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THE OPTIMAL TIME DECAY RATES FOR INCOMPRESSIBLE VISCOELASTIC FLUIDS IN CRITICAL FRAMEWORK

Abstract. We study the Cauchy problem for multi-dimensional incompressible viscoelastic fluids in the whole space. The optimal time decay rates of strong solution constructed by Qian (2010), and Zhang (2011) in L^2-critical regularity framework are obtained for low frequencies of the data under a suitable additional condition. The proof relies on an application of Fourier analysis to a mixed parabolic-hyperbolic system, and on a refined time-weighted energy functional. As a by-product, time decay rates of L^q-L^r type are also captured in the critical framework.

1. Introduction and main results. In this paper, we consider the following multi-dimensional ($N \geq 2$) incompressible viscoelastic flow:

$$(1.1) \qquad \begin{cases} \partial_t U + v \cdot \nabla U = \nabla v U, \\ \partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v + \nabla \cdot (UU^T), \\ \nabla \cdot v = 0 \end{cases}$$

with the initial data

$$(1.2) \qquad U|_{t=0} = U_0, \quad v|_{t=0} = v_0.$$

Here $U(x,t) = (U^{ij}(x,t))_{N \times N}$ denotes the deformation tensor, $v(x,t)$ is the fluid velocity and $p(x,t)$ is the hydrodynamic pressure. The system (1.1) is one of the basic macroscopic models for viscoelastic flows, which corresponds to the so-called Hookean linear elasticity. For more physical background of this system we refer to [9, 11].

Let us assume that U_0 and v_0 satisfy the constraints

(1.3)
$$\begin{cases} \nabla \cdot v_0 = 0, \\ \det U_0 = 1, \\ \nabla \cdot U_0^T = 0, \\ U_0^{lk}\nabla_l U_0^{ij} - U_0^{lj}\nabla_l U_0^{jk} = 0. \end{cases}$$

The system (1.1) has been extensively studied. In the framework of Sobolev spaces, Chen and Zhang [4] and Lei et al. [12] proved the well-posedness for the Cauchy problem and the periodic problem, respectively. In 2014, Hu and Wu [10] obtained optimal L^2 time decay rates for global smooth solutions and their spatial derivatives and established the weak-strong uniqueness property in the class of finite energy weak solutions. In the framework of Besov spaces, Qian [15] and Zhang [16] independently proved the existence and uniqueness of a local solution, which was also shown to exist globally in time provided the initial data is small in certain norms. However, there is no result on time decay rates for global strong solutions constructed in [15, 16].

The main motivation of this paper is to consider this question. Our main ideas are based on the known global well-posedness from [15, 16], an application of Fourier analysis to a linearized parabolic-hyperbolic system, and a refined time-weighted energy functional. In low frequencies, employing decay estimates for the Fourier transform of Green's function for the linearized system, which are similar to the decay behavior of the heat kernel, and combining this with Duhamel's principle, one can obtain the desired time-weighted energy estimates. In high frequencies, employing the Fourier localization technology, the symmetrizer methods and classical nonlinear estimates including product estimates and commutator estimates, we can obtain decay estimates for the high frequencies of $(\nabla H, v)$. Finally, in order to close the energy estimates, we further exploit some decay estimates with gain of regularity for the high frequencies of Λv. With the help of these tools, we finally establish time decay estimates for the strong solution to the Cauchy problem (1.1)–(1.2).

Let us first recall a global-in-time well-posedness result for the Cauchy problem (1.1)–(1.2), which is very useful for our proof.

THEOREM 1.1 ([15, 16]). *Assume* $(U_0 - I, v_0) \in \dot{B}_{2,1}^{N/2} \cap \dot{B}_{2,1}^{N/2-1} \times \dot{B}_{2,1}^{N/2-1}$ *and* (U_0, v_0) *satisfies* (1.3). *Then there exist constants* c *and* M *such that if*

(1.4)
$$\|U_0 - I\|_{\dot{B}_{2,1}^{N/2}\cap\dot{B}_{2,1}^{N/2-1}} + \|v_0\|_{\dot{B}_{2,1}^{N/2-1}} \leq c,$$

then the Cauchy problem (1.1)–(1.2) *has a unique global solution* $(U - I, v)$ *such that*

(1.5)
$$X(t) \leq M(\|U_0 - I\|_{\dot{B}_{2,1}^{N/2}\cap\dot{B}_{2,1}^{N/2-1}} + \|v_0\|_{\dot{B}_{2,1}^{N/2-1}}),$$

where

$$(1.6) \quad X(t) := \|U - I\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{N/2-1}_{2,1}) \cap L^1_t(\dot{B}^{N/2+1}_{2,1})} + \|U - I\|^{h}_{\widetilde{L}^{\infty}_t(\dot{B}^{N/2}_{2,1}) \cap L^1_t(\dot{B}^{N/2}_{2,1})}$$
$$+ \|v\|_{\widetilde{L}^{\infty}_t(\dot{B}^{N/2-1}_{2,1})} + \|v\|_{L^1_t(\dot{B}^{N/2+1}_{2,1})}.$$

Now we state our main result.

THEOREM 1.2. *Suppose the data* $(U_0 - I, v_0)$ *satisfy the assumptions of Theorem* 1.1. *Denote* $\langle\tau\rangle := \sqrt{1 + \tau^2}$ *and* $\alpha := \min(N/4 + 2, 1/2 + N/2 - \varepsilon)$ *with* $\varepsilon > 0$ *arbitrarily small. There exists a positive constant* c *such that if in addition*

$$(1.7) \qquad D_0 := \|(U_0 - I, v_0)\|^{\ell}_{\dot{B}^{-N/2}_{2,\infty}} \le c,$$

then the global solution $(U - I, v)$ *given by Theorem* 1.1 *satisfies, for all* $t \ge 0$,

$$(1.8) \qquad D(t) \le C(D_0 + \|(\Lambda(U_0 - I), v_0)\|^{h}_{\dot{B}^{N/2-1}_{2,1}}),$$

where

$$D(t) := \sup_{s \in (-N/2, 2]} \|\langle\tau\rangle^{N/4+s/2}(U-I, v)\|^{\ell}_{L^{\infty}_t(\dot{B}^s_{2,1})} + \|\langle\tau\rangle^{\alpha}(\nabla U, v)\|^{h}_{\widetilde{L}^{\infty}_t(\dot{B}^{N/2-1}_{2,1})}$$
$$+ \|\tau \nabla v\|^{h}_{\widetilde{L}^{\infty}_t(\dot{B}^{N/2}_{2,1})}.$$

As an application of Theorem 1.2, we can obtain the following L^q-L^r type decay rates for the Cauchy problem (1.1)–(1.2).

COROLLARY 1.3. *For the solution* $(U - I, v)$ *constructed in Theorem* 1.1, *we have*

(i) *if* $-N/2 < s \le N/2$, *then*

$$\|\Lambda^s(U - I)\|_{L^2} \lesssim \langle t\rangle^{-(N/4+s/2)}\big(D_0 + \|(\Lambda(U_0 - I), v_0)\|^{h}_{\dot{B}^{N/2-1}_{2,1}}\big),$$

(ii) *if* $-N/2 < s \le N/2 - 1$, *then*

$$\|\Lambda^s v\|_{L^2} \lesssim \langle t\rangle^{-(N/4+s/2)}\big(D_0 + \|(\Lambda(U_0 - I), v_0)\|^{h}_{\dot{B}^{N/2-1}_{2,1}}\big),$$

where the fractional derivative operator Λ^{ℓ} *is defined by*

$$\Lambda^{\ell} f := \mathcal{F}^{-1}(|\cdot|^{\ell} \mathcal{F} f).$$

Proof. Employing the homogeneous Littlewood–Paley decomposition for $\Lambda^s a$, we have

$$\Lambda^s(U - I) = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q \Lambda^s(U - I).$$

Thus

$$\|\Lambda^s(U-I)\|_{L^2} \lesssim \sum_{q\in\mathbb{Z}} \|\Delta_q\Lambda^s(U-I)\|_{L^2} = \|\Lambda^s(U-I)\|_{\dot B^0_{2,1}}.$$

Based on the low-high frequencies decomposition, we may write

$$\sup_{t\in[0,T]} \langle t\rangle^{N/4+s/2}\|\Lambda^s(U-I)\|_{\dot B^0_{2,1}}$$
$$\lesssim \|\langle t\rangle^{N/4+s/2}H\|^{\ell}_{L^\infty_T(\dot B^s_{2,1})} + \|\langle t\rangle^{N/4+s/2}(U-I)\|^{h}_{L^\infty_T(\dot B^s_{2,1})}.$$

By (1.8) and the definition of $D(t)$, we get

$$\|\langle t\rangle^{N/4+s/2}(U-I)\|^{\ell}_{L^\infty_T(\dot B^s_{2,1})} \lesssim D_0 + \|(\Lambda(U_0-I),v_0)\|^{h}_{\dot B^{N/2-1}_{2,1}}.$$

If $-N/2 < s \le N/2$, then $\alpha \ge N/4 + s/2$, and we obtain

$$\|\langle t\rangle^{N/4+s/2}(U-I)\|^{h}_{L^\infty_T(\dot B^s_{2,1})} \lesssim D_0 + \|(\Lambda(U_0-I),v_0)\|^{h}_{\dot B^{N/2-1}_{2,1}}.$$

Thus,

$$\|\Lambda^s(U-I)\|_{L^2} \lesssim \langle t\rangle^{-(N/4+s/2)}\big(D_0 + \|(\Lambda(U_0-I),v_0)\|^{h}_{\dot B^{N/2-1}_{2,1}}\big).$$

Similarly, we also obtain an estimate of v for which we need the stronger conditions $-N/2 < s \le N/2 - 1$ for the high frequencies. ∎

REMARK 1.4. Compared with [10], we establish the decay rates of the strong solution in the so-called critical Besov spaces in any dimension $N \ge 2$ and not only for $N = 3$. From Corollary 1.3, we obtain the decay rates of the solution when $s = 0, 1$ and $N = 3$, which coincides with the result of [10]. Additionally, the decay index s can take both negative and nonnegative values, rather than only nonnegative integer values, which improves the classical decay results, as in [10].

