

EXTENSIONS OF VALUATIONS TO THE HENSELIZATION AND COMPLETION OF A LOCAL RING

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Abstract. We give counterexamples to a question on the existence of good extensions of valuations to the Henselization and to a quotient of the completion of a birational extension of a local domain. We give an outline of our construction and proof in our paper [Acta Math. Vietnam. 44 (2019), 159–172].

1. Introduction. Suppose that K is a field. Associated to a valuation ν of K is a value group Φ_ν and a valuation ring V_ν with maximal ideal m_ν . Let R be a local domain with quotient field K . We say that ν dominates R if $R \subset V_\nu$ and $m_\nu \cap R = m_R$ where m_R is the maximal ideal of R . We have an associated semigroup $S^R(\nu) = \{\nu(f) \mid f \in R\}$ (with the convention that $\nu(0) = \infty$ is larger than any element of Φ_ν), as well as the associated graded ring along the valuation

$$(1) \quad \text{gr}_\nu(R) = \bigoplus_{\gamma \in \Phi_\nu} \mathcal{P}_\gamma(R) / \mathcal{P}_\gamma^+(R) = \bigoplus_{\gamma \in S^R(\nu)} \mathcal{P}_\gamma(R) / \mathcal{P}_\gamma^+(R)$$

which is defined by Teissier in [24]. Here

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geq \gamma\} \quad \text{and} \quad \mathcal{P}_\gamma^+(R) = \{f \in R \mid \nu(f) > \gamma\}.$$

This ring plays an important role in local uniformization of singularities ([24] and [25]). The ring $\text{gr}_\nu(R)$ is a domain, but it is often not Noetherian, even when R is.

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Suppose that $K \rightarrow K^*$ is a finite extension of fields and ν^* is a valuation which is an extension of ν to K^* . We have the classical indices

$$e(\nu^*/\nu) = [\Phi_{\nu^*} : \Phi_{\nu}] \quad \text{and} \quad f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_{\nu}/m_{\nu}]$$

as well as the defect $\delta(\nu^*/\nu)$ of the extension. Ramification of valuations and the defect are discussed in Chapter VI of [27], [13] and Kuhlmann's papers [17] and [18]. A survey is given in Section 7.1 of [10]. By Ostrowski's lemma, if ν^* is the unique extension of ν to K^* , we have

$$(2) \quad [K^* : K] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}$$

where p is the characteristic of the residue field V_{ν}/m_{ν} . From this formula, the defect can be computed using Galois theory in an arbitrary finite extension. If V_{ν}/m_{ν} has characteristic 0, then $\delta(\nu^*/\nu) = 0$ and $p^{\delta(\nu^*/\nu)} = 1$, so there is no defect.

In this article, we answer the following natural question.

QUESTION 1.1. Suppose that R is a Noetherian (excellent) local domain which is dominated by a valuation ν . Does there exist a regular local ring R' of the quotient field K of R such that ν dominates R' and R' dominates R , a prime ideal P of the m_R -adic completion \widehat{R}' such that $P \cap R' = (0)$ and an extension $\hat{\nu}$ of ν to the quotient field of \widehat{R}'/P which dominates \widehat{R}'/P such that

$$\text{gr}_{\nu}(R') \cong \text{gr}_{\hat{\nu}}(\widehat{R}'/P)?$$

The statement of Question 1.1 is summarized in the following commutative diagram:

$$\begin{array}{ccc} V_{\nu} & \longrightarrow & V_{\hat{\nu}} \\ \uparrow & & \uparrow \\ R' & \longrightarrow & \widehat{R}'/P \\ \uparrow & & \\ R & & \end{array}$$

If ν has rank 1, then setting

$$P(\widehat{R})_{\infty} = \{f \in \widehat{R} \mid \nu(f) = \infty\}$$

we infer that $P(\widehat{R})_{\infty}$ is a prime ideal in \widehat{R} and

$$\text{gr}_{\nu}(R) \cong \text{gr}_{\hat{\nu}}(\widehat{R}/P(\widehat{R})_{\infty}),$$

so if ν has rank 1, then Question 1.1 has a positive answer for local domains R and rank 1 valuations ν which admit local uniformization. If R is essentially of finite type over a field of characteristic 0, then we can even take P so that \widehat{R}'/P is a regular local ring ([9]).

