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Continued fractions for strong Engel series and Lüroth series with signs

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1. Introduction. Given a sequence of positive integers (x_n) such that $x_n \mid x_{n+1}$ for all $n \geq 1$, the sum of the reciprocals is the *Engel series*

$$S = \sum_{j=1}^{\infty} \frac{1}{x_j} \tag{1.1}$$

and the alternating sum of the reciprocals is the *Pierce series* [23]

$$S' = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{x_j}. \tag{1.2}$$

(To ensure convergence it should be assumed that (x_n) is eventually increasing, i.e. for all n there is some $n' > n$ with $x_{n'} > x_n$.) Every positive real number admits both an Engel expansion, of the form (1.1), and a Pierce expansion (1.2) [6, 11], and these S, S' are unique: after removing the integer part it is sufficient to consider numbers in the interval $(0, 1)$, and then (x_n) is strictly increasing with $x_1 \geq 2$ in (1.1) and $x_1 \geq 1$ in (1.2). Although they are not quite so well known, Engel expansions and Pierce expansions have much in common with continued fraction expansions, both in the way that they are determined recursively, and from a metrical point of view; for instance, see [10] for the case of Engel series and [29] for Pierce series.

In recent work [12], the first author presented a family of sequences (x_n) generated by certain non-linear recurrences of second order, of the form

$$x_{n+2}x_n = x_{n+1}^2(1 + x_{n+1}\, G(x_{n+1})), \qquad n \geq 1,\ G(x) \in \mathbb{Z}[x], \tag{1.3}$$

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where $G(x) > 0$ for $x > 0$, such that the corresponding Engel series (1.1) yields a transcendental number with a regular continued fraction expansion

$$S = [a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

whose coefficients (partial quotients) $a_0, a_1, \dots$ are explicitly given in terms of the x_n.

More recently [30], the second author proved that, when the sequence (x_n) is generated by the same sort of non-linear recurrence as (1.3), an analogous result holds for the associated Pierce series (1.2), although the structure of the corresponding continued fractions is different. It was subsequently noted in [14, 15] that the recurrence (1.3) could be further generalized, and at the same time allow the explicit continued fraction expansion to be determined for the sum of an arbitrary rational number $r = p/q$ and an Engel series, that is,

$$\frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j}, \qquad \text{with } q = x_1 \,|\, x_2, \tag{1.4}$$

and similarly for the case of $p/q \pm$ a Pierce series.

The key to the results in [12, 14, 15, 30] was that, subject to a recurrence like (1.3), the truncation of the particular series (1.1) or (1.2) at the nth term yields a convergent of the continued fraction of S or S', whose length depends linearly on n. Continuing in this vein, Duverney et al. showed in [7] that a finite sum

$$\sum_{j=1}^{n} \frac{y_j}{x_j} \tag{1.5}$$

can be expanded as a continued fraction of general type, of length $2n$, i.e.

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\cdots + \cfrac{a_{2n}}{b_{2n}}}}}, \tag{1.6}$$

where a_j, b_j are explicit rational functions of the indeterminates x_j, y_j, and they presented a similar formula for the alternating sum $\sum_{j=1}^{n} (-1)^{j-1} y_j / x_j$ as a general continued fraction of length $3n - 4$.

The formulae for the continued fraction (1.6) are conveniently written in terms of the "exponentiated shift" operator θ from [7], defined by

$$\theta[u_n] = \frac{u_{n+1}}{u_n}, \tag{1.7}$$

and if it is assumed that (x_n) is an increasing sequence of positive integers with $x_1 \geq 2$ and (y_n) is another sequence of positive integers, then (writing $\theta^2[x_n] = \theta[\theta[x_n]]$) the recurrence

$$\theta^2[x_n] - \theta^2[y_n] = \alpha_n x_n, \tag{1.8}$$

for an arbitrary sequence (α_n) consisting of positive integers, provides a natural generalization of (1.3), with the results on Engel and Pierce series in [12, 14, 30] corresponding to the special case $y_n = 1$ for all n. Moreover, the irrationality exponents of transcendental numbers given by suitable Engel and Pierce series were explicitly calculated in [15]; and in [8] (based on a result from [9]) this was further extended to find the irrationality exponents of the limits $n \to \infty$ for more general series (1.5), subject to the recurrence (1.8) with appropriate assumptions on the growth of the sequences (x_n), (y_n) and (α_n).

In a separate development [13], the first author showed that if the denominators in an Engel series satisfy the stronger divisibility property

$$x_n^2 \mid x_{n+1}, \qquad n \geq 1, \tag{1.9}$$

then the continued fraction expansion of (1.1) can be written explicitly in terms of the integers z_j defined by

$$z_1 = x_1, \qquad z_{j+1} = \frac{x_{j+1}}{x_j^2} \in \mathbb{Z}_{>0}, \quad j \geq 1. \tag{1.10}$$

Henceforth we refer to a series (1.1) with the property (1.9) as a *strong Engel series.*

Included in this class of strong Engel series is the set of numbers

$$\sum_{n=0}^{\infty} \frac{1}{u^{2^n}} \tag{1.11}$$

for integer $u \geq 2$, with the case $u = 2$ being known as the *Kempner number* [1]. All of these numbers are transcendental, with irrationality exponent 2 (see [3]). Their continued fraction expansions were found in recursive form in [26], with a non-recursive representation described in [28]; further generalizations with a similar folded recursive structure were given in [27] and later [24]. The series that are treated in the latter works all depend on a single integer parameter, e.g. the integer u in (1.11), whereas strong Engel series and their generalizations to be considered in this paper depend on the infinite set of parameters z_j.

