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STUDY OF MULTIPLE STRUCTURES ON PROJECTIVE SUBVARIETIES

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Abstract. Let k an algebraically closed field, $\operatorname{char} k = 0$.

We study multiplicity-r structures on varieties for $r \in \mathbb{N}$, $r \geq 2$. Let Z be a reduced irreducible nonsingular $(N-2)$-dimensional variety such that $rZ = X \cap F$, where X is a normal $(N-1)$-fold of degree n, F is a smooth $(N-1)$-fold of degree m in $\mathbb{P}^N$, such that $r \in \mathbb{N}$, $r \geq 2$, $Z \cap \operatorname{Sing}(X) \neq \emptyset$. There are effective divisors V and D_1 on Z such that $O_Z(V - (r-1)D_1) \simeq {\omega_Z}^r(-rm - n + (N+1)r)$, where ω_Z is the canonical sheaf of Z.

Let $Z \subset \mathbb{P}^N$ be a reduced irreducible subvariety of codimension 2. Let Y be an irreducible hypersurface in $\mathbb{P}^N$, $Z \subset Y$. Let ${\omega^o}_Z$ be the dualizing sheaf of Z. Then, there exists a hypersurface X in $\mathbb{P}^N$ such that $Z = Y \cap X$ is a scheme-theoretical complete intersection if and only if

- ${\omega^o}_Z \simeq \omega_{\mathbb{P}^N} \otimes \wedge^2 \mathcal{N}_Z|_{\mathbb{P}^N}$.
- $\deg Y$ divides $\deg Z$.
- ${\omega^o}_Z \simeq O_Z(\deg Y + (\frac{\deg Z}{\deg Y}) - N - 1)$.

Introduction. There are two sections in this paper. In Section 1 we study multiplicity-r structures on varieties $r \in \mathbb{N}$.

Let k be an algebraically closed field of characteristic 0. Most of our results could also be extended to $\operatorname{char} k = p$, $p > 0$. Let Y be a nonsingular variety in $\mathbb{P}^N$, with ideal sheaf I_Y. A non-reduced structure $\tilde{Y}$ is a *multiplicity-r structure* on Y if

- $I_{\tilde{Y}} \subset I_Y$;
- $\tilde{Y}$ is locally a complete intersection;

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- $\tilde{Y}$ has multiplicity r, i.e. for each point $P \in Y$ and a general hyperplane H through P, the local intersection multiplicity is

$$i(P;\tilde{Y},H) = \dim \frac{O_P}{I(\tilde{Y} \cap H)} = r.$$

Multiplicity-2 structures on nonsingular varieties appear in several instances; for example, when studying nonsingular curves on a Kummer surface in $\mathbb{P}^3$, passing through some of its nodes. To define a multiplicity-2 structure $\tilde{Y}$ on a codimension 2 nonsingular variety Y is, under some conditions, equivalent to defining a subbundle $L \subset N_{Y|\mathbb{P}^n}$, assuming that $I_Y/I_{\tilde{Y}}$ is locally free. W. Barth gave a construction of the Horrocks–Mumford bundle assuming the existence of a nonsingular irreducible curve with certain properties; among them a double structure on the curve (see [2]). The Horrocks–Mumford bundle is a stable indecomposable rank 2 vector bundle over $\mathbb{P}^4$. The study of varieties which are complete intersections with a non-reduced structure on them could be used in the construction of vector bundles in $\mathbb{P}^n$.

Multiple structures on curves ('thickenings of curves') appear in the enumerative geometry of Calabi–Yau n-folds. In particular, Gromov–Witten theory can be used to define an enumerative geometry of curves in Calabi–Yau 5-folds (see [6]). Also thickenings of rational curves, analytically contractible, on Calabi–Yau 3-folds are used to study the genus zero Gopakumar–Vafa invariants (see [5]).

Let Z be a reduced irreducible nonsingular $(N-2)$-dimensional variety such that $rZ = X \cap F$, $r \in \mathbb{N}$, $r \geq 2$, where F is a $(N-1)$-fold in $\mathbb{P}^N$, X is a normal $(N-1)$-fold with $Z \cap \mathrm{Sing}(X) \neq \emptyset$. We prove that there are effective divisors V and D_1 on Z such that $O_Z(V-(r-1)D_1) \simeq {\omega_Z}^r(-rm-n+(N+1)r)$, where ω_Z is the canonical sheaf of Z.