We also consider the following related question.

QUESTION 1.2. Suppose R is a Noetherian (excellent) local domain which is dominated by a valuation ν . Does there exist a regular local ring R' of the quotient field K of R such that ν dominates R' and R' dominates R , and an extension ν^h of ν to the quotient field of the Henselization $(R')^h$ of R' which dominates $(R')^h$ such that

$$\text{gr}_{\nu}(R') \cong \text{gr}_{\nu^h}((R')^h)?$$

The statement of Question 1.2 is summarized in the following commutative diagram:

$$\begin{array}{ccc}
 V_\nu & \longrightarrow & V_{\nu^h} \\
 \uparrow & & \uparrow \\
 R' & \longrightarrow & (R')^h \\
 \uparrow & & \\
 R & &
 \end{array}$$

If an answer to Question 1.1 is positive then so is an answer to Question 1.2. A start on answering Question 1.2 is the following proposition.

PROPOSITION 1.3 ([6], Proposition 1.7). *Suppose that R and S are normal local rings such that R is excellent, S lies over R and S is unramified over R , $\tilde{\nu}$ is a valuation of the quotient field L of S which dominates S and ν is the restriction of $\tilde{\nu}$ to the quotient field K of R . Suppose L is finite over K . Then there exists a normal local ring R' of K which is dominated by ν and dominates R' such that if R'' is a normal local ring of K which is dominated by ν and dominates R' and S'' is the local ring of the integral closure of R'' in L which is dominated by $\tilde{\nu}$, then $R'' \rightarrow S''$ is unramified and*

$$\mathrm{gr}_{\tilde{\nu}}(S'') \cong \mathrm{gr}_{\nu}(R'') \otimes_{R''/m_{R''}} S''/m_{S''}.$$

The statement of Proposition 1.3 is summarized in the following commutative diagram:

$$\begin{array}{ccc}
 V_\nu & \longrightarrow & V_{\tilde{\nu}} \\
 \uparrow & & \uparrow \\
 R'' & \longrightarrow & S'' \\
 \uparrow & & \uparrow \\
 R' & & \\
 \uparrow & & \\
 R & \longrightarrow & S.
 \end{array}$$

Questions 1.1 and 1.2 have a negative answer in general (even in equicharacteristic 0). We give an outline of the proof from [7].

A very interesting related problem, which is still open, is [15, Conjecture 1.1] on the existence of “scalewise birational” extensions of associated graded rings. [15, Conjecture 1.1] is a refinement of [24, Statement 5.19]. The results of Proposition 1.3 are counterexamples to possible hopes of improving the statements of Conjecture 1.11 and Theorem 7.1 of [15].

Our proof relies on the construction of generating sequences, using the algorithm of [11], which is a generalization of the algorithm of [23]. The construction of generating sequences in a local domain which is dominated by a valuation which provide enough information to determine the associated graded ring along the valuation is an important problem. Some recent papers addressing this are [12], [16], [20] and [22].

2. An outline of the proof. In this section we give an outline of the proof of the following theorem, from which the existence of the counterexamples to Questions 1.1 and 1.2 follow.

THEOREM 2.1 ([7], Theorems 1.5 and 1.6). *Suppose that k is an algebraically closed field. Then there exists a three-dimensional regular local ring T_0 , which is a localization of a finite type k -algebra, with residue field k , and a valuation φ of the quotient field K of T_0 which dominates T_0 and whose residue field is k , such that if T is a regular local ring of K which is dominated by φ and dominates T_0 , T^h is the Henselization of T and φ^h is an extension of φ to the quotient field of T^h which dominates T^h , then*

$$S^{T^h}(\varphi^h) \neq S^T(\varphi)$$

under the natural inclusion $S^T(\varphi) \subset S^{T^h}(\varphi^h)$.