In the next section we generalize the results of [13] to the case of a series

$$\sum_{j=1}^{\infty} \frac{\epsilon_j}{x_j}, \qquad \epsilon_j = \pm 1, \tag{1.12}$$

where the denominators satisfy the strong Engel property (1.9) and the sequence of signs ϵ_j is arbitrary. In fact, using the same approach as in [15] we provide the explicit continued fraction expansion for a sum of the form p/q + a strong Engel series. In the third section we give an application of these results to a family of *Lüroth series*, that is, series of the type

$$\frac{1}{u_1} + \sum_{j=2}^{\infty} \frac{1}{u_1(u_1-1)\cdots u_{j-1}(u_{j-1}-1)u_j}, \tag{1.13}$$

where we impose the condition that the sequence (u_n) satisfies a non-linear recurrence of second order analogous to (1.3). We also consider one of the alternating analogues of Lüroth series introduced in [16], and other generalizations along similar lines. The final section is mostly devoted to calculating exact irrationality exponents for certain families of series of generalized Lüroth type, defined by particular recurrences of second order for u_n. Inspired by [8], these recurrences are given in terms of "pseudo-polynomials" with arbitrary real exponents (see (4.2) below), rather than polynomials like G in (1.3). We conclude with a conjecture about transcendence of strong Engel series with signs.

2. Some explicit continued fractions. To fix our notation, we briefly recall some standard facts about continued fractions. In what follows, we denote a finite regular continued fraction by

$$[a_0; a_1, a_2, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}, \tag{2.1}$$

where $a_0 \in \mathbb{Z}$, $a_j \in \mathbb{Z}_{>0}$ and the convergent p_n/q_n is in lowest terms with $q_n > 0$. We define the *length* of (2.1) to be the index of the final partial quotient, written as

$$\ell([a_0; a_1, a_2, \ldots, a_n]) = n;$$

so we ignore the integer part a_0 when counting the length. Every $r \in \mathbb{Q}$ can be written as a finite continued fraction (2.1), although this representation is not unique (there is both an odd and an even length representation), but each $\xi \in \mathbb{R} \setminus \mathbb{Q}$ is given uniquely by an infinite continued fraction with convergents p_n/q_n of the form (2.1), that is (with $a_0 = \lfloor \xi \rfloor$),

$$\xi = [a_0; a_1, a_2, \dots] = \lim_{n\to\infty} [a_0; a_1, a_2, \dots, a_n] = \lim_{n\to\infty} \frac{p_n}{q_n}. \tag{2.2}$$

The three-term recurrence relation satisfied by the numerators and denominators of the convergents is equivalent to the matrix relation

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

for all $n \geq -1$, with

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking determinants in (2.3) yields the standard identity

$$p_j q_{j-1} - p_{j-1} q_j = (-1)^{j-1}, \qquad j \geq -1. \tag{2.4}$$

Given a finite continued fraction (2.1), written as $[a_0; \mathbf{a}]$, where $\mathbf{a} = (a_1, a_2, \dots, a_n)$ is the word of length n defining the fractional part, let $\mathbf{a}^R = (a_n, a_{n-1}, \dots, a_1)$ denote the reversed word, and introduce the modified word of length $n+1$ given by

$$\tilde{\mathbf{a}} = (a_1, a_2, \dots, a_{n-1}, a_n - 1, 1),$$

together with its reversal $\tilde{\mathbf{a}}^R$. Then it is convenient to define the following two families of transformations, labelled by a parameter z:

$$\varphi_z^{(+1)} : [a_0; \mathbf{a}] \mapsto [a_0; \mathbf{a}, z-1, \tilde{\mathbf{a}}^R], \quad \varphi_z^{(-1)} : [a_0; \mathbf{a}] \mapsto [a_0; \tilde{\mathbf{a}}, z-1, \mathbf{a}^R]. \tag{2.5}$$

These operators are analogous to the folding maps employed in [24]; with a slightly different notation, the operator $\varphi_z^{(+1)}$ was defined previously in [15]. For words of length zero, the action of these operators is defined by

$$\varphi_z^{(+1)}([a_0]) = [a_0; z-1, 1], \qquad \varphi_z^{(-1)}([a_0]) = [a_0 - 1; 1, z-1]. \tag{2.6}$$

(In what follows, we will sometimes omit the subscript z, but the implicit dependence on a parameter z should be understood.)

For each z, starting from a continued fraction of length $\ell([a_0; \mathbf{a}]) = n$, each of the operators $\varphi_z^{(\pm1)}$ generically produces a new continued fraction of length $\ell(\varphi_z^{(\pm1)}([a_0; \mathbf{a}])) = 2n+2$. However, if it happens that $z = 1$ or $a_n = 1$ in (2.5), then a zero coefficient will appear in the continued fraction obtained by applying one of these operators, and so in order to obtain only positive partial quotients one must use the concatenation operation

$$[\dots, A, 0, B, \dots] \mapsto [\dots, A+B, \dots] \tag{2.7}$$

(see e.g. [24, Proposition 3]), which reduces the length by 2. Henceforth we will assume that, whenever this occurs, the action of $\varphi_z^{(\pm1)}$ is understood as producing the result of concatenation of any zero that appears.