In Section 2 we give conditions for a reduced irreducible subvariety Z, in $\mathbb{P}^N$, of codimension 2, to be a scheme-theoretical complete intersection. Our motivation is the extension of a type of Noether–Lefschetz Theorem from surfaces to n-folds (see [4]).

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1. Let Z be a reduced irreducible nonsingular $(N-2)$-dimensional variety such that $rZ = X \cap F$, where X is a normal $(N-1)$-fold, F is a smooth $(N-1)$-fold in $\mathbb{P}^N$, such that $r \in \mathbb{N}$, $r \geq 2$, $Z \cap \mathrm{Sing}(X) \neq \emptyset$. Let m be the degree of F and n be the degree of X. Let I_Z be the ideal sheaf of Z in $\mathbb{P}^N$.

Let us write $T = rZ$, $r \in \mathbb{N}$, $r \geq 2$. Consider the Cohen–Macaulay filtration

$$Z = T_1 \subset T_2 \subset \ldots \subset T_r = rZ, \quad \text{where} \quad T_i = iZ,\ 1 \leq i \leq r.$$

Let $J_i = {I_Z}^i + O_{\mathbb{P}^N}(-F)$, $1 \leq i \leq r$. Note that $I_Z = J_1$, ${I_Z}^2 \subset J_2 \subset I_Z$. $\frac{J_i}{J_{(i+1)}}$ is a rank 1 locally free O_Z-module. $\frac{J_i}{I_Z J_i}$ is a rank 2 locally free O_Z-module, $1 \leq i \leq r$.

We generalize some results of [1].

Let $\mathcal{L} = \frac{I_Z}{J_2}$. We consider the exact sequence

$$0 \to \frac{J_2}{{I_Z}^2} \to \frac{I_Z}{{I_Z}^2} \to \mathcal{L} \to 0 \tag{1}$$

$\mathcal{L}$ is a quotient of $\frac{I_Z}{{I_Z}^2}$, the conormal bundle of Z in $\mathbb{P}^N$. $\frac{I_Z}{{I_Z}^2}$ is a locally free sheaf of rank 2 over $O_Z = O_{\mathbb{P}^N}/I_Z$.

Since Z is contained in the hypersurface F of degree m, for $I_F = O_{\mathbb{P}^N}(-m)$, we have

$$0 \to O_{\mathbb{P}^N}(-m) \to I_Z,$$

which, by restriction to Z, gives $0 \to O_Z(-m) \to \frac{I_Z}{{I_Z}^2}$, since $\mathrm{Tor}_1(\frac{I_Z}{O_{\mathbb{P}^N}(-m)}, O_Z) = 0$.

We also have

$$0 \to O_{\mathbb{P}^N}(-m) \to J_2 \to I_Z, \tag{2}$$

Restricting (2) to Z, we have $O_Z(-m) \to \frac{J_2}{I_Z J_2} \to \frac{I_Z}{{I_Z}^2}$.

Let $M_1 =: \frac{J_2}{{I_Z}^2}$. It is a rank 1 locally free sheaf on Z since so it is $\mathcal{L}$ and (1) is exact. Let $\alpha : O_Z(-m) \to \frac{I_Z}{{I_Z}^2}$. Let $\delta : \frac{I_Z}{{I_Z}^2} \to \mathrm{Coker}\,\alpha$.

We also have

$$0 \to O_Z(-m) \to \frac{J_2}{{I_Z}^2} \to \frac{M_1}{O_Z(-m)} \to 0,$$

$$0 \to \frac{M_1}{O_Z(-m)} \to \mathrm{Coker}\,\alpha \to \mathcal{L} \to 0.$$

Let $\gamma : \mathrm{Coker}\,\alpha \to \mathcal{L} \to 0$. Let us consider $\gamma \circ \delta$. Hence, there exists an effective divisor D_1 on Z such that

$$0 \to O_Z(-m + D_1) \to \frac{I_Z}{{I_Z}^2} \to \mathcal{L} \to 0. \tag{3}$$

Thus, $M_1 = O_Z(D_1 - m)$.

Taking exterior powers in the exact sequence

$$0 \to \frac{I_Z}{{I_Z}^2} \to \Omega|_{\mathbb{P}^N} \otimes O_Z \to \omega_Z \to 0,$$

we obtain $\wedge^2(\frac{I_Z}{{I_Z}^2}) \simeq {\omega_Z}^{-1}(-(N+1))$.

