Abstract. We introduce highly efficient solvers of nonlinear equations involving Banach space valued operators. The local convergence is based only on the first Fréchet derivative in contrast to earlier works using derivatives up to order seven to show the sixth order of convergence. Hence, we extend the applicability of these methods. Numerical examples are used to test the conditions of the theoretical results.

1. Introduction. One of the greatest challenges in computational mathematics is to find a solution $x_*$ of the equation

$$F(x) = 0,$$

(1.1)

where $F : \Omega \to \mathcal{E}_2$ is a Fréchet differentiable operator. Here and below, $\Omega \subset \mathcal{E}_1$ is a nonempty, open set, and $\mathcal{E}_1, \mathcal{E}_2$ are Banach spaces.

In this study, we are concerned with the local convergence of the sixth order method given as

$$x_0 \in \Omega, \quad y_n = x_n - \frac{2}{3} F'(x_n)^{-1} F(x_n),$$

$$z_n = x_n - \left[ a_1 I + a_2 (F'(y_n)^{-1} F'(x_n))^2 \right] F'(x_n)^{-1} F(y_n),$$

$$x_{n+1} = z_n - \left[ (b_2 F'(x_n) + b_3 F'(y_n))^{-1} (F'(x_n) + b_1 F'(y_n)) \right] \cdot F'(x_n)^{-1} F(z_n)$$

(1.2)

where $a_1, a_2, b_1, b_2, b_3$ are given real numbers satisfying

$$a_1 = 1 - a_2, \quad a_2 = \frac{3}{8}, b_2 = b_1 - b_3 + 1, \quad b_3 = \frac{1}{2} (5b_1 + 3).$$
and \( b_1 \neq -1 \) is a free parameter. Method (1.2) was studied in [20], but for the case \( \mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^k \) (\( k \) a natural number). Using conditions on high order derivative, and Taylor series (although these derivatives do not appear in method (1.2)), convergence order was established in [20]. The hypotheses on higher order derivatives limit the usage of method (1.2).

As an academic example, let \( \mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R} \) and \( \Omega = [-1/2, 3/2] \). Define \( F \) on \( \Omega \) by
\[
F(x) = x^3 \log(x^2) + x^5 - x^4.
\]
Then we have \( x^* = 1 \), and
\[
\begin{align*}
F'(x) &= 3x^2 \log(x^2) + 5x^4 - 4x^3 + 2x^2, \\
F''(x) &= 6x \log(x^2) + 20x^3 - 12x^2 + 10x, \\
F'''(x) &= 6 \log(x^2) + 60x^2 - 24x + 22.
\end{align*}
\]
Obviously \( F'''(x) \) is not bounded on \( \Omega \). So, the convergence of method (1.2) is not guaranteed by the analysis in [17, 18, 20].

Other problems with the usage of method (1.2) are: no information on how to choose \( x_0 \); bounds on \( \|x_n - x^*\| \) and information on the location of \( x^* \). All these are addressed in this paper by only using conditions on the first derivative, and in the more general setting of Banach space valued operators. To avoid the use of Taylor series and high convergence order derivatives, we rely on the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [1, 2, 4].

The rest of the paper includes: the local convergence analysis in Section 2, and an example in Section 3.