Our interest in the above transformations is due to

LEMMA 2.1.

$$\frac{p_n}{q_n} \pm \frac{(-1)^n}{zq_n^2} = \varphi_z^{(\pm 1)}([a_0; \mathbf{a}]). \tag{2.8}$$

Proof. This is a version of what is referred to as the Folding Lemma in [1], where it is attributed to Mendès France [22]. The formula for $\varphi_z^{(+1)}$ is [15, Lemma 4.1], and is also a corollary of [24, Proposition 2], so we just give details of the proof for $\varphi_z^{(-1)}$. Using matrix identities, similarly to the proof of [13, Proposition 2.1], we define

$$\mathbf{A}_a := \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

so that

$$\mathbf{M}_n := \mathbf{A}_{a_0}\mathbf{A}_{a_1}\cdots\mathbf{A}_{a_n} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

by (2.3). Then the continued fraction $\varphi_z^{(-1)}([a_0; \mathbf{a}]) = [a_0; \tilde{\mathbf{a}}, z-1, \mathbf{a}^R]$ of length $2n+2$ corresponds to the matrix product

$$\begin{aligned} \tilde{\mathbf{M}}_{2n+2} &:= \mathbf{A}_{a_0}\mathbf{A}_{a_1}\cdots\mathbf{A}_{a_{n-1}}\mathbf{A}_{a_n-1}\mathbf{A}_1\mathbf{A}_{z-1}\mathbf{A}_{a_n}\mathbf{A}_{a_{n-1}}\cdots\mathbf{A}_{a_1} \\ &= \mathbf{M}_n\mathbf{A}_{a_n}^{-1}\mathbf{A}_{a_n-1}\mathbf{A}_1\mathbf{A}_{z-1}\mathbf{M}_n^T\mathbf{A}_{a_0}^{-1}, \end{aligned}$$

which simplifies further to give

$$\begin{aligned} \tilde{\mathbf{M}}_{2n+2} &= \begin{pmatrix} \tilde{p}_{2n+2} & \tilde{p}_{2n+1} \\ \tilde{q}_{2n+2} & \tilde{q}_{2n+1} \end{pmatrix} \\ &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}\begin{pmatrix} z & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q_n & p_n - a_0q_n \\ q_{n-1} & p_{n-1} - a_0q_{n-1} \end{pmatrix}, \end{aligned}$$

where the entries in the first column of $\tilde{\mathbf{M}}_{2n+2}$ are

$$\tilde{p}_{2n+2} = zp_nq_n + p_nq_{n-1} - p_{n-1}q_n, \qquad \tilde{q}_{2n+2} = zq_n^2.$$

Then, from the determinant formula (2.4) it follows that the final convergent of the finite continued fraction $[a_0; \tilde{\mathbf{a}}, z-1, \mathbf{a}^R]$ is

$$\frac{\tilde{p}_{2n+2}}{\tilde{q}_{2n+2}} = \frac{zp_nq_n + (-1)^{n-1}}{zq_n^2} = \frac{p_n}{q_n} - \frac{(-1)^n}{zq_n^2},$$

as required. ∎

We now have the necessary tools to describe the continued fraction expansion of a series (1.12), subject to the strong Engel property (1.9). For

$\epsilon_1 = \pm 1$, after subtracting the integer part, we arrive at a series of the form

$$S = \frac{p}{q} + \sum_{n=2}^{\infty} \frac{\epsilon_n}{x_n}, \qquad q = x_1,\ \epsilon_n = \pm 1, \tag{2.9}$$

with $0 < p/q < 1$; so we proceed to describe the continued fraction expansion for a sum of the form (2.9) with an arbitrary positive rational number p/q. Due to the property of finite continued fractions that

$$[a_0; \mathbf{a}, b, 1] = [a_0; \mathbf{a}, b+1], \tag{2.10}$$

we can always write p/q in the form

$$\frac{p}{q} = [a_0; a_1, \ldots, a_k], \qquad a_k > 1. \tag{2.11}$$

The main result of this section is the following generalization of [24, Theorem 1].

Theorem 2.2. *Given a rational number p/q in lowest terms, if it is an integer then write $p/q = a_0 = [a_0]$, or otherwise let $[a_0; \mathbf{a}]$ be its continued fraction expansion* (2.11) *of length $k > 0$. Then, subject to the strong Engel property* (1.9)*, the continued fraction expansion of the series* (2.9) *is given in terms of $[a_0; \mathbf{a}]$ and the integer parameters $z_j = x_j/x_{j-1}^2 > 0$ for $j \geq 2$ by*

$$S = \prod_{j=3}^{\infty} \varphi_{z_j}^{(\epsilon_j)} (\varphi_{z_2}^{(\epsilon_2(-1)^k)}([a_0; \mathbf{a}])) = \cdots \varphi_{z_4}^{(\epsilon_4)} \varphi_{z_3}^{(\epsilon_3)} \varphi_{z_2}^{(\epsilon_2(-1)^k)}([a_0; \mathbf{a}]). \tag{2.12}$$

Suppose further that $k > 0$ and $z_j > 1$ for all j, with $a_1 > 2$ if $k = 1$, and $a_1 > 1$ otherwise. Then the length $\ell_n = \ell(S_n)$ of the continued fraction for S_n, the nth partial sum of the series, is

$$\ell_n = (k+2)2^{n-1} - 2. \tag{2.13}$$

Proof. The formula (2.12) for S follows by induction from Lemma 2.1, using the fact that

$$S_{n+1} = S_n + \frac{\epsilon_{n+1}}{x_{n+1}} = \frac{p_{\ell_n}}{q_{\ell_n}} + \frac{\epsilon_{n+1}}{z_{n+1} q_{\ell_n}^2},$$

so the continued fraction for S_{n+1} is obtained by applying the operator $\varphi_{z_{n+1}}^{(\epsilon_{n+1})}$ to the continued fraction for S_n, except if $n = 1$ and k is odd, when one should apply $\varphi_{z_2}^{(-\epsilon_2)}$ instead; and note that the length ℓ_n is even for $n > 1$, since the operators $\varphi_z^{(\pm 1)}$ always produce an even length continued fraction. The exact expression (2.13) for the lengths in the case of $k > 0$ and generic parameters follows immediately from the recurrence $\ell_{n+1} = 2\ell_n + 2$ with the initial value $\ell_1 = k$. ∎

Remark 2.3. Theorem 2.3 in [13] covers the special case $p/q = 1$ with $\epsilon_j = +1$ for all j, while Theorem 4.2 in [15] is the case of k even with all ϵ_j equal to $+1$.

