEOREM 2 (V. P. Havin). *Let $F \subset \mathbb{C}$ be compact and let g analytic on $\mathbb{C} \setminus F$ and $g(\infty) = 0$. There is a measure ν on F such that $g(z) = \hat{\nu}(z)$ for all $z \notin F$ if and only if there is C_g such that for all functions h which are analytic in a neighborhood of F ,*

$$|T_g(h)| \leq C_g \|h\|_F$$

where $\|h\|_F = \sup_{z \in F} |h(z)|$ and

$$T_g(h) = -\frac{1}{2\pi i} \int_{\partial U} g(z)h(z) dz$$

where U is any small enough neighborhood of F such that ∂U consists of finitely many analytic curves that surround E in positive direction. When this is the case, we may take $C_g = \|\nu\|$, the total variation of ν .

THEOREM 3 (G. G. Tumarkin, [21]). *Let g be analytic in $\mathbb{C} \setminus S^1$ and $g(\infty) = 0$. Then $g = \hat{\eta}$ for some measure η (with $\text{supp}(\eta) \subset S^1$) if and only if*

$$\sup_{0 < r < 1} \int_{S^1} |g(r\theta) - g(\theta/r)| |d\theta| < \infty.$$

COMMENT 1. Notice the particular case when $g = 0$ off $\overline{\mathbb{D}}$.

For completeness, we prove this statement for the direction we need. So let $g = \hat{\eta}$ for a measure η on S^1 . Given $\zeta = e^{i\theta}$, $w = e^{it} \in S^1$ and $0 < r < 1$ we have

$$\frac{1}{w - r\zeta} - \frac{1}{w - \zeta/r} = -\frac{\zeta(r^2 - 1)}{w\zeta(1 - r\zeta/w)(rw/\zeta - 1)} = e^{-it} P_r(\theta - t)$$

where $P_r(\theta - t) = (1 - r^2)/(1 + r^2 - 2r \cos(\theta - t))$ is the Poisson kernel. Therefore,

$$\hat{\eta}(r\zeta) - \hat{\eta}(\zeta/r) = \int_0^{2\pi} P_r(\theta - t) e^{-it} d\eta(e^{it})$$

and

$$\begin{aligned} \int_{S^1} |g(r\zeta) - g(\zeta/r)| |d\zeta| &\leq \int_0^{2\pi} d\theta \int_0^{2\pi} P_r(\theta - t) d|\eta|(e^{it}) \\ &\leq \int_0^{2\pi} d|\eta|(e^{it}) \int_0^{2\pi} P_r(\theta - t) d\theta = 2\pi \|\eta\| \end{aligned}$$

where $\|\eta\|$ is the total variation of the measure η .

The next two statements are most probably well known too, though we are not aware of precise references and for completeness give the proofs (cf. [4]–[5]). For the first one, recall the following definition. Let D be a bounded or unbounded circular domain, i.e., $\partial D = S_1 \cup \dots \cup S_p$ where S_1, \dots, S_p are pairwise disjoint circles. The Hardy space $H^1(D)$ is the set of all holomorphic functions F in D with $F(\infty) = 0$ if $\infty \in D$ such that

$$\|F\|_{H^1(D)} := \limsup_{S \in \mathcal{S}} \int_S |F(w)| |dw| < \infty$$

where \mathcal{S} is the collection of all circles $S \subset D$ in a small neighborhood of ∂D that are concentric with one of S_j , $j = 1, \dots, p$. It is well known (see e.g. [14]) that any $F \in H^1(D)$ has a non-tangential limit $F(w)$ at almost every $w \in \partial D$ with respect to the Lebesgue (arc) measure on ∂D . We also need the following representation for $F \in H^1(D)$ assuming D is bounded and S_1 is the outer boundary of D :

$$(7) \quad F = F_1 + \dots + F_p$$

where $F_j \in D_j$, D_1 is a bounded domain with boundary S_1 , and D_j , $j = 2, \dots, p$, is an unbounded domain with $\partial D_j = S_j$. This representation follows essentially from the Cauchy formula (see [14]). Note that each F_j is holomorphic in a domain that contains all other S_k , $k \neq j$.