2. Local convergence analysis. We base the local convergence analysis of method (1.2) on some real functions and parameters. Consider a continuous increasing function \( w_0 : [0, \infty) \to [0, \infty) \) satisfying \( w_0(0) = 0 \). Assume that the equation
\[
(2.1) \quad w_0(t) = 1
\]
has a minimal positive solution denoted by \( \rho_0 \). Consider also continuous increasing functions \( w : [0, \rho_0) \to [0, \infty) \) and \( v : [0, \rho_0) \to [0, \infty) \) with \( w(0) = 0 \). Define functions \( g_1 \) and \( h_1 \) on the interval \( [0, \rho_0) \) by
\[
g_1(t) = \int_{0}^{1} w((1 - \theta)t) d\theta + \frac{1}{3} \int_{0}^{1} v(\theta) d\theta, \quad h_1(t) = g_1(t) - 1.
\]
Assume that
\[
(2.2) \quad \frac{1}{3} v(0) - 1 < 0.
\]
By these definitions and (2.1), we have \( h_1(0) = v(0)/3 - 1 < 0 \) and \( h_1(t) \to \infty \) as \( t \to \rho_0^- \). The mean value theorem then ensures that the equation \( h_1(t) = 0 \)}
has at least one solution in the interval \((0, \rho_0)\). Denote the minimal such solution by \(r_1\). Assume that the equation
\[ w_0(g_1(t)t) = 1 \]  
(2.3)  
has a minimal positive solution \(\rho\). Set \(\rho_1 = \min\{\rho_0, \rho\}\). Moreover, define functions \(g_2\) and \(h_2\) on \([0, \rho_1]\) by
\[ g_2(t) = \frac{\int_0^1 w((1 - \theta)t) \, d\theta}{1 - w_0(t)} \]
and \(h_2(t) = g_2(t) - 1\). By these definitions \(h_2(0) = -1\) and \(h_2(t) \to \infty\) as \(t \to \rho_1^-\). Denote by \(r_2\) the minimal solution of \(h_2(t) = 0\) in \((0, \rho_1)\). Assume that the equations
\[ w_0(g_2(t)t) = 1 \]  
(2.4)  
and
\[ p(t) = 1 \]  
(2.5)  
have minimal positive solutions \(\rho_2\), and \(\rho_p\), respectively, where
\[ p(t) = \frac{l}{|b_2|w_0(t) + b_3w_0(g_1(t)t), \quad b_2 + b_3 \neq 0. \]
Set \(\rho_3 = \min\{\rho_1, \rho_2, \rho_p\}\). Furthermore, define functions \(g_3\) and \(h_3\) on \([0, \rho_3]\) by
\[ g_3(t) = \left[ \frac{\int_0^1 w((1 - \theta)g_2(t)t) \, d\theta}{1 - w_0(g_2(t))} \right] \frac{(w_0(t) + w_0(g_2(t)t)) \int_0^1 v(\theta g_2(t)t) \, d\theta}{(1 - w_0(g_2(t)))^2} + \frac{|b_2 - 1|}{|b_2 + b_3|} \]
\[ h_3(t) = g_3(t) - 1. \]
By these definitions, \(h_3(0) = -1\) and \(h_3(t) \to \infty\) as \(t \to \rho_3^-\). Denote by \(r_3\) the minimal solution of the equation \(h_3(t) = 0\) in \((0, \rho_3)\). Finally, define a radius of convergence denoted by \(r\) as
\[ r = \min\{r_j\}, \quad j = 1, 2, 3. \]  
(2.6)  
It follows that
\[ 0 \leq w_0(t) < 1, \]  
(2.7)  
\[ 0 \leq w_0(g_1(t)t) < 1, \]  
(2.8)
\(0 \leq w_0(g_2(t)t) < 1,\)
\(0 \leq p(t) < 1,\)
\(0 \leq g_j(t) < 1,\)
for all \(t \in [0, r).\)

The local convergence analysis of method \(1.2\) also uses the following conditions \((A)\):

\(\text{(A1)}\) \(F : \Omega \to \mathcal{E}_2\) is continuously differentiable and there exists \(x_* \in \Omega\) such that \(F(x_*) = 0\) and \(F'(x_*)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)\).

\(\text{(A2)}\) The function \(w_0 : [0, \infty) \to [0, \infty)\) is continuous, increasing and satisfies \(w_0(0) = 0\).

Set \(\Omega_0 = \Omega \cap S(x_*, \rho_0)\), where \(\rho_0\) is given in \((2.1)\).

\(\text{(A3)}\) The functions \(w, v : [0, \rho_0) \to [0, \infty)\) are continuous, increasing, \(w(0) = 0\), and for each \(x, y \in \Omega_0\),
\[\|F'(x_*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|),\]
\[\|F'(x_*)F'(x)\| \leq v(\|x - x_*\|).\]

\(\text{(A4)}\) \(\tilde{S}(x_*, r) \subseteq \Omega, \rho_0, \rho, \rho_1, \rho_2, \rho_P, \rho_3, r\) exist, and are defined as previously.

