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# Rays to renormalizations

by

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**Summary.** Let $K_P$ be the filled Julia set of a polynomial $P$, and $K_f$ the filled Julia set of a renormalization $f$ of $P$. We show, loosely speaking, that there is a finite-to-one function $\lambda$ from the set of $P$-external rays having limit points in $K_f$ onto the set of $f$-external rays to $K_f$ such that $R$ and $\lambda(R)$ share the same limit set. In particular, if a point of the Julia set $J_f = \partial K_f$ of a renormalization is accessible from $\mathbb{C} \setminus K_f$ then it is accessible through an external ray of $P$ (the converse is obvious). Another interesting corollary is that a component of $K_P \setminus K_f$ can meet $K_f$ only in a single (pre-)periodic point. We also study a correspondence induced by $\lambda$ on arguments of rays. These results are generalizations to all polynomials (covering notably the case of connected Julia set $K_P$) of some results of Levin and Przytycki (1996), Blokh et al. (2016) and Petersen and Zakeri (2019) where it is assumed that $K_P$ is disconnected and $K_f$ is a periodic component of $K_P$.

## 1. Introduction

**1.1. Polynomial external rays.** Let $Q : \mathbb{C} \to \mathbb{C}$ be a non-linear polynomial considered as a dynamical system. Conjugating $Q$ if necessary by a linear transformation, one can assume without loss of generality that $Q$ is monic centered, i.e., $Q(z) = z^{\deg(Q)} + az^{\deg(Q)-2} + \cdots$ .

We briefly recall the necessary definitions (see e.g. [DH1], [CG], [Mil0], [LS91] for details). The *filled Julia set* $K_Q$ of $Q$ is the complement $\mathbb{C} \setminus A_Q$ to the *basin of infinity* $A_Q = \{z : Q^n(z) \to \infty \text{ as } n \to \infty\}$, and $J_Q = \partial A_Q = \partial K_Q$ is the *Julia set* (here and below $Q^n(z)$ is the image of $z$ by the $n$-iterate $Q^n$ of $Q$ for $n$ non-negative and the full preimage of $z$ by $Q^{|n|}$ for $n$ negative).

Let $u_Q : A_Q \to \mathbb{R}_+$ be Green's function in $A_Q$ such that $u_Q(z) \sim \log|z| + o(1)$ as $z \to \infty$. For all $z$ in some neighborhood $W$ of $\infty$, $u_Q(z) = \log|B_Q(z)|$

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where $B_Q$ is the *Böttcher coordinate* of $Q$ at $\infty$, i.e., a univalent function from $W$ onto $\{w : |w| > R\}$, for some $R > 1$, such that $B_Q(Q(z)) = B_Q(z)^{\deg Q}$ for $z \in W$ and $B_Q(z)/z \to 1$ as $z \to \infty$.

An *equipotential* of $Q$ of level $b > 0$ is the level set $\{z : u_Q(z) = b\}$. Alternatively, the equipotential containing a point $z \in A_Q$ is the closure of the union $\bigcup_{n>0} Q^{-n}(Q^n(z))$ and $u_Q(z) = \lim_{n\to\infty}(\deg(Q))^{-n} \log |Q^n(z)|$ is the *level* of this equipotential where $b = u_Q(z)$ is called the $Q$-*level* of $z \in A_Q$. Note that $u_Q(Q(z)) = (\deg Q)u_Q(z)$ for all $z \in A_Q$.

The gradient flow for Green's function (potential) $u_Q$ equipped with direction from $\infty$ to $J_Q$ defines $Q$-external rays. More specifically, the gradient flow has singularities precisely at the critical points of $u_Q$ which are preimages by $Q^n$, $n = 0, 1, \dots,$ of critical points of $Q$ that lie in the basin of infinity $A_Q$. If a trajectory $R$ of the flow that starts at $\infty$ does not meet a critical point of $u_Q$, it extends as a smooth (analytic) curve, *external ray* $R$, up to $J_Q$. If $R$ does meet a critical point of $u_Q$, one should consider instead two corresponding (non-smooth) left and right external rays as left and right limits of smooth external rays tending to $R$ (for a visualization of such rays, see e.g., Figures 1(a-b) of [LP96] or images in [PZ19]–[PZ20]; to get an impression about the geometry of the Julia set of renormalizable polynomials, see e.g. the computer images of [Pict]). Each external ray $R$ is parameterized by the level of equipotential $b \in (+\infty, 0)$.

The *argument* $\tau \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ of an external ray $R$ is the argument of the curve $R$ asymptotically at $\infty$. Informally, $\tau$ is the argument at which $R$ crosses the "circle at infinity". The correspondence between external rays and their arguments is one-to-one on smooth rays and two-to-one on non-smooth ones. If $R$ is a $Q$-external ray of argument $\tau$ then $Q(R)$ is also a ray of argument $\sigma_{\deg(Q)}(\tau)$ where $\sigma_k(t) = tk \pmod 1$. Note that, for any $b$ large enough, $B_Q$ maps the equiponential of level $b$ onto the round circle $\{|w| = e^b\}$ and arcs of external rays from this equipotential to $\infty$ onto standard rays that are orthogonal to this circle. Finally, $K_Q$ is connected if and only if $B_Q$ extends as a univalent function to the basin of infinity $A_Q$, if and only if all external rays of $Q$ are smooth.

Let $\mathbb{S} = \{|z| = 1\}$ be the unit circle which we identify—when this is not confusing—with $\mathbb{T}$ via the exponential $\mathbb{T} \ni t \mapsto \exp(2\pi i t) \in \mathbb{S}$.

**1.2. Polynomial-like maps and renormalization.** Let us recall [DH2] that a triple $(W, W_1, f)$ is a *polynomial-like map* if $W, W_1$ are topological discs, $\overline{W_1} \subset W$ and $f : W_1 \to W$ is a proper holomorphic map of some degree $m \geq 2$. The set of non-escaping points $K_f = \bigcap_{n=1}^{\infty} f^{-n}(W)$ is called the *filled Julia set* of $(W, W_1, f)$. By the Straightening Theorem [DH2], there exists a monic centered polynomial $G$ of degree $m$ which is *hybrid equivalent* to $f$, i.e., there is a quasiconformal homeomorphism $h : \mathbb{C} \to \mathbb{C}$ which is conformal a.e. on $K_f$, such that $G \circ h = h \circ f$ near $K_f$. The map $h$ is called a *straightening*. This implies in particular that $K_f$ is the set of limit points of $\bigcup_{n \geq 0} f^{-n}(z)$ for any $z \in W$ with, perhaps, at most one exception.

We say that another polynomial-like map $(\tilde{W}, \tilde{W}_1, \tilde{f})$ of the *same degree* $m$ is *equivalent* to $(W, W_1, f)$ if there is a component $E$ of $W \cap \tilde{W}$ such that $K_f \subset E$ and $f = \tilde{f}$ in a neighborhood of $K_f$. Taking a point $z$ as above close to $J_f = \partial K_f$, it follows (cf. [McM, Theorem 5.11]) that $K_f = K_{\tilde{f}}$ and that this is indeed an equivalence relation for polynomial-like maps. Denote by $\mathbf{f}$ the equivalence class of the polynomial-like map $(W, W_1, f)$, by $K_{\mathbf{f}}$, $J_{\mathbf{f}}$ the corresponding filled Julia set and Julia set of (any representative of) $\mathbf{f}$, and by $f$ the restriction to a neighborhood of $K_{\mathbf{f}}$ of an $\mathbf{f}$-representative (i.e., for any two representatives $(W^{(i)}, W_1^{(i)}, f_i)$, $i = 1, 2$, we have $f_1 = f_2 = f$ in a neighborhood of $K_{\mathbf{f}}$).

