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WEIGHTED FRACTIONAL AND HARDY TYPE OPERATORS IN ORLICZ–MORREY SPACES

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Abstract. We prove boundedness of the Riesz fractional integral operator between distinct Orlicz–Morrey spaces, which is a generalization of the Adams type result. Moreover, we investigate boundedness of some weighted Hardy type operators and weighted Riesz fractional integral operators between distinct Orlicz–Morrey spaces.

1. Introduction. Morrey spaces were introduced in 1938 by C. Morrey [17] to study local behavior of solutions of second order elliptic partial differential equations. Since then this space has been systematically investigated by many authors. *Morrey spaces* are defined as

$$M^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \colon \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p\,dy < \infty \right\},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$. They are Banach ideal spaces on $\mathbb{R}^n$ with respect to the norm

$$\|f\|_{p,\lambda} := \sup_{x\in\mathbb{R}^n,\, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p\,dy \right)^{1/p}.$$

Here and below, $B(x,r)$ denotes the open ball with center at $x \in \mathbb{R}^n$ and radius $r > 0$, that is, $\{y \in \mathbb{R}^n \colon |y-x| < r\}$. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$, which is $|B(x,r)| = v_n r^n$ with $v_n = |B(0,1)|$.

Morrey spaces are generalizations of L^p-spaces since $M^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Moreover, the space $M^{p,\lambda}(\mathbb{R}^n)$ is trivial when $\lambda > n$, that is, $M^{p,\lambda}(\mathbb{R}^n) = \{0\}$ (the set of all functions equivalent to 0 on $\mathbb{R}^n$—see [4, Lemma 1]) and $M^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ by the Lebesgue differentiation theorem (for the proof we refer to [12, Theorem 4.3.6]).

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Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a measurable function satisfying the following assumptions:

$$\lim_{r\to 0^+} \varphi(r) = \varphi(0) = 0, \qquad \varphi(r) = 0 \Leftrightarrow r = 0, \tag{1.1}$$

$$\varphi(r) \geq Cr^n \quad \text{for some constant } C > 0 \text{ and all } 0 < r \leq 1. \tag{1.2}$$

Replacing r^λ by such a function $\varphi(r)$ in the definition of Morrey spaces $M^{p,\lambda}(\mathbb{R}^n)$ we obtain *generalized Morrey spaces*

$$M^{p,\varphi}(\mathbb{R}^n) = \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)\colon \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p\, dy < \infty \Big\},$$

with the norm defined by

$$\|f\|_{p,\varphi} := \sup_{x\in\mathbb{R}^n,\, r>0} \bigg(\frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p\, dy \bigg)^{1/p}.$$

For properties of Morrey-type spaces we refer, for instance, to [2], [3], [12], [23] and the references therein.

We will use Orlicz–Morrey spaces, therefore we need the definition of Orlicz spaces on $\mathbb{R}^n$. These spaces were introduced by Orlicz [21], [22] as a generalization of L^p-spaces.

A function $\Phi\colon [0,+\infty) \to [0,+\infty)$ is called an *Orlicz function* if it is an increasing, continuous and convex function with $\Phi(0) = 0$.

For any Orlicz function Φ the *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is defined in the following way:

$$L^\Phi(\mathbb{R}^n) = \Big\{ f \in L^0(\mathbb{R}^n)\colon \int_{\mathbb{R}^n} \Phi(k|f(y)|)\, dy < \infty \text{ for some } k > 0 \Big\}.$$

These spaces are Banach ideal spaces with the norm

$$\|f\|_{L^\Phi} = \inf\Big\{ \lambda > 0\colon \int_{\mathbb{R}^n} \Phi(|f(y)|/\lambda)\, dy \leq 1 \Big\}.$$

For further properties of Orlicz spaces we refer, for instance, to [11], [13] and [24].

The study of boundedness of the *Riesz fractional integral operator* I_α, $0 < \alpha < n$, defined for $x \in \mathbb{R}^n$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

between L^p-spaces was initiated by Sobolev [27] in 1938. He proved that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < n/\alpha$ if and only if $1/q = 1/p - \alpha/n$ (cf. [28, Theorem 1, pp. 119–121]). The boundedness of the Riesz fractional integral operator between Orlicz spaces was proved by Simonenko [26] and later on by Cianchi [7]. The results on boundedness of the Riesz fractional integral operator from $M^{p,\lambda}(\mathbb{R}^n)$ to $M^{q,\mu}(\mathbb{R}^n)$ were first obtained by S. Spanne with the Sobolev exponent $1/q = 1/p - \alpha/n$, and this result was published by Peetre [23]. A stronger result with a better exponent $1/q = 1/p - \alpha/(n - \lambda)$ was obtained by Adams [1] (see also [6]). Nakai [18] extended Spanne's result and proved the boundedness of I_α in generalized Morrey spaces. Eridani and Gunawan [8] obtained an Adams-type result for generalized Morrey spaces.

Then it was natural to consider the boundedness of I_α in Orlicz–Morrey spaces. The Orlicz–Morrey spaces $M^{\Phi,\varphi}(\mathbb{R}^n)$, introduced in [19], unify Orlicz and Morrey spaces. In [20] Nakai studied the $M^{\Phi,\varphi}(\mathbb{R}^n) \to M^{\Psi,\psi}(\mathbb{R}^n)$ boundedness of the generalized fractional integral operator and obtained an Adams-type result. Mizuta and Shimomura [16] extended Nakai's result to generalized Morrey spaces of integral form. We also refer to [15], where the boundedness of generalized Riesz potentials was considered on an open bounded set G on $\mathbb{R}^n$ from a generalized Morrey space $M^{1,\varphi}(G)$ to an Orlicz–Morrey space $M^{\Phi,\psi}(G)$ and also between distinct Orlicz spaces.