NOTATIONS. We let C be a positive generic constant that may vary at different places, and denote $A \le CB$ by $A \lesssim B$. We shall also need the notations, for some $k_0 \in \mathbb{Z}$,

$$z^{\ell} := \sum_{j\le k_0} \dot\Delta_j z, \quad z^h := z - z^{\ell},$$

$$\|z\|^{\ell}_{\dot B^s_{p,1}} := \sum_{j\le k_0} 2^{js}\|\dot\Delta_j z\|_{L^p}, \quad \|z\|^{h}_{\dot B^s_{p,1}} := \sum_{j\ge k_0} 2^{js}\|\dot\Delta_j z\|_{L^p}.$$

Noting the small overlap between low and high frequencies, we have

$$\|z^{\ell}\|_{\dot B^s_{p,1}} \lesssim \|z\|^{\ell}_{\dot B^s_{p,1}} \quad \text{and} \quad \|z^h\|_{\dot B^s_{p,1}} \lesssim \|z\|^{h}_{\dot B^s_{p,1}}.$$

2. Littlewood–Paley theory and some useful lemmas. In this section, we introduce some common notations and basic theorems about the Littlewood–Paley decomposition, and present some auxiliary lemmas.

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz class of rapidly decreasing function. Given $f \in \mathcal{S}(\mathbb{R}^N)$, its Fourier transform $\mathcal{F}f = \widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx.$$

Let (χ, φ) be a couple of smooth functions valued in $[0, 1]$ such that χ is supported in the ball $\{\xi \in \mathbb{R}^N : |\xi| \leq 4/3\}$, φ is supported in the shell $\{\xi \in \mathbb{R}^N : 3/4 \leq |\xi| \leq 8/3\}$, $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$ and

$$(2.1) \qquad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N,$$

$$(2.2) \qquad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

The homogeneous frequency localization operators $\dot{\Delta}_j$ and $\dot{S}_j$ are defined by

$$\dot{\Delta}_j f := \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), \quad \dot{S}_j f := \sum_{q \leq j-1} \dot{\Delta}_q f \quad \text{for } j \in \mathbb{Z}.$$

We denote by $\mathcal{S}'_h(\mathbb{R}^N)$ the dual space of $\mathcal{Z}(\mathbb{R}^N) = \{f \in \mathcal{S}(\mathbb{R}^N) : D^\alpha \widehat{f}(0) = 0$, where α is any multi-index$\}$; it can also be identified with the quotient space $\mathcal{S}'(\mathbb{R}^N)/\mathbb{P}$ with $\mathbb{P}$ the polynomial space. The formal equality

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f$$

holds true for $f \in \mathcal{S}'_h(\mathbb{R}^N)$ and is called the *homogeneous Littlewood–Paley decomposition*. One easily verifies that with our choice of φ,

$$\dot{\Delta}_j \dot{\Delta}_q f \equiv 0 \quad \text{if } |j - q| \geq 2 \quad \text{and} \quad \dot{\Delta}_j(\dot{S}_{q-1} f \dot{\Delta}_q f) \equiv 0 \quad \text{if } |j - q| \geq 5.$$

Let us recall the definition of homogeneous Besov spaces and some of their properties (see [1, 5, 7, 8]).

DEFINITION 2.1. Let $\mathcal{S}'$ be the space of all tempered distributions. For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the *homogeneous Besov space* $\dot{B}^s_{p,1}$ to be

$$\dot{B}^s_{p,1} = \{f \in \mathcal{S}'_h : \|f\|_{\dot{B}^s_{p,1}} < \infty\}$$

with

$$\mathcal{S}'_h = \left\{f \in \mathcal{S}' : \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f = f \in \mathcal{S}'\right\} \quad \text{and} \quad \|f\|_{\dot{B}^s_{p,1}} = \sum_{j \in \mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p}.$$

DEFINITION 2.2. Let $s \in \mathbb{R}$, $1 \leq p, \rho, r \leq \infty$, $0 < T \leq \infty$. The *homogeneous space-time Besov space* $\tilde{L}^\rho_T \dot{B}^s_{p,r}$ is defined by

$$\tilde{L}^\rho_T \dot{B}^s_{p,r} = \{f \in \mathbb{R}_+ \times \mathcal{S}'(\mathbb{R}^N) : \|f\|_{\tilde{L}^\rho_T \dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\tilde{L}^\rho_T \dot{B}^s_{p,r}} := \left\|2^{js}\|\dot{\Delta}_j f(t)\|_{L^\rho(0,T;L^p)}\right\|_{\ell^r}.$$

Obviously, $\tilde{L}_T^1 \dot{B}_{p,1}^s = L_T^1 \dot{B}_{p,1}^s$. By a direct application of Minkowski's inequality, we have the following relations between these spaces:

$$L_T^\rho \dot{B}_{p,r}^s \hookrightarrow \tilde{L}_T^\rho \dot{B}_{p,r}^s, \quad r \geq \rho,$$

$$\tilde{L}_T^\rho \dot{B}_{p,r}^s \hookrightarrow L_T^\rho \dot{B}_{p,r}^s, \quad \rho \geq r.$$

The following Bernstein lemma will be frequently used.

LEMMA 2.3 ([2]). *Let $1 \leq p \leq q \leq \infty$. Assume that $f \in L^p(\mathbb{R}^N)$. Then for any $\gamma \in (\mathbb{N} \cup \{0\})^N$, there exist constants C_1, C_2 independent of f, j such that*

$$\operatorname{supp} \widehat{f} \subseteq \{|\xi| \leq A_0 2^j\} \implies \|\partial^\gamma f\|_q \leq C_1 2^{j|\gamma| + jN(1/p - 1/q)} \|f\|_p,$$

$$\operatorname{supp} \widehat{f} \subseteq \{A_1 2^j \leq |\xi| \leq A_2 2^j\} \implies \|f\|_p \leq C_2 2^{-j|\gamma|} \sup_{|\beta| = |\gamma|} \|\partial^\beta f\|_p.$$

We have the following properties of the product in Besov spaces.

LEMMA 2.4 ([7, 8]). *For all $1 \leq r, p, p_1, p_2 \leq \infty$, there exists a universal positive constant C such that*

$$\|fg\|_{\dot{B}_{p,r}^s} \leq C \|f\|_{L^\infty} \|g\|_{\dot{B}_{p,r}^s} + C \|g\|_{L^\infty} \|f\|_{\dot{B}_{p,r}^s} \quad \text{if } s > 0;$$

$$\|fg\|_{\dot{B}_{p,r}^{s_1 + s_2 - N/p}} \leq C \|f\|_{\dot{B}_{p,r}^{s_1}} \|g\|_{\dot{B}_{p,\infty}^{s_2}} \quad \begin{array}{l} \text{if } s_1, s_2 < N/p \text{ and} \\ \quad s_1 + s_2 + N \min\{0, 1 - 2/p\} > 0; \end{array}$$

$$\|fg\|_{\dot{B}_{p,r}^s} \leq C \|f\|_{\dot{B}_{p,r}^s} \|g\|_{\dot{B}_{p,\infty}^{N/p} \cap L^\infty} \quad \text{if } |s| < N/p;$$

$$\|fg\|_{\dot{B}_{2,1}^{s_1 + s_2 - N/2}} \leq C \|f\|_{\dot{B}_{2,1}^{s_1}} \|g\|_{\dot{B}_{2,1}^{s_2}} \quad \text{if } s_1, s_2 \leq N/2 \text{ and } s_1 + s_2 > 0.$$

Some embedding properties and interpolation inequalities for Besov spaces are given in

LEMMA 2.5.