\(\text{(A5)}\) There exists \(\tilde{r} \geq r\) such that \(\int_0^1 w_0(\theta \tilde{r}) d\theta < 1\).

Set \(\Omega_1 = \Omega \cap \tilde{S}(x_*, \tilde{r})\).

Next, we present the local convergence analysis of method \(1.2\) using conditions \((A)\) and the preceding notation.

**Theorem 2.1.** Under conditions \((A)\) and for \(x_0 \in S(x_*, r) - \{x_*\}\), the following hold:

\(\text{(2.12)}\) \(\{x_n\} \subseteq S(x_*, r),\)
\(\text{(2.13)}\) \(\lim_{n \to \infty} x_n = x_*,\)
\(\text{(2.14)}\) \(\|y_n - x_*\| \leq g_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < r,\)
\(\text{(2.15)}\) \(\|z_n - x_*\| \leq g_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|,\)
\(\text{(2.16)}\) \(\|x_{n+1} - x_*\| \leq g_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|,\)

and \(x_*\) is the only solution of the equation \(F(x)\) in the set \(\Omega_1\) where \(r, g_j\) and \(\Omega_1\) are defined as previously.

**Proof.** By conditions \((A_2), (2.6), (2.7), (A_4)\) and \(x \in S(x_*, r)\), we get
\(\text{(2.17)}\) \(\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\|x - x_*\|) < w_0(r) < 1,\)

which together with the Banach lemma on invertible operators \(4.15\) implies that \(F'(x_*)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)\), and
\(\text{(2.18)}\) \(\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - w_0(\|x - x_*\|)}.\)
The point $y_0$ is well defined by the condition $x_0 \in S(x_*, r) \setminus \{x_*\}$, and for $x = x_0$.

Using the first substep of method (1.2), (A_1), (2.6), (2.11) (for $j = 1$, (A_3) and (2.18) for $x = x_0$, we get

\[
(2.19) \quad \|y - x_*\| = \|x_0 - x_* - F'(x_0)^{-1}F(x_0) + \frac{1}{3}F'(x_0)^{-1}F(x_0)\| \\
\leq \|F'(x_0)^{-1}F(x_*)\| \\
\times \left[ \frac{1}{0} F'(x_*)^{-1} \left( F'(x_*) + t(x_0 - x_*) \right) - F(x_0) \right] d\theta \left( x_0 - x_* \right) \| \\
\times \left[ \frac{1}{3} \|F'(x_0)^{-1}F(x_*)\| \|F'(x_*)^{-1}F(x_0)\| \right] \\
\leq \frac{1}{0} w((1 - \theta)\|x_0 - x_*\|) d\theta + \frac{1}{2} \int_0^1 v(\theta \|x_0 - x_*\|) d\theta \\
\|x_0 - x_*\| \\
\leq g_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r,
\]

which shows (2.14) for $n = 0$ and $y_0 \in S(x_*, r)$, where we have also used

\[
(2.20) \quad \|F'(x_0)^{-1}F(x)\| = \|F'(x_*)^{-1}(F(x) - F(x_*))\| \\
= \left[ \int_0^1 F'(x_*)^{-1}F'(x_* + \theta(x - x_*)) d\theta \right] \|x - x_*\| \\
= \int_0^1 r(\theta \|x - x_*\|) d\theta \|x - x_*\|
\]

for $x = x_0$.