From the latter and the exact sequence (1),

$$\mathcal{L} = \frac{I_Z}{J_2} \simeq \wedge^2\left(\frac{I_Z}{{I_Z}^2}\right) \otimes {M_1}^{\otimes -1} \simeq {\omega_Z}^{-1}(-(N+1)) \otimes O_Z(m - D_1).$$

We can obtain similar result from (3).

Consider the exact sequence

$$0 \to \frac{{I_Z}^2}{I_Z J_2} \to \frac{J_2}{I_Z J_2} \to \frac{J_2}{{I_Z}^2} \to 0.$$

$$\wedge^2\left(\frac{J_2}{I_Z J_2}\right) \simeq \frac{{I_Z}^2}{I_Z J_2} \otimes \frac{J_2}{{I_Z}^2} \simeq {\omega_Z}^{-2}(-2(N+1)) \otimes O_Z(m - D_1),$$

since $(\frac{I_Z}{J_2})^{\otimes 2} \simeq \frac{{I_Z}^2}{I_Z J_2}$ and $(\frac{I_Z}{J_2})^{\otimes 2} \simeq {\omega_Z}^{-2}(-2(N+1)) \otimes O_Z(2m - 2D_1)$.

We also have

$$0 \to O_{\mathbb{P}^N}(-m) \to J_3 \to J_2 \to I_Z, \tag{4}$$

After restricting to Z, we have $O_Z(-m) \to \frac{J_3}{I_Z J_3} \to \frac{J_2}{I_Z J_2}$. Let μ_1 be their composition; $\mu_1 : O_Z(-m) \to \frac{J_2}{I_Z J_2}$. Consider $\mu_{11} : \frac{J_2}{I_Z J_2} \to \operatorname{Coker} \mu_1$.

Consider the exact sequence

$$0 \to \frac{J_3}{I_Z J_2} \to \frac{J_2}{I_Z J_2} \to \frac{J_2}{J_3} \to 0. \tag{5}$$

Then $M_2 \simeq \frac{J_3}{I_Z J_2}$ is a rank 1 locally free sheaf on Z. Let $\mu_{12} : O_Z(-m) \to \frac{J_3}{I_Z J_2}$. The image of μ_1 maps to zero in $\frac{J_2}{J_3}$. We have an exact sequence

$$0 \to \operatorname{Coker} \mu_{12} \to \operatorname{Coker} \mu_1 \to \frac{J_2}{J_3} \to 0.$$

Then there exists an effective divisor D_2 such that $M_2 = O_Z(D_2 - m)$ since

$$0 \to O_Z(-m) \to M_2 \to \frac{M_2}{O_Z(-m)} \to 0,$$

and

$$0 \to O_Z(-m + D_2) \to \frac{J_2}{I_Z J_2} \to \frac{J_2}{J_3} \to 0.$$

D_2 is the divisor associated to the torsion subsheaf $\frac{M_2}{O_Z(-m)}$.

We have

$$\frac{J_2}{J_3} \simeq \wedge^2 \left(\frac{J_2}{I_Z J_2} \right) \otimes {M_2}^{\otimes -1},$$

$$\frac{J_2}{J_3} \simeq {\omega_Z}^{-2}(-2(N+1)) \otimes O_Z(2m - D_1 - D_2).$$

Notice that $(\frac{I_Z}{J_2})^{\otimes 2} \simeq \frac{{I_Z}^2}{I_Z J_2}$. We have

$$0 \to \frac{{I_Z}^2}{I_Z J_2} \to \frac{J_2}{I_Z J_2}.$$

Consider the exact sequence (5). There exists an effective divisor N_1 on Z such that

$$\frac{J_2}{J_3} \simeq \left(\frac{I_Z}{J_2} \right)^{\otimes 2} (N_1).$$

Let $M_{r-1} \simeq \frac{J_r}{I_Z J_{(r-1)}}$ which is a rank 1 locally free sheaf on Z. As in (4), we also have

$$0 \to O_{\mathbb{P}^N}(-m) \to J_r \to J_{(r-1)} \to I_Z\,.$$

After restricting to Z the map $0 \to O_{\mathbb{P}^N}(-m) \to J_{(r-1)}$, we have $\mu_{(r-2)} : O_Z(-m) \to \frac{J_{(r-1)}}{I_Z J_{(r-1)}}$.