- *For any $p \in [1, \infty]$ we have the continuous embedding*

$$\dot{B}_{p,1}^0 \hookrightarrow L^p \hookrightarrow \dot{B}_{p,\infty}^0.$$

- *If $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, then*

$$\dot{B}_{p_1,r_1}^s \hookrightarrow \dot{B}_{p_2,r_2}^{s - d(1/p_1 - 1/p_2)}.$$

- *The space $\dot{B}_{p,1}^{d/p}$ is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if $p < \infty$).*
- *If $1 \leq p, r_1, r_2, r \leq \infty$, $\sigma_1 \neq \sigma_2$ and $\theta \in (0, 1)$, then*

$$\|f\|_{\dot{B}_{p,r}^{\theta \sigma_2 + (1 - \theta)\sigma_1}} \leq C \|f\|_{\dot{B}_{p,r_1}^{\sigma_1}}^{1 - \theta} \|f\|_{\dot{B}_{p,r_2}^{\sigma_2}}^{\theta}.$$

LEMMA 2.6 ([8]). *Let $1 \leq p, p_1 \leq \infty$ and $\sigma \in \mathbb{R}$. There exists a constant $C > 0$ depending only on σ such that for all $q \in \mathbb{Z}$, we have*

$$\|[v \cdot \nabla, \partial_\ell \dot{\Delta}_q]a\|_{L^p} \leq Cc_q 2^{-q(\sigma-1)}\|\nabla v\|_{\dot{B}_{p_1,1}^{N/p_1}}\|\nabla a\|_{\dot{B}_{p,1}^{\sigma-1}}$$

$$for \ -\min(N/p_1, N/p') < \sigma \leq 1 + \min(N/p_1, N/p),$$

$$\|[v \cdot \nabla, \dot{\Delta}_q]a\|_{L^p} \leq Cc_q 2^{-q\sigma}\|\nabla v\|_{\dot{B}_{p_1,\infty}^{N/p_1}\cap L^\infty}\|a\|_{\dot{B}_{p,1}^{\sigma}}$$

$$for \ -\min(N/p_1, N/p') < \sigma < 1 + N/p_1,$$

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$ and $(c_q)_{q\in\mathbb{Z}}$ denotes a sequence such that $\sum_{q\in\mathbb{Z}} c_q \leq 1$.

PROPOSITION 2.7 ([1]). *Assume $\mu > 0$, $\sigma \in \mathbb{R}, (p, r) \in [1, \infty]^2$ and $1 \leq \rho_2 \leq \rho_1 \leq \infty$. Let u satisfy*

$$(2.3) \qquad \begin{cases} \partial_t u - \mu \Delta u = f, \\ u|_{t=0} = u_0. \end{cases}$$

Then for all $T > 0$ the following a priori estimate is fulfilled:

$$(2.4) \qquad \mu^{1/\rho_1}\|u\|_{\widetilde{L}_T^{\rho_1}(\dot{B}_{p,r}^{\sigma+2/\rho_1})} \lesssim \|u_0\|_{\dot{B}_{p,r}^{\sigma}} + \mu^{1/\rho_2-1}\|f\|_{\widetilde{L}_T^{\rho_2}(\dot{B}_{p,r}^{\sigma-2+2/\rho_2})}.$$

REMARK 2.8. The solutions to the Lamé system

$$(2.5) \qquad \begin{cases} \partial_t u - \mathcal{A}u = f, \\ u|_{t=0} = u_0, \end{cases}$$

also fulfill (2.4).

We finish this subsection by recalling an elementary but useful inequality.

LEMMA 2.9 ([14]). *Let $r_1, r_2 > 0$ satisfy $\max\{r_1, r_2\} > 1$. Then*

$$\int_0^t (1 + t - \tau)^{-r_1}(1 + \tau)^{-r_2}\, d\tau \leq C(r_1, r_2)(1 + t)^{-\min\{r_1, r_2\}}.$$

3. Time decay estimates. In this section, we exhibit optimal time decay estimates of strong solutions to the system (1.1)–(1.2) for initial data close to a stable equilibrium state in critical regularity framework. We divide the argument into several steps.

STEP 1: *Reformulation of the original system* (1.1)–(1.2). With $H = U - I$, the system (1.1)–(1.2) can be reformulated as follows:

$$(3.1) \qquad \begin{cases} \partial_t H - \nabla v = -v \cdot \nabla H + \nabla v H, \\ \partial_t v - \mu \Delta v - \nabla \cdot H + \nabla p = -v \cdot \nabla v + \nabla \cdot (HH^T), \\ \nabla \cdot v = 0, \\ H|_{t=0} = H_0, \quad v|_{t=0} = v_0. \end{cases}$$

As in [15], define $d^{ij} = -\Lambda^{-1}\nabla_j v^i$; then $v^i = \Lambda^{-1}\nabla_j d^{ij}$. From (3.1) we can deduce the following system for (H^{ij}, d^{ij}):

$$(3.2) \quad \begin{cases} \partial_t H^{ij} + \Lambda d^{ij} = F - v \cdot \nabla H^{ij} =: f, \\ \partial_t d^{ij} - \mu\Delta d^{ij} - \Lambda H^{ij} = G - v \cdot \nabla d^{ij} =: g, \end{cases}$$

where

$$F = \nabla_k v^i H^{kj},$$
$$G = v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}(v \cdot \nabla v^i - H^{lk}\nabla_l H^{ik})$$
$$\quad + \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij}).$$

STEP 2: *Bounds for the low frequencies.* Denoting by $A(D)$ the semigroup associated to (3.2) we have, for all $q \in \mathbb{Z}$,

$$(3.3) \quad \begin{pmatrix} \dot{\Delta}_q H^{ij}(t) \\ \dot{\Delta}_q d^{ij}(t) \end{pmatrix} = e^{tA(D)} \begin{pmatrix} \dot{\Delta}_q H_0^{ij} \\ \dot{\Delta}_q d_0^{ij} \end{pmatrix} + \int_0^t e^{(t-\tau)A(D)} \begin{pmatrix} \dot{\Delta}_q f(\tau) \\ \dot{\Delta}_q g(\tau) \end{pmatrix} d\tau.$$

From an explicit computation of the action of $e^{tA(D)}$ in Fourier variables (see e.g. [13, 3]), we discover that there exist positive constants c_0 and C depending only on q_0 and such that

$$|\mathcal{F}(e^{tA(D)}W)(\xi)| \leq Ce^{-c_0 t|\xi|^2}|\mathcal{F}W(\xi)| \quad \text{for all } |\xi| \leq 2^{q_0}.$$

Therefore, using Parseval's equality and the definition of $\dot{\Delta}_q$ we get, for all $q \leq q_0$,

$$\|e^{tA(D)}\dot{\Delta}_q W\|_{L^2} \lesssim e^{-c_0 2^{2q}t}\|\dot{\Delta}_q W\|_{L^2}.$$

Hence, multiplying by $t^{N/4+s/2}2^{qs}$ and summing over $q \leq q_0$ yields

$$(3.4) \quad t^{N/4+s/2}\sum_{q \leq q_0} 2^{qs}\|e^{tA(D)}\dot{\Delta}_q W\|_{L^2}$$
$$\lesssim \sum_{q \leq q_0} 2^{qs}e^{-c_0 2^{2q}t}\|\dot{\Delta}_q W\|_{L^2}t^{N/4+s/2}$$
$$\lesssim \sum_{q \leq q_0} 2^{q(s+N/2)}e^{-c_0 2^{2q}t}\|\dot{\Delta}_q W\|_{L^2}2^{q(-N/2)}t^{N/4+s/2}$$
$$\lesssim \|W\|_{\dot{B}_{2,\infty}^{-N/2}}^{\ell}\sum_{q \leq q_0} 2^{q(s+N/2)}e^{-c_0 2^{2q}t}t^{\frac{1}{2}(s+N/2)}.$$

For any $\sigma > 0$ there exists a constant C_σ such that

$$(3.5) \quad \sup_{t \geq 0}\sum_{q \in \mathbb{Z}} t^{\sigma/2}2^{q\sigma}e^{-c_0 2^{2q}t} \leq C_\sigma.$$

We infer from (3.4) and (3.5) that for $s > -N/2$,

$$\sup_{t \geq 0} t^{N/4+s/2}\|e^{tA(D)}W\|_{\dot{B}_{2,1}^s}^{\ell} \lesssim \|W\|_{\dot{B}_{2,\infty}^{-N/2}}^{\ell}.$$

It is also obvious that for $s > -N/2$,

$$\|e^{tA(D)}W\|^{\ell}_{\dot{B}^s_{2,1}} \lesssim \|W\|^{\ell}_{\dot{B}^{-N/2}_{2,\infty}} \sum_{q \leq q_0} 2^{q(s+N/2)} \lesssim \|W\|^{\ell}_{\dot{B}^{-N/2}_{2,\infty}}.$$

So, setting $\langle t \rangle := \sqrt{1+t^2}$, we arrive at

$$(3.6) \qquad \sup_{t \geq 0} \langle t \rangle^{N/4+s/2} \|e^{tA(D)}W\|^{\ell}_{\dot{B}^s_{2,1}} \lesssim \|W\|^{\ell}_{\dot{B}^{-N/2}_{2,\infty}},$$

and thus, taking advantage of Duhamel's formula, we get

$$(3.7) \qquad \left\| \int_0^t e^{(t-\tau)A(D)}(f,g)(\tau)\, d\tau \right\|^{\ell}_{\dot{B}^s_{2,1}}$$
$$\lesssim \int_0^t \langle t-\tau \rangle^{-(N/4+s/2)} \|(f,g)(\tau)\|^{\ell}_{\dot{B}^{-N/2}_{2,\infty}}\, d\tau.$$

We claim that for all $s \in (-N/2, 2]$ and all $t \geq 0$,

$$(3.8) \int_0^t \langle t-\tau \rangle^{-(N/4+s/2)} \|(f,g)(\tau)\|^{\ell}_{\dot{B}^{-N/2}_{2,\infty}}\, d\tau \lesssim \langle t \rangle^{-(N/4+s/2)}(X(t)^2 + D(t)^2).$$

Owing to the embedding $L^1 \hookrightarrow \dot{B}^{-N/2}_{2,\infty}$, it suffices to prove (3.8) with $\|(f,g)(\tau)\|^{\ell}_{L^1}$ instead of $\|(f,g)(\tau)\|^{\ell}_{\dot{B}^{-N/2}_{2,\infty}}$.