Theorem 2.1 immediately gives a counterexample to Question 1.2, and gives a counterexample to Question 1.1, since if T is a regular local ring which dominates T_0 and is dominated by φ and P is a prime ideal in \hat{T} such that $P \cap T = (0)$, then $T \subset T^h \subset \hat{T}/P$.

We now give an outline of the proof of Theorem 2.1. Let $R_0 = k[x, y, t]_{(x, y)} \cong k(t)[x, y]_{(x, y)}$. We will define a valuation ν dominating R_0 by constructing a generating sequence

$$P_0 = x, P_1 = y, P_2, \dots$$

Let $\bar{p}_1, \bar{p}_2, \dots$ be the sequence of prime numbers, excluding the characteristic of k . Define

$$a_1 = \bar{p}_1 + 1$$

and inductively define a_i by

$$a_{i+1} = \bar{p}_i \bar{p}_{i+1} a_i + 1.$$

Define

$$P_{i+1} = P_i^{\bar{p}_i^2} - (1+t)x^{\bar{p}_i a_i}$$

for $i \geq 1$. Set $\nu(x) = 1$, $\nu(P_i) = \frac{a_i}{\bar{p}_i}$ for $i \geq 1$. Then the valuation group of ν is

$$\Phi_\nu = \bigcup_{i \geq 1} \frac{1}{\bar{p}_1 \bar{p}_2 \cdots \bar{p}_i} \mathbb{Z}.$$

Let \bar{k} be an algebraic closure of $k(t)$, and $\alpha_i \in \bar{k}$ be a root of

$$f_i(u) = u^{\bar{p}_i} - (1+t) \in k[u]$$

for $i \geq 1$. Then $f_i(u)$ is the minimal polynomial of α_i over $k(\alpha_1, \dots, \alpha_{i-1})$. We have

$$V_\nu/m_\nu = k(\{\alpha_i \mid i \geq 1\}) = k(\{(1+t)^{1/\bar{p}_i} \mid i \geq 1\})$$

and

$$\alpha_i = \left[\frac{P_i^{\bar{p}_i}}{x^{a_i}} \right].$$

Suppose that A is a regular local ring of the quotient field K of R_0 which is dominated by ν and dominates R_0 . Now $R_0 \rightarrow A$ factors as a sequence of quadratic transforms (local rings of blowups of maximal ideals in regular local rings) by [1, Theorem 3]. An inductive construction of generating sequences of ν in an intermediate sequence of quadratic

transforms between R_0 and A is derived in formulas (6), (7) and (8) of [7], using [11, Theorem 7.1]. It follows that there exist regular parameters z and w in A such that

$$Q_0 = z, Q_1 = w, Q_2 = w^{\bar{p}c} - (1+t)\tau^{\bar{p}}z^{\bar{p}e}, \dots$$

is the first part of a generating sequence of ν in A . Here $\bar{p} = \bar{p}_{1+l}$ for some l and τ is a unit in A . Further, $c, e \in \mathbb{Z}_+$ satisfy $\gcd(c, e) = 1$.

Let λ be a \bar{p} -th root of $1+t$ in an algebraic closure of K , $L = K(\lambda)$ and $\bar{\nu}$ be an extension of ν to L . Let $\varepsilon \in k$ be a primitive \bar{p} -th root of unity.

Let $B = A[\lambda]$ and $C = B_{m_{\bar{\nu}} \cap B}$. Then $A \rightarrow C$ is unramified, so C is a regular local ring with regular parameters z, w .