LEMMA 3.1. *Let Ω be a finitely connected bounded domain with no isolated points of the boundary and let $\kappa : \Omega \rightarrow \mathbb{C}$ be holomorphic. Assume that*

$$(8) \quad \|\kappa\| := \limsup_{\epsilon \rightarrow 0} \int_{\partial\Omega_\epsilon} |\kappa(w)| |dw| < \infty$$

where $\Omega_\epsilon = \psi_\Omega(\{w : \text{dist}(w, \partial\Delta_\Omega) > \epsilon\})$. Then there is a measure ν^κ supported on $\partial\Omega$ such that $\hat{\nu}^\kappa(z) = \kappa(z)$ for $z \in \Omega$ and $\hat{\nu}^\kappa(z) = 0$ for $z \notin \overline{\Omega}$. Furthermore, $\nu^\kappa \ll \omega_\Omega$ with $\|\nu^\kappa\| = \|\kappa\|$.

Proof. Choose $\epsilon_n \rightarrow 0$ and given n define a measure ν_n on $\partial\Omega_{\epsilon_n}$ by $d\nu_n(z) = \frac{1}{2\pi i} \kappa(z) dz$. By (8), $\sup_n \|\nu_n\| < \infty$. Let ν^κ be a weak* limit of the sequence $\{\nu_n\}$. By the Cauchy formula,

$$\hat{\nu}^\kappa(z) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\partial\Omega_{\epsilon_n}} \frac{\kappa(\zeta) d\zeta}{\zeta - z}$$

is equal to $\kappa(z)$ for $z \in \Omega$ and 0 outside $\overline{\Omega}$. Denote $\psi = \psi_\Omega$, $\Delta = \Delta_\Omega$, $\Gamma_n = \{w : \text{dist}(w, \partial\Delta) = \epsilon_n\}$ and $\tilde{\kappa} = (\kappa \circ \psi)\psi'$. Note that (8) is equivalent to $\tilde{\kappa} \in H^1(\Delta)$. Let $\psi(w)$ and $\tilde{\kappa}(w)$, $w \in \partial\Delta$, denote also the corresponding limits (existing almost everywhere) of $\psi(u)$ and $\tilde{\kappa}(u)$ as $u \rightarrow w$ non-tangentially. We have to prove that $\nu^\kappa \ll \omega_\Omega$. For this, it is enough to show that for any continuous compactly supported function h on \mathbb{C} ,

$$(9) \quad \int h d\nu^\kappa = \frac{1}{2\pi i} \int_{\partial\Delta} h(\psi(w)) \tilde{\kappa}(w) dw.$$

This would imply that $\nu^\kappa \ll \omega_\Omega$ and that $\|\nu^\kappa\| = \|\kappa\|$. One can check (9) separately for each component of $\partial\Delta$. Let us do this for the outer component S_1 of $\partial\Delta = S_1 \cup \dots \cup S_p$ (for the other components, the proof is the same with straightforward modifications). One can assume $S_1 = S^1$, the unit circle.

Since $\int h d\nu^\kappa = \lim_{n \rightarrow \infty} \int h d\nu_n$, we have to check that, for $r_n = 1 - \epsilon_n$,

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{|w|=r_n} h(\psi(w)) \tilde{\kappa}(w) dw = \frac{1}{2\pi i} \int_{|w|=1} h(\psi(w)) \tilde{\kappa}(w) dw.$$

As $\tilde{\kappa} \in H^1(\Delta)$, let $\tilde{\kappa} = \sum_{j=1}^p \tilde{\kappa}_j$ be the corresponding representation as in (7). Then

$$\int_{|w|=r_n} h(\psi(w)) \tilde{\kappa}(w) dw = \sum_{j=1}^p \int_{|w|=r_n} h(\psi(w)) \tilde{\kappa}_j(w) dw.$$

Let $j > 1$. Since $\tilde{\kappa}_j$, $j \neq 1$, is a holomorphic function in a domain that contains S^1 , $h(\psi(w)) \tilde{\kappa}_j(w)$ is bounded in $\{r < |w| < 1\}$ for some $r < 1$, hence, one can apply the Lebesgue dominated convergence theorem:

$$\lim_{n \rightarrow \infty} \int_{|w|=r_n} h(\psi(w)) \tilde{\kappa}_j(w) dw = \int_{|w|=1} h(\psi(w)) \tilde{\kappa}_j(w) dw.$$

It remains to check that

$$\lim_{n \rightarrow \infty} \int_{|w|=r_n} h(\psi(w)) \tilde{\kappa}_1(w) dw = \int_{|w|=1} h(\psi(w)) \tilde{\kappa}_1(w) dw.$$