It is worth briefly commenting on the non-generic cases, when the formula (2.13) is no longer valid. If $z_j = 1$ for some $j \geq 2$ then a zero appears and concatenation reduces the length of the continued fraction by 2. When $k = 0$, starting from $[a_0] = a_0$ we see from (2.6) that $\varphi_{z_2}^{(+1)}([a_0])$ and $\varphi_{z_2}^{(-1)}([a_0])$ include 1 as a partial quotient, which means that in the first case a single application of $\varphi^{(+1)}$ or $\varphi^{(-1)}$ produces a zero, while in the second case applying $\varphi^{(+1)}$ or $\varphi^{(-1)}$ moves the 1 to the end, so that a zero appears at the next step.

Otherwise, for $k \geq 1$ the initial application of $\varphi_{z_2}^{(\pm 1)}$ moves the coefficient a_1 to the end of the continued fraction, where it remains thereafter, so if $a_1 = 1$ then a zero appears at the next step. In the particular case $k = 1$, applying $\varphi^{(+1)}$ twice sends

$$\begin{aligned}[a_0; a_1] &\mapsto [a_0; a_1, z_2 - 1, 1, a_1 - 1] \\ &\mapsto [a_0; a_1, z_2 - 1, 1, a_1 - 1, z_3 - 1, 1, a_1 - 2, 1, z_2 - 1, a_1],\end{aligned}$$

and the situation is similar when $\varphi^{(+1)}$ is followed by $\varphi^{(-1)}$, while $\varphi^{(-1)}$ followed by $\varphi^{(+1)}$ sends

$$\begin{aligned}[a_0; a_1] &\mapsto [a_0; a_1 - 1, 1, z_2 - 1, a_1] \\ &\mapsto [a_0; a_1 - 1, 1, z_2 - 1, a_1, z_3 - 1, 1, a_1 - 1, z_2 - 1, 1, a_1 - 1],\end{aligned}$$

so that $a_1 - 2$ appears at the next step, and similarly when $\varphi^{(-1)}$ is applied twice. So the case $a_1 = 2$ is also degenerate when $k = 1$.

The latter considerations allow us to state the following corollary of Theorem 2.2, which will be relevant to the series of Lüroth type treated in the next section.

Corollary 2.4. *For a strong Engel series with signs, that is,*

$$\frac{1}{x_1} + \sum_{j=2}^{\infty} \frac{\epsilon_j}{x_j}, \qquad \epsilon_j = \pm 1, \tag{2.14}$$

with $z_1 = x_1 > 1$, $z_j = x_j/x_{j-1}^2 > 1$ for all $j \geq 2$, the nth partial sum S_n has length

$$\ell_n = 3 \cdot 2^{n-1} - 2, \qquad n \geq 1, \tag{2.15}$$

in the generic case that $x_1 > 2$. In the special case $x_1 = 2$, for a strong Engel series with $\epsilon_j = 1$ for all j, the formula should be modified to

$$\ell_n = 5 \cdot 2^{n-2}, \qquad n \geq 3, \tag{2.16}$$

while for a strong Pierce series with $\epsilon_j = (-1)^{j-1}$, the formula becomes

$$\ell_n = 5 \cdot 2^{n-2} - 2, \qquad n \geq 3. \tag{2.17}$$

Proof. This is the case $k = 1$ of Theorem 2.2 with $a_0 = 0$ and $a_1 = x_1 = z_1$, so the generic length formula (2.13) yields (2.15) immediately. In the degenerate case $x_1 = a_1 = 2$, for a strong Engel series one must start by applying $\varphi^{(-1)}$ followed by $\varphi^{(+1)}$, which sends

$$[0; 2] \mapsto [0; 1, 1, z_2 - 1, 2] \mapsto [0; 1, 1, z_2 - 1, 2, z_3 - 1, 1, 1, z_2 - 1, 1, 1],$$

while at the next stage $\varphi^{(+1)}$ together with (2.7) produces

$$[0; 1, 1, z_2-1, 2, z_3-1, 1, 1, z_2-1, 1, 1, z_4-1, 2, z_2-1, 1, 1, z_3-1, 2, z_2-1, 1, 1].$$

Then, because this and all subsequent continued fractions begin $[0; 1, 1, \ldots]$, each successive application of $\varphi^{(+1)}$ requires concatenation, so the recurrence for the lengths is modified to $\ell_{n+1} = 2\ell_n$ for $n \geq 3$, which gives the formula (2.16). For a strong Pierce series one should repeatedly apply $\varphi^{(+1)}$ followed by $\varphi^{(-1)}$, so that with $x_1 = 2$ the sequence begins

$$[0; 2] \mapsto [0; 2, z_2 - 1, 1, 1] \mapsto [0; 2, z_2 - 1, 2, z_3 - 1, 1, 1, z_2 - 1, 2],$$

where we used (2.7) in the second step, and thereafter there is always a 2 at the beginning and end of each continued fraction, so no further concatenation is required and we just have the generic recursion $\ell_{n+1} = 2\ell_n + 2$ for $n \geq 3$, giving (2.17). ■

Remark 2.5. Note that in the strong Engel case with $x_1 = 2$ the continued fractions from $n = 3$ onwards all end with $[\ldots, 1, 1]$, so we could instead use (2.10) to make them end with $[\ldots, 2]$ and write a reduced length formula

$$\ell_n = 5 \cdot 2^{n-2} - 1$$

in that case. However, the non-reduced continued fractions maintain the property of having even length at each stage, which we prefer to keep. Similarly, in the strong Pierce case we instead take the reduced continued fractions ending in 2 for the same reason.