The operator I_α plays an important role in real and harmonic analysis with applications (see, e.g., [2, Chapter 15]). In particular, in [9] it was shown that various operators can be estimated from above by Riesz potentials and the boundedness of those operators in generalized Morrey spaces was proved.

In this paper we first prove $M^{\Phi,\varphi}(\mathbb{R}^n) \to M^{\Psi,\psi}(\mathbb{R}^n)$ boundedness of the operator I_α using Hedberg's method [10] and we obtain an Adams-type result. In [20] Nakai described conditions for the boundedness of fractional integral operators between distinct Orlicz–Morrey spaces in integral terms. We provide conditions on the $M^{\Phi,\varphi}(\mathbb{R}^n) \to M^{\Psi,\psi}(\mathbb{R}^n)$ boundedness of the operator I_α in a more suitable way for our further needs. We also prove the boundedness of some weighted Hardy operators between distinct Orlicz–Morrey spaces and finally using pointwise estimates we investigate the boundedness of a weighted Riesz fractional integral operator from the Orlicz–Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ to the Orlicz–Morrey space $M^{\Psi,\psi}(\mathbb{R}^n)$ for some classes of weights.

Throughout this paper, we will let C denote a positive constant whose value may change from line to line, but which is independent of essential parameters.

2. Preliminaries

2.1. Orlicz–Morrey spaces. Let Φ be an Orlicz function and let φ satisfy conditions (1.1)–(1.2). We define the generalized Orlicz–Morrey spaces $M^{\Phi,\varphi}(\mathbb{R}^n)$ in the following way:

$$M^{\Phi,\varphi}(\mathbb{R}^n) = \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \colon \|f\|_{M^{\Phi,\varphi}} = \sup_{B=B(x,r)} \|f\|_{\Phi,\varphi,B} < \infty \Big\},$$

where $r > 0$, $x \in \mathbb{R}^n$ and

$$\|f\|_{\Phi,\varphi,B} = \inf\left\{\lambda > 0\colon \frac{1}{\varphi(r)} \int\limits_{B(x,r)} \Phi(|f(y)|/\lambda)\, dy \le 1\right\}.$$

In the case $\Phi(u) = u^p$, $1 \le p < \infty$, the Orlicz–Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ turns into the generalized Morrey space $M^{p,\varphi}(\mathbb{R}^n)$.

To each Orlicz function Φ one can associate a complementary function Φ^*, defined for $v \ge 0$ by

$$\Phi^*(v) = \sup_{u>0}\, [uv - \Phi(u)].$$

We say that an Orlicz function Φ satisfies the Δ_2-*condition*, and we write $\Phi \in \Delta_2$, if there exists a constant $C \ge 1$ such that $\Phi(2u) \le C\Phi(u)$ for all $u > 0$.

For any ball $B = B(x,r)$ the generalized Hölder inequality holds:

$$\int\limits_{B(x,r)} |f(y)|\, |g(y)|\, dy \le 2\varphi(r)\|f\|_{\Phi,\varphi,B}\|g\|_{\Phi^*,\varphi,B}.$$

In particular,

$$\int\limits_{B(x,r)} |f(y)|\, dy \le 2r^n \Phi^{-1}(\varphi(r)/r^n)\|f\|_{M^{\Phi,\varphi}}. \tag{2.1}$$

For further properties of the Orlicz–Morrey spaces we refer, for instance, to [14] and [20].

2.2. Almost increasing and almost decreasing functions. A non-negative function g on $(0,\infty)$ is said to be *almost increasing* (resp. *almost decreasing*) on $(0,\infty)$ if there exists a constant $C \ge 1$ such that $g(x) \le Cg(y)$ for all $0 < x \le y$ (resp. all $x \ge y > 0$). We will also need the following technical lemma:

Lemma 1.

(i) *Let $g\colon (0,\infty) \to (0,\infty)$ be a measurable almost decreasing function. Then there exists a constant $C > 0$ such that*

$$\sum_{k=0}^{\infty} g(2^{k+1}r) \le C \int\limits_r^{\infty} \frac{g(t)}{t}\, dt \quad \textit{for all } r > 0.$$

(ii) *Let $g\colon (0,\infty) \to (0,\infty)$ be a measurable almost increasing function. Then there exists a constant $C > 0$ such that*

$$\sum_{k=0}^{\infty} g(2^{-k-1}r) \le C \int\limits_0^{r} \frac{g(t)}{t}\, dt \quad \textit{for all } r > 0.$$

Proof. (i) Since g is almost decreasing,

$$\int_r^\infty \frac{g(t)}{t}\,dt = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{g(t)}{t}\,dt \ge C\sum_{k=0}^\infty g(2^{k+1}r)\int_{2^k r}^{2^{k+1}r}\frac{dt}{t}$$
$$= C\ln 2\sum_{k=0}^\infty g(2^{k+1}r).$$