From now on, let us fix a monic centered polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $d > 1$.

We say that $\mathbf{f}$ is a *renormalization* of $P$ (cf. [McM], [Inou]) if $\mathbf{f}$ is an equivalence class of polynomial-like maps such that $K_{\mathbf{f}}$ is a connected proper subset of $K_P$ and, for some $r \geq 1$, $f = P^r$ in a neighborhood of $K_{\mathbf{f}}$.

**1.3. Assumptions.** Suppose that

(p1) $\mathbf{f}$ *is a renormalization of* $P$.

To avoid a situation when an external ray of $P$ can have a limit point in $J_{\mathbf{f}}$ as well as a limit point off $J_{\mathbf{f}}$, we introduce another condition:

(p2) *There exists a representative* $(W^*, W_1^*, f)$ *of the renormalization* $\mathbf{f}$ *of* $P$ *and some* $b_* > 0$ *as follows. If* $z \in \partial W_1^*$ *belongs to an external ray of* $P$ *which has a limit point in* $K_{\mathbf{f}}$ *then the* $P$*-level of* $z$ *is at least* $b_*$*, i.e.,* $u_P(z) \geq b_*$.

Let us stress that external rays of $P$ as in (p2) can cross the boundaries of $W^*$, $W_1^*$ many times (or e.g. have joint arcs with the boundaries).

This condition holds if $W^*$ is obtained by the following frequently used construction that we only indicate here; see [Mil1], [McM], [Inou] for details. In the first step, a simply connected domain $W_0$ is built using an appropriate Yoccoz puzzle so that $\partial W_0 = L_{\text{hor}} \cup L_{\text{vert}} \cup F$ where $L_{\text{hor}}$ is a union of finitely many arcs of a fixed equipotential of $P$, $L_{\text{vert}}$ is a union of finitely many arcs of external rays of $P$ between ends of arcs of $L_{\text{hor}}$, and $F$ is a finite set of repelling periodic points of $J_P$ or/and their preimages such that $K_{\mathbf{f}} \subset W_0 \cup F$ and $f : f^{-1}(W_0) \to W_0$ is a branched covering. By the construction, every external ray of $P$ to $J_{\mathbf{f}} \setminus F$ must cross the "horizontal" part $L_{\text{hor}}$ so that (p2) is obviously satisfied for the set of those rays. If either $L_{\text{vert}} = F = \emptyset$ (as in Example 1 that follows) or $F \cap K_{\mathbf{f}} = \emptyset$, one can take $W^* = W_0$ so that (p2) holds for $W_1^* = f^{-1}(W^*)$. If $F \subset J_{\mathbf{f}}$, then $W_0 \setminus f^{-1}(W_0)$ is a degenerate annulus. Then, in the second step, $W^*$ is modified from $W_0$ by "thickening" [Mil1, p. 12] around points of the set $F$, which adds only finitely many rays (tending to $F$). Then (p2) holds for $W_1^* = f^{-1}(W^*)$ as well.

EXAMPLE 1. Assume that the Julia set of the polynomial $P$ is disconnected and $K$ is a component of $K_P$ different from a point. In this case $K = K_{\mathbf{f}}$ for some renormalization $\mathbf{f}$ of $P$ and conditions (p1)–(p2) are fulfilled. The boundary of $W^*$ (hence of $W_1^*$, too) can be chosen to be merely a component of an equipotential that encloses $K$. With such a choice, each intersection point of an external ray of $P$ with $\partial W^*$ has a fixed level so every external ray can cross the boundaries of $W^*$ and $W_1^*$ at most once.

Our goal is to study a correspondence between external rays of $P$ that have limit points in $J_{\mathbf{f}}$, on the one hand, and external, or polynomial-like rays of the renormalization $\mathbf{f}$, on the other (up to a change of straightening, see below). In the case of disconnected Julia set $J_P$ and the renormalization $\mathbf{f}$ as in Example 1 this has been done in [LP96], [ABC16, Sect. 6], and [PZ19].

**1.4. Polynomial-like rays.** For a curve $\alpha : [0,1) \to \bar{\mathbb{C}}$, the *limit* (or *principal*, or *accumulation*) set of $\alpha$ is $\Pr(\alpha) = \overline{\alpha} \setminus \alpha$.

Let us define external rays of the renormalization $\mathbf{f}$. By [DH2], since $K_{\mathbf{f}}$ is connected, the monic centered polynomial $G$ of degree $m$ which is hybrid equivalent to any representative of $\mathbf{f}$ is uniquely defined by $\mathbf{f}$. Let $h$ be a *straightening* of $\mathbf{f}$. By this we mean a quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$ which is conformal a.e. on $K_{\mathbf{f}}$ and satisfies $G \circ h = h \circ f$ on some neighborhood of $K_{\mathbf{f}}$. One can also assume that $h$ is conformal at $\infty$ such that $h'(\infty) \neq 0$.

As the filled Julia set $K_G$ is connected, given $t \in \mathbb{T}$ there is a unique external ray of $G$ of argument $t$, denoted by $R_{t,G}$. Its $h^{-1}$-image $l_t^h := h^{-1}(R_{t,G})$ is called the *polynomial-like ray to* $K_{\mathbf{f}}$ *of argument* $t$. As $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, $\Pr(l_t^h) = h^{-1}(\Pr(R_{t,G}))$. Note that the straightening $h$ is not unique. However, the polynomial $G$ is unique, and if $\tilde{h}$ is another straightening, although $\tilde{h}$ defines another system of polynomial-like rays, the homeomorphism $\tilde{h}^{-1} \circ h : \mathbb{C} \to \mathbb{C}$ maps $l_t^h$ onto $l_t^{\tilde{h}}$ and $\Pr(l_t^h)$ onto $\Pr(l_t^{\tilde{h}})$).

In what follows we fix a straightening map $h : \mathbb{C} \to \mathbb{C}$ (see Theorem 3(e) and its proof though). Then the set $\{l_t\}$ of polynomial-like rays is fixed, too (where we omit $h$ in $l_t^h$ as $h$ is fixed). For brevity, $P$-external rays are called *$P$-rays*, or just *rays*, and polynomial-like rays to $K_{\mathbf{f}}$ are *$f$-rays, or polynomial-like rays.*

**1.5. Main results.** Given a connected compact set $K \subset \mathbb{C}$ which is different from a point, we say that a curve $\gamma : [0,1) \to \Omega := \mathbb{C} \setminus K$ *converges to a prime end* $\hat{P}$ of $K$ if, for a conformal homeomorphism $\psi : \mathbb{C} \setminus K \to \{|z| > 1\}$, the curve $\psi\circ\gamma:[0,1)\to\{|z|>1\}$ converges to a single point $P\in\mathbb{S}$; we say that $\gamma$ converges to the prime end $\hat{P}$ *non-tangentially* if moreover $\psi\circ\gamma$ converges to the point $P$ non-tangentially, i.e., the set $\psi\circ\gamma((1-\epsilon,1))$ lies inside a sector (Stolz angle) $\{z:|\arg(z-P)-\arg P|\le\alpha\}$ for some $\epsilon>0$ and $\alpha\in(0,\pi/2)$. Furthermore, we say that two curves $\gamma_1,\gamma_2:[0,1)\to\Omega$ are *K-equivalent* if they both converge to the same prime end and moreover have the same limit sets $\mathrm{Pr}(\gamma_1)=\mathrm{Pr}(\gamma_2)$ in $\partial K$ ([1]). By Lindelöf's theorem (see e.g. [Pom, Theorem 2.16]), if two curves converge to the same prime end non-tangentially, they share the same limit set. Therefore, if $\gamma_1,\gamma_2$ converge to the same prime end of $K$ non-tangentially, then $\gamma_1,\gamma_2$ are also $K$-equivalent.