3. Lüroth series and generalizations. It was shown by Lüroth [20] that every real number in the interval $(0, 1)$ admits an expansion of the form (1.13) for a certain sequence of integers $u_j \geq 2$. As well as being used for Diophantine approximation of real numbers [4, 5], Lüroth series have also been employed in the context of rational function approximations of power series [17, 18].

In [16], Kalpazidou et al. introduced two different alternating analogues of Lüroth series, each of which provides a unique representation of a real number. For a number in $(0, 1)$, the first type of alternating Lüroth expansion defined in [16] takes the form

$$\frac{1}{u_1} + \sum_{j=2}^{\infty} \frac{(-1)^{j-1}}{u_1(u_1 + 1) \cdots u_{j-1}(u_{j-1} + 1)u_j}, \tag{3.1}$$

for a sequence of integers $u_j \geq 1$.

It is clear from the form of (1.13) that if we set $x_1 = u_1$ and

$$x_j = u_j \prod_{k=1}^{j-1} u_k(u_k - 1), \qquad j \geq 2,$$

then $x_j \mid x_{j+1}$, so a Lüroth series is a particular example of an Engel series, and similarly an alternating Lüroth series (3.1) is a particular type of Pierce series.

In order to consider these two examples together it is convenient to start with a more general family of series, of the form

$$S' = \frac{1}{u_1} + \sum_{j=2}^{\infty} \frac{\epsilon_j}{u_1 v_1 \cdots u_{j-1} v_{j-1} u_j}, \qquad \epsilon_j = \pm 1, \tag{3.2}$$

for sequences of integers $u_j, v_j \in \mathbb{Z}_{>0}$. If we now take

$$x_1 = u_1, \qquad x_j = u_j \prod_{k=1}^{j-1} u_k v_k, \quad j \geq 2, \tag{3.3}$$

then it turns out we can further obtain the strong Engel property (1.9) for the sequence (x_n) whenever (u_n) satisfies certain recurrence relations of second order, analogous to (1.3).

Proposition 3.1. *Suppose that the sequence (u_n) satisfies either the recurrence*

$$u_{n+2} u_n = \alpha_n u_{n+1}^3 v_{n+1}, \qquad n \geq 1, \tag{3.4}$$

where $u_2 = m u_1^2 v_1$, or

$$u_{n+2} = \alpha_n u_{n+1}^2 v_n, \qquad n \geq 1, \tag{3.5}$$

where $u_2 = m u_1$, and in each case (α_n) is an arbitrary sequence of positive integers, with $u_1, m \in \mathbb{Z}_{>0}$ arbitrary. Then the associated sequence (x_n) defined by (3.3) *has the strong Engel property, that is, $z_j = x_j / x_{j-1}^2 \in \mathbb{Z}$ for all $j \geq 2$.*

Proof. For the proof it is convenient to write the various relations between u_n, v_n, α_n and x_n in terms of the exponentiated shift operator, as in (1.7). From (3.3) we have

$$\theta[x_n] = u_n v_n \theta[u_n], \qquad n \geq 1,$$

which implies

$$\theta^2[x_n] = \theta[u_n]\theta[v_n]\theta^2[u_n]$$

also holds for $n \geq 1$. Then, by rewriting the first recurrence (3.4) as

$$\theta^2[u_n] = \alpha_n u_{n+1} v_{n+1},$$

we calculate

$$z_2 = \frac{x_2}{x_1^2} = \frac{u_2 v_1}{u_1} = m u_1 v_1^2 \in \mathbb{Z}_{>0},$$

while for $n \geq 1$ we have

$$z_{n+2} = \alpha_n u_{n+1} v_{n+1}^2 \rho_n,$$

with

$$\rho_n := \frac{\theta[u_n]}{x_n v_n}, \qquad n \geq 1.$$

The latter definition gives $\rho_1 = u_2/(u_1^2 v_1) = m$ and

$$\theta[\rho_n] = \frac{\theta^2[u_n]}{\theta[x_n]\theta[v_n]} = \alpha_n, \tag{3.6}$$

which just says that $\rho_{n+1} = \alpha_n \rho_n$, so by induction we have $\rho_n \in \mathbb{Z}_{>0}$ for all $n \geq 1$, and this implies $z_j \in \mathbb{Z}_{>0}$ for all $j \geq 2$, as required. Similarly, we rewrite the second recurrence (3.5) as

$$\theta^2[u_n] = \alpha_n u_n v_n,$$

then calculate

$$z_2 = \frac{u_2 v_1}{u_1} = m v_1 \in \mathbb{Z}_{>0},$$

and for $n \geq 1$ we find

$$z_{n+2} = \alpha_n v_{n+1} \rho_n,$$

where in this case we instead take the definition

$$\rho_n := \frac{u_n \theta[u_n]}{x_n}, \qquad n \geq 1.$$

The latter definition gives $\rho_1 = m$ once again, and also

$$\theta[\rho_n] = \frac{\theta[u_n]\theta^2[u_n]}{\theta[x_n]} = \alpha_n,$$

which is the same final result for $\theta[\rho_n]$ as in (3.6), so the conclusion is the same. ∎