To bound the term with f, we use the decomposition

$$f = -v \cdot \nabla H^{ij} + H^{kj}(\nabla_k v^i)^{\ell} + H^{kj}(\nabla_k v^i)^h.$$

Now, from Hölder's inequality, the embedding $\dot{B}^0_{2,1} \hookrightarrow L^2$, the definitions of $D(t)$, α and Lemma 2.9, one may write, for all $s \in (-N/2, 2]$,

$$(3.9) \qquad \int_0^t \langle t-\tau \rangle^{-(N/4+s/2)} \|(v \cdot \nabla H^{ij})(\tau)\|^{\ell}_{L^1}\, d\tau$$
$$\lesssim \int_0^t \langle t-\tau \rangle^{-(N/4+s/2)} \|v\|_{L^2} \|\nabla H^{ij}\|_{L^2}\, d\tau$$
$$\lesssim \int_0^t \langle t-\tau \rangle^{-(N/4+s/2)} \|v\|_{\dot{B}^0_{2,1}} \|\nabla H^{ij}\|_{\dot{B}^0_{2,1}}\, d\tau$$
$$\lesssim \int_0^t \langle t-\tau \rangle^{-(N/4+s/2)} (\|v\|^{\ell}_{\dot{B}^0_{2,1}} + \|v\|^h_{\dot{B}^0_{2,1}})(\|\nabla H^{ij}|^{\ell}_{\dot{B}^0_{2,1}} + \|\nabla H^{ij}\|^h_{\dot{B}^0_{2,1}})\, d\tau$$

$$\lesssim \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{N/4}\|v(\tau)\|_{\dot{B}_{2,1}^0}^{\ell} \right) \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{N/4+1/2}\|\nabla H^{ij}\|_{\dot{B}_{2,1}^0}^{\ell} \right)$$
$$\times \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-(N/2+1/2)}\, d\tau$$

$$+ \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{N/4}\|v(\tau)\|_{\dot{B}_{2,1}^0}^{\ell} \right) \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{\alpha}\|\nabla H^{ij}(\tau)\|_{\dot{B}_{2,1}^{N/2-1}}^{h} \right)$$
$$\times \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-(N/4+\alpha)}\, d\tau$$

$$+ \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{\alpha}\|v(\tau)\|_{\dot{B}_{2,1}^{N/2-1}}^{h} \right) \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{N/4+1/2}\|\nabla H^{ij}\|_{\dot{B}_{2,1}^0}^{\ell} \right)$$
$$\times \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-(\alpha+N/4+1/2)}\, d\tau$$

$$+ \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{\alpha}\|v(\tau)\|_{\dot{B}_{2,1}^{N/2-1}}^{h} \right) \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{\alpha}\|\nabla H^{ij}\|_{\dot{B}_{2,1}^{N/2-1}}^{h} \right)$$
$$\times \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-2\alpha}\, d\tau$$

$$\lesssim D(t)^2 \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-\min(N/2+1/2,\,\alpha+N/4,\,2\alpha)}\, d\tau$$
$$\lesssim \langle t\rangle^{-(N/4+s/2)}D(t)^2.$$

The term $H^{kj}(\nabla_k v^i)^{\ell}$ may be treated along the same lines:

$$(3.10) \quad \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|H^{kj}(\nabla_k v^i)^{\ell}\|_{L^1}\, d\tau$$
$$\lesssim \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}(\|H\|_{\dot{B}_{2,1}^0}^{\ell} + \|H\|_{\dot{B}_{2,1}^0}^{h})\|\nabla v\|_{\dot{B}_{2,1}^0}^{\ell}\, d\tau$$
$$\lesssim \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{N/4}\|H(\tau)\|_{\dot{B}_{2,1}^0}^{\ell} \right) \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{N/4+1/2}\|v(\tau)\|_{\dot{B}_{2,1}^1}^{\ell} \right)$$
$$\times \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-(N/2+1/2)}\, d\tau$$

$$+ \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{\alpha}\|H(\tau)\|_{\dot{B}_{2,1}^{N/2}}^{h} \right) \left(\sup_{0\leq\tau\leq t} \langle\tau\rangle^{N/4+1/2}\|v(\tau)\|_{\dot{B}_{2,1}^1}^{\ell} \right)$$
$$\times \int_0^t \langle t-\tau\rangle^{-(N/4+s/4)}\langle\tau\rangle^{-(\alpha+N/4+1/2)}\, d\tau$$

$$\lesssim D(t)^2 \int_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-\min(N/2+1/2,\,\alpha+N/4+1/2)}\, d\tau$$
$$\lesssim \langle t\rangle^{-(N/4+s/2)}D(t)^2.$$

Regarding the term with $H^{kj}(\nabla_k v^i)^h$, we use the fact that for $t \geq 2$,

$$\int_0^t \langle t - \tau \rangle^{-(N/4+s/2)} \|H^{kj}(\nabla_k v^i)^h(\tau)\|_{L^1} \, d\tau$$

$$\lesssim \int_0^t \langle t - \tau \rangle^{-(N/4+s/2)} \|H(\tau)\|_{\dot{B}_{2,1}^0} \|\nabla v(\tau)\|_{\dot{B}_{2,1}^0}^h \, d\tau$$

$$\lesssim \int_0^1 \langle t - \tau \rangle^{-(N/4+s/2)} \|H(\tau)\|_{\dot{B}_{2,1}^0} \|\nabla v(\tau)\|_{\dot{B}_{2,1}^0}^h \, d\tau$$

$$+ \int_1^t \langle t - \tau \rangle^{-(N/4+s/2)} \|H(\tau)\|_{\dot{B}_{2,1}^0} \|\nabla v(\tau)\|_{\dot{B}_{2,1}^0}^h \, d\tau$$

$$=: I_1 + I_2.$$

Recalling the definitions of $X(t)$ and $D(t)$, we obtain

$$I_1 = \int_0^1 \langle t - \tau \rangle^{-(N/4+s/2)} \|H(\tau)\|_{\dot{B}_{2,1}^0} \|\nabla v(\tau)\|_{\dot{B}_{2,1}^0}^h \, d\tau$$

$$\lesssim \langle t \rangle^{-(N/4+s/2)} \sup_{0 \leq \tau \leq 1} \|H(\tau)\|_{\dot{B}_{2,1}^0} \int_0^1 \|\nabla v(\tau)\|_{\dot{B}_{2,1}^0}^h \, d\tau$$

$$\lesssim \langle t \rangle^{-(N/4+s/2)} \sup_{0 \leq \tau \leq 1} \|H(\tau)\|_{\dot{B}_{2,1}^0} \int_0^1 \|v(\tau)\|_{\dot{B}_{2,1}^{N/2+1}}^h \, d\tau$$

$$\lesssim \langle t \rangle^{-(N/4+s/2)} D(1) X(1)$$

and, using the fact that $\langle \tau \rangle \approx \tau$ when $\tau \geq 1$,

$$I_2 = \int_1^t \langle t - \tau \rangle^{-(N/4+s/2)} \|H(\tau)\|_{\dot{B}_{2,1}^0} \|\nabla v(\tau)\|_{\dot{B}_{2,1}^0}^h \, d\tau$$

$$\lesssim \int_1^t \langle t - \tau \rangle^{-(N/4+s/2)} (\|H(\tau)\|_{\dot{B}_{2,1}^0}^\ell + \|H(\tau)\|_{\dot{B}_{2,1}^0}^h) \|\nabla v(\tau)\|_{\dot{B}_{2,1}^0}^h \, d\tau$$

$$\lesssim \left(\sup_{1 \leq \tau \leq t} \langle \tau \rangle^{N/4} \|H(\tau)\|_{\dot{B}_{2,1}^0}^\ell \right) \left(\sup_{1 \leq \tau \leq t} \|\tau \nabla v(\tau)\|_{\dot{B}_{2,1}^{N/2}}^h \right)$$

$$\times \int_1^t \langle t - \tau \rangle^{-(N/4+s/2)} \langle \tau \rangle^{-(N/4+1)} \, d\tau$$

$$+ \left(\sup_{1 \leq \tau \leq t} \langle \tau \rangle^\alpha \|H(\tau)\|_{\dot{B}_{2,1}^{N/2}}^h \right) \left(\sup_{1 \leq \tau \leq t} \|\tau \nabla v(\tau)\|_{\dot{B}_{2,1}^{N/2}}^h \right)$$

$$\times \int_1^t \langle t - \tau \rangle^{-(N/4+s/2)} \langle \tau \rangle^{-(\alpha+1)} \, d\tau$$

$$\lesssim D(t)^2 \int\limits_1^t \langle t-\tau\rangle^{-(N/4+s/2)}\langle\tau\rangle^{-\min(\alpha+1,N/4+1)}\,d\tau$$

$$\lesssim \langle t\rangle^{-(N/4+s/2)}D(t)^2.$$

Therefore, for $t \geq 2$, we arrive at

(3.11) $$\int\limits_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|H^{kj}(\nabla_k v^i)^h(\tau)\|_{L^1}\,d\tau$$

$$\lesssim \langle t\rangle^{-(N/4+s/2)}(D(t)^2 + X(t)^2).$$

The case $t \leq 2$ is obvious as $\langle t\rangle \approx 1$ and $\langle t-\tau\rangle \approx 1$ for $0 \leq \tau \leq t \leq 2$, and