(ii) Since g is almost increasing,

$$\int_0^r \frac{g(t)}{t}\,dt = \sum_{k=0}^\infty \int_{2^{-k-1}r}^{2^{-k}r} \frac{g(t)}{t}\,dt \ge C\sum_{k=0}^\infty g(2^{-k-1}r)\int_{2^{-k-1}r}^{2^{-k}r}\frac{dt}{t}$$
$$= C\ln 2\sum_{k=0}^\infty g(2^{-k-1}r). \ \blacksquare$$

3. Boundedness of the Riesz fractional integral operator between distinct Orlicz–Morrey spaces. Throughout this paper we assume that Φ is an Orlicz function, and that φ and ψ satisfy assumptions (1.1)–(1.2) and the following conditions:

(3.1) $\qquad \varphi$ is increasing on $(0,\infty)$ and $\varphi(r)/r^n$ is decreasing on $(0,\infty)$,

(3.2) $\qquad \psi$ is almost increasing on $(0,\infty)$,

(3.3) $\qquad \varphi(r)\le A\psi(r)$ for some constant $A>0$ and any $r>0$.

We will use the following notation:

$$U(r)=\frac{\varphi(r)}{r^n},\qquad g(r)=r^\alpha\Phi^{-1}(U(r)),\qquad V(r)=\Phi^{-1}(r)[U^{-1}(r)]^\alpha. \tag{3.4}$$

Also we always assume that the function V defined in (3.4) satisfies the following conditions:

V is continuous, increasing, unbounded, concave on $[0,\infty)$ with $V(0)=0$.

Then the function $\Psi=V^{-1}$ is an Orlicz function which defines our target space $M^{\Psi,\psi}(\mathbb{R}^n)$.

Note that

$$V(U(r))=\Phi^{-1}(U(r))[U^{-1}(U(r))]^\alpha=g(r). \tag{3.5}$$

The Hardy–Littlewood maximal operator M is defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy.$$

This operator is bounded in $M^{\Phi,\varphi}(\mathbb{R}^n)$ provided $\Phi^*\in\Delta_2$, and

$$\|Mf\|_{M^{\Phi,\varphi}}\le C_0\|f\|_{M^{\Phi,\varphi}} \tag{3.6}$$

with some constant $C_0 > 0$ independent of f (the proof is given for example in [14] and [20]).

In our first theorem we prove boundedness of the Riesz fractional integral operator I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$. In the proof we use a pointwise estimate of $I_\alpha f$ by the maximal operator Mf and boundedness of the maximal operator Mf in $M^{\Phi,\varphi}(\mathbb{R}^n)$. This method of proof was introduced by Hedberg [10].

THEOREM 1. *Let $0 < \alpha < n$ and let Φ be an Orlicz function with $\Phi^* \in \Delta_2$. If the function g defined in* (3.4) *is almost decreasing and there exists a constant $C > 0$ such that*

$$\int_r^\infty g(t)\,\frac{dt}{t} \le Cg(r) \quad \text{for all } r > 0, \tag{3.7}$$

then the Riesz fractional integral operator I_α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$.

Proof. We follow Hedberg's approach and split I_α into two integrals,

$$I_\alpha f(x) = \int_{|x-y|\le r} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy + \int_{|x-y|>r} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy =: I_1(x,r) + I_2(x,r),$$

with $r \in (0,\infty)$ to be chosen later. As shown in [10],

$$|I_1(x,r)| \le cr^\alpha Mf(x).$$

The integral $I_2(x,r)$ is estimated in a similar way. Applying (2.1) we obtain

$$\begin{aligned}
|I_2(x,r)| &\le \sum_{k=0}^\infty \int_{2^k r<|x-y|\le 2^{k+1}r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \\
&\le 2\|f\|_{M^{\Phi,\varphi}} \sum_{k=0}^\infty \frac{(2^{k+1}r)^n}{(2^k r)^{n-\alpha}} \Phi^{-1}\left(\frac{\varphi(2^{k+1}r)}{(2^{k+1}r)^n}\right) \\
&= 2^{n-\alpha+1}\|f\|_{M^{\Phi,\varphi}} \sum_{k=0}^\infty (2^{k+1}r)^\alpha \Phi^{-1}\left(\frac{\varphi(2^{k+1}r)}{(2^{k+1}r)^n}\right) \\
&= C\|f\|_{M^{\Phi,\varphi}} \sum_{k=0}^\infty g(2^{k+1}r),
\end{aligned}$$

where g is defined in (3.4) and the constant $C > 0$ depends only on n and α. Since the function $g(t) = t^\alpha \Phi^{-1}(\varphi(t)/t^n)$ is almost decreasing on $(0,\infty)$ it follows from the first inequality in Lemma 1 that

$$|I_2(x,r)| \le C\|f\|_{M^{\Phi,\varphi}} \int_r^\infty g(t)\,\frac{dt}{t},$$

and by (3.7) we obtain

$$|I_2(x,r)| \le C\|f\|_{M^{\Phi,\varphi}} g(r) = C\|f\|_{M^{\Phi,\varphi}} r^\alpha \Phi^{-1}(U(r)) \quad \text{for any } r>0.$$

Combining the estimates of $I_1(x,r)$ and $I_2(x,r)$ we get

$$|I_\alpha f(x)| \le Cr^\alpha[Mf(x) + \|f\|_{M^{\Phi,\varphi}} \Phi^{-1}(U(r))] \quad \text{for any } r>0.$$