Here we have to use the fact that $\tilde{\kappa}_1 \in H^1(\mathbb{D})$, the Hardy space in the unit disk. We have

$$\begin{aligned} & \left| \int_{|w|=r_n} h(\psi(w)) \tilde{\kappa}_1(w) dw - \int_{|w|=1} h(\psi(w)) \tilde{\kappa}_1(w) dw \right| \\ & \leq \int_0^{2\pi} |r_n h(\psi(r_n e^{it})) - h(\psi(e^{it}))| |\tilde{\kappa}_1(r_n e^{it})| dt \\ & \quad + \int_0^{2\pi} |h(\psi(e^{it}))| |\tilde{\kappa}_1(r_n e^{it}) - \tilde{\kappa}_1(e^{it})| dt \rightarrow 0. \end{aligned}$$

Here are some details. Let I_n be the first integral on the RHS, and J_n the second one. Then

$$J_n \leq \sup_{t \in [0, 2\pi]} |h(\psi(e^{it}))| \int_0^{2\pi} |\tilde{\kappa}_1(r_n e^{it}) - \tilde{\kappa}_1(e^{it})| dt \rightarrow 0$$

because $\tilde{\kappa}_1 \in H^1(\mathbb{D})$ [17, Ch. II, B, 2°]. Let us prove that $\lim_n I_n = 0$. Let $I := \limsup_n I_n$. Since $r\psi(re^{it}) \rightarrow \psi(e^{it})$ as $r \rightarrow 1$ a.e. in t and h is continuous, for every $\sigma > 0$ there is $E_\sigma \subset [0, 2\pi]$ such that $l(E_\sigma) > 2\pi - \sigma$ (where l is the Lebesgue measure on $(0, 2\pi)$) and $r_n h(\psi(r_n e^{it})) - h(\psi(e^{it})) \rightarrow 0$ uniformly in $t \in E_\sigma$. Since also $\tilde{\kappa}_1 \in H^1(\mathbb{D})$,

$$\lim_n \int_{E_\sigma} |r_n h(\psi(r_n e^{it})) - h(\psi(e^{it}))| |\tilde{\kappa}_1(r_n e^{it})| dt = 0.$$

Hence,

$$I \leq 2 \sup |h| \limsup_n \int_{F_\sigma} |\tilde{\kappa}_1(r_n e^{it})| dt$$

where $F_\sigma = [0, 2\pi] \setminus E_\sigma$ so that $l(F_\sigma) \leq \sigma$. On the other hand,

$$\begin{aligned} \left| \int_{F_\sigma} |\tilde{\kappa}_1(r_n e^{it})| dt - \int_{F_\sigma} |\tilde{\kappa}_1(e^{it})| dt \right| &\leq \int_0^{2\pi} \chi_{F_\sigma} |\tilde{\kappa}_1(r_n e^{it}) - \tilde{\kappa}_1(e^{it})| dt \\ &\leq \int_0^{2\pi} |\tilde{\kappa}_1(r_n e^{it}) - \tilde{\kappa}_1(e^{it})| dt \rightarrow 0 \end{aligned}$$

as $r \rightarrow 1$. Thus

$$I \leq 2 \sup |h| \int_{F_\sigma} |\tilde{\kappa}_1(e^{it})| dt.$$

By the absolute continuity of the Lebesgue integral, for every $\epsilon > 0$ there is $\sigma > 0$ such that for every $F_\sigma \subset [0, 2\pi]$ with $l(F_\sigma) < \sigma$,

$$I \leq 2 \sup |h| \int_{F_\sigma} |\tilde{\kappa}_1(e^{it})| dt < 2 \sup |h| \epsilon.$$

As $\epsilon > 0$ is arbitrary, $I = 0$. ■

In the following, for a bounded function $h : X \rightarrow \mathbb{C}$, let

$$\|h\|_X = \sup_{x \in X} |h(x)|.$$

LEMMA 3.2. *Let $g \not\equiv 0$ be analytic in a bounded circular domain Δ . Assume that there is $C_g > 0$ such that for every function h holomorphic in a neighborhood of $\partial\Delta$ and for all $\epsilon > 0$ small enough,*

$$(10) \quad \left| \int_{\Gamma_\epsilon} g(w)h(w) dw \right| \leq C_g \|h\|_{\partial\Delta}$$

where $\Gamma_\epsilon = \{w \in \Delta : \text{dist}(w, \partial\Delta) = \epsilon\}$. Then (a) there is a measure η which is absolutely continuous with respect to the Lebesgue (arc) measure on $\partial\Delta$ such that $\hat{\eta}$ is g in Δ and 0 off $\overline{\Delta}$, and (b) we have

$$(11) \quad \|g\| := \limsup_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} |g(w)| |dw| < \infty.$$

Moreover, $\|g\| = \|\eta\|$, the total variation of η , and there are a sequence $\{h_j\}$ of locally analytic on $\partial\Delta$ functions and a sequence $\epsilon_j \rightarrow 0$ such that $\|h_j\|_{\partial\Delta} \rightarrow 1$ as $j \rightarrow \infty$ and

$$(12) \quad \lim_{j \rightarrow \infty} \int_{\Gamma_{\epsilon_j}} g(w)h_j(w) dw = \|\eta\| = \|g\|.$$

Conversely, (11) (obviously) implies (10).