PROPOSITION 2.2. *We have an inequality $S^C(\bar{\nu}) \neq S^A(\nu)$.*

Proof. The semigroup $S^A(\nu) = S(\{\nu(Q_i) \mid i \geq 0\})$, where

$$Q_0 = z, Q_1 = w, Q_2 = w^{\bar{p}c} - (1+t)\tau^{\bar{p}}z^{\bar{p}e}, \dots$$

We have

$$\gamma_1 = \left[\frac{w^c}{z^e} \right] \in V_\nu/m_\nu \subset V_{\bar{\nu}}/m_{\bar{\nu}}.$$

Let

$$0 \neq \beta = [\lambda\tau] \in V_{\bar{\nu}}/m_{\bar{\nu}}$$

and

$$h_j = w^c - \varepsilon^j \lambda \tau z^e \in C.$$

If $\varepsilon^j \beta \neq \gamma_1$, then $\bar{\nu}(h_j) = e\nu(z)$, so

$$\sum_{j=1}^{\bar{p}} \bar{\nu}(h_j) = \nu(Q_2) > \bar{p}e\nu(z)$$

implies that there exists a unique value of j such that $\varepsilon^j \beta = \gamma_1$ and $\bar{\nu}(h_j) > e\nu(z)$. If $\bar{\nu}(h_j) \in S^A(\nu)$, then

$$\bar{\nu}(h_j) \in S(\nu(z), \nu(w))$$

since

$$\bar{\nu}(h_j) = \nu(Q_2) - (\bar{p} - 1)e\nu(z) < \nu(Q_2).$$

Thus $\nu(Q_2) \in G(\nu(z), \nu(w))$, a contradiction. ■

Let μ be a valuation of

$$V_\nu/m_\nu = k(t)[\{1+t\}^{1/\bar{p}i} \mid i \geq 1]$$

which is an extension of the (t) -adic valuation on $k[t]_{(t)}$. The value group of μ is \mathbb{Z} . Let φ be the composite valuation of ν and μ on K , so that $V_\varphi = \pi^{-1}(V_\mu)$, where $\pi : V_\nu \rightarrow V_\nu/m_\nu$ is the residue map. We have

$$V_\varphi/m_\varphi = V_\mu/m_\mu = k.$$

Let $T_0 = k[t, x, y]_{(t, x, y)}$ which is dominated by φ .

Proposition 2.2 implies the following proposition.

PROPOSITION 2.3. *Suppose that T is a regular local ring of K which dominates T_0 and is dominated by φ . Then there exists a finite separable extension field L of K such that T is unramified in L . Further, if $\bar{\varphi}$ is an extension of φ to L and if U is the normal local ring of L which lies over T and is dominated by $\bar{\varphi}$, then*

- 1) U is a regular local ring,
- 2) $T \rightarrow U$ is unramified with no residue field extension,
- 3) $S^U(\bar{\varphi}) \neq S^T(\varphi)$.

We can now give the proof of Theorem 2.1. We first review the construction of the Henselization T^h of a normal local ring T (after Nagata, [21]). Let N be a separable closure of K . N is an (infinite) Galois extension of K with Galois group $G(N/K)$. Let E be a local ring of the integral closure of T in N . The splitting group is

$$G^s(E/T) = \{\sigma \in G(N/K) \mid \sigma(E) = E\}.$$

We define

$$T^h = E^{G^s(E/T)}$$

which has the quotient field $M = N^{G^s(E/T)}$.

Let $K \rightarrow L$ be the field extension of Proposition 2.3. Choose an embedding $K \rightarrow L \rightarrow N$. Let U be the local ring of the integral closure of T in L which is dominated by E . Then U is unramified over T with no residue field extension, so $L \subset M$ and U is dominated by T^h . Let $\bar{\varphi} = \varphi^h|_L$. Then $\bar{\varphi}$ dominates U and T^h dominates U , so $S^U(\bar{\varphi}) \subset S^{T^h}(\varphi^h)$. But $S^U(\bar{\varphi}) \neq S^T(\varphi)$ by Proposition 2.3, so $S^{T^h}(\varphi^h) \neq S^T(\varphi)$.

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