Since the function $\Phi^{-1}(U(r))$ is surjective, we can choose $r>0$ such that $Mf(x) = C_0\|f\|_{M^{\Phi,\varphi}}\Phi^{-1}(U(r))$, which implies $\Phi\big(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\varphi}}}\big) = U(r)$, where the constant C_0 is defined in (3.6). Since the function $\frac{\varphi(r)}{r^n} = U(r)$ is decreasing it follows that

$$r = U^{-1}\bigg(\Phi\bigg(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\varphi}}}\bigg)\bigg).$$

Finally,

$$\begin{aligned}|I_\alpha f(x)| &\le 2C\bigg[U^{-1}\bigg(\Phi\bigg(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\varphi}}}\bigg)\bigg)\bigg]^\alpha Mf(x)\\ &= C_1\|f\|_{M^{\Phi,\varphi}}\bigg[U^{-1}\bigg(\Phi\bigg(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\varphi}}}\bigg)\bigg)\bigg]^\alpha (\Phi^{-1}\circ\Phi)\bigg(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\varphi}}}\bigg)\\ &= C_1\|f\|_{M^{\Phi,\varphi}} V\bigg(\Phi\bigg(\frac{Mf(x)}{C_0\|f\|_{M^{\Phi,\varphi}}}\bigg)\bigg).\end{aligned}$$

Since $\Psi = V^{-1}$ and $\varphi(r) \le A\psi(r)$ it follows that for any ball $B(x,r)$,

$$\begin{aligned}\frac{1}{\psi(r)}\int_{B(x,r)} \Psi\bigg(\frac{|I_\alpha f(y)|}{C_1\|f\|_{M^{\Phi,\varphi}}}\bigg)dy &\le \frac{1}{\psi(r)}\int_{B(x,r)} \Phi\bigg(\frac{Mf(y)}{C_0\|f\|_{M^{\Phi,\varphi}}}\bigg)dy\\ &\le \frac{A}{\varphi(r)}\int_{B(x,r)} \Phi\bigg(\frac{Mf(y)}{\|Mf\|_{M^{\Phi,\varphi}}}\bigg)dy \le A,\end{aligned}$$

where the last inequality follows from the boundedness of the maximal operator M from $M^{\Phi,\varphi}(\mathbb{R}^n)$ into itself provided $\Phi^* \in \Delta_2$. Hence for $A \ge 1$, by the convexity of Ψ, we obtain

$$\frac{1}{\psi(r)}\int_{B(x,r)} \Psi\bigg(\frac{|I_\alpha f(y)|}{AC_1\|f\|_{M^{\Phi,\varphi}}}\bigg)dy \le 1,$$

which proves boundedness of I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ and $\|I_\alpha f\|_{M^{\Psi,\psi}} \le AC_1\|f\|_{M^{\Phi,\varphi}}$ for any $f \in M^{\Phi,\varphi}(\mathbb{R}^n)$. ■

We now show by examples that Theorem 1 generalizes Adams's result.

EXAMPLE 1. Let $0<\alpha<n, 1<p<q<\infty$ and

$$\Phi(u) = u^p, \quad \Psi(u) = u^q, \quad \varphi(r) = \psi(r) = r^\lambda.$$

Then from Theorem 1 we get the result of Adams [1], that is, the operator I_α is bounded from $M^{p,\lambda}(\mathbb{R}^n)$ to $M^{q,\lambda}(\mathbb{R}^n)$ under the conditions

$$0<\lambda<n-\alpha p \quad \text{and} \quad \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}. \tag{3.8}$$

Indeed, since the function $g(r)=r^{\alpha+(\lambda-n)/p}$ is almost decreasing it follows that $\lambda\le n-\alpha p$, and together with requirement (3.7) we arrive at the first condition in (3.8). The second condition in (3.8) follows from the fact that $\Psi=V^{-1}$, where V is defined in (3.4) and in this case $V(r)=r^{\frac{1}{p}+\frac{\alpha}{\lambda-n}}$.

Example 2. Let $p>1, 0<\alpha<n, 0<\lambda<n-\alpha p$ and $\varphi(r)=\psi(r)=r^\lambda$. For $\beta_1,\beta_2,\gamma_1,\gamma_2\ge 0$ let

$$\Phi(u)=\begin{cases} u^p\big(\ln\frac{1}{u}\big)^{-\beta_1}\big(\ln\ln\frac{1}{u}\big)^{-\beta_2} & \text{for small } u>0,\\ u^p(\ln u)^{\gamma_1}(\ln\ln u)^{\gamma_2} & \text{for large } u>0,\end{cases}$$

$$\Psi(u)\approx\begin{cases} u^{pc}\big(\ln\frac{1}{u}\big)^{-\beta_1 c}\big(\ln\ln\frac{1}{u}\big)^{-\beta_2 c} & \text{for small } u>0,\\ u^{pc}(\ln u)^{\gamma_1 c}(\ln\ln u)^{\gamma_2 c} & \text{for large } u>0,\end{cases}$$

where $c=\frac{n-\lambda}{n-\lambda-\alpha p}$. Then the operator I_α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\varphi}(\mathbb{R}^n)$.