Proof. (10) means that the conditions of Theorem 2 of Havin are satisfied for the compact $F = \partial\Delta$ and the function which is g in Δ and 0 outside $\overline{\Delta}$. Hence, there exists a measure η supported on $\partial\Delta$ such that $\hat{\eta} = g$ in Δ and $\hat{\eta} = 0$ in $\mathbb{C} \setminus \overline{\Delta}$.

Let us prove (b) first. We can assume that $\partial\Delta$ is the union of $S_1 = S^1$ and finitely many disjoint circles S_k , $k = 2, \dots, p$, inside the unit circle S^1 . Let $\eta_k = \eta|_{S_k}$, $1 \leq k \leq p$. If $w \in \Delta$ is close to the component $S_1 = S^1$ of $\partial\Delta$, so that $w = r\zeta$, $\zeta \in S^1$, we have

$$g(r\zeta) = (\hat{\eta}_1(r\zeta) - \hat{\eta}_1(\zeta/r)) + \delta(r, \zeta)$$

where $\delta(r, \zeta) = \sum_{k=2}^p (\hat{\eta}_k(r\zeta) - \hat{\eta}_k(\zeta/r))$ tends to 0 uniformly in ζ as $r \rightarrow 1$ because $\hat{\eta}_k$ is analytic off S_k . Hence, applying Theorem 3 to the measure η_1 we get (11) for a component Γ_ϵ which is near S_1 . The proof for the other components of Γ_ϵ is very similar. This proves (11).

In turn, (11) means that the condition of Lemma 3.1 is satisfied where $\Omega = \Delta$ (so that $\Delta_\Omega = \Delta$ and $\psi_\Omega = \text{id}$) and $\kappa = g$. By Lemma 3.1, there exists a measure ν^g supported on $\partial\Delta$ and absolutely continuous with respect to the arc measure on $\partial\Delta$ such that $\hat{\nu}^g = g$ in Δ and $\hat{\nu}^g = 0$ outside of $\overline{\Delta}$. But then the Cauchy transform of the measure $\eta - \nu^g$ vanishes outside of $\partial\Delta$. Since $\partial\Delta$ is a C-compact, we conclude that $\nu^g = \eta$. This proves part (a) along with $\|\eta\| = \|g\|_{H^1}$.

It remains to find a sequence $\{h_j\}$ as in (12). Note that since Δ consists of finitely many components (which are circles), it is enough to find $\{h_j\}$ for each component separately. So let

$$S_b(R) = \{w = b + Re^{it} : 0 \leq t \leq 2\pi\}$$

be such a component. As $g \in H^1(\Delta)$ and $g \not\equiv 0$, the non-tangential limit \tilde{g} of g exists and is non-zero for almost every $w \in S_b(R)$. Note that for $w \in S_b(R)$, $|dw| = \frac{\alpha}{w-b} dw$ where $\alpha = \frac{R}{i}$. Since $\tilde{g} \neq 0$ almost everywhere, the function $H(w) = \frac{\alpha}{w-b} \frac{|\tilde{g}(w)|}{\tilde{g}(w)}$ is in $L^\infty(S_b(R))$. By Luzin's theorem, given $\delta > 0$, there is a continuous function H^δ on $S_b(R)$ such that $\sup_w |H^\delta(w)| \leq \sup_w |H(w)| = 1$ and $l(\{w : H(w) \neq H^\delta(w)\}) < \delta$ where l is the Lebesgue (arc) measure on $S_b(R)$. In turn, let h^δ be a locally holomorphic function on $S_b(R)$ such that $\sup_{w \in S_b(R)} |H^\delta(w) - h^\delta(w)| < \delta$. Then the sequence of functions $h_j := h^{1/j}$, defined for all j large enough, and a sequence ϵ_j tending to zero fast enough, work.