The following statement was proved in [ABC16] ([2]) in the set up of Example 1.

THEOREM 1 (cf. [ABC16, Theorem 6.9]). *Assume* (p1)–(p2) *hold. For each $P$-ray $R$ that has an accumulation point in $K_{\mathbf{f}}$ we have* $\mathrm{Pr}(R)\subset J_{\mathbf{f}}$ *and there is a unique polynomial-like ray $l=\lambda(R)$ such that the curves $l$, $R$ are $K_{\mathbf{f}}$-equivalent. Moreover, $l$, $R$ converge to a single prime end of $K_{\mathbf{f}}$ non-tangentially. Furthermore, $\lambda:R\mapsto l$ maps the set of $P$-rays to $K_{\mathbf{f}}$* ***onto*** *the set of polynomial-like rays, and is "almost injective": $\lambda$ is one-to-one except when one and only one of the following* (i)–(ii) *holds. Suppose that $\lambda^{-1}(\ell)=\{R_1,\dots,R_k\}$ with $k>1$.*

(i) *$k=2$ and both rays $R_1,R_2$ are non-smooth and share a common arc starting at a critical point of Green's function $u_P$ to $J_{\mathbf{f}}$, or*
(ii) *there is $z\in J_{\mathbf{f}}$ such that $\mathrm{Pr}(R_i)=\{z\}$, $i=1,\dots,k$, at least two of the rays $R_1,\dots,R_k$ are disjoint, and, for some $n\ge0$, $P^{rn}(z)\in Y$ where $Y\subset J_{\mathbf{f}}$ is a finite collection of repelling or parabolic periodic points of $P$ that depends merely on $K_{\mathbf{f}}$.*

*If $K_P$ is connected then* (i) *is not possible.*

Note that in case (ii) any two disjoint $P$-rays completed by the joint limit point $z$ split the plane into two domains such that one of them contains $K_{\mathbf{f}}\setminus\{z\}$, and the other one, points from $K_P\setminus K_{\mathbf{f}}$. In particular, if $K_P$ is connected, the second domain must contain a component of $K_P\setminus K_{\mathbf{f}}$ that goes all the way to a pre-periodic point $z\in J_{\mathbf{f}}$. In fact, this is "if and only if": see Theorem 2(b) below.

For an illustration, see e.g. pictures in [McM, p. 116, explained in Example IV, p. 115] of a "dragon" filled Julia set of a quadratic polynomial $P$ admitting three renormalizations; the maps $\lambda$ corresponding to theses renormalizations are one-to-one except at countably many polynomial-like rays

([1]) One can show that if $\gamma_1$ converges to a single point $a\in\partial K$, then $\gamma_2$ is $K$-equivalent to $\gamma_1$ if and only if $\gamma_1$, $\gamma_2$ are homotopic through a family of curves in $\Omega$ converging to $a$.

([2]) In [ABC16], a different terminology is used.

where $\lambda$ is 6-to-1 in the top picture, 2-to-1 in the left bottom and 3-to-1 in the right bottom. In all three cases, the landing points of rays where $\lambda$ is not one-to-one are (pre-)periodic to a fixed point of $P$ where six $P$-rays land.

The next two theorems are consequences of the proof of Theorem 1.

THEOREM 2. *Assume* (p1)–(p2).

(a) *If a point $a \in J_{\mathbf{f}}$ is accessible along a curve $s$ in $\mathbb{C} \setminus K_{\mathbf{f}}$, then $a$ is the landing point of a $P$-ray $R$; moreover the curves $s$, $R$ are $K_{\mathbf{f}}$-equivalent.*
(b) *There exists a finite set $Y \subset J_{\mathbf{f}}$ of repelling or parabolic periodic points of $f$, as follows. Let $S$ be a component of $K_P \setminus K_{\mathbf{f}}$ such that $(\overline{S} \setminus S) \cap J_{\mathbf{f}} \neq \emptyset$. Then $\overline{S} \setminus S$ is a single point $b \in J_{\mathbf{f}}$, and moreover $f^n(b) \in Y$ for some $n \geq 0$.*

Note that part (a) is in fact an easy corollary of Lemma 2.1 similar to a result of [LP96]. Part (b) is void if (and only if) $K_{\mathbf{f}}$ is itself a component of $K_P$.

For the next statement, we introduce the following notations. Let $\Lambda \subset \mathbb{T}$ be the set of arguments of all $P$-rays that have their limit points in $J_{\mathbf{f}}$. Observe that by Theorem 1 the whole limit sets of such rays are in $J_{\mathbf{f}}$ and, given $\tau \in \Lambda$, there is a unique $P$-ray, denoted by $R_{\tau,P}$, which has its limit set in $J_{\mathbf{f}}$. Indeed, this is obvious if the $P$-ray of argument $\tau$ is smooth. On the other hand, if there are two $P$-rays, left and right, of argument $\tau$, only one of them can have its limit point in $J_{\mathbf{f}}$ because the other one must go to another component of $K_P$. Now, the map $\lambda$ of Theorem 1 induces a map $p : \Lambda \to \mathbb{T}$ such that for all $\tau \in \Lambda$,

$$\lambda(R_{\tau,P}) = l_{p(\tau)}.$$

By Theorem 1, $\Pr(l_{p(\tau)}) = \Pr(R_{\tau,P})$, and moreover $R_{\tau,P}$, $l_{p(\tau)}$ are $K_{\mathbf{f}}$-equivalent.

Given a positive integer $k$, let $\sigma_k : \mathbb{T} \to \mathbb{T}$, $\sigma_k(t) = kt \pmod 1$. Recall that $\deg(f) = m$. Let $D := \deg(P^r) = d^r$.

THEOREM 3 (cf. [PZ19]).

(a) *$\Lambda$ is a compact nowhere dense subset of $\mathbb{T}$ which is invariant under $\sigma_D$.*
(b) *$\sigma_m \circ p = p \circ \sigma_D$ on $\Lambda$.*
(c) *The map $p : \Lambda \to \mathbb{T}$ is surjective and finite-to-one, and moreover "almost injective" as defined in Theorem 1.*
(d) *$p : \Lambda \to \mathbb{T}$ extends to a continuous monotone degree one map $\tilde{p} : \mathbb{T} \to \mathbb{T}$.*
(e) *The map $p$ is unique in the following sense: if $\tilde{p} : \Lambda \to \mathbb{T}$ corresponds to another straightening $\tilde{h}$, then $\tilde{p}(t) = p(t) + k/(m-1) \pmod 1$ for some $k = 0, 1, \ldots, m-1$.*

In the set up of Example 1, i.e., when $K_P$ is disconnected and $K_f$ is a periodic component of $K_P$, Theorem 3 was proved in [PZ19] (by a different method), with part (c) replaced by an explicit bound for the cardinality of fibers of the map $p$ as well as with an extra statement about the Hausdorff dimension of the set $\Lambda$.