Note that in Example 2 we can consider a more general function $\Phi(u)$, defined for $p>1$, $\beta_i,\gamma_i\ge 0, i=1,\dots,n$, as

$$\Phi(u)=\begin{cases} u^p\big(\ln\frac{1}{u}\big)^{-\beta_1}\big(\ln\ln\frac{1}{u}\big)^{-\beta_2}\cdots\big(\ln\ldots\ln\frac{1}{u}\big)^{-\beta_n} & \text{for small } u>0,\\ u^p(\ln u)^{\gamma_1}(\ln\ln u)^{\gamma_2}\cdots(\ln\ldots\ln u)^{\gamma_n} & \text{for large } u>0.\end{cases}$$

Example 3. Let $0<\lambda<n$, $\alpha>0$ and $\varphi(r)=\psi(r)=r^\lambda$. For $a>1$ and $b\ge 0$ such that $a+b<\frac{n-\lambda}{\alpha}$ let

$$\Phi(u)=u^a(\ln(1+u))^b\approx\begin{cases} u^{a+b} & \text{for small } u>0,\\ u^a(\ln u)^b & \text{for large } u>0,\end{cases}$$

$$\Psi(u)\approx\begin{cases} u^{(a+b)\theta(a+b)} & \text{for small } u>0,\\ u^{a\theta(a)}(\ln u)^{b\theta(a)} & \text{for large } u>0,\end{cases}$$

where $\theta(r)=\frac{n-\lambda}{n-\lambda-\alpha r}$. Then I_α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\varphi}(\mathbb{R}^n)$.

4. Boundedness of weighted Hardy operators between distinct Orlicz–Morrey spaces. Let $w\colon(0,\infty)\to(0,\infty)$ be a continuous measurable function such that $w(2r)\le Cw(r)$ for some constant $C>0$, and $w(r)/r^a$ is almost increasing on $(0,\infty)$ for some $a\in\mathbb{R}$. We consider the following *weighted Hardy operators*:

$$H_w^\alpha f(x) = |x|^{\alpha-n} w(|x|) \int\limits_{|y|\le|x|} \frac{f(y)}{w(|y|)}\, dy,$$

$$\mathcal{H}_w^\alpha f(x) = |x|^{\alpha} w(|x|) \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n w(|y|)}\, dy.$$

Theorem 2. *Let* $0<\alpha<n$. *Suppose* $\frac{\psi(r)}{r^n}$ *is almost decreasing on* $(0,\infty)$ *and*

$$\int_0^r \varphi(t)\,\frac{dt}{t} \le C\varphi(r)$$

for some constant $C>0$ *and all* $r>0$.

(i) *The Hardy operator* H_w^α *is bounded from* $M^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $M^{\Psi,\psi}(\mathbb{R}^n)$ *provided*

$$\begin{aligned} &\frac{r^{n-\alpha}}{w(r)} g(r) \text{ is almost increasing on } (0,\infty) \text{ and} \\ &\int_0^r \frac{t^{n-\alpha}}{w(t)}\,\frac{g(t)}{t}\,dt \le C_1 \frac{r^{n-\alpha}}{w(r)} g(r) \end{aligned} \tag{4.1}$$

for some constant $C_1>0$ *and all* $r>0$, *where* g *is defined in* (3.4).

(ii) *The Hardy operator* $\mathcal{H}_w^\alpha$ *is bounded from* $M^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $M^{\Psi,\psi}(\mathbb{R}^n)$ *provided*

$$\begin{aligned} &\frac{g(r)}{r^\alpha w(r)} \text{ is almost decreasing on } (0,\infty) \text{ and} \\ &\int_r^\infty \frac{g(t)}{t^\alpha w(t)}\,\frac{dt}{t} \le C_2 \frac{g(r)}{r^\alpha w(r)} \end{aligned} \tag{4.2}$$

for some constant $C_2>0$ *and all* $r>0$, *where* g *is defined in* (3.4).

In order to prove Theorem 2 we need the following lemma.

Lemma 2. *Let* Ψ *be an Orlicz function. Suppose the function* ψ *defined by* (1.1)–(1.2) *is almost increasing and such that* $\psi(r)/r^n$ *is almost decreasing on* $(0,\infty)$. *Assume that* $f\colon \mathbb{R}^n \to \mathbb{R}$ *and* $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ *are given measurable functions. If there exists a constant* $C>0$ *such that* $|f(x)| \le Cg(|x|)$ *for all* $x\in\mathbb{R}^n$ *and*

(i) $\Psi\circ g$ *is almost decreasing on* $(0,\infty)$,
(ii) $\int_0^r t^{n-1}(\Psi\circ g)(t)\,dt \le C\psi(r)$ *for all* $r>0$,
(iii) $r^n(\Psi\circ g)(r) \le C\psi(r)$ *for all* $r>0$,

then $f\in M^{\Psi,\psi}(\mathbb{R}^n)$.

Proof. Let $y \in B(x,r)$. We consider two cases separately: $|x| \le 2r$ and $|x| > 2r$.

Let first $|x| \le 2r$. Then $|y| \le |y-x| + |x| \le 3r$ and therefore $B(x,r) \subset B(0,3r)$. Since $\varPsi(u)$ is increasing and $|f(y)| \le Cg(|y|)$ for all $y \in \mathbb{R}^n$, it follows that

$$\int\limits_{B(x,r)} \varPsi\left(\frac{|f(y)|}{C}\right) dy \le \int\limits_{B(x,r)} \varPsi\left(g(|y|)\right) dy \le \int\limits_{B(0,3r)} \varPsi\left(g(|y|)\right) dy$$
$$= v_n \int\limits_0^{3r} \rho^{n-1} \varPsi\left(g(\rho)\right) d\rho \le v_n C \psi(3r),$$

where in the last inequality we have applied condition (ii) of this lemma.