By the second substep of method (1.2), for $n = 0$,

\[
(2.21) \quad z_0 - x_* = x_0 - x_* - F'(x_0)^{-1}F(x_0) + F'(x_0)^{-1}F(x_0) - a_1F'(x_0)^{-1}F(x_0) \\
= (x_0 - x_* - F'(x_0)^{-1}F(x_0)) \\
+ (a_1 - 1)F'(y_0)^{-1}\left[ (F'(x_0) - F'(x_*)) + (F'(x_*) - F'(y_0)) \right] \\
\times F'(y_0)^{-1}(F'(x_0) + F'(y_0))F'(x_0)^{-1}F(x_0),
\]

where we have used $a_1 = 1 - a_2$, and

\[
(2.22) \quad F'(x_0)^{-1}F(x_0) - a_1F'(x_0)^{-1}F(x_0) \left[ (F'(y_0)^{-1}F'(x_0)) + F'(y_0) \right] \\
= \left[ (a_1 - 1)[(F'(y_0)^{-1}F'(x_0)) + F'(y_0)] \right] \\
= \left[ (a_1 - 1)[F'(y_0)^{-1}F'(x_0) - I][F'(y_0)^{-1}F'(x_0) + I]F'(x_0)^{-1}F(x_0) \right] \\
= \left[ (a_1 - 1)[F'(y_0)^{-1}\left( (F'(x_0) - F'(x_*)) + (F'(x_*) + F'(y_0)) \right) \right] \\
\times (F'(x_0) + F'(y_0))F'(x_0)^{-1}F(x_0).
\]

As in (2.18) for $x = y_0$, and using (2.8), (2.19) we get $F'(y_0)^{-1} \in \mathcal{L}(E_2, E_1)$
and
\[
(2.23) \quad \|F'(y_0)^{-1}F'(x_*)\| \leq \frac{1}{1 - w_0(\|y_0 - x_*\|)} \\
\quad \leq \frac{1}{1 - w_0(g_1(\|x_0 - x_*\|)\|x_0 - x_*\|)}
\]
and \(z_0\) is well defined by the second substep of method (1.2). Then, using
(2.6), (2.11) (for \(j = 2\)), (2.18) (for \(x = x_0\)), (2.19), (2.20) (for \(x = x_0, y_0\))
and (2.23), we obtain in turn
\[
(2.24) \quad \|z_0 - x_*\|
\leq \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\|
+ |a_1 - 1||F'(y_0)^{-1}F'(x_*)[\|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\|
+ \|F'(x_*)^{-1}(F'(x_0) - F'(y_0))\|\|F'(y_0)^{-1}F'(x_*)\|
\times \left[\|F'(x_*)^{-1}F'(x_0)\| + \|F'(x_*)^{-1}F'(y_0)\|\|F'(x_0)^{-1}F'(x_*)\|
\times \|F'(x_*)^{-1}F'(x_0)\|\right]
\leq \left[\int_0^1 w((1 - \theta)\|x_0 - x_*\|) d\theta \right]\frac{1}{1 - w_0(\|x_0 - x_*\|)} + |a_1 - 1|
\times \frac{w_0(\|x_0 - x_*\|) + w_0(\|y_0 - x_*\|)\left(v(\|x_0 - x_*\|) + v(\|y_0 - x_*\|)\right)}{(1 - w_0(g_1(\|x_0 - x_*\|)\|x_0 - x_*\|)^2(1 - w_0(\|x_0 - x_*\|))}
\times \int_0^1 v(\theta\|x_0 - x_*\|) d\theta \right]\|x_0 - x_*\|
\leq g_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|,
\]
so \(z_0 \in S(x_*, r)\), and (2.15) holds for \(n = 0\).

Next, we show that \((b_2F'(x_0) + b_3F'(y_0))^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)\).