A detailed proof of the main Theorem 1 is given in Sect. 2 and the proofs of Theorems 2–3 are in Sect. 3. The proof of Theorem 1 follows rather closely the proofs of [LP96, Lemma 2.1] and [ABC16, Theorems 6.8–6.9]. An essential difference is that we have to adapt the proofs to the situation that external rays of $P$ can cross the boundary of $W_1^*$ as in (p2) many times.

**2. Proof of Theorem 1.** Let $f : W_1^* \to W^*$ be a representative of $\mathbf{f}$ as in (p2). As $K_f$ is connected, all the critical points of $f$ are in $K_f$. Hence, for each $k$, $f^k : f^{-k}(W_1^* \setminus K_f) \to W_1^* \setminus K_f$ is an unbranched (degree $m^k$) map. Therefore, $L_k := f^{-k}(\partial W_1^*)$ is the boundary of a simply connected domain $f^{-k}(W_1^*)$.

Let $\mathcal{R}$ denote the set of all $P$-rays $R$ such that $R$ has a limit point in $J_f$. First, we show that all limit points of $R \in \mathcal{R}$ are in $J_f$, introducing some notations along the way. Let

$$b_{*,k} = \inf\{u_P(z) : z \in R \cap L_k,\ R \in \mathcal{R}\}.$$

By (p2), $b_{*,0} > 0$. As $R \in \mathcal{R}$ implies $P^r(R) \in \mathcal{R}$, we have $b_{*,k} \geq b_{*,0}/D^k$, hence $b_{*,k} > 0$, for all $k$. Let $R \in \mathcal{R}$ and $k \geq 0$. Since $R \cap L_k$ is a closed set and $b_{*,k} > 0$, there exists a unique point $z_k(R) \in R \cap L_k$ such that $u_P(z_k(R)) = \inf\{u_P(z) : z \in R \cap L_k\}$. Observe that the arc $\Gamma_{k,R}$ of $R$ from $z_k(R)$ down to $J_P$ lies entirely in $\overline{f^{-k}(W_1^*)}$. As $\bigcap_{k\geq 0} \overline{f^{-k}(W_1^*)} = K_f$, we see immediately that the limit set of $R$, which is $\bigcap_{k\geq 0} \overline{\Gamma_{k,R}}$, is a subset of $J_f$.

Before proceeding with more notations and the main lemma, let us note that $b_{*,k} = b_{*,0}/D^k$, $k = 1, 2, \ldots$. Indeed, as $f^k : f^{-k}(W_1^* \setminus K_f) \to W_1^* \setminus K_f$ is an unbranched covering, each component of $f^{-k}(R)$ is an arc of some ray from $\mathcal{R}$. This implies that $b_{*,k} \leq b_{*,0}/D^k$. The opposite inequality was seen before.

Now, choose a conformal isomorphism $\psi$ from $\mathbf{C} \setminus K_f$ onto $\mathbf{D}^* = \{|z| > 1\}$ such that $\psi(z)/z \to e$ as $z \to \infty$, for some $e > 0$. A curve $\tilde{R}$ in $\mathbf{D}^*$ with limit set in $\mathbb{S} = \{|z| = 1\}$ is called a $K$-*related ray* if its preimage $\psi^{-1}(\tilde{R})$ is a $P$-ray $R \in \mathcal{R}$, i.e., $R$ has its limit set in $K_f$. The argument of $\tilde{R}$ is said to be the argument of the ray $\psi^{-1}(\tilde{R})$. Let $A_K = \psi(W^* \setminus K_f)$ be an "annulus" with boundary curves $\psi(\partial W^*)$ and $\mathbb{S}$. Denote $\tilde{z}_k(\tilde{R}) = \psi(z_k(R)$. Note that $\tilde{z}_k(\tilde{R}) \in \psi(L_k) \cap \tilde{R}$ and the arc of the $R$-related ray $\tilde{R}$ from $\tilde{z}_k(\tilde{R})$ to $\mathbb{S}$ is contained in the "annulus" between $\psi(L_k)$ and $\mathbb{S}$. An arc of a $K$-related ray $\tilde{R} = \psi(R)$ from the point $\tilde{z}_0(\tilde{R}) = \psi(z_0(R)) \in \psi(L_0)$ to $\mathbb{S}$ is called a $K$-*related arc*. Its argument is the argument of the corresponding ray. The following main lemma and its proof are minor adaptations of the ones of [LP96, Lemma 2.1].

Lemma 2.1.

1° *Every $K$-related arc has a finite length, and hence converges to a unique point of $\mathbb{S}$.*

2° *For every closed arc $I \subset \mathbb{S}$ (in particular a point), the set $K(I)$ of arguments of all $K$-related arcs converging to a point of $I$ is a non-empty compact set.*

3° *The set of all $K$-related arcs in $\{z : 1 < |z| < 1+\epsilon\}$ converging to a point $z_0$ lies in a Stolz angle*

$$\{z : |\arg(z - z_0) - \arg z_0| \leq \alpha\},$$

*where $\alpha \in (0, \pi/2)$ and $\epsilon$ do not depend on $z_0 \in \mathbb{S}$.*

*Proof.* 1° Let $B_{*,k} = \sup\{u_P(z) : z \in L_k\}$. For every $k \geq 0$ there is a number $C_k$ such that, for every ray $R \in \mathcal{R}$, the length of the arc $R_k$ of $R$ between the points $z_k(R)$ and $z_{k+1}(R)$ is bounded by $C_k$. This is because the latter arc is an arc of a $P$-ray that joins two equipotentials of positive levels $B_{*,k}$, $b_{*,k}$. Denote $\tilde{L}_k = \psi(L_k)$. Then $\tilde{L}_k$ is a compact subset of $A_K$ which surrounds $\mathbb{S}$. By the above, every $K$-related arc $\tilde{R}$ splits into arcs $\tilde{R}_k = \psi(R_k)$, $k \geq 0$, i.e., $\tilde{R}_k$ is the arc of $\tilde{R}$ joining $\tilde{z}_k(\tilde{R})$ and $\tilde{z}_{k+1}(\tilde{R})$. For every $k$, the supremum of the lengths over all arcs $\tilde{R}_k$ of the $K$-related rays $\tilde{R}$ is bounded by

$$\tilde{C}_k = C_k \sup\{|\psi'(z)| : z \in \overline{W}_1^* \setminus f^{-k-2}(W_1^*)\}.$$

Let $A_{1,K} = \psi(W_1^* \setminus K_f)$ and $g = \psi \circ f \circ \psi^{-1} : A_{1,K} \to A_K$. Then $z$ tends to $\mathbb{S}$ if and only if $g(z)$ tends to $\mathbb{S}$. It is well-known (see e.g. [P86]) that $g$ extends to an expanding holomorphic map in an annulus $U_0 = \{z : 1 - \rho_0 < |z| < 1 + \rho_0\}$ for some $\rho_0 > 0$. This means that after passing if necessary to an iterate of $g$ (which we also denote $g$) we have

$$|(g^{-1})'(z)| < c < 1 \tag{1}$$

for every $z \in U_0$ and for every branch $g^{-1}$ such that $g^{-1}(z) \in U_0$.