Example 3.2. Taking $\alpha_n = 1$, $v_n = u_n - 1$ for all $n \geq 1$, the recurrence (3.4) becomes

$$u_{n+2} u_n = u_{n+1}^3 (u_{n+1} - 1), \tag{3.7}$$

and setting $u_1 = 3$, $m = 1$ gives $u_2 = u_1^2(u_1 - 1) = 18$, so the sequence (u_n) begins with

$$3, 18, 33048, 66266659938624768, \ldots. \tag{3.8}$$

We have $x_1 = 3$, $x_{n+1} = x_n u_{n+1}(u_n - 1)$ for $n \geq 1$. Hence

$$3, 108, 60676128, 1328755210427661807382195322 88, \ldots$$

is the beginning of the sequence (x_n). Then we find $z_2 = 108/3^2 = 12$, $z_3 = 60676128/108^2 = 5202$, $z_4 = 132875521042766180738219532288/60676128^2 = 36091859899032$, and so on.

EXAMPLE 3.3. Taking $\alpha_n = 1$, $v_n = u_n + 1$ for all $n \geq 1$, the recurrence (3.5) becomes

$$u_{n+2} = u_{n+1}^2(u_n + 1), \tag{3.9}$$

and taking $u_1 = 2$, $m = 1$ gives $u_2 = u_1 = 2$, so the sequence (u_n) begins with

$$2, 2, 12, 432, 2426112, \ldots. \tag{3.10}$$

We have $x_1 = 2$, $x_{n+1} = x_n u_{n+1}(u_n + 1)$ for $n \geq 1$. Hence

$$2, 12, 432, 2426112, 2548646416023552, \ldots$$

is the beginning of the sequence (x_n), and it is not hard to show that in fact $x_n = u_{n+1}$ for all $n \geq 1$, with this particular choice of initial values for (3.9). Then we find $z_2 = 12/2^2 = 3$, $z_3 = 432/12^2 = 3$, $z_4 = 2426112/432^2 = 13$, and in general $z_n = u_{n-1} + 1$ for all $n \geq 2$.

We can now combine the results in Section 2 with Proposition 3.1 to describe the continued fraction expansion of certain Lüroth series with the strong Engel property, not just of the form (1.13) but also with arbitrary signs inserted.

THEOREM 3.4. *Suppose that the sequence (u_n) satisfies either the recurrence*

$$u_{n+2}u_n = \alpha_n u_{n+1}^3(u_{n+1} - 1), \qquad n \geq 1, \tag{3.11}$$

where $u_2 = m u_1^2(u_1 - 1)$, or

$$u_{n+2} = \alpha_n u_{n+1}^2(u_n - 1), \qquad n \geq 1, \tag{3.12}$$

where $u_2 = m u_1$, and in each case (α_n) is an arbitrary sequence of positive integers, with $u_1 \in \mathbb{Z}_{>1}$, $m \in \mathbb{Z}_{>0}$ arbitrary. Then the continued fraction expansion of the sum

$$S = \frac{1}{u_1} + \sum_{j=2}^{\infty} \frac{\epsilon_j}{u_1(u_1 - 1)\cdots u_{j-1}(u_{j-1} - 1)u_j}, \qquad \epsilon_j = \pm 1, \tag{3.13}$$

is given by

$$S = \prod_{j=3}^{\infty} \varphi_{z_j}^{(\epsilon_j)}(\varphi_{z_2}^{(-\epsilon_2)}([0; u_1])) = \cdots \varphi_{z_4}^{(\epsilon_4)} \varphi_{z_3}^{(\epsilon_3)} \varphi_{z_2}^{(-\epsilon_2)}([0; u_1]), \tag{3.14}$$

where either $z_{n+1} = u_n(u_n - 1)^2\rho_n$ when (3.11) *holds, or $z_{n+1} = (u_n - 1)\rho_n$*

when (3.12) *holds, with*

$$\rho_n = m \prod_{k=1}^{n-1} \alpha_k \tag{3.15}$$

in both cases, for all $n \geq 1$.

Proof. This follows by combining Theorem 2.2 for $k = 1$ with Proposition 3.1, in the particular case that $v_n = u_n - 1$ for all n, and noting that from (3.6) we have $\rho_{n+1} = \alpha_n \rho_n$ for $n \geq 1$ with the initial value $\rho_1 = m$, which yields both (3.15) and the appropriate formula for z_{n+1} according to whether (3.11) or (3.12) holds. ∎

Example 3.5. As a continuation of Example 3.2, it follows that the number $S \approx 0.34259260907$, whose Lüroth series is defined by the sequence (3.8), that is,

$$S = \frac{1}{3} + \frac{1}{108} + \frac{1}{60676128} + \frac{1}{1328755210427661807382195322288} + \cdots,$$

has continued fraction expansion

$$[0; 2, 1, 11, 3, 5201, 1, 2, 11, 1, 2, 36091859899031, 1, 1, 1, 11, 2, 1, 5201, 3, 11, 1, 2, \ldots].$$

The infinite continued fraction is obtained by folding the sequence of finite continued fractions for the nth truncation of the series, that is,

$$[0; 3] \mapsto [0; 2, 1, 11, 3] \mapsto [0; 2, 1, 11, 3, 5201, 1, 2, 11, 1, 2] \mapsto \cdots,$$

where the lengths are given by (2.15), and this pattern of lengths remains the same if arbitrary signs are inserted in S.