Here are the details. Assuming for simplicity $S_b(R) = S^1$, for each $j \geq 1$ choose $\epsilon_j > 0$ such that $|h^{1/j}(r_j e^{it}) - h^{1/j}(e^{it})| < 1/j$ for all $t \in [0, 2\pi]$ and for $r_j = 1 - \epsilon_j$. Write

$$\left| \int_{|w|=r_j} g(w) h^{1/j}(w) dw - \int_{|w|=1} |\tilde{g}(w)| |dw| \right| \leq A_j + B_j + C_j$$

where

$$A_j := \left| \int_{|w|=r_j} g(w)h^{1/j}(w) dw - \int_{|w|=1} \tilde{g}(w)h^{1/j}(w) dw \right| \rightarrow 0$$

because

$$\begin{aligned} A_j &\leq \int_0^{2\pi} |r_j g(r_j e^{it}) - \tilde{g}(e^{it})| |h^{1/j}(r_j e^{it})| dt \\ &\quad + \int_0^{2\pi} |\tilde{g}(e^{it})| |h^{1/j}(r_j e^{it}) - h^{1/j}(e^{it})| dt \rightarrow 0 \end{aligned}$$

as $\int_0^{2\pi} |g(re^{it}) - \tilde{g}(e^{it})| dt \rightarrow 0$ with $r \rightarrow 1$ and by the choice of ϵ_j ,

$$B_j := \left| \int_{|w|=1} (\tilde{g}(w)h^{1/j}(w) - \tilde{g}(w)H^{1/j}(w)) dw \right| \rightarrow 0$$

as $\sup_{|w|=1} |H^{1/j}(w) - h^{1/j}(w)| \rightarrow 0$ as $j \rightarrow \infty$, and

$$\begin{aligned} C_j &:= \left| \int_{|w|=1} \tilde{g}(w)H^{1/j}(w) dw - \int_{|w|=1} |\tilde{g}(w)| |dw| \right| \\ &= \left| \int_{|w|=1} (\tilde{g}(w)H^{1/j}(w) - \tilde{g}(w)H(w)) dw \right| \rightarrow 0 \end{aligned}$$

as $l(\{w : H(w) \neq H^{1/j}(w)\}) \rightarrow 0$ as $j \rightarrow \infty$. ■

Another result we are going to use belongs to approximation theory. Note that we do not use it in full generality—see comments right after the statement.

THEOREM 4 (A. Davie [7], [8], Zhijian Qiu [20]). *Let U be a bounded open subset of \mathbb{C} such that each of its components is finitely connected and the complement of U contains no isolated points. Then the following conditions are equivalent:*

- (i) *the collection of components of U is a D -collection,*
- (ii) *$A(U)$ is strongly boundedly pointwise dense in $H^\infty(U)$: each bounded analytic function h on U is a pointwise limit of a sequence $h_n \in A(U)$ with $\|h_n\|_{\overline{U}} \leq \|h\|_U$.*

Here are some comments on Theorem 4 and the way we apply it:

- (1) We need the implication (i) \Rightarrow (ii) only.
- (2) By Davie [8], (ii) is equivalent to a seemingly weaker statement (ii'): *each bounded analytic function h on U is a pointwise limit of a bounded sequence $\{h_n\} \subset A(U)$.*
- (3) Davie [7] (see preceding [6] though) proved that (i) and (ii') are equivalent when every component of U is simply connected.

(4) The case when all but finitely many components of U are simply connected can be reduced easily to that in (3) with the help of a simple geometric construction (by covering every finitely connected component by finitely many simply connected ones using only inner smooth cuts and then applying a local criterion of [11] for $A(U)$ to be pointwise boundedly dense in $H^\infty(U)$; see [20, Lemma 2.1] for the details).

(5) We employ Theorem 4 only to functions h which are non-zero on finitely many components U_1, \dots, U_m of U . This case can be reduced to the case (4) as follows. Firstly, since $h = \sum_{i=1}^m h\chi_{U_i}$, it is enough to prove the claim for each $h\chi_{U_i}$ separately, i.e., when h is non-zero only on a single component, say U_1 . Now, if U_2, U_3, \dots are all other components of U , let us modify U to get a larger open set \tilde{U} roughly by joining to each U_j ($j > 1$) some components of $\mathbb{C} \setminus U_j$ disjoint from U_1 to turn it into a simply connected domain; see details in [20, proof of Theorem 2.1]. Then $h\chi_{U_1}$ is bounded analytic in \tilde{U} and $A(\tilde{U}) \subset A(U)$, and we apply the case (4) to \tilde{U} .