Fix a set $\tilde{L}_m \subset U = A_K \cap U_0$ for some $m$ large enough. Then, for each $n = 1, 2, \ldots$, $\tilde{L}_{n+m} = \{z \in U : g^n(z) \in \tilde{L}_{n+m}\}$. Denote by $l_n$ the supremum of the lengths of $\tilde{R}_{n+m}$ over all $R \in \mathcal{R}$. Note that each $l_n$ is finite, because $l_n \leq \tilde{C}_{m+n}$. In fact, much more is true: as $g^n(\tilde{R}_{n+m})$ is $\tilde{S}_m$ for some ray $S \in \mathcal{R}$, (1) yields $l_n < c^n l_0$. Given a $K$-related ray $\tilde{R}$, the length of its arc from the point $\tilde{z}_m(\tilde{R})$ to $\mathbb{S}$, which is in the component of $\mathbf{C} \setminus \tilde{\gamma}_0$ containing $\mathbb{S}$, is bounded from above by $\sum_{n=0}^{\infty} c^n l_0 < \infty$. Moreover, the same argument shows the following

Claim 1. *The lengths of the arcs of $K$-related rays $\tilde{R}$ between $\tilde{z}_k(\tilde{R})$ and $\mathbb{S}$ tend uniformly to zero (exponentially in $k$).*

2° Fix a closed non-degenerate arc $I \subset \mathbb{S}$. There exists a $K$-related ray converging to a point of $I$. Indeed, otherwise no $K$-related ray ends in the arc $g^n(I)$, for any $n$. This is impossible because $g^n(I) = \mathbb{S}$ for large $n$ and the set of $K$-related rays is non-empty (for example, it contains images by $\psi$ of $P$-rays landing at repelling periodic points of the polynomial-like map $f : W_1^* \to W^*$; for the existence of such $P$-rays, see [Mil0], [EL89], [LP96]). We need to show that the set $K(I)$ of arguments of all $K$-related rays ending in $I$ is closed.

This is an immediate consequence of the next claim which follows, basically, from Claim 1 and will also be useful later on. Given a $K$-related ray $\tilde{R}_t$ of argument $t$ (i.e., $t \in \Lambda$) consider its arc $\hat{r}_t$ between $\tilde{L}_0$ and $\mathbb{S}$, parameterized as a curve $\tilde{r}_t : [b_{*,0}, 0] \to A_K \cup \mathbb{S}$ as follows. For any $b \in [b_{*,0}, 0)$, define the point $r_t(x) \in A_K$ to be such that $\psi^{-1}(r_t(x))$ is a point of a $P$-ray of argument $t$ and equipotential level $b$. Finally, let $\tilde{r}_t(0) = \lim_{b\to 0} \tilde{r}_t(x) \in \mathbb{S}$ where the limit exists by 1°.

Claim 2. *The family* $\tilde{\mathcal{R}} = \{\tilde{r}_t\}_{t\in\Lambda}$ *is a compact subset of* $C[b_{*,0}, 0]$.

Let us first show that this family is equicontinuous. In view of Claim 1, this will follow from the equicontinuity of the restricted family $\tilde{\mathcal{R}}_m = \{\hat{r}_t : [b_{*,0}, b_{*,0}/D^m] \to A_K\}$ for each integer $m > 1$. Fix $m$ and consider two objects: a compact set $E_m \subset \mathbb{C}$ bounded by the equipotential of levels $b_{*,0}$ and $b_{*,0}/D^m$ of $P$ and a family $\mathcal{R}_m$ of (closed) arcs in $E_m$ of all $P$-rays that join the equipotential levels $b_{*,0}$, $b_{*,0}/D^m$ and are parameterized by the equipotential level $b \in [b_{*,0}, b_{*,0}/D^m]$. It is easy to see that this is a compact subset of $C[b_{*,0}, b_{*,0}/D^m]$ (indeed, map this family by a fixed high iterate of $P$ to a family of smooth arcs of $P$-rays which are preimages of segments of standard rays by the Böttcher coordinate $B_P$ at infinity; hence, this new family is compact; then pull it back). As $\mathcal{R}_m \subset C[b_{*,0}, b_{*,0}/D^m]$ is compact, it is equicontinuous. In turn, since $\psi^{-1}$ is a homeomorphism on $E_m$ (onto its image) and each $\psi^{-1}(\tilde{r}_t)$ is in $\mathcal{R}_m$, the family $\tilde{\mathcal{R}}_m$ is equicontinuous too. Thus $\tilde{\mathcal{R}}$ is an equicontinuous family.

It remains to prove that it is closed. So suppose a sequence $\hat{r}_{t_n}$ converges uniformly in $[b_{*,0}, 0]$. In particular, $\hat{r}_{t_n}$ crosses $\tilde{L}_k$ for each $k$ large enough. One can assume that $t_n$ tends to some $t$. Then the sequence of arcs of $P$-rays $\psi^{-1} \circ \tilde{r}_{t_n}$, on the one hand, tends, uniformly on each interval $[b_{*,0}, b_{*,0}/D^m]$, to an arc $r$ of a $P$-ray of argument $t$, on the other hand, crosses each $L_k$ with $k$ large. Hence, $r$ has a limit point in $K_f$. Applying $\psi$ we find that the limit of $\tilde{r}_{t_n}$ is a $K$-related arc, which ends the proof of the claim.

This proves 2° when $I$ is not a single point. By the intersection of compacta, 2° also holds if $I$ is a point.

3° Every branch of $g^{-n}$ is a well defined univalent function in every disc contained in $U_0$. Hence, by the Koebe distortion theorem (see e.g. [Gol]), one can choose $0 < \rho' < \rho_0$ such that for every

$$z \in U' = \{z : 1 - \rho' < |z| < 1 + \rho'\},$$

every $n = 1, 2, \dots$ and every branch $g^{-n}$,

$$\left| \frac{(g^{-n})'(x)}{(g^{-n})'(y)} \right| < 2 \tag{2}$$

whenever $|z - x| < \rho'$ and $|z - y| < \rho'$.

We reduce $U'$ further as follows. By Claim 1, fix $m_0 > m$ such that the length of the arc of any $K$-related ray $\tilde{R}$ between $\tilde{z}_{m_0}(\tilde{R})$ and $\mathbb{S}$ is less than $\rho'$. On the other hand, if $z$ lies in an unbounded component of $R \setminus z_{m_0}(R)$, i.e., in the arc of $R$ between $z_{m_0}(R)$ and $\infty$, then $u_P(z) \geq b_{*,m_0}$, in particular, there is $r > 0$ independent of $z$ and $R$ as above such that the distance between $z$ and $J_P$ is at least $r$. Therefore, there exists some $\rho_1 \in (0, \rho')$ such that for every $z \in \{z : 1 < |z| < 1 + \rho_1\}$, if $z$ belongs to a $K$-related ray $\tilde{R}$ then $z$ lies in an arc of $\tilde{R}$ between $\tilde{z}_{m_0}(\tilde{R})$ and $\mathbb{S}$. Let

$$U_1 = \{z : 1 - \rho_1 < |z| < 1 + \rho_1\}.$$

We introduce the following notations:

Given $x \in U_1$, denote by $l_x$ the part of the $K$-related ray passing through $x$ between $x$ and $\mathbb{S}$ (if such a ray exists). This notation is correct: as already noted before, if another $K$-related ray passes through $x$ and next ramifies from $l_x$, it goes to a component of $\psi(J(f))$, not to $\mathbb{S}$. So it is not $K$-related.

Denote by $h_x$ the interval which joins $x$ and $\mathbb{S}$, orthogonal to $\mathbb{S}$. Denote by $l(x)$ and $h(x)$ the corresponding Euclidean lengths. Find a large enough $N$ such that $\tilde{\gamma}_0 := \tilde{L}_N$ in $U_1$. By the choice of $U_1$,

$$l(x) < \rho' \quad \text{for all } x \text{ between } \tilde{\gamma}_0 \text{ and } \mathbb{S}. \tag{3}$$

Let $\tilde{\gamma}_1 = g^{-1}(\tilde{\gamma}_0)$. There exists a positive $\beta_0$ less than 1 such that

$$\frac{h(x)}{l(x)} > \beta_0 \tag{4}$$

for all points $x$ in the annulus $V$ between $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$.