(3.12) $$\int\limits_0^t \|H^{kj}(\nabla_k v^i)^h\|_{L^1}\,d\tau$$

$$\lesssim \|H\|_{L_t^\infty(L^2)}\|\nabla v\|_{L_t^1(L^2)}^h \lesssim \|H\|_{L_t^\infty(\dot{B}_{2,1}^0)}\|\nabla v\|_{L_t^1(\dot{B}_{2,1}^0)}^h$$

$$\lesssim \|H\|_{L_t^\infty(\dot{B}_{2,1}^0)}\|v\|_{L_t^1(\dot{B}_{2,1}^{N/2+1})}^h \lesssim X(t)D(t).$$

From (3.9)–(3.12), we get

$$\int\limits_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|f(\tau)\|_{\dot{B}_{2,\infty}^{-N/2}}^\ell\,d\tau \lesssim \langle t\rangle^{-(N/4+s/2)}\big(X(t)^2 + D(t)^2\big).$$

Next, in order to bound the term of (3.8) corresponding to g, we use the decomposition

$$g := G - v \cdot \nabla d^{ij}$$

$$= -v \cdot \nabla d^{ij} + v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}(v \cdot \nabla v^i - H^{lk}\nabla_l H^{ik})$$

$$+ \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij}).$$

Similar to (3.10)–(3.12), we have

$$\int\limits_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|v \cdot (\nabla d^{ij})^\ell\|_{L^1}\,d\tau \lesssim \langle t\rangle^{-(N/4+s/2)}D(t)^2$$

and

$$\int\limits_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|v \cdot (\nabla d^{ij})^h\|_{L^1}\,d\tau \lesssim \langle t\rangle^{-(N/4+s/2)}\big(D(t)^2 + X(t)^2\big).$$

Similar to $v \cdot \nabla d^{ij}$, we have

$$\int\limits_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}v \cdot \nabla v^i\|_{L^1}\,d\tau$$

$$\lesssim \langle t\rangle^{-(N/4+s/2)}(D(t)^2 + X(t)^2).$$

For $\Lambda^{-1}\nabla_j\mathcal{P}H^{lk}\nabla_lH^{ik} + \Lambda^{-1}\nabla_k(H^{lj}\nabla_lH^{ik} - H^{lk}\nabla_lH^{ij})$, similar to (3.9), we have

$$\int\limits_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|\Lambda^{-1}\nabla_j\mathcal{P}H^{lk}\nabla_lH^{ik}+\Lambda^{-1}\nabla_k(H^{lj}\nabla_lH^{ik}-H^{lk}\nabla_lH^{ij})\|_{L^1}d\tau$$
$$\lesssim \langle t\rangle^{-(N/4+s/2)}D(t)^2.$$

Thus,

$$\int\limits_0^t \langle t-\tau\rangle^{-(N/4+s/2)}\|g(\tau)\|_{\dot{B}_{2,\infty}^{-N/2}}^\ell\, d\tau \lesssim \langle t\rangle^{-(N/4+s/2)}(X(t)^2+D(t)^2),$$

and, the proof of (3.8) is complete.

Combining (3.6) and (3.8), we conclude that for all $t \geq 0$ and $s \in (-N/2, 2]$,

$$(3.13) \qquad \langle t\rangle^{N/4+s/2}\|(H,v)\|_{\dot{B}_{2,1}^s}^\ell \lesssim D_0 + X(t)^2 + D(t)^2.$$

STEP 3: *Decay estimates for the high frequencies of* $(\nabla H, v)$. Applying the operator $\dot{\Delta}_q$ to the system (3.2), we deduce that $(\dot{\Delta}_qH, \dot{\Delta}_qv)$ satisfies

$$(3.14) \qquad \begin{cases} \partial_t\dot{\Delta}_qH^{ij} + \dot{\Delta}_q\Lambda d^{ij} = -\dot{\Delta}_q(v\cdot\nabla H^{ij}) + \dot{\Delta}_qF, \\ \partial_t\dot{\Delta}_qd^{ij} - \mu\Delta\dot{\Delta}_qd^{ij} - \dot{\Delta}_q\Lambda H^{ij} = -\dot{\Delta}_q(v\cdot\nabla d^{ij}) + \dot{\Delta}_qG. \end{cases}$$

Taking the L^2-scalar product of the first equation of (3.14) with $\dot{\Delta}_qH^{ij}$ and $\mu^2/2\Lambda^2\dot{\Delta}_qH^{ij}$, and the second equation with $\dot{\Delta}_qd^{ij}$, we obtain the following five identities:

$$(3.15) \quad \frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_qH^{ij}\|_{L^2}^2 + (\dot{\Delta}_q\Lambda d^{ij}\,|\,\dot{\Delta}_qH^{ij})$$
$$= -(\dot{\Delta}_q(v\cdot\nabla H^{ij})\,|\,\dot{\Delta}_qH^{ij}) + (\dot{\Delta}_qF\,|\,\dot{\Delta}_qH^{ij}),$$

$$(3.16) \quad \frac{\mu^2}{4}\frac{d}{dt}\|\Lambda\dot{\Delta}_qH^{ij}\|_{L^2}^2 + \frac{\mu^2}{2}(\Lambda\dot{\Delta}_qd^{ij}\,|\,\Lambda^2\dot{\Delta}_qH^{ij})$$
$$= -\frac{\mu^2}{2}(\dot{\Delta}_q(v\cdot\nabla H^{ij})\,|\,\Lambda^2\dot{\Delta}_qH^{ij}) + \frac{\mu^2}{2}(\dot{\Delta}_qF\,|\,\Lambda^2\dot{\Delta}_qH^{ij}),$$

and

$$(3.17) \quad \frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_qd^{ij}\|_{L^2}^2 + \mu\|\Lambda\dot{\Delta}_qd^{ij}\|_{L^2}^2 - (\Lambda\dot{\Delta}_qH^{ij}\,|\,\dot{\Delta}_qd^{ij})$$
$$= -(\dot{\Delta}_q(v\cdot\nabla d^{ij})\,|\,\dot{\Delta}_qd^{ij}) + (\dot{\Delta}_qG\,|\,\dot{\Delta}_qd^{ij}).$$

Applying the operation Λ to the first equation in (3.14) and taking the L^2

scalar product with $\dot{\Delta}_q d^{ij}$, and the second one with $\dot{\Delta}_q \Lambda H^{ij}$, we get

$$(3.18) \quad -\frac{\mu}{2}\frac{d}{dt}(\dot{\Delta}_q \Lambda H^{ij} \,|\, \dot{\Delta}_q d^{ij}) - \frac{\mu}{2}\|\Lambda\dot{\Delta}_q d^{ij}\|_{L^2}^2$$

$$-\frac{\mu^2}{2}(\Lambda\dot{\Delta}_q d^{ij} \,|\, \Lambda^2\dot{\Delta}_q H^{ij}) + \frac{\mu}{2}\|\Lambda\dot{\Delta}_q H^{ij}\|_{L^2}^2$$

$$= -\frac{\mu}{2}(\dot{\Delta}_q \Lambda F \,|\, \dot{\Delta}_q d^{ij}) - \frac{\mu}{2}(\dot{\Delta}_q G \,|\, \Lambda\dot{\Delta}_q H^{ij}) + \frac{\mu}{2}(\dot{\Delta}_q \Lambda(v\cdot\nabla H^{ij}) \,|\, \dot{\Delta}_q d^{ij})$$

$$+ \frac{\mu}{2}(\dot{\Delta}_q(v\cdot\nabla d^{ij}) \,|\, \Lambda\dot{\Delta}_q H^{ij}).$$

Define

$$\alpha_q^2 = \|\dot{\Delta}_q H^{ij}\|_{L^2}^2 + \frac{\mu^2}{2}\|\Lambda\dot{\Delta}_q H^{ij}\|_{L^2}^2 + \|\dot{\Delta}_q d^{ij}\|_{L^2}^2 - \mu(\dot{\Delta}_q \Lambda H^{ij} \,|\, \dot{\Delta}_q d^{ij}).$$