It is straightforward to state the analogue of Theorem 3.4 for the case of an alternating Lüroth series (3.1), also with the inclusion of arbitrary signs. The proof is essentially the same so is omitted.

Theorem 3.6. *Suppose that the sequence* (u_n) *satisfies either the recurrence*

$$u_{n+2} u_n = \alpha_n u_{n+1}^3 (u_{n+1} + 1), \qquad n \geq 1, \tag{3.16}$$

where $u_2 = m u_1^2 (u_1 + 1)$, *or*

$$u_{n+2} = \alpha_n u_{n+1}^2 (u_n + 1), \qquad n \geq 1, \tag{3.17}$$

where $u_2 = m u_1$, *and in each case* (α_n) *is an arbitrary sequence of positive integers, with* $u_1 \in \mathbb{Z}_{>1}$ *and* $m \in \mathbb{Z}_{>0}$ *arbitrary. Then the continued fraction expansion of the sum*

$$S' = \frac{1}{u_1} + \sum_{j=2}^{\infty} \frac{\epsilon_j}{u_1(u_1+1) \cdots u_{j-1}(u_{j-1}+1) u_j}, \qquad \epsilon_j = \pm 1, \tag{3.18}$$

is given by the same formula as for S *in* (3.14)*, but with* $z_{n+1} = u_n(u_n+1)^2\rho_n$ *when* (3.16) *holds, or* $z_{n+1} = (u_n+1)\rho_n$ *when* (3.17) *holds, with* $\rho_n = m\prod_{k=1}^{n-1}\alpha_k$ *in both cases, for all* $n \geq 1$.

EXAMPLE 3.7. As a continuation of Example 3.3, it follows that the number $S' \approx 0.418981069299$, whose alternating Lüroth expansion is defined by the sequence (3.10), that is,

$$S' = \frac{1}{2} - \frac{1}{12} + \frac{1}{432} - \frac{1}{2426112} + \frac{1}{2548646416023552} - \cdots,$$

has continued fraction expansion

$$[0; 2, 2, 1, 1, 2, 2, 2, 1, 1, 12, 2, 2, 2, 2, 1, 1, 2, 2, 432, 1, 1, 2, 1, 1, 2, 2, 2, 2, 12, 1, 1, \dots].$$

The infinite continued fraction is obtained by folding the sequence of finite continued fractions for the nth truncation of the series S', that is,

$$\begin{aligned}[0; 2] \mapsto [0; 2, 2, 1, 1] &\mapsto [0; 2, 2, 1, 1, 2, 2, 2, 2] \\ &\mapsto [0; 2, 2, 1, 1, 2, 2, 2, 1, 1, 12, 2, 2, 2, 2, 1, 1, 2, 2]\end{aligned}$$

etc., and since $k = 1$ and $a_1 = x_1 = 2$, this is a non-generic case, with the lengths being given by (2.17) for $n \geq 3$.

The result of Proposition 3.1 requires the sequence (u_n) to satisfy one of the recurrences (3.4) or (3.5), which depend on how the sequence (v_n) is specified, for instance, imposing $v_n = u_n - 1$ for a Lüroth series (1.13), or $v_n = u_n + 1$ for an alternating Lüroth series (3.1), as above. However, there is another way to obtain the strong Engel property, by imposing independent conditions on the sequences (u_n) and (v_n).

PROPOSITION 3.8. *Suppose that the sequences* (u_n) *and* (v_n) *satisfy*

$$u_n = \beta_n \prod_{k=1}^{n-1} u_k, \quad v_n = \gamma_n \prod_{k=1}^{n-1} v_k, \qquad n \geq 2, \tag{3.19}$$

where (β_n) *and* (γ_n) *are arbitrary sequences of positive integers, with arbitrary* $u_1, v_1 \in \mathbb{Z}_{>0}$. *Then the associated sequence* (x_n) *defined by* (3.3) *has the strong Engel property, that is,* $z_j = x_j/x_{j-1}^2 \in \mathbb{Z}$ *for all* $j \geq 2$.

Proof. We have

$$z_2 = \frac{x_2}{x_1^2} = \frac{u_2u_1v_1}{u_1^2} = \frac{u_2v_1}{u_1} = \beta_2v_1,$$

while for $j \geq 2$ we see that

$$z_{j+1} = \frac{u_{j+1}\prod_{k=1}^{j}u_kv_k}{u_j^2(\prod_{k=1}^{j-1}u_kv_k)^2} = \frac{u_{j+1}v_j}{u_j\prod_{k=1}^{j-1}u_kv_k} = \beta_{j+1}\gamma_j,$$

and the result follows. ∎

EXAMPLE 3.9. Upon setting $\beta_n = n$, $\gamma_n = 1$ for all $n \geq 2$ and $u_1 = v_1 = 1$, we have $v_n = 1$ for all n, and we find

$$u_n = n \prod_{k=1}^{n-2} (n-k)^{2^{k-1}}, \qquad x_n = \prod_{k=0}^{n-2} (n-k)^{2^k},$$

where $x_n = \prod_{k=1}^{n} u_k$ in this case, which implies that $z_n = \beta_n = n$ for $n \geq 2$. So the sequence (u_n) begins with $1, 2, 6, 48, 2880, 9953280, \ldots$, and (x_n) begins with $1, 2, 12, 576, 1658880, 16511297126400, \ldots$. The alternating sum $\sum_{j\geq 1} (-1)^{j-1}/x_j$ is the strong Pierce series

$$S' = 1 - \frac{1}{2} + \frac{1}{12} - \frac{1}{576} + \frac{1}{1658880} - \frac{1}{16511297126400} + \cdots, \tag{3.20}$$

and the continued fraction expansion of $S' \approx 0.5815978250$ is

$$[0; 1, 1, 2, 1, 1, 3, 2, 2, 1, 1, 4, 2, 2, 2, 3, 1, 1, 2, 2, 5, 1, 1, 2, 1, 1, 3, 2, 2, 2, 4, 1, 1, 2, 2, \ldots].$$