We need the following consequence of Theorem 4:

COROLLARY 3.1. *Let V and $(\Omega_i)_{i=1}^N$ be as in (P1) of Theorem 1, i.e., conditions (I)–(II) hold and \bar{V} is an A-compact. Fix $1 \leq m \leq N$. Given $\epsilon > 0$ small enough, let h be a bounded analytic function in $D_\epsilon := \bigcup_{k=1}^m \Omega_k \setminus \bar{\Omega}_{k,\epsilon}$. Then given a positive sequence $\sigma_n \rightarrow 0$, there is a sequence of rational functions R_n with poles outside of $E_\epsilon := \bar{V} \setminus \bigcup_{k=1}^m \Omega_{k,\epsilon}$ such that $\|R_n\|_{E_\epsilon} \leq \|h\|_{D_\epsilon} + \sigma_n$, $R_n(z) \rightarrow h(z)$ for every $z \in D_\epsilon$ and $R_n \rightarrow 0$ in $V \setminus \bigcup_{k=1}^m \Omega_k$.*

Proof. Since \bar{V} is an A-compact and the boundary of E_ϵ is the disjoint union of ∂V and finitely many analytic curves $\partial\Omega_{k,\epsilon}$, $1 \leq k \leq m$, E_ϵ is an A-compact as well. This follows e.g. from Vitushkin's theorem [22]. Now, given $\epsilon > 0$, $\{\sigma_n\}$ and h as in the statement, let us extend h from D_ϵ to a (bounded analytic) function h^1 in $V_\epsilon := \bigcup_{k=1}^N \Omega_k \setminus \bigcup_{k=1}^m \bar{\Omega}_{k,\epsilon}$ by defining $h^1 = 0$ on $\bigcup_{k=m+1}^N \Omega_k$. In view of condition (I), by Theorem 4, there is a sequence $h_n \in A(V_\epsilon)$ such that $h_n(z) \rightarrow h^1(z)$ as $n \rightarrow \infty$ for all $z \in V_\epsilon$ and $\|h_n\|_{\bar{V}_\epsilon} \leq \|h^1\|_{V_\epsilon} = \|h\|_{D_\epsilon}$. By condition (II), $h_n \in A(\bar{V}_\epsilon) = A(E_\epsilon)$. Since E_ϵ is an A-compact, for each n there is a rational function R_n with poles outside of E_ϵ such that $\|R_n - h_n\|_{E_\epsilon} < \sigma_n$.

The sequence R_n is as required. Indeed, for any $z \in E_\epsilon$,

$$|R_n(z) - h^1(z)| \leq |R_n(z) - h_n(z)| + |h_n(z) - h^1(z)| < \sigma_n + |h_n(z) - h^1(z)|,$$

hence $\{R_n\}$ tends to $h^1(z) = h(z)$ on D_ϵ and to $h^1(z) = 0$ on $\bigcup_{k=m+1}^N \Omega_k$. At the same time, $\|R_n\|_{E_\epsilon} \leq \|h_n\|_{E_\epsilon} + \sigma_n \leq \|h\|_{D_\epsilon} + \sigma_n$. ■

3.2. Proof of Theorem 1. Let us start with the more difficult part (P1): assume that there is a measure ν on E such that (1) holds and prove (b). First of all, since $\hat{\nu} = 0$ off $E \cup \bar{V}$ and E is a C-compact by

condition (III)(ii), Lemma 2.1 immediately tells us that $\nu = 0$ on $E \setminus \partial V$. In other words, one can assume from the beginning that

$$E = \partial V.$$

Note at this point that since \bar{V} is an A-compact by (III)(i), its boundary E is a C-compact (see [10, p. 227]).

Now, let $\tilde{C} = C_g$ be the constant guaranteed by Theorem 2 for the compact $E = \partial V$ and the function $g = g_\kappa$ where

$$(13) \quad g_\kappa(z) = \begin{cases} \kappa_i(z) & \text{if } z \in \Omega_i, \text{ for each } i, \\ 0 & \text{if } z \in \mathbb{C} \setminus \bar{V}. \end{cases}$$

Let us fix a collection of uniformizations $\psi_{\Omega_k} : \Delta_{\Omega_k} \rightarrow \Omega_k$, where the closures of the circular domains Δ_{Ω_k} are pairwise disjoint. Write $\psi_k = \psi_{\Omega_k}$ and $\Delta_k = \Delta_{\Omega_k}$. Let

$$\tilde{\kappa}_k(w) = \kappa_k(\psi_k(w))\psi'_k(w)$$

for $w \in \Delta_k$. Fix $1 \leq m \leq N$.