Employing Young's inequality, we have

$$\left|\mu(\dot{\Delta}_q \Lambda H^{ij} \,|\, \dot{\Delta}_q d^{ij})\right| \le \frac{\mu^2}{2}\|\Lambda\dot{\Delta}_q H^{ij}\|_{L^2}^2 + \frac{1}{2}\|\dot{\Delta}_q d^{ij}\|_{L^2}^2.$$

Hence, there exist positive constants c_1 and c_2 such that

$$c_1\alpha_q^2 \le \|\dot{\Delta}_q H^{ij}\|_{L^2}^2 + \|\Lambda\dot{\Delta}_q H^{ij}\|_{L^2}^2 + \|\dot{\Delta}_q d^{ij}\|_{L^2}^2 \le c_2\alpha_q^2.$$

Thus,

$$\alpha_q \approx \|\Lambda\dot{\Delta}_q H^{ij}\|_{L^2} + \|\dot{\Delta}_q d^{ij}\|_{L^2} \quad \text{for } q > q_0.$$

Combined with (3.15)–(3.18), this yields

$$(3.19) \quad \frac{1}{2}\frac{d}{dt}\alpha_q^2 + \frac{\mu}{2}\|\Lambda\dot{\Delta}_q d^{ij}\|_{L^2}^2 + \frac{\mu}{2}\|\Lambda\dot{\Delta}_q H^{ij}\|_{L^2}^2$$

$$= -(\dot{\Delta}_q(v\cdot\nabla H^{ij}) \,|\, \dot{\Delta}_q H^{ij}) - \frac{\mu^2}{2}(\dot{\Delta}_q(v\cdot\nabla H^{ij}) \,|\, \Lambda^2\dot{\Delta}_q H^{ij})$$

$$- (\dot{\Delta}_q(v\cdot\nabla d^{ij}) \,|\, \dot{\Delta}_q d^{ij})$$

$$+ \frac{\mu}{2}(\dot{\Delta}_q \Lambda(v\cdot\nabla H^{ij}) \,|\, \dot{\Delta}_q d^{ij}) + \frac{\mu}{2}(\dot{\Delta}_q(v\cdot\nabla d^{ij}) \,|\, \Lambda\dot{\Delta}_q H^{ij})$$

$$+ (\dot{\Delta}_q F \,|\, \dot{\Delta}_q H^{ij}) + \frac{\mu^2}{2}(\dot{\Delta}_q F \,|\, \Lambda^2\dot{\Delta}_q H^{ij}) + (\dot{\Delta}_q G \,|\, \dot{\Delta}_q d^{ij})$$

$$- \frac{\mu}{2}(\dot{\Delta}_q \Lambda F \,|\, \dot{\Delta}_q d^{ij}) - \frac{\mu}{2}(\dot{\Delta}_q G \,|\, \Lambda\dot{\Delta}_q H^{ij})$$

$$\lesssim \alpha_q\big(\|\dot{\Delta}_q(v\cdot\nabla H^{ij})\|_{L^2} + \|\Lambda\dot{\Delta}_q(v\cdot\nabla H^{ij})\|_{L^2} + \|\dot{\Delta}_q(v\cdot\nabla d^{ij})\|_{L^2}$$

$$+ \|\dot{\Delta}_q F\|_{L^2} + \|\dot{\Delta}_q \Lambda F\|_{L^2} + \|\dot{\Delta}_q G\|_{L^2}\big).$$

For $q \geq q_0$ and for some $c_0 = c(q_0) > 0$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\alpha_q^2 + c_0\alpha_q^2 \\
&\lesssim \alpha_q\big(\|\dot{\Delta}_q(v\cdot\nabla H^{ij})\|_{L^2} + \|\Lambda\dot{\Delta}_q(v\cdot\nabla H^{ij})\|_{L^2} + \|\dot{\Delta}_q(v\cdot\nabla d^{ij})\|_{L^2} \\
&\quad + \|\dot{\Delta}_q F\|_{L^2} + \|\dot{\Delta}_q\Lambda F\|_{L^2} + \|\dot{\Delta}_q G\|_{L^2}\big) \\
&\lesssim \big(\|(F_q,\Lambda F_q,G_q)\|_{L^2} + \|R_q(v,H^{ij})\|_{L^2} + \|R_q(v,d^{ij})\|_{L^2} \\
&\quad + \|\widetilde{R}_k(v,H^{ij})\|_{L^2} + \|\nabla v\|_{L^\infty}\alpha_q\big)\alpha_q,
\end{aligned}
$$

where

$$
F_q = \dot{\Delta}_q F, \qquad G_q = \dot{\Delta}_q G,
$$

with

$$
\begin{aligned}
R_q(v,b) &:= [v\cdot\nabla,\dot{\Delta}_q]b = v\cdot\nabla\dot{\Delta}_q b - \dot{\Delta}_q(v\cdot\nabla b) \quad \text{for } b \in \{H^{ij}, b^{ij}\}, \\
\widetilde{R}_q^i(v,H^{ij}) &:= [v\cdot\nabla,\partial_i\dot{\Delta}_q]H^{ij} = v\cdot\nabla\partial_i\dot{\Delta}_q H^{ij} - \partial_i\dot{\Delta}_q(v\cdot\nabla H^{ij}).
\end{aligned}
$$

After time integration, we discover that

$$
\begin{aligned}
e^{c_0 t}\alpha_q(t) \leq \alpha_q(0) + \int_0^t e^{c_0\tau}\big(\|(F_q,\Lambda F_q,G_q)\|_{L^2} + \|R_q(v,H^{ij})\|_{L^2} \\
+ \|R_q(v,d^{ij})\|_{L^2} + \|\widetilde{R}_k(v,H^{ij})\|_{L^2} + \|\nabla v\|_{L^\infty}\alpha_q\big)\,d\tau.
\end{aligned}
$$

For $q \geq q_0$, we have $\alpha_q \approx \|(\Lambda\dot{\Delta}_q H^{ij}, \dot{\Delta}_q d^{ij})\|_{L^2}$. Then

$$
\begin{aligned}
\langle t\rangle^\alpha\|(\Lambda\dot{\Delta}_q H^{ij}, \dot{\Delta}_q d^{ij})(t)\|_{L^2} &\lesssim \langle t\rangle^\alpha e^{-c_0 t}\|(\Lambda\dot{\Delta}_q H^{ij}, \dot{\Delta}_q d^{ij})(0)\|_{L^2} \\
&\quad + \langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}\big(\|(F_q,\Lambda F_q,G_q)\|_{L^2} + \|R_q(v,H^{ij})\|_{L^2} + \|R_q(v,d^{ij})\|_{L^2} \\
&\quad + \|\widetilde{R}_k(v,H^{ij})\|_{L^2} + \|\nabla v\|_{L^\infty}\alpha_q\big)\,d\tau,
\end{aligned}
$$

and multiplying both sides by $2^{(N/2-1)q}$, taking the supremum on $[0,T]$, and summing over $q \geq q_0$, we get

$$
\begin{aligned}
(3.20) \quad \|\langle t\rangle^\alpha(\Lambda H^{ij}, d^{ij})\|_{\widetilde{L}_T^\infty(\dot{B}_{2,1}^{N/2-1})}^h \\
\lesssim \|(\Lambda H_0^{ij}, d_0^{ij})\|_{\dot{B}_{2,1}^{N/2-1}}^h + \sum_{q\geq q_0}\sup_{0\leq t\leq T}\langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}2^{(N/2-1)q}S_q\,d\tau
\end{aligned}
$$

with $S_q := \sum_{i=1}^5 S_q^i$ and

$$
\begin{aligned}
S_q^1 &:= \|(F_q,\Lambda F_q,G_q)\|_{L^2}, \quad S_q^2 := \|R_q(v,H^{ij})\|_{L^2}, \quad S_q^3 := \|R_q(v,d^{ij})\|_{L^2}, \\
S_q^4 &:= \|\widetilde{R}_k(v,H^{ij})\|_{L^2}, \qquad S_q^5 := \|\nabla v\|_{L^\infty}\|(\Lambda\dot{\Delta}_q H^{ij}, \dot{\Delta}_q d^{ij})\|_{L^2}.
\end{aligned}
$$