Consider $\mu_{(r-2)1} : \frac{J_{(r-1)}}{I_Z J_{(r-1)}} \to \operatorname{Coker} \mu_{(r-2)}$. Let $\mu_{(r-2)2} : O_Z(-m) \to \frac{J_r}{I_Z J_{(r-1)}}$. The image of $\mu_{(r-2)}$ maps to zero in $\frac{J_{(r-1)}}{J_r}$.

We have an exact sequence $0 \to \operatorname{Coker} \mu_{(r-2)2} \to \operatorname{Coker} \mu_{(r-2)} \to \frac{J_{(r-1)}}{J_r} \to 0$. We deduce that there exists an effective divisor $D_{(r-1)}$ such that $M_{(r-1)} = O_Z(D_{(r-1)} - m)$.

Proposition 1. *$\frac{J_{(r-1)}}{J_r} \simeq (\frac{I_Z}{J_2})^{\otimes (r-1)}(N_{(r-2)})$, $r \geq 3$, where $N_{(r-2)}$ is an effective divisor on Z. Also, $\wedge^2(\frac{J_{(r-1)}}{I_Z J_{(r-1)}}) \simeq {\omega_Z}^{-(r-1)}(-(N+1)(r-1)) \otimes O_Z((r-2)m - (r-1)D_1 + D_{(r-1)})$, where D_1 and $D_{(r-1)}$ are effective divisors on Z.*

Proof. For $N = 3$, see [1].

Let $N \geq 3$. We have an exact sequence

$$0 \to \frac{J_r}{I_Z J_{(r-1)}} \to \frac{J_{(r-1)}}{I_Z J_{(r-1)}} \to \frac{J_{(r-1)}}{J_r} \to 0. \tag{6}$$

Notice that $(\frac{I_Z}{J_2})^{\otimes(r-1)} \simeq \frac{{I_Z}^{(r-1)}}{{I_Z}^{(r-2)} J_2}$.

$$0 \to \frac{{I_Z}^{(r-1)}}{{I_Z}^{(r-2)} J_2} \to \frac{J_{(r-1)}}{{I_Z}^{(r-2)} J_2} \to \frac{J_{(r-1)}}{J_2 J_{(r-2)}} \to \frac{J_{(r-1)}}{J_r} \to 0.$$

Since ${I_Z}^{(r-2)} J_2 \subset J_2 J_{(r-2)}$ the map $\frac{J_{(r-1)}}{{I_Z}^{(r-2)} J_2} \to \frac{J_{(r-1)}}{J_2 J_{(r-2)}}$ is surjective. Also $\frac{J_{(r-1)}}{J_2 J_{(r-2)}} \to \frac{J_{(r-1)}}{J_r} \to 0$, since $J_2 J_{(r-2)} \subset J_r$. There exists an effective divisor $N_{(r-2)}$ on Z such that

$$\frac{J_{(r-1)}}{J_r} \simeq \left(\frac{I_Z}{J_2}\right)^{\otimes(r-1)} (N_{(r-2)}).$$

From (6) we deduce $\frac{J_{(r-1)}}{J_r} \simeq \wedge^2(\frac{J_{(r-1)}}{I_Z J_{(r-1)}}) \otimes {M_{(r-1)}}^{\otimes -1}$. Thus,

$$\wedge^2\left(\frac{J_{(r-1)}}{I_Z J_{(r-1)}}\right) \simeq \frac{J_{(r-1)}}{J_r} \otimes M_{(r-1)},$$

$$\wedge^2\left(\frac{J_{(r-1)}}{I_C J_{(r-1)}}\right)$$
$$\simeq {\omega_Z}^{-(r-1)}(-(N+1)(r-1)) \otimes O_Z((r-2)m - (r-1)D_1 + D_{(r-1)}). \ \blacksquare$$

PROPOSITION 2. *Let Z be an irreducible nonsingular $(N-2)$-subvariety in $\mathbb{P}^N$, $rZ = F \cap X$, $r \in \mathbb{N}$, $r \geq 2$, where F is a smooth hypersurface of degree m, X is a normal hypersurface of degree n, $Z \cap \mathrm{Sing}(X) \neq \emptyset$. Then $\frac{J_{(r-1)}}{J_r} \simeq \omega_Z(-(m+n-(N+1)))$. There are effective divisors V and D_1 on Z such that $O_Z(V - (r-1)D_1) \simeq {\omega_Z}^r(-rm - n + (N+1)r)$.*

Proof. Since $rZ = F \cap X$, J_r is a locally complete intersection ideal. We see that $\omega_{rZ} \simeq O_{rZ}(m + n - (N+1))$.