CLAIM. *For every function \tilde{h} which is holomorphic in a small enough neighborhood of $\Gamma := \bigcup_{k=1}^m \partial\Delta_k$ and every $\epsilon > 0$ small enough,*

$$(14) \quad \left| \sum_{k=1}^m \int_{\Gamma_{k,\epsilon}} \tilde{\kappa}_k(w)\tilde{h}(w) dw \right| \leq \tilde{C}\|\tilde{h}\|_\Gamma$$

where $\Gamma_{k,\epsilon} = \{w \in \Delta_k : \text{dist}(w, \partial\Delta_k) = \epsilon\}$.

Proof of the Claim. Fix any $\epsilon > 0$ such that \tilde{h} is holomorphic in a neighborhood of $\bigcup_{k=1}^m \{w \in \Delta_k : \text{dist}(w, \partial\Delta_k) \leq \epsilon\}$. Then, for each $\tilde{\epsilon} \in (0, \epsilon)$ and $k \in \{1, \dots, m\}$,

$$(15) \quad \int_{\Gamma_{\tilde{\epsilon}}} \tilde{\kappa}_k(w)\tilde{h}(w) dw = \int_{\Gamma_\epsilon} \tilde{\kappa}_k(w)\tilde{h}(w) dw.$$

Let $h = \tilde{h} \circ \phi_k$ where $\phi_k = \psi_k^{-1} : \Omega_k \rightarrow \Delta_k$. Let $\epsilon_0 \in (\tilde{\epsilon}, \epsilon)$. Then h is defined in $D_{\epsilon_0} = \bigcup_{k=1}^m \Omega_k \setminus \bar{\Omega}_{k,\epsilon_0}$. Moreover, h is holomorphic and bounded in D_{ϵ_0} . Let $E_{\epsilon_0} = \bar{V} \setminus \bigcup_{k=1}^m \Omega_{k,\epsilon_0}$. Fix a positive sequence $\sigma_n \rightarrow 0$ and, by Corollary 3.1, find a sequence of rational functions R_n with poles outside E_{ϵ_0} such that $R_n(z) \rightarrow h(z)$ for all $z \in D_{\epsilon_0}$, $R_n(z) \rightarrow 0$ for all $z \in \bigcup_{k=m+1}^N \Omega_k$ and $\|R_n\|_{E_{\epsilon_0}} \leq \|h\|_{D_{\epsilon_0}} + \sigma_n$ for all n . Hence, for $k = 1, \dots, m$ and fixed $\tilde{\epsilon}$,

$$(16) \quad \int_{\Gamma_{\tilde{\epsilon}}} \tilde{\kappa}_k(w)\tilde{h}(w) dw = \int_{\partial\Omega_{k,\tilde{\epsilon}}} \kappa_k(z)h(z) dz = \lim_{n \rightarrow \infty} \int_{\partial\Omega_{k,\tilde{\epsilon}}} \kappa_k(z)R_n(z) dz.$$

Consider a finite collection $A_{\tilde{\epsilon}} := \{\partial\Omega_{k,\tilde{\epsilon}}\}_{k=1}^m$ of closed analytic curves. Let us complete it by a finite collection $B_{\tilde{\epsilon}}$ of pairwise disjoint closed analytic curves in $\mathbb{C} \setminus \bigcup_{k=1}^m \bar{\Omega}_k$ such that altogether $A_{\tilde{\epsilon}} \cup B_{\tilde{\epsilon}} = \partial U_{\tilde{\epsilon}}$ where $U_{\tilde{\epsilon}}$ is a (small)

neighborhood of E . Note that $R_n \rightarrow h$ on $A_{\tilde{\epsilon}}$, $R_n \rightarrow 0$ on $B_{\tilde{\epsilon}} \cap \bigcup_{k=m+1}^N \Omega_k$ while $\hat{\nu} = 0$ on $B_{\tilde{\epsilon}} \cap (\mathbb{C} \setminus \overline{V})$. Therefore, by the choice of \tilde{C} ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left| \sum_{k=1}^m \int_{\partial \Omega_{k,\tilde{\epsilon}}} \kappa_k(z) R_n(z) dz \right| \\ &= \lim_{n \rightarrow \infty} \left| \int_{\partial U_{\tilde{\epsilon}}} g_{\kappa}(z) R_n(z) dz \right| \leq \limsup_n \tilde{C} \|R_n\|_{E_{\epsilon_0}} \\ &\leq \lim_n \tilde{C} (\|h\|_{D_{\epsilon_0}} + \sigma_n) = \tilde{C} \|h\|_{D_{\epsilon_0}} = \tilde{C} \|\tilde{h}\|_{\bigcup_{k=1}^m \{w \in \Delta_k : \text{dist}(w, \partial \Delta_k) \leq \epsilon_0\}}. \end{aligned}$$