Since $\psi(r)/r^n$ is almost decreasing on $(0,\infty)$, we have $\frac{\psi(3r)}{\psi(r)} \le 3^n C$ and

$$\frac{1}{\psi(r)} \int\limits_{B(x,r)} \varPsi(|f(y)|/C)\, dy \le C_n.$$

Therefore, for $C_n > 1$, by the convexity of $\varPsi$,

$$\frac{1}{\psi(r)} \int\limits_{B(x,r)} \varPsi\left(\frac{|f(y)|}{CC_n}\right) dy \le 1,$$

which gives $f \in M^{\varPsi,\psi}(\mathbb{R}^n)$.

Let now $|x| > 2r$. Then $|y| \ge \big||x| - |x-y|\big| > r$ and applying conditions (i) and (iii) of the lemma, we get

$$\int\limits_{B(x,r)} \varPsi\left(\frac{|f(y)|}{C}\right) dy \le \int\limits_{B(x,r)} \varPsi\left(g(|y|)\right) dy \le C v_n r^n \left(\varPsi \circ g\right)(r) \le C v_n \psi(r).$$

Thus, for $C_n > 1$, again by the convexity of $\varPsi$, we have

$$\frac{1}{\psi(r)} \int\limits_{B(x,r)} \varPsi\left(\frac{|f(y)|}{CC_n}\right) dy \le 1,$$

which shows that $f \in M^{\varPsi,\psi}(\mathbb{R}^n)$. The proof is complete. ∎

COROLLARY 1. *If $\psi(r)/r^n$ is almost decreasing on $(0,\infty)$ and there exists a constant $C > 0$ such that*

$$\int\limits_0^r \varphi(t)\,\frac{dt}{t} \le C\varphi(r) \quad \textit{for all } r > 0, \tag{4.3}$$

then the function $g(r)$ defined in (3.4) satisfies conditions (i)–(iii) of Lemma 2.

Proof. Note that from (3.5) and $\varPsi = V^{-1}$ we get

$$(\varPsi \circ g)(r) = \varPsi(V(U(r))) = U(r) = \varphi(r)/r^n,$$

which implies condition (i) of Lemma 2, since $\varphi(r)/r^n$ is decreasing on $(0,\infty)$.

By (3.3) we have

$$r^n(\Psi\circ g)(r)=\frac{r^n\varphi(r)}{r^n}\le A\psi(r),$$

which gives condition (iii) of Lemma 2.

Applying (4.3), we obtain

$$\int_0^r t^{n-1}(\Psi\circ g)(t)\,dt=\int_0^r t^{n-1}\frac{\varphi(t)}{t^n}\,dt=\int_0^r\frac{\varphi(t)}{t}\,dt\le C\varphi(r)\le AC\psi(r),$$

which shows that (ii) is also satisfied. ∎

Proof of Theorem 2. (i) We have

$$\begin{aligned}|H_w^\alpha f(x)|&\le|x|^{\alpha-n}w(|x|)\int_{|y|\le|x|}\frac{|f(y)|}{w(|y|)}\,dy\\&=|x|^{\alpha-n}w(|x|)\sum_{k=0}^{\infty}\int_{2^{-k-1}|x|<|y|\le 2^{-k}|x|}\frac{|f(y)|}{w(|y|)}\,dy.\end{aligned}$$

Since $w(t)/t^a$ is almost increasing on $(0,\infty)$ for some $a\in\mathbb{R}$, for $|y|>2^{-k-1}|x|$ we have

$$\begin{aligned}w(|y|)&\ge C\left(\frac{|y|}{2^{-k-1}|x|}\right)^a w(2^{-k-1}|x|)\\&>C\left(\frac{2^{-k-1}|x|}{2^{-k-1}|x|}\right)^a w(2^{-k-1}|x|)=Cw(2^{-k-1}|x|).\end{aligned}$$

Thus, applying (2.1), we get

$$\begin{aligned}|H_w^\alpha f(x)|&\le C|x|^{\alpha-n}w(|x|)\sum_{k=0}^{\infty}\frac{1}{w(2^{-k-1}|x|)}\int_{|y|\le 2^{-k}|x|}|f(y)|\,dy\\&\le 2C\|f\|_{M^{\Phi,\varphi}}|x|^{\alpha-n}w(|x|)\sum_{k=0}^{\infty}\frac{(2^{-k}|x|)^n}{w(2^{-k-1}|x|)}\Phi^{-1}\left(\frac{\varphi(2^{-k}|x|)}{(2^{-k}|x|)^n}\right)\\&\le 2^{n+1}C\|f\|_{M^{\Phi,\varphi}}|x|^{\alpha-n}w(|x|)\sum_{k=0}^{\infty}\frac{(2^{-k-1}|x|)^n}{w(2^{-k-1}|x|)}\Phi^{-1}\left(\frac{\varphi(2^{-k-1}|x|)}{(2^{-k-1}|x|)^n}\right),\end{aligned}$$

where in the last inequality we have used the fact that $\frac{\varphi(r)}{r^n}$ is decreasing on $(0,\infty)$ and $\Phi^{-1}(u)$ is increasing on $(0,\infty)$. Since the function $\frac{r^{n-\alpha}}{w(r)}g(r)=\frac{r^n}{w(r)}\Phi^{-1}\big(\frac{\varphi(r)}{r^n}\big)$ is almost increasing on $(0,\infty)$ we can apply the second inequality from Lemma 1:

$$|H_w^\alpha f(x)|\le 2^{n+1}C\|f\|_{M^{\Phi,\varphi}}|x|^{\alpha-n}w(|x|)\int_0^{|x|}\frac{t^{n-\alpha}g(t)}{w(t)}\,\frac{dt}{t}.$$

By (4.1), we obtain

$$|H_w^\alpha f(x)| \le C g(|x|),$$

where g is defined in (3.4). Consequently, by Lemma 2 and Corollary 1 we get $H_w^\alpha f \in M^{\Psi,\psi}(\mathbb{R}^n)$.