The corresponding sequence of foldings of finite continued fractions begins

$$\begin{aligned} [1] &\mapsto [0; 1, 1] = [0; 2] \\ &\mapsto [0; 1, 1, 2, 2] \\ &\mapsto [0; 1, 1, 2, 1, 1, 3, 2, 2, 1, 1] \\ &\mapsto [0; 1, 1, 2, 1, 1, 3, 2, 2, 1, 1, 4, 2, 2, 2, 3, 1, 1, 2, 1, 1], \end{aligned}$$

and so on, viewed as corresponding to $k = 0$ in Theorem 2.2, or to $k = 1$ if we combine the first two terms so that $S' = 1/x'_1 + \sum_{j\geq 2} (-1)^j/x'_j = 1/2 + 1/12 - 1/576 + 1/1658880 - \cdots$, with $x'_1 = 2$, $x'_j = x_{j+1}$ for $j \geq 2$, and then the sequence of lengths is given by (2.16).

4. Irrationality exponents and transcendence. In this final section we compute the irrationality exponents of certain families of transcendental numbers defined by series of Lüroth/alternating Lüroth type, with arbitrary signs, that have the strong Engel property, before concluding with a conjecture concerning the whole family of series in Theorem 2.2.

Recall that the *irrationality exponent* $\mu(\xi)$ of a real number ξ is defined to be the supremum of the set of real numbers μ such that there are infinitely many rational approximations p/q satisfying the inequality

$$|\xi - p/q| < 1/q^{\mu}.$$

For an irrational number ξ we have $\mu(\xi) \geq 2$, and the irrationality exponent is given in terms of q_n, the denominators of the convergents of the continued fraction expansion of ξ, by the formula

$$\mu(\xi) = 1 + \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}. \tag{4.1}$$

If $\mu(\xi) > 2$ then ξ is transcendental, by Roth's theorem [25].

THEOREM 4.1. *Suppose that a number $\xi \in \mathbb{R}_{>0}$ is defined by either a series of the Lüroth type* (3.13) *with arbitrary signs, subject to a recurrence of the form* (3.11)*, or of the alternating Lüroth type* (3.18) *with arbitrary signs, subject to a recurrence of the form* (3.16)*, where in each case α_n is given by*

$$\alpha_n = \lceil \exp(C\nu^n) P(u_n, u_{n+1}) \rceil \quad \text{with} \quad P(X,Y) = \sum_{i=0}^{M} \sum_{j=0}^{N} c_{ij} X^{r_i} Y^{s_j}, \tag{4.2}$$

for non-negative integers M, N, positive real numbers C, c_{ij}, ν, and non-negative real exponents $r = r_M > r_{M-1} > \cdots > r_0 \geq 0$, $s = s_N > s_{N-1} > \cdots > s_0 \geq 0$. Then ξ is transcendental with irrationality exponent

$$\mu(\xi) = \max\big(\nu, \tfrac{1}{2}\big(s + 4 + \sqrt{(s+4)^2 + 4(r-1)}\,\big)\big). \tag{4.3}$$

Similarly, if ξ is defined by one of the series (3.13) *or* (3.18)*, with arbitrary signs, subject to a recurrence of the form* (3.12) *or* (3.17)*, respectively, with α_n as in* (4.2)*, then it is transcendental with irrationality exponent*

$$\mu(\xi) = \max\big(\nu, \tfrac{1}{2}\big(s + 2 + \sqrt{(s+2)^2 + 4(r+1)}\,\big)\big). \tag{4.4}$$

Proof. For the sake of simplicity, we assume that the series (3.13) or (3.18) being considered is generic, in the sense described in Corollary 2.4, which means imposing the requirement $u_1 \geq 3$, but if this is not the case then the same method of proof applies with only minor modifications. Clearly (u_n) is an increasing sequence of positive integers. Upon setting $\Lambda_n = \log u_n$ and taking logarithms in either (3.11) or (3.16), subject to (4.2), we find

$$\Lambda_{n+2} - (s+4)\Lambda_{n+1} + (1-r)\Lambda_n = \Delta_n, \qquad \Delta_n = C\nu^n + o(1). \tag{4.5}$$

By applying the method of Aho and Sloane [2], the inhomogeneous linear equation (4.5) can be solved "explicitly" to yield

$$\Lambda_n = A\lambda^n + B\bar{\lambda}^n + F_n, \qquad F_n = C'\nu^n(1 + o(1)), \tag{4.6}$$

for certain constants A, B, C', where

$$\lambda = \tfrac{1}{2}\big(s + 4 + \sqrt{(s+4)^2 + 4(r-1)}\,\big) \geq 2 + \sqrt{3}, \tag{4.7}$$

and $\bar{\lambda}$ is the conjugate root of the characteristic quadratic for (4.5). More details of the precise form of A, B and F_n can be found in [12] (see also [15]), but are not needed here; the formula (4.6) is not really an explicit solution, because F_n and A, B depend implicitly on the sequence (u_n). Then taking

$$\mu = \max(\nu, \lambda),$$

we see that

$$\Lambda_n = D\mu^n(1 + o(1)), \qquad D > 0, \tag{4.8}$$

where D is either C' or A depending on which of ν or λ is the greater.