As we have seen in Section 1, letting D_1 be the associated divisor to the torsion subsheaf $\frac{M_1}{O_Z(-m)}$, we have $M_1 = O_Z(D_1 - m)$. We shall see that $\frac{J_{(r-1)}}{J_r} \simeq \omega_Z(-(m+n-(N+1)))$.

Let $S = \frac{J_{(r-1)}}{J_r}$. The exact sequence

$$0 \to S \to \frac{O_{\mathbb{P}^N}}{J_r} \to \frac{O_{\mathbb{P}^N}}{J_{(r-1)}} \to 0$$

induces the surjective map

$$\omega_{rZ} \simeq \mathcal{E}xt^2(O_{rZ}, \omega_{\mathbb{P}^N}) \to \mathcal{E}xt^2(S, \omega_{\mathbb{P}^N}),$$

since $O_{(r-1)Z}$ is locally Cohen–Macaulay and hence $\mathcal{E}xt^3(O_{(r-1)Z}, \omega_{\mathbb{P}^N}) = 0$.

$\mathcal{E}xt^2(S, \omega_{\mathbb{P}^N})$ is a rank 1 locally free sheaf on Z.

$$S \simeq \mathcal{E}xt^2(\mathcal{E}xt^2(S, \omega_{\mathbb{P}^N}), \omega_{\mathbb{P}^N}) \simeq \mathcal{E}xt^2((\omega_{rZ})|_Z, \omega_{\mathbb{P}^N})$$
$$\simeq \mathcal{E}xt^2(O_Z, \omega_{\mathbb{P}^N})(-l) \simeq \omega_Z(-l),$$

where $l = m + n - (N+1)$.

The functor $\mathcal{E}xt^2(-,\omega_{\mathbb{P}^N})$ is exact and reflexive on the category of Cohen–Macaulay $O_{\mathbb{P}^N}$-modules of codimension 2. Thus, $\frac{J_{(r-1)}}{J_r} \simeq \omega_Z(-(m+n-(N+1)))$.

By Proposition 1, $\frac{J_{(r-1)}}{J_r} \simeq (\frac{I_Z}{J_2})^{\otimes(r-1)}(N_{(r-2)})$, so

$$\omega_Z(-(m+n-(N+1))) \simeq {\omega_Z}^{-r+1}((r-1)(m-(N+1))) \otimes \mathcal{O}_Z(-(r-1)D_1) \otimes O_Z(N_{(r-2)}).$$

Let $V := N_{(r-2)}$. It follows $O_Z(V-(r-1)D_1) \simeq {\omega_Z}^r(-rm-n+(N+1)r)$. ∎

2.

Definition 3. A closed subscheme $Z \subset \mathbb{P}^N$ of codimension k is a *complete intersection* if there are k hypersurfaces (i.e. locally principal subschemes of codimension 1) F_1, $F_2,\dots,F_k$, such that $Z = F_1 \cap \dots \cap F_k$, as schemes.

Proposition 4. *Let $Z \subset \mathbb{P}^N$ be a reduced irreducible subvariety of codimension 2. Let Y be an irreducible hypersurface in $\mathbb{P}^N$, $Z \subset Y$. Let ${\omega^o}_Z$ be the dualizing sheaf of Z. Then there exists a hypersurface X in $\mathbb{P}^N$ such that $Z = Y \cap X$ is a scheme-theoretical complete intersection if and only if Z satisfies the following properties*

- ${\omega^o}_Z \simeq \omega_{\mathbb{P}^N} \otimes \wedge^2 \mathcal{N}_Z|_{\mathbb{P}^N}$.
- $\deg Y$ *divides* $\deg Z$.
- ${\omega^o}_Z \simeq O_Z(\deg Y + (\frac{\deg Z}{\deg Y}) - N - 1)$.