Indeed by (2.6), (2.10), (A2) and \(b_2 + b_3 \neq 0\), we have in turn
\[
(2.25) \quad \|((b_2 + b_3)F'(x_*))^{-1}(b_2F'(x_0) + b_3F'(y_0) - b_2F'(x_*) - b_3F'(x_*))\|
\leq \frac{1}{|b_2 + b_3|}\left[|b_2||F'(x_*)^{-1}(F'(x_0) - F'(x_*))|
+ |b_3||F'(x_*)^{-1}(F'(y_0) - F'(x_*))|\right]
\leq \frac{1}{|b_2 + b_3|}\left[|b_2|w_0(\|x_0 - x_*\|) + |b_3|w_0(\|y_0 - x_*\|)\right] \leq p(\|x_0 - x_*\|) < 1,
\]
so
\[
(2.26) \quad \|(b_2F'(x_0) + b_3F'(y_0))^{-1}F'(x_*)\| \leq \frac{1}{|b_2 + b_3|(1 - p(\|x_0 - x_*\|))}.
\]
Similarly by (2.6), (2.9), (2.18) (for \( x = z_0 \)) and (2.24), we have \( F'(z_0)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1) \), and
\[
\| F'(z_0)^{-1} F'(x_*) \| \leq \frac{1}{1 - w_0(\| z_0 - x_* \| )} \leq \frac{1}{1 - w_0(g_2(\| x_0 - x_* \| )\| x_0 - x_* \| )},
\]
so \( x_1 \) is well defined by the third substep of method (1.2). Next, using (2.6), (2.11) (for \( j = 3 \)), (2.18) (for \( x = x_0, y_0, z_0 \)), (2.19), (2.24), (2.25), (2.26), \( b_2 = b_1 - b_3 + 1 \) and the triangle inequality
\[
(2.27) \quad x_1 - x_* = (z_0 - x_* - F'(z_0)^{-1} F(z_0)) + F'(z_0)^{-1} F(x_0)
- (b_2 F'(x_0) + b_3 F'(y_0))^{-1}(F'(x_0) + b_1 F'(y_0)) F'(x_0)^{-1} F(x_0)
= (z_0 - x_* - F'(z_0)^{-1} F(z_0)) + (F'(z_0)^{-1} F(z_0) - F'(x_0)^{-1} F(x_0))
+ [I - (b_2 F'(x_0) + b_3 F'(y_0))^{-1}(F'(x_0) + b_1 F'(y_0)) F'(x_0)^{-1} F(x_0)] F'(x_0)^{-1} F(z_0)
= (z_0 - x_* - F'(z_0)^{-1} F(z_0)) + F'(z_0)^{-1} (F'(x_0) - F'(z_0)) F'(x_0)^{-1} F(z_0)
+ (b_2 F'(x_0) + b_3 F'(y_0)^{-1}) [(b_2 - 1) F'(x_0) + (b_2 - b_1) F'(y_0)] F'(x_0)^{-1} F(z_0)
= (z_0 - x_* - F'(z_0)^{-1} F(z_0)) + F'(z_0)^{-1} (F'(x_0) - F'(z_0)) F'(x_0)^{-1} F(z_0)
+ (b_2 - 1)(b_2 F'(x_0) + b_3 F'(y_0))^{-1} [(F'(x_0) - F'(x_*))]
+ (F'(x_*) - F'(y_0)] F'(x_0)^{-1} F(z_0)
\]
leading to
\[
(2.28) \quad \| x_1 - x_* \| \leq \left[ \frac{t \int_0^1 w((1 - \theta) \| z_0 - x_* \| ) \, d\theta}{1 - w_0(\| z_0 - x_* \| )} \right. \\
+ \frac{(w_0 \| x_0 - x_* \| + w_0 \| z_0 - x_* \| ) \int_0^1 v(\theta \| z_0 - x_* \| ) \, d\theta}{1 - w_0(\| z_0 - x_* \| ) (1 - w_0(\| x_0 - x_* \| ))} \\
+ \frac{\| b_2 - 1 \| (w_0 \| x_0 - x_* \| + w_0 \| y_0 - x_* \| ) \int_0^1 v(\theta \| z - x_* \| ) \, d\theta}{\| b_2 + b_3 \| (1 - p(\| x_0 - x_* \| ) (1 - w_0(\| x_0 - x_* \| )))} \right] \| z_0 - x_* \|
\leq g_3(\| x_0 - x_* \| ) \| x_0 - x_* \| \leq \| x_0 - x_* \|,
\]
so \( x_1 \in S(x_*, r) \) and (2.16) holds for \( n = 0 \). The induction for estimations (2.14)–(2.16) is terminated if \( x_i, y_i, z_i, x_{i+1} \) replace \( x_0, y_0, z_0, x_1 \) respectively in the preceding computations. Then, by the estimations
\[
(2.29) \quad \| x_{i+1} - x_* \| \leq \varsigma \| x_i - x_* \| < r,
\]
where \( \varsigma = g_3(\|x_0 - x_*\|) \in [0, 1] \), we conclude that \( \lim_{i \to \infty} x_i = x_* \) and \( x_{i+1} \in S(x_*, r) \).