Bounding the sum, for $0 \leq t \leq 2$, and taking advantage of Proposition 2.6, we end up with

$$(3.21) \quad \sum_{q \geq q_0} \sup_{0 \leq t \leq 2} \langle t \rangle^\alpha \int_0^t e^{c_0(\tau - t)} 2^{(N/2-1)q} S_q(\tau) \, d\tau \lesssim \int_0^2 \sum_{q \geq q_0} 2^{(N/2-1)q} S_q(\tau) \, d\tau$$

$$\lesssim \int_0^2 \left(\|(F, \Lambda F, G)\|_{\dot{B}_{2,1}^{N/2-1}}^h + \|\nabla v\|_{\dot{B}_{2,1}^{N/2}} \|(H^{ij}, d^{ij}, \nabla H^{ij})\|_{\dot{B}_{2,1}^{N/2-1}} \right) d\tau$$

$$\lesssim \int_0^2 \left(\|(\Lambda F, G)\|_{\dot{B}_{2,1}^{N/2-1}}^h + \|\nabla v\|_{\dot{B}_{2,1}^{N/2}} \|(H^{ij}, d^{ij}, \nabla H^{ij})\|_{\dot{B}_{2,1}^{N/2-1}} \right) d\tau$$

$$=: Q_1 + Q_2.$$

From Lemma 2.4, we bound the terms Q_1 and Q_2 as follows:

$$\int_0^2 \|\Lambda F\|_{\dot{B}_{2,1}^{N/2-1}}^h \, d\tau \lesssim \int_0^2 \|F\|_{\dot{B}_{2,1}^{N/2}}^h \, d\tau \lesssim \int_0^2 \|\nabla_k v^i H^{kj}\|_{\dot{B}_{2,1}^{N/2}}^h \, d\tau$$

$$\lesssim \int_0^2 \|\nabla_k v^i\|_{\dot{B}_{2,1}^{N/2}} \|H^{kj}\|_{\dot{B}_{2,1}^{N/2}} \, d\tau$$

$$\lesssim \|H^{kj}\|_{L_t^\infty(\dot{B}_{2,1}^{N/2})} \|\nabla_k v^i\|_{L_t^1(\dot{B}_{2,1}^{N/2})} \lesssim X(2)^2,$$

$$\int_0^2 \|G\|_{\dot{B}_{2,1}^{N/2-1}}^h \, d\tau$$

$$\lesssim \int_0^2 \|v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}v \cdot \nabla v^i\|_{\dot{B}_{2,1}^{N/2-1}}^h \, d\tau$$

$$+ \int_0^2 \|\Lambda^{-1}\nabla_j \mathcal{P}H^{lk}\nabla_l H^{ik} + \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij})\|_{\dot{B}_{2,1}^{N/2-1}}^h \, d\tau$$

$$\lesssim \int_0^2 \|v\|_{\dot{B}_{2,1}^{N/2-1}} \|\nabla v\|_{\dot{B}_{2,1}^{N/2}} \, d\tau + \int_0^2 \|H\|_{\dot{B}_{2,1}^{N/2}} \|\nabla H\|_{\dot{B}_{2,1}^{N/2-1}} \, d\tau$$

$$\lesssim \|v\|_{L_t^\infty(\dot{B}_{2,1}^{N/2-1})} \|\nabla v\|_{L_t^1(\dot{B}_{2,1}^{N/2})} + \|H\|_{L_t^2(\dot{B}_{2,1}^{N/2})} \|\nabla H\|_{L_t^2(\dot{B}_{2,1}^{N/2-1})}$$

$$\lesssim X(2)^2,$$

and

$$\int_0^2 \|\nabla v\|_{\dot{B}_{2,1}^{N/2}} \|(H^{ij}, d^{ij}, \nabla H^{ij})\|_{\dot{B}_{2,1}^{N/2-1}} \, d\tau$$

$$\lesssim \|\nabla v\|_{L_T^1(\dot{B}_{2,1}^{N/2})} \|(H^{ij}, d^{ij}, \nabla H^{ij})\|_{L_T^\infty(\dot{B}_{2,1}^{N/2-1})} \lesssim X(2)^2.$$

Therefore, for $t \leq 2$,

$$
(3.22) \qquad \sum_{q \geq q_0} \sup_{0 \leq t \leq 2} \langle t \rangle^\alpha \int_0^t e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q \, d\tau \lesssim X(2)^2.
$$

To bound the supremum on $[2, T]$, we split the integral on $[0, t]$ into integrals on $[0, 1]$ and $[1, t]$, respectively. The $[0, 1]$ part is easy to handle:

$$
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^\alpha \int_0^1 e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q \, d\tau \leq \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^\alpha e^{-c_0/2t} \int_0^1 2^{(N/2-1)q} S_q \, d\tau
$$

$$
\lesssim \int_0^1 \sum_{q \geq q_0} 2^{(N/2-1)q} S_q \, d\tau.
$$

Hence

$$
(3.23) \qquad \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^\alpha \int_0^1 e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q(\tau) \, d\tau \lesssim X(1)^2.
$$

Let us finally consider the $[1, t]$ part of the integral for $2 \leq t \leq T$. We shall use repeatedly the inequalities

$$
(3.24) \qquad \|\tau \nabla v\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{N/2})} \lesssim D(t),
$$

which are straightforward for the high frequencies, while for the low frequencies they stem from

$$
\|\tau \nabla v\|_{\widetilde{L}_t^\infty(\dot{B}_{2,1}^{N/2})}^\ell \lesssim \|\langle \tau \rangle^{N/4+1/2} v\|_{L_t^\infty(\dot{B}_{2,1}^1)}^\ell \lesssim D(t).
$$

Regarding the contribution of S_q^1, by Lemma 2.9 we first notice that

$$
\sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^\alpha \int_1^t e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q^1(\tau) \, d\tau \lesssim \|\tau^\alpha (\Lambda F, G)\|_{\widetilde{L}_T^\infty(\dot{B}_{2,1}^{N/2-1})}^h.
$$

Now, product laws in tilde spaces ensure that

$$
\|\tau^\alpha \Lambda F\|_{\widetilde{L}_T^\infty(\dot{B}_{2,1}^{N/2-1})}^h \lesssim \|\tau^{\alpha-1} H\|_{\widetilde{L}_T^\infty(\dot{B}_{2,1}^{N/2})} \|\tau \nabla v\|_{\widetilde{L}_T^\infty(\dot{B}_{2,1}^{N/2})}.
$$

The high frequencies of the first term are obviously bounded by $D(T)$:

$$
(3.25) \qquad \|\tau^{\alpha-1} H\|_{\widetilde{L}_T^\infty(\dot{B}_{2,1}^{N/2})}^h \lesssim \|\tau^\alpha H\|_{\widetilde{L}_T^\infty(\dot{B}_{2,1}^{N/2})}^h \lesssim D(T).
$$

As for the low frequencies of the first term, we notice that if $N \leq 4$ then for

all small enough $\varepsilon > 0$,

$$
(3.26) \qquad \|\tau^{\alpha-1}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim \|\tau^{\alpha-1}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{N/2-2\varepsilon}_{2,1})}
$$

$$
\lesssim \|\tau^{\alpha-1-N/2+\varepsilon}\tau^{N/2-\varepsilon}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{N/2-2\varepsilon}_{2,1})}
$$

$$
\lesssim \|\tau^{N/2-\varepsilon}c\|^{\ell}_{L^{\infty}_T(\dot{B}^{N/2-2\varepsilon}_{2,1})} \lesssim D(T),
$$

and if $N \geq 5$ then

$$
(3.27) \qquad \|\tau^{\alpha-1}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim \|\tau^{\alpha-1}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{2}_{2,1})}
$$

$$
\lesssim \|\tau^{\alpha-2-N/4}\tau^{N/4+1}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{2}_{2,1})}
$$

$$
\lesssim \|\tau^{N/4+1}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{2}_{2,1})} \lesssim D(T).
$$

Combining (3.25)–(3.27), we obtain

$$
(3.28) \qquad\qquad \|\tau^{\alpha-1}H\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim D(T).
$$

Therefore, using (3.24) and (3.28) we get

$$
\|\tau^{\alpha}\Lambda F\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \lesssim D(T)^2.
$$

Noticing that $G = v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}(v \cdot \nabla v^i - H^{lk}\nabla_l H^{ik}) + \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij})$, similar to (3.28) we get

$$
(3.29) \qquad\qquad \|\tau^{\alpha-1}v\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \lesssim D(T).
$$

Employing (3.24) and (3.29), we obtain

$$
\|\tau^{\alpha}\big(v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}(v \cdot \nabla v^i)\big)\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})}
$$

$$
\lesssim \|\tau^{\alpha-1}v\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \|\tau\nabla v\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim D(T)^2.
$$

Product laws in tilde spaces ensure that

$$
\|\tau^{\alpha}\big(\Lambda^{-1}\nabla_j \mathcal{P}(H^{lk}\nabla_l H^{ik}) + \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij})\big)\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})}
$$

$$
\lesssim \|\tau^{\alpha/2}H\|^{2}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})}.
$$

The high frequencies of the first term are obviously bounded by $D(T)$:

$$
(3.30) \qquad\qquad \|\tau^{\alpha/2}H\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim \|\tau^{\alpha}H\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim D(T).
$$

As for the low frequencies of the first term, we notice that if $N \leq 4$ then for

all small enough $\varepsilon > 0$,

$$(3.31) \qquad \|\tau^{\alpha/2}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim \|\tau^{\alpha/2}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{N/2-2\varepsilon}_{2,1})}$$
$$\lesssim \|\tau^{\alpha/2-N/2+\varepsilon N/2-\varepsilon}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{N/2-2\varepsilon}_{2,1})} \lesssim D(T),$$

and if $N \geq 5$ then

$$(3.32) \qquad \|\tau^{\alpha/2}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim \|\tau^{\alpha/2}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{2}_{2,1})}$$
$$\lesssim \|\tau^{\alpha/2-N/4-1}\tau^{N/4+1}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{2}_{2,1})} \lesssim D(T).$$

Combining (3.30)–(3.32), we obtain

$$(3.33) \qquad \|\tau^{\alpha/2}H\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \lesssim D(T).$$