Thus, by the latter inequality along with (15) and (16), for every $\epsilon_0 \in (0, \epsilon)$,

$$\left| \sum_{k=1}^m \int_{\Gamma_{\tilde{\epsilon}}} \tilde{\kappa}_k(w) \tilde{h}(w) dw \right| \leq \tilde{C} \|\tilde{h}\|_{\bigcup_{k=1}^m \{w \in \Delta_k : \text{dist}(w, \partial \Delta_k) \leq \epsilon_0\}}.$$

But $\lim_{\epsilon_0 \rightarrow 0} \|\tilde{h}\|_{\{w \in \Delta_k : \text{dist}(w, \partial \Delta_k) \leq \epsilon_0\}} = \|\tilde{h}\|_{\partial \Delta_k}$ because \tilde{h} is continuous up to $\partial \Delta_k$. This proves (14) and the Claim.

We now proceed as follows. Since the closures of Δ_k , $1 \leq k \leq m$ (where m is finite) are pairwise disjoint, given $k \in \{1, \dots, m\}$, the Claim immediately implies that the condition of Lemma 3.2 holds for $\Delta = \Delta_k$ and $g = \tilde{\kappa}_k(w)$. We conclude there exist a measure η_k which is absolutely continuous with respect to the Lebesgue measure on $\partial \Delta_k$, a sequence $\{h_{k,j}\}_{j=1}^{\infty}$ of functions locally analytic near $\partial \Delta_k$ and a sequence $\epsilon_j \rightarrow 0$ such that $\|h_{k,j}\|_{\partial \Delta_k} \rightarrow 1$ as $j \rightarrow \infty$ and

$$(17) \quad \lim_{j \rightarrow \infty} \int_{\Gamma_{k,\epsilon_j}} \kappa_k(\psi_k(w)) \psi_k'(w) h_{k,j}(w) dw = \|\eta_k\| = \|\kappa_k\|,$$

where $\|\kappa_k\| = \limsup_{\epsilon \rightarrow 0} \int_{\partial \Omega_{k,\epsilon}} |\kappa_k(z)| |dz|$. Now, apply the Claim with \tilde{h}_j being $\tilde{h}_{k,j}$ near $\partial \Delta_k$, $1 \leq k \leq m$. By Lemma 3.2 and the Claim,

$$\sum_{k=1}^m \|\kappa_k\| = \sum_{k=1}^m \|\eta_k\| = \sum_{k=1}^m \limsup_{\epsilon \rightarrow 0} \int_{\Gamma_{k,\epsilon}} |\kappa_k(\psi_k(w)) \psi_k'(w)| |dw| \leq \tilde{C}.$$

Since \tilde{C} is independent of m , this proves (3), that is, $\sum_{k=1}^N \|\kappa_k\| < \infty$. This allows us to easily finish the proof that (a) implies (b) as follows. For each $1 \leq k \leq N$, by Lemma 3.1 (with $\Omega = \Omega_k$ and $\kappa = \kappa_k$) there is a measure ν^{κ_k} such that $\hat{\nu}^{\kappa_k} = \kappa_k$ in Ω_k and $\hat{\nu}^{\kappa_k} = 0$ off $\overline{\Omega_k}$, and moreover $\nu^{\kappa_k} \ll \omega_{\Omega_k}$ and $\|\nu^{\kappa_k}\| = \|\kappa_k\|$. Define $\nu_w = \sum_{k=1}^N \nu^{\kappa_k}$. Then $\|\nu_w\| = \sum_{k=1}^N \|\nu^{\kappa_k}\| < \infty$. Since by condition (I) harmonic measures of different Ω_k are pairwise singular, $\{\nu^{\kappa_k}\}_{k=1}^N$ are pairwise singular as well. Now, $\hat{\nu}_w = \sum_{k=1}^N \hat{\nu}^{\kappa_k}$ is equal to κ_k in Ω_k for each k , and 0 off \overline{V} . Let us compare the measures ν and ν_w . For the difference measure $\tau = \nu - \nu_w$, we have $\hat{\tau} = 0$ off E where E is a

C-compact. Hence, $\tau = \nu - \nu_\omega = 0$, and (a) \Rightarrow (b) is proved. In fact, the above considerations also prove (P2), by using again Lemma 3.1.

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