Consider \( y_* \in \Omega_1 \) with \( F(y_*) = 0 \). Set \( T = \int_0^1 F'(y_* + \theta(x_* - y_*)) d\theta \). By \((A_2)\) and \((A_3)\), we get

\[
\|F'(x_*)^{-1}(T - F'(x_*))\| \leq \frac{1}{0} w_0(\theta \|y_* - x_*\|) d\theta \leq \frac{1}{0} w_0(\theta \bar{r}) d\theta < 1,
\]

so \( T^{-1} \in L(E_2, E_1) \). Finally, by the identity \( 0 = F(x_*) - F(y_*) = T(x_* - y_*) \), we deduce that \( x_* = y_* \). \( \blacksquare \)

**Remark 2.2.** (a) In the case when \( w_0(t) = L_0 t, w(t) = Lt \), the radius \( \rho_A = \frac{2}{2L_0 + L} \) was obtained by Argyros \([2]\) as the convergence radius for Newton’s method under conditions \([2.7] - [2.9]\). Notice that the convergence radius for Newton’s method given independently by Rheinboldt \([16]\) and Traub \([21]\) is given by

\[
\rho_{TR} = \frac{2}{3L_1} < \rho_A,
\]

where \( L_1 \) is the Lipschitz constant on \( \Omega \). As an example, let us consider the function \( F(x) = e^x - 1 \). Then \( x_* = 0 \). Set \( \Omega = B(0, 1) \). Then we have \( L_0 = e - 1 < L = e^{1/L_0} < L_1 = e \), so \( \rho_{TR} = 0.24252961 < \rho_A = 0.3826919122323857 \).

(b) The local results can be used for projection methods such as Arnoldi’s method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy \([2, 4, 11]\).

(c) The results can also be used to solve equations where the operator \( F' \) satisfies the autonomous differential equation \([11]\):

\[
F'(x) = P(F(x)),
\]

where \( P : \mathcal{E}_2 \to \mathcal{E}_2 \) is a known continuous operator. Since \( F'(x_*) = P(F(x_*)) = P(0) \), we can apply the results without actually knowing the solution \( x_* \). As an example let \( F(x) = e^x - 1 \). Then we can choose \( P(x) = x + 1 \) and \( x_* = 0 \).

(d) It is worth noticing that method \([1.2]\) does not change when we use the conditions of the preceding theorem instead of the stronger conditions used in \([18, 20]\). Moreover, we can compute the computational order of convergence (COC) defined as

\[
\xi = \lim_{n \to \infty} \left[ \ln\left( \frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|} \right) / \ln\left( \frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|} \right) \right].
\]
or the approximate computational order of convergence (ACOC) \[6\]
\[
\xi_1 = \lim_{n \to \infty} \left[ \ln \left( \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \right) / \ln \left( \frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|} \right) \right].
\]
This way we obtain in practice the order of convergence, but no higher order derivatives are used.

3. Numerical example. We present the following example to test the convergence criteria.

**Example 3.1.** Let \( \mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^3 \), \( \Omega = U(0,1) \), \( x^* = (0,0,0)^T \) and define \( F \) on \( \Omega \) by

\[
(3.1) \quad F(x) = F(u_1, u_2, u_3) = \left( e^{u_1} - 1, \frac{e - 1}{2} u_2^2 + u_2, u_3 \right)^T.
\]

For \( u = (u_1, u_2, u_3)^T \), the Fréchet derivative is given by

\[
F'(u) = \begin{pmatrix}
  e^{u_1} & 0 & 0 \\
  0 & (e - 1)u_2 + 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}.
\]

Using the norm of the maximum of the rows \( x^* = (0,0,0)^T \) and since \( F'(x^*) = \text{diag}(1,1,1) \), by conditions (H) we get \( w_0(t) = (e - 1)t \), \( w(t) = e^{\frac{1}{e-1}} t \), \( w_1(t) = v(t) = e^{\frac{1}{e-1}} \), \( r_1 = 0.0154407 \), \( r_2 = 0.461425 \), \( r_3 = 0.00179951 \), \( r = r_3 \).