Fix the maximal $\epsilon_0 > 0$ such that

$$U_2 = \{z : 1 - \epsilon_0 < |z| < 1 + \epsilon_0\}$$

does not intersect $\tilde{\gamma}_1$. We intend to prove assertion $3^{\circ}$ of our lemma with

$$\alpha = \arccos\left(\frac{\beta_0}{8L}\right)$$

where $L = \sup\{|g'(z)| : z \in U_0\}$ and with $\epsilon$ between 0 and $\epsilon_0$ so small that $1 < |z| < 1 + \epsilon$ and $h(z)/|z - z_0| \geq 2\cos\alpha$ implies $|\arg(z - z_0) - \arg z_0| \leq \alpha$.

It is enough to prove that

$$\frac{h(x)}{l(x)} > \beta = \frac{\beta_0}{4L} \tag{5}$$

for all $x \in U_2$. Assume the contrary: there exists $x_* \in U_2$ that belongs to some $K$-related ray $\tilde{R}$ with

$$h(x_*)/l(x_*) \le \beta. \tag{6}$$

Choose the minimal $n \ge 1$ such that $g^n(x_*) \in V$.

The lengths $h^{(i)}$ and $l^{(i)}$ of the curves $g^i(h_{x_*})$ and $g^i(l_{x_*})$ cannot exceed $\rho'$ for all $i = 0, 1, \ldots, n$. This holds for $l^{(i)}$ by (3), because $g^i(x_*)$ is between $\tilde{\gamma}_0$ and $\mathbb{S}$. We cope with $h^{(i)}$'s by induction: Length$(h^{(0)}) < \rho$ by the definition of $U_1$. If it holds for all $i \le j-1$ then by (2),

$$\frac{h^{(j-1)}}{l^{(j-1)}} \le 4\beta = \beta_0/L.$$

Then

$$h^{(j)} \le Lh^{(j-1)} \le \beta_0 \cdot l^{(j-1)} < l^{(j-1)} < \rho'.$$

Now we use the assumption (6) and again apply (2) to obtain, for $z_* = g^n(x_*) \in \tilde{S}_N$,

$$\frac{h(z_*)}{l(z_*)} \le \frac{h^{(n)}}{l^{(n)}} \le 4\beta = \beta_0/L < \beta_0.$$

This contradicts (4). ∎

Comment. The key bound (5) can also be seen directly from (4) (with, for instance, $\beta = \beta_0/10$) by applying, besides the Koebe distortion bound (2), another distortion bound: there is a function $\epsilon : (0,1) \to (0,+\infty)$, with $\epsilon(r) \to 0$ as $r \to 0$, such that for any univalent function $\varphi$ on the unit disc, if $\varphi(0) = 0$ and $\varphi'(0) = 1$, then

$$\left|\log \frac{\varphi(z)}{z}\right| < \epsilon(|z|)$$

(see e.g. [Gol]). This bound is applied to the function

$$\varphi(z) = \frac{g^{-n}(w + \rho_0 z) - g^{-n}(w)}{(g^{-n})'(w)\rho_0}$$

where $n$ is minimal such that $g^n(x) \in V$, and $w \in \mathbb{S}$ is the projection of $g^n(x)$ to $\mathbb{S}$ and reducing $\rho'$. Note that $(g^{-n})'(w) > 0$ because $g$ preserves $\mathbb{S}$.

We continue as follows (cf. [ABC16, proof of Theorem 6.9]). Recall that a straightening $h : \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism which is holomorphic at $\infty$ and $h'(\infty) \ne 0$. It conjugates the polynomial-like map $f$ to the polynomial $G$ near their filled Julia sets $K_f$ and $K_G$. Let $B_G : A_G \to \mathbb{D}^*$ be the Böttcher coordinate of $G$ such that $B_G(z)/z \to 1$ as $z \to \infty$, which is well defined in the basin of infinity $A_G = \mathbb{C} \setminus K_G$ as $K_G$ is connected.

We have the following picture:

$$\mathbb{D}^* \xrightarrow{\psi^{-1}} \mathbb{C} \setminus K_f \xrightarrow{h} \mathbb{C} \setminus K_G \xrightarrow{B_G} \mathbb{D}^*. \tag{7}$$

Consider the map $\Psi := \psi \circ h^{-1} \circ B_G^{-1} : \mathbb{D}^* \to \mathbb{D}^*$ from the uniformization plane of the polynomial $G$ to the $g$-plane of $K$-related rays. It is a quasiconformal homeomorphism which is holomorphic at $\infty$. For $u \in \mathbb{S}$, let $L_u = \Psi(r_u \cap \mathbb{D}^*)$ where $r_u = \{tu : t > 0\}$ is a standard ray in the uniformization plane of $G$ (3).

LEMMA 2.2. *The curve $L_u$ converges non-tangentially to a unique point $z_0 = z_0(u)$ of the unit circle $\mathbb{S}$. Moreover, there is $\beta \in (0, \pi/2)$ such that, for any $u \in \mathbb{S}$ and all $z \in L_u$ close enough to $\mathbb{S}$,*

$$|\arg(z - z_0) - \arg z_0| \le \beta. \tag{8}$$

*Here $\beta$ depends only on the quasiconformal deformation of the straightening map $h$. Furthermore, for every $z_0 \in \mathbb{S}$ there exists a unique $u$ such that $L_u$ lands at $z_0$.*

*Proof of Lemma 2.2* (cf. [ABC16, Section 6]). The map $\Psi : \mathbb{D}^* \to \mathbb{D}^*$ extends to a homeomorphism of the closures and then to a quasiconformal homeomorphism $\Psi^*$ of $\mathbb{C}$ by $\overline{\Psi^*(z)} = 1/\Psi^*(1/\bar{z})$ (see [Ahl]). Note that the quasiconformal deformations of $\Psi$ and $\Psi^*$ are the same, equal to the quasiconformal deformation $M$ of the straightening map $h$. Consider the curve $L_u^* = \Psi^*(r_u)$. It is an extension of the curve $L_u$, which crosses $\mathbb{S}$ at a point $z_0 = \Psi^*(u)$. As a quasiconformal image of a straight line, the curve $L_u^*$ has the following property [Ahl]: there exists $C = C(M) > 0$ such that

$$|z - z_0|/|z - 1/\overline{z}| < C \quad \text{for every } z \in L_u^*.$$

Therefore, $L_u^*$ tends to $z_0$ non-tangentially; moreover, (8) holds for some $\beta = \beta(C(M))$. The last claim follows from the fact that $\Psi^*$ is a homeomorphism. ■

Now, define the correspondence $\lambda$ as follows (having in mind (7)). Let $R$ be a $P$-ray to $K_f$. By Lemma 2.1, the $K$-related ray $\tilde{R} = \psi(R)$ tends to a point $z_0 \in \mathbb{S}$. By Lemma 2.2, there exists a unique $L_u$ which tends to $z_0$. The curve $\psi^{-1}(L_u) = h^{-1} \circ B_G^{-1}(\{tu : t > 1\})$ is a polynomial-like ray $l_\tau$ where $u = e^{2\pi i \tau}$. Let

$$\lambda(R) := \psi^{-1}(L_u).$$

The correspondence $\lambda$ is "onto" by the first claim of Lemma 2.2 along with Lemma 2.1(2°).