(ii) We have

$$\begin{aligned}|\mathcal{H}_w^\alpha f(x)| &\le |x|^\alpha w(|x|) \int\limits_{|y|>|x|} \frac{|f(y)|}{|y|^n w(|y|)}\,dy \\ &= |x|^\alpha w(|x|) \sum_{k=0}^{\infty} \int\limits_{2^k|x|<|y|\le 2^{k+1}|x|} \frac{|f(y)|}{|y|^n w(|y|)}\,dy.\end{aligned}$$

Since $w(t)/t^a$ is almost increasing on $(0,\infty)$ for some $a \in \mathbb{R}$, and since $w(2t) \le C w(t)$, for $|y| > 2^k|x|$ we have

$$w(|y|) \ge C\left(\frac{|y|}{2^k|x|}\right)^a w(2^k|x|) \ge C\left(\frac{2^k|x|}{2^k|x|}\right)^a w(2^k|x|) \ge C w(2^{k+1}|x|).$$

Thus, applying (2.1), we get

$$\begin{aligned}|\mathcal{H}_w^\alpha f(x)| &\le C|x|^\alpha w(|x|) \sum_{k=0}^{\infty} \frac{1}{(2^k|x|)^n w(2^{k+1}|x|)} \int\limits_{|y|\le 2^{k+1}|x|} |f(y)|\,dy \\ &\le 2C\|f\|_{M^{\Phi,\varphi}} |x|^\alpha w(|x|) \sum_{k=0}^{\infty} \frac{(2^{k+1}|x|)^n}{(2^k|x|)^n w(2^{k+1}|x|)} \Phi^{-1}\left(\frac{\varphi(2^{k+1}|x|)}{(2^{k+1}|x|)^n}\right) \\ &= 2^{n+1}C\|f\|_{M^{\Phi,\varphi}} |x|^\alpha w(|x|) \sum_{k=0}^{\infty} \frac{1}{w(2^{k+1}|x|)} \Phi^{-1}\left(\frac{\varphi(2^{k+1}|x|)}{(2^{k+1}|x|)^n}\right).\end{aligned}$$

The function $\frac{g(r)}{r^\alpha w(r)} = \frac{1}{w(r)}\Phi^{-1}\big(\frac{\varphi(r)}{r^n}\big)$ is almost decreasing on $(0,\infty)$ and therefore we can apply the first inequality of Lemma 1:

$$\begin{aligned}|\mathcal{H}_w^\alpha f(x)| &\le 2^{n+1}C\|f\|_{M^{\Phi,\varphi}} |x|^\alpha w(|x|) \int\limits_{|x|}^{\infty} \frac{1}{w(t)} \Phi^{-1}\left(\frac{\varphi(t)}{t^n}\right) \frac{dt}{t} \\ &= 2^{n+1}C\|f\|_{M^{\Phi,\varphi}} |x|^\alpha w(|x|) \int\limits_{|x|}^{\infty} \frac{g(t)}{t^\alpha w(t)}\,\frac{dt}{t}.\end{aligned}$$

Applying (4.2), we obtain

$$|\mathcal{H}_w^\alpha f(x)| \le C g(|x|),$$

where g is defined in (3.4). Thus, by Lemma 2 and Corollary 1 we conclude that $\mathcal{H}_w^\alpha f \in M^{\Psi,\psi}(\mathbb{R}^n)$. ∎

When $w(t) \equiv 1$ we do not need to require the condition (4.1) for the $M^{\Phi,\varphi}(\mathbb{R}^n) \to M^{\Psi,\psi}(\mathbb{R}^n)$ boundedness of $H^\alpha = H_w^\alpha|_{w\equiv 1}$.

REMARK 1. Let $0 < \alpha < n$. Suppose $\frac{\psi(r)}{r^n}$ is almost decreasing on $(0,\infty)$ and φ satisfies (4.3). Then the Hardy operator H^α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$.

Proof. From (2.1) we get

$$\begin{aligned}|H^\alpha f(x)| &\le |x|^{\alpha-n} \int\limits_{|y|\le|x|} |f(y)|\,dy \le 2\|f\|_{M^{\Phi,\varphi}} |x|^\alpha \Phi^{-1}\left(\frac{\varphi(|x|)}{|x|^n}\right) \\ &= 2\|f\|_{M^{\Phi,\varphi}} g(|x|),\end{aligned}$$

where g is defined in (3.4). Thus, $H^\alpha f \in M^{\Psi,\psi}(\mathbb{R}^n)$ by Lemma 2 and Corollary 1. ∎

Boundedness of the Hardy operator H^α in Morrey-type spaces was also proved in [5]. In particular, in the case of classical Morrey spaces we have (see also [5, Theorem 7]):

EXAMPLE 4. Let $1 < p < q < \infty, 0 < \alpha, \lambda < n$ and

$$\varphi(r) = \psi(r) = r^\lambda, \quad \Phi(u) = u^p, \quad \Psi(u) = u^q.$$

If $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$, then H^α is bounded from $M^{p,\lambda}(\mathbb{R}^n)$ to $M^{q,\lambda}(\mathbb{R}^n)$.