**Example 3.2.** Let \( \mathcal{E}_1 = \mathcal{E}_2 = C[0,1] \), \( \Omega = \bar{U}(0,1) \). Define a function \( F \) on \( \Omega \) by

\[
F(w)(x) = w(x) - 5 \int_0^1 x\theta w(\theta)^3 d\theta.
\]

The Fréchet derivative is given by

\[
F'(w(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta w(\theta)^2 \xi(\theta) d\theta \quad \text{for each} \quad \xi \in \Omega.
\]

Then we have \( x^* = 0 \), \( w_0(t) = L_0 t \), \( w(t) = L t \), \( w_1(t) = v(t) = 2 \), \( L_0 = 7.5 \) \( < L = 15 \). Thus, the radii of convergence are given by

\[
r_1 = 0.02222, \quad r_2 = 0.0678682, \quad r_3 = 0.0313983, \quad r = r_1.
\]

**Example 3.3.** Returning to the motivational example of the introduction, we can choose \( w_0(t) = w(t) = 96.662907t \), \( w_1(t) = v(t) = 1.0631 \). Then the radii of convergence are given by

\[
r_1 = 0.00445282, \quad r_2 = 0.00689712, \quad r_3 = 0.00355681, \quad r = r_3.
\]
Example 3.4. Consider the Hammerstein integral equation (see [14, pp. 19–20]) defined by

\begin{equation}
(3.2) \quad x(s) = \frac{1}{5} \int_{0}^{1} S(s,t)x(t)^3 \, dt, \quad x \in C[0,1], \quad s,t \in [0,1],
\end{equation}

where the kernel \( S \) is

\[ S(s,t) = \begin{cases} s(1-t), & s \leq t, \\ (1-s)t, & t \leq s. \end{cases} \]

We use \( \int_{0}^{1} \phi(t) \, dt \approx \sum_{k=1}^{8} w_k \phi(t_k) \) in (3.2), where \( t_k \) and \( w_k \) are the abscissas and weights, respectively. Denoting the approximations of \( x(t_i) \) by \( x_i \) (\( i = 1, \ldots, 8 \)), we get the following \( 8 \times 8 \) system of nonlinear equations:

\[ 5x_i - 5 - \sum_{k=1}^{8} a_{ik} x_k^3 = 0, \quad i = 1, \ldots, 8, \]

\[ a_{ik} = \begin{cases} w_k t_k (1-t_i), & k \leq i, \\ w_k t_i (1-t_k), & i < k. \end{cases} \]

By the Gauss–Legendre quadrature formula, we obtain the values of \( t_k \) and \( w_k \) when \( k = 8 \), listed in Table 1. We follow the stopping criteria for programming \( \|F(x_n)\| \leq 10^{-100} \) and \( \|x_{n+1} - x_n\| \leq 10^{-100} \); the solution after seven iterations with precision \( 10^{-6} \) is

\[ x_* = (1.002096 \ldots, 1.009900 \ldots, 1.019727 \ldots, 1.026436 \ldots, 1.026436 \ldots, 1.019727 \ldots, 1.009900 \ldots, 1.002096 \ldots)^T, \]

\[ \xi = 5.9432173 \text{ and } \xi_1 = 5.6203419. \]

We get \( w_0(t) = w(t) = \frac{3}{40} t \) and \( v(t) = 1 + w_0(t) \). The radius of convergence is given by

\[ r_1 = 8.88889, \quad r_2 = 7.12148, \quad r_3 = 6.73523, \quad r = 1, \]

since \( r \) can be at most 1.
References


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