Now, both curves $\tilde{R}$, $L_u$ in $\mathbb{D}^*$ tend to the point $z_0 \in \mathbb{S}$ non-tangentially, by Lemmas 2.1 and 2.2 respectively. Then, by definition, the $P$-ray $R$ and the

(3) Note that the curve $L_u$ lies in the left-hand disc $\mathbb{D}^*$ of (7) while the point $u$ is at the boundary of the right-hand disc there.

polynomial-like ray $\lambda(R)$ converge to a single prime end of $K_{\mathbf{f}}$ non-tangentially, hence $R$ and $\lambda(R)$ are also $K_{\mathbf{f}}$-equivalent. Finally, the condition that $R$ and $\lambda(R)$ are $K_{\mathbf{f}}$-equivalent uniquely determines the polynomial-like ray $\lambda(R)$.

It remains to prove the "almost injectivity" of $\lambda$. This is a direct consequence of the one-to-one correspondence between $K$-related rays and curves $L_u$ established above and the following claim whose proof is identical to the one of [ABC16, Theorem 6.8] (for completeness, we reproduce it below with obvious changes in notation). While passing from $K$-related rays to $P$-rays we use the fact that if a $K$-related ray is periodic, the corresponding $P$-ray converges to a periodic point of $P$ which is either repelling or parabolic (by the Snail Lemma [Mil0], it cannot be irrationally indifferent).

LEMMA 2.3. *Any point $w \in \mathbb{S}$ is the landing point of precisely one $K$-related ray, except when one and only one of the following holds:*

(i) *$w$ is the landing point of exactly two $K$-related rays which are non-smooth and have a common smooth arc that goes to $w$;*
(ii) *$w$ is a landing point of at least two disjoint $K$-related rays, in which case $w$ is a (pre)periodic point of $g$ and some iterate $g^n(w)$ belongs to a finite set $\hat{Y}$ (depending only on $K$) of $g|_{\mathbb{S}}$-periodic points each of which is the landing point of finitely many, but at least two, $K$-related rays, which are periodic of the same period depending merely on the landing point $w$ (4).*

*Moreover, if $w$ is periodic then* (i) *cannot hold.*

*Proof.* Assume that there are two $K$-related rays landing at a point $w \in \mathbb{S}$ and that (i) does not hold. We need to prove that then (ii) holds. Since (i) does not hold, there exist disjoint $K$-related rays landing at $w$. Let us study this case in detail.

Associate to any such pair of rays $\hat{R}_t$, $\hat{R}_{t'}$ an open arc $(\hat{R}_t, \hat{R}_{t'})$ of $\mathbb{S}$ as follows. Two points of $\mathbb{S}^1$ with the arguments $t, t'$ split $\mathbb{S}$ into two arcs. Let $(\hat{R}_t, \hat{R}_{t'})$ be the one that contains no arguments of $K$-related rays except possibly for those that land at $w$. Geometrically, this means the following. The $K$-related rays $\hat{R}_t, \hat{R}_{t'}$ together with $w \in \mathbb{S}$ split the plane into two domains. The arc $(\hat{R}_t, \hat{R}_{t'})$ corresponds to one of them, disjoint from $\mathbb{S}$. Let $L(\hat{R}_t, \hat{R}_{t'}) = \delta$ be the angular length of $(\hat{R}_t, \hat{R}_{t'})$. Clearly, $0 < \delta < 1$. Now we make a few observations.

(1) *If $K$-related disjoint rays of arguments $t_1, t'_1$ land at a common point $w_1$ while $K$-related disjoint rays of arguments $t_2, t'_2$ land at a point $w_2 \neq w_1$, then the arcs $(\hat{R}_{t_1}, \hat{R}_{t'_1})$, $(\hat{R}_{t_2}, \hat{R}_{t'_2})$ are disjoint.*

(4) In [ABC16, Theorem 6.8(ii)], it is claimed erroneously that all $K$-related rays to the point $w$ are smooth (cf. [PZ20]). Note that this claim is not relevant to the rest of [ABC16].

The above follows from the definition of the arc $(\hat{R}_t, \hat{R}_t)$.

(2) *If disjoint $K$-related rays $\hat{R}_t, \hat{R}_{t'}$ of arguments $t, t'$ land at a common point $w$, then the $K$-related rays $g(\hat{R}_t), g(\hat{R}_{t'})$ are also disjoint and land at the common point $g(w)$. Moreover,*

$$L(g(\hat{R}_t), g(\hat{R}_{t'})) \geq \min\{D\delta \pmod 1, 1 - D\delta \pmod 1\} > 0.$$

Indeed, the images $g(\hat{R}_t), g(\hat{R}_{t'})$ are disjoint near $g(w)$, because $g$ is locally one-to-one. Hence, $g(\hat{R}_t) \cap g(\hat{R}_{t'}) = \emptyset$, because otherwise the corresponding $P$-rays would have their limit sets in different components of $K_P$, a contradiction since both rays $g(\hat{R}_t)$, $g(\hat{R}_{t'})$ are $K$-related. Since the argument of $g(\hat{R}_t)$ is represented by the point $Dt \pmod 1 \in (0, 1)$, we get the inequality of (2).

Let us consider the following set $\hat{Z}(K)$ of points in $\mathbb{S}$: $w \in \hat{Z}(K)$ if and only if there is a pair of disjoint $K$-related rays $\hat{R}, \hat{R}'$ which both land at $w$ and satisfy $L(\hat{R}, \hat{R}') \geq 1/(2D)$. Denote by $\hat{Y}(K)$ the set of periodic points which are in forward images of the points of $\hat{Z}(K)$.

(3) *If the set $\hat{Z}(K)$ is non-empty, then it is finite, and consists of* (*pre*)-*periodic points.*

Indeed, $\hat{Z}(K)$ is finite by (1). Assume $w \in \hat{Z}(K)$. Then by (2) some iterate $g^n(w)$ must hit $\hat{Z}(K)$ again.

To complete the proof, choose disjoint $K$-related rays $\hat{R}_t, \hat{R}_{t'}$ landing at $w \in \mathbb{S}$ and use this to prove that all claims of (ii) hold.

*We show that the orbit $w, g(w), \ldots$ cannot be infinite.* Indeed, otherwise by (1)–(2), we have a sequence of non-degenerate pairwise disjoint arcs $(g^n(\hat{R}_t), g^n(\hat{R}_{t'})) \subset \mathbb{S}$, $n = 0, 1, \ldots$. By (2), some iterates of $w$ must hit the finite set $\hat{Z}(K)$ and hence $\hat{Y}(K)$ (which are therefore non-empty), a contradiction.

*Hence for some $0 \leq n < l$, $g^n(w) = g^l(w)$; let us verify that the other claims of* (ii) *hold.* Replacing $w$ by $g^n(w)$, we may assume that $w$ is a (repelling) periodic point of $g$ of period $k = l - n$. By (2), $w \in \hat{Y}(K)$. By [LP96, Theorem 1], the set of $K$-related rays landing at $w$ is finite, and each $K$-related ray landing at $w$ is periodic with the same period. Hence, (ii) holds. Finally, the last claim of the lemma follows because a periodic non-smooth ray must have infinitely many broken points, hence, no other ray can have a common arc with it that goes up to the Julia set; see [ABC16, Lemma 6.1] for details. ∎

## 3. Proofs of Theorems 2–3

**3.1. Theorem 2.** Part (a) is an immediate corollary of Lemma 2.1 and Lindelöf's theorem, as in [LP96]. Indeed, since a curve $s \subset W \setminus K_{\mathbf{f}}$ converges to a point $a \in K_{\mathbf{f}}$, the curve $\tilde{s} = \psi(s)$ converges to a point $z_0 \in \mathbb{S}$, and the limit of the function $\psi^{-1}$ along the curve $\tilde{s}$ exists and equals $a$. By Lemma 2.1, there is a $K$-related ray $\tilde{R}$ that tends to $z_0$, and it tends non-tangentially. Then, by [Pom, Corollary 2.17], the $P$-ray $R$ converges to the same point $a$. By definition, the curves $s$, $R$ are $K_{\mathbf{f}}$-equivalent.