Therefore,

$$\|\tau^{\alpha}\big(\Lambda^{-1}\nabla_j\mathcal{P}(H^{lk}\nabla_l H^{ik}) + \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij})\big)\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})}$$
$$\lesssim D(T)^2.$$

Thus,

$$\|\tau^{\alpha}G\|^{h}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \lesssim D(T)^2.$$

We end up with

$$(3.34) \qquad \sum_{q\geq q_0}\sup_{2\leq t\leq T}\langle t\rangle^{\alpha}\int_1^t e^{c_0(\tau-t)}2^{(N/2-1)q}S^1_q(\tau)\,d\tau \lesssim X(T)^2 + D(T)^2.$$

To bound the term with S^2_q, we use the fact that

$$\int_1^t e^{c_0(\tau-t)}\|R_q(v, H^{ij})\|_{L^2}\,d\tau \leq \|R_q(\tau v, \tau^{\alpha-1}H^{ij})\|_{L^{\infty}_t(L^2)}\int_1^t e^{c_0(\tau-t)}\tau^{-\alpha}\,d\tau.$$

Hence, thanks to Lemma 2.9 and Proposition 2.6,

$$\sum_{q\geq q_0}\sup_{2\leq t\leq T}\langle t\rangle^{\alpha}\int_1^t e^{c_0(\tau-t)}2^{(N/2-1)q}S^2_q(\tau)\,d\tau$$
$$\lesssim \sum_{q\geq q_0}2^{(N/2-1)q}\|R_q(\tau v, \tau^{\alpha-1}H^{ij})\|_{L^{\infty}_t(L^2)}$$
$$\lesssim \|\tau\nabla v\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})}\|\tau^{\alpha-1}H\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})}.$$

The first term on the right side may be bounded thanks to (3.24), and the high frequencies of the last term on the right side are obviously bounded by $D(T)$. As for the low frequencies of the last term, we notice that if $N \leq 6$

then for all small enough $\varepsilon > 0$,

$$\|\tau^{\alpha-1}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \lesssim \|\tau^{\alpha-1}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{N/2-1-2\varepsilon}_{2,1})}$$

$$\lesssim \|\tau^{\alpha-1/2-N/2+\varepsilon}\tau^{N/2-1/2-\varepsilon}H\|^{\ell}_{L^{\infty}_T(\dot{B}^{N/2-1-2\varepsilon}_{2,1})} \lesssim D(T),$$

and if $N \geq 7$ then

$$\|\tau^{\alpha-1}H\|^{\ell}_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \lesssim \|\tau^{\alpha-1}H\|^{\ell}_{L^{\infty}_T(\dot{B}^2_{2,1})}$$

$$\lesssim \|\tau^{\alpha-N/4-2}\tau^{N/4+1}H\|^{\ell}_{L^{\infty}_T(\dot{B}^2_{2,1})} \lesssim D(T).$$

We eventually get

$$(3.35) \qquad \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^{\alpha} \int_1^t e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q^2(\tau)\, d\tau \lesssim D(T)^2.$$

Similarly, we have

$$(3.36) \qquad \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^{\alpha} \int_1^t e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q^3(\tau)\, d\tau \lesssim D(T)^2.$$

Finally, using product laws, (3.24), (3.28), (3.29) and Lemma 2.9, we obtain

$$(3.37) \qquad \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^{\alpha} \int_1^t e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q^4(\tau)\, d\tau$$

$$\lesssim \|\tau\nabla v\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \|\tau^{\alpha-1}\nabla H\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \sup_{2 \leq t \leq T} t^{\alpha} \int_1^t e^{c_0(\tau-t)} \tau^{-\alpha}\, d\tau$$

$$\lesssim D(T)^2$$

and

$$(3.38) \qquad \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^{\alpha} \int_1^t e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q^5(\tau)\, d\tau$$

$$\lesssim \|\tau\nabla v\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2}_{2,1})} \|\tau^{\alpha-1}(\nabla H, d)\|_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \sup_{2 \leq t \leq T} t^{\alpha} \int_1^t e^{c_0(\tau-t)} \tau^{-\alpha}\, d\tau$$

$$\lesssim D(T)^2.$$

Putting all the above inequalities (3.34)–(3.38) together, we conclude that

$$(3.39) \qquad \sum_{q \geq q_0} \sup_{2 \leq t \leq T} \langle t \rangle^{\alpha} \int_1^t e^{c_0(\tau-t)} 2^{(N/2-1)q} S_q(\tau)\, d\tau \lesssim X(T)^2 + D(T)^2.$$

Then plugging (3.22), (3.23) and (3.39) into (3.20) yields

$$(3.40) \quad \|\langle\tau\rangle^{\alpha}(\Lambda H^{ij}, d^{ij})\|^h_{\widetilde{L}^{\infty}_T(\dot{B}^{N/2-1}_{2,1})} \lesssim \|(\Lambda H_0^{ij}, d_0^{ij})\|^h_{\dot{B}^{N/2-1}_{2,1}} + X(T)^2 + D(T)^2.$$

STEP 4: *Decay estimates with gain of regularity for the high frequencies of* ∇d^{ij}. In order to close the estimates of the time-weighted energy functional, this step is devoted to bounding the last terms of $D(t)$. We shall use the fact that the velocity d^{ij} satisfies the equation

$$(3.41) \qquad \partial_t d^{ij} - \mu \Delta d^{ij} = \Lambda H^{ij} + G - v \cdot \nabla d^{ij} =: h.$$

Therefore,

$$\partial_t(t\Delta d^{ij}) - \mu \Delta(t\Delta d^{ij}) = \Delta d^{ij} + t\Delta h.$$

We deduce from Proposition 2.7 that

$$\|\tau \Delta d^{ij}\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})} \lesssim \|\Delta d^{ij}\|^h_{L^1_t(\dot B^{N/2-1}_{2,1})} + \|\tau \Delta h\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-3}_{2,1})},$$

whence, using the bounds given by Theorem 1.1,

$$(3.42) \qquad \|\tau \nabla d^{ij}\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2}_{2,1})} \lesssim X(0) + \|\tau h\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})}.$$

In order to bound the first term of h, we notice that, because $\alpha \geq 1$ and according to (3.40), we have

$$\|\tau \Lambda H^{ij}\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})} \lesssim \|\langle \tau \rangle^\alpha \Lambda H^{ij}\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})} \lesssim X(0) + X(t)^2 + D(t)^2.$$

Noticing that $G = v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}(v \cdot \nabla v^i - H^{lk}\nabla_l H^{ik}) + \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij})$, from (3.24), (3.33) and the definition of $X(t)$ we have

$$(3.43) \quad \|\tau\big(v \cdot \nabla(-\Lambda^{-1}\nabla_j v^i) + \Lambda^{-1}\nabla_j \mathcal{P}(v \cdot \nabla v^i)\big)\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})}$$
$$\lesssim \|v\|_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})} \|\tau \nabla v\|_{\widetilde{L}^\infty_t(\dot B^{N/2}_{2,1})} \lesssim X(t)D(t)$$

and

$$(3.44)$$
$$\|\tau(\Lambda^{-1}\nabla_j \mathcal{P}(H^{lk}\nabla_l H^{ik}) + \Lambda^{-1}\nabla_k(H^{lj}\nabla_l H^{ik} - H^{lk}\nabla_l H^{ij}))\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})}$$
$$\lesssim \|\tau H \cdot \nabla H\|_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})} \lesssim \|\tau^{1/2} H\|^2_{\widetilde{L}^\infty_t(\dot B^{N/2}_{2,1})} \lesssim D(t)^2.$$

The third term of h is similar to (3.43), and we obtain

$$\|\tau(v \cdot \nabla d^{ij})\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2-1}_{2,1})} \lesssim X(t)D(t).$$

Hence, reverting to (3.42), we get

$$(3.45) \qquad \|\tau \nabla d^{ij}\|^h_{\widetilde{L}^\infty_t(\dot B^{N/2}_{2,1})} \lesssim X(0) + X(t)^2 + D(t)^2.$$

Finally, adding the resulting inequality to (3.13) and (3.40) yields, for all $t \geq 0$,

$$D(t) \lesssim X(0) + D_0 + \|(\Lambda H_0^{ij}, d_0^{ij})\|_{\dot{B}_{2,1}^{N/2-1}}^h + X(t)^2 + D(t)^2$$

$$\lesssim D_0 + \|(\Lambda H_0^{ij}, d_0^{ij})\|_{\dot{B}_{2,1}^{N/2-1}}^h + X(t)^2 + D(t)^2,$$

where we have used $X(0)^\ell = \|(H_0, v_0)\|_{\dot{B}_{2,1}^{N/2-1}}^\ell \lesssim \|(H_0, v_0)\|_{\dot{B}_{2,\infty}^{-N/2}}^\ell$. As Theorem 1.1 ensures that $X(t)$ is small, one can conclude that (1.8) is fulfilled for all time if D_0 and $\|(\Lambda H_0^{ij}, d_0^{ij})\|_{\dot{B}_{2,1}^{N/2-1}}^h$ are small enough. This completes the proof of Theorem 1.2.

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