Proof. For $N = 3$ see [1] and [3]. Let us assume $\deg Y = m$. Let ${\omega^o}_Z \simeq \omega_{\mathbb{P}^N} \otimes \wedge^2 \mathcal{N}_Z|_{\mathbb{P}^N}$. Let $\frac{\deg Z}{\deg Y} = n$ and ${\omega^o}_Z \simeq O_Z(m+n-N-1)$.

Let $\mathcal{N}$ be the normal bundle of Z in $\mathbb{P}^N$. Let $\mathcal{E}$ be a rank 2 vector bundle in $\mathbb{P}^N$ obtained from Z via Hartshorne–Serre correspondence so that $Z \subset \mathbb{P}^N$ is the set of zeroes of a section $\tilde{s} \in H^0(\mathbb{P}^N,\mathcal{E})$ and $\mathcal{E}|_Z \simeq \mathcal{N}$. Let $\mathcal{E}^*$ be the dual of $\mathcal{E}$. Let I_Z be the ideal sheaf of Z in $\mathbb{P}^N$.

We have an exact sequence

$$0 \to \det \mathcal{E}^* \to \mathcal{E}^* \to I_Z \to 0 \tag{7}$$

Since ${\omega^o}_Z \simeq \omega_{\mathbb{P}^N} \otimes \wedge^2 \mathcal{N}_Z|_{\mathbb{P}^N}$,

$${\omega^o}_Z \simeq \wedge^2 \left(\frac{I_Z}{{I_Z}^2}\right)^*(-N-1).$$

Since ${\omega^o}_Z \simeq O_Z(m+n-N-1)$,

$$\wedge^2 \left(\frac{I_Z}{{I_Z}^2}\right)^* \simeq O_Z(m+n),$$

$$\wedge^2 \mathcal{E}^*|_Z \simeq \wedge^2 \left(\frac{I_Z}{{I_Z}^2}\right) \simeq O_Z(-m-n),$$

$$\wedge^2 \mathcal{E}^* \simeq O_{\mathbb{P}^N}(-m-n).$$

The first Chern class of $\mathcal{E}^*$ is $-m-n$, the second Chern class is mn.

From (7) we have

$$0 \to O_{\mathbb{P}^N}(-m-n) \to \mathcal{E}^* \to I_Z \to 0.$$

Tensoring with $-\otimes O_{\mathbb{P}^N}(m)$, we obtain

$$0 \to O_{\mathbb{P}^N}(-n) \to \mathcal{E}^*(m) \to I_Z(m) \to 0.$$

Computing cohomology,

$$H^0(\mathcal{E}^*(m)) \to H^0(I_Z(m)) \to 0,$$

since $H^1(O_{\mathbb{P}^N}(-n)) = 0$. Let $s_1 \in H^0(\mathcal{E}^*(m))$ be an element whose image is equal to Y in $H^0(I_Z(m))$. For every effective divisor D in $\mathbb{P}^N$, there exists a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{E}^*(m-D)) & \longrightarrow & H^0(I_Z(m-D)) \\ \big\downarrow \alpha & & \big\downarrow \beta \\ H^0(\mathcal{E}^*(m)) & \longrightarrow & H^0(I_Z(m)) \end{array}$$

with vertical arrows $\alpha : H^0(\mathcal{E}^*(m-D)) \to H^0(\mathcal{E}^*(m))$ and $\beta : H^0(I_Z(m-D)) \to H^0(I_Z(m))$ for $s_1 \in H^0(\mathcal{E}^*(m))$ is not in $\operatorname{Im}\alpha$ since Y is irreducible. Thus, the scheme of zeroes of s_1 is empty in codimension 1 components.

$c_2(\mathcal{E}^*(m)) = c_2(\mathcal{E}^*) + mc_1(\mathcal{E}^*) + m^2 = 0$, so the scheme of zeroes of s_1 is empty. Note that $c_1(\mathcal{E}^*(m)) = c_1(\mathcal{E}^*) + 2m = m - n$. Thus, $\mathcal{E}^*(m)$ is decomposable into the direct summands $O_{\mathbb{P}^N}$ and $O_{\mathbb{P}^N}(m-n)$.

Hence, we deduce that $\mathcal{E}^*$ is decomposable into the direct summands $O_{\mathbb{P}^N}(-m)$ and $O_{\mathbb{P}^N}(-n)$. Therefore, there exists a hypersurface X of degree n such that $Z = Y \cap X$.

The necessity condition is obvious since $Z = Y \cap X$ is a scheme-theoretical complete intersection. ∎

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