Let us prove part (b). The closed set $S \cup K_f$ is connected and so too is its complement (by the Maximum Principle). Consider the set $\hat{S} = \psi(S) \subset \mathbb{D}^*$. Let $I = \overline{\hat{S}} \setminus \hat{S}$. Then $I$ is a connected closed subset of the unit circle $\mathbb{S}$.

Let us prove $I$ is a single point. Otherwise there is an interior point $x \in I$ which is $g$-periodic. Let $\beta$ be a $K$-related ray that lands at $x$. Notice that since $x$ is an interior point of $I$, $\beta$ must cross $\hat{S}$. Now, since $x$ is $g$-periodic, $R = \psi^{-1}(\beta)$ is a periodic $P$-ray, hence it converges to a periodic point $a \in \overline{S} \setminus S$ of $P$ and crosses $S$, a contradiction since $S \subset K_P$. This proves that $I$ is a single point; denote it by $z_0$.

Choose two sequences $z_n', z_n''$ of $\mathbb{S}$ tending to $z_0$ from the left and from the right respectively, and two sequences of $K$-related rays $l_n', l_n''$ so that $l_n'$ lands at $z_n'$ and $l_n''$ lands at $z_n''$. Then, passing perhaps to subsequences, by Claim 2 in the proof of Lemma 2.1, the sequence $l_n'$ tends to a $K$-related ray $l'$ and $l_n''$ tends to a $K$-related ray $l''$, where $l'$ and $l''$ land at the same $z_0$. By the above, $l', l''$ are disjoint.

Now we apply Lemma 2.3 to conclude that $z_0$ is $g$-(pre-)periodic, and some iterate of $z_0$ lies in a finite set $\hat{Y} \subset \mathbb{S}$ of periodic points, which is independent of $z_0$. Hence, the point $a$ is $P$-(pre-)periodic, and some iterate of $a$ lies in a finite set $Y \subset J_f$ of periodic points, which is independent of $a$. As every point of $Y$ is a landing point of a periodic ray, it can be either repelling or parabolic.

**3.2. Theorem 3.** Proof of (b), (c): It follows from the definition of $\Lambda$ that $\sigma_D(\Lambda) = \Lambda$ and $\sigma_m \circ p = p \circ \sigma_D$ on $\Lambda$. By invariance and since $\Lambda \neq \mathbb{T}$, the set $\Lambda$ contains no intervals; (c) is a reformulation of a part of the statement of Theorem 1.

Proof of (a), (d): Considering $\Lambda$ as a subset of $\mathbb{S} = \{|z| = 1\}$ define a new map $p_K : \Lambda \to \mathbb{S}$ as follows: for $\tau \in \Lambda$, let $p_K(\tau) \in \mathbb{S}$ be the landing point of a $K$-related ray of argument $\tau$. Recall the map $\Psi = \psi \circ h^{-1} \circ B_G^{-1} : \mathbb{D}^* \to \mathbb{D}^*$ introduced in the proof of Theorem 1, and its quasi-conformal extension $\Psi^* : \mathbb{C} \to \mathbb{C}$. By Lemma 2.2 and the definition of the maps $\lambda$ and $p$, we have

$$p_K = \Psi^*|_{\mathbb{S}} \circ p. \tag{9}$$

Since $\Psi^* : \mathbb{S} \to \mathbb{S}$ is an orientation preserving homeomorphism, it is enough to prove (a), (d) with $p$ replaced by $p_K$. By Lemma 2(2°), $p_K^{-1}(I)$ is closed in $\mathbb{S}$ for any closed arc $I \subset \mathbb{S}$. Therefore, $\Lambda = p_K^{-1}(\mathbb{S})$ is closed and the map $p_K : \Lambda \to \mathbb{S}$ is continuous. To show (d), define an extension $\tilde{p}_K : \mathbb{S} \to \mathbb{S}$ of

$p_K : \Lambda \to \mathbb{S}$ in an obvious way as follows. Let $J := (t_1, t_2)$ be a component of $\mathbb{S} \setminus \Lambda$. Then $p_K(t_1) = p_K(t_2) =: w_J$ because otherwise there would be a point of $\mathbb{S}$ with no $K$-related rays landing at it. Let $\tilde{p}_K(\tau) = w_J$ for all $\tau \in J$. Then $\tilde{p}_K : \mathbb{S} \to \mathbb{S}$ is continuous. Now, given $t \in \mathbb{S}$, the set $\tilde{p}_K^{-1}(\{t\})$ is either a singleton or a non-trivial closed arc. This follows from the definition of $\tilde{p}_K$ and because $K$-related rays with different arguments do not intersect unless case (i) of Theorem 2.3 takes place. Therefore, $\tilde{p}_K : \mathbb{S} \to \mathbb{S}$ is monotone and of degree one.

Proof of (e): Let $\tilde{h}$ be another straightening, $\tilde{\Psi} : \mathbb{D}^* \to \mathbb{D}^*$ the corresponding quasiconformal map and $\tilde{\Psi}^* : \mathbb{C} \to \mathbb{C}$ its quasiconformal extension. As $p_K : \mathbb{S} \to \mathbb{S}$ is independent of the straightening, by (9) we have $\tilde{p} = T|_{\mathbb{S}} \circ p$ where $T = (\tilde{\Psi}^*)^{-1} \circ \Psi^*$. On the other hand, on $\mathbb{D}^*$, $T = (B_G \circ \tilde{h}) \circ (B_G \circ h)^{-1}$, hence $T$ commutes with $z \mapsto z^m$ for $|z| > 1$ near $\mathbb{S}$, by definitions of $h, B_G$. Therefore, the homeomorphism $\nu := T|_{\mathbb{S}} : \mathbb{S} \to \mathbb{S}$ commutes with $z \mapsto z^m$ on $\mathbb{S}$, too. It is then well known that $\nu(z) = vz$ for some $v \in \mathbb{C}$ with modulus 1 such that $v^m = v$ (proof: as $\nu(1)^m = \nu(1)$ let $v = \nu(1)$, so that a homeomorphism $\nu_0 = v^{-1}\nu : \mathbb{S} \to \mathbb{S}$ commutes with $z \mapsto z^m$ too and $\nu_0(1) = 1$; then there is a lift $\tilde{\nu}_0 : \mathbb{R} \to \mathbb{R}$ of $\nu_0$ such that $\tilde{\nu}_0(0) = 0$, $\tilde{\nu}_0 - 1$ is 1-periodic and $\tilde{\nu}_0(mx) = m\tilde{\nu}_0(x)$ for all $x \in \mathbb{R}$, which in turn implies $\tilde{\nu}_0(n/m^k) = n/m^k$ for all $n, k \in \mathbb{Z}_{>0}$; by continuity, $\tilde{\nu}_0(x) = x$ for all $x$).

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