5. Boundedness of weighted Riesz fractional integral operators between distinct Orlicz–Morrey spaces. We deal with the following classes of weights:

DEFINITION 1. Let $0 < \mu \le 1$. We denote by $V_\pm^\mu$ the class of weight functions $w\colon (0,\infty) \to (0,\infty)$ which are continuous and satisfy the following conditions:

$$\begin{aligned} V_+^\mu: &\quad \frac{|w(x)-w(y)|}{|x-y|^\mu} \le C\frac{w(\max\{x,y\})}{(\max\{x,y\})^\mu}, \\ V_-^\mu: &\quad \frac{|w(x)-w(y)|}{|x-y|^\mu} \le C\frac{w(\min\{x,y\})}{(\max\{x,y\})^\mu}, \end{aligned}$$

where $x, y > 0$ and $x \ne y$.

Observe that if $w(t) \in V_+^\mu$, then $1/w(t) \in V_-^\mu$. Typical examples of such weights are

(i) $w(t) = t^\beta \in V_+^\mu$ and $w(t) = t^{-\beta} \in V_-^\mu$, where $\beta \ge 0$.
(ii) $w(t) = t^\beta (\ln(e+t))^\gamma \in V_+^\mu$ and $w(t) = t^{-\beta}(\ln(e+t))^{-\gamma} \in V_-^\mu$, where $\beta > 1$ and $\gamma \ge 0$.

The weighted Riesz fractional integral operator $w(|x|)(I_\alpha \frac{1}{w} f)(x)$ can be estimated by the non-weighted Riesz fractional integral operator and weighted Hardy type operators. We will use the following pointwise estimate, obtained in [25]:

Lemma 3. *Let $0 < \alpha < n$, $w \in V_-^\mu \cup V_+^\mu$ with $\mu = \min\{1, n-\alpha\}$ be a weight and $f\colon \mathbb{R}^n \to [0,\infty)$ be a given measurable function. Then the following pointwise estimate holds:*

$$w(|x|)\left(I_\alpha \frac{1}{w} f\right)(x) \le I_\alpha f(x) + c \begin{cases} H_w^\alpha f(x) + \mathcal{H}_{-\alpha}^\alpha f(x) & \text{if } w \in V_+^\mu, \\ H^\alpha f(x) + \mathcal{H}_{w_\alpha}^\alpha f(x) & \text{if } w \in V_-^\mu, \end{cases}$$

where $\mathcal{H}_{-\alpha}^\alpha = \mathcal{H}_w^\alpha|_{w=|x|^{-\alpha}}$ and $w_\alpha(|x|) = |x|^{-\alpha} w(|x|)$.

In the next theorem we will additionally require that w satisfies $w(2r) \le Cw(r)$ for some constant $C > 0$ and all $r > 0$, and $w(r)/r^a$ is almost increasing for some $a \in \mathbb{R}$. Note, that if $w \in V_+^\mu$, then w is almost increasing on $(0,\infty)$. Therefore we do not need to require that $w(r)/r^a$ be almost increasing since it is obviously satisfied with $a = 0$.

Theorem 3. *Let $0 < \alpha < n$ and let Φ be an Orlicz function with $\Phi^* \in \Delta_2$. Assume that $\mu = \min\{1, n-\alpha\}$ and $w \in V_-^\mu \cup V_+^\mu$. Assume also that $\psi(r)/r^n$ is almost decreasing on $(0,\infty)$, φ satisfies condition (4.3), and the function g defined in (3.4) is almost decreasing on $(0,\infty)$ and satisfies (3.7).*

(i) *If $w \in V_+^\mu$, then the operator $w(|x|)\big(I_\alpha \frac{1}{w} f\big)(x)$ is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ provided conditions (4.1) hold.*

(ii) *If $w \in V_-^\mu$, then the operator $w(|x|)\big(I_\alpha \frac{1}{w} f\big)(x)$ is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ provided*

$$\frac{g(r)}{w(r)} \text{ is almost decreasing on } (0,\infty) \text{ and } \int_r^\infty \frac{g(t)}{w(t)}\frac{dt}{t} \le C\frac{g(r)}{w(r)} \tag{5.1}$$

for some constant $C > 0$ and all $r > 0$.

Proof. First of all note that I_α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ since all conditions of Theorem 1 are satisfied.

(i) Let $w \in V_+^\mu$. It was shown in Theorem 2 that the Hardy operator H_w^α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ provided that conditions (4.1) hold. By (4.2) the Hardy operator $\mathcal{H}_{-\alpha}^\alpha$ is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ if

$$g(r) = r^\alpha \Phi^{-1}\left(\frac{\varphi(r)}{r^n}\right) \text{ is almost decreasing on } (0,\infty) \text{ and (3.7) holds,}$$

which is satisfied by assumption.

(ii) Let $w \in V_-^\mu$. As shown in Remark 1, H^α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ under the present assumptions. Applying the condition (4.2) for the weight $w_\alpha(|x|) = |x|^{-\alpha} w(|x|)$, we see that $\mathcal{H}_{w_\alpha}^\alpha$ is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\psi}(\mathbb{R}^n)$ provided (5.1) holds. This completes the proof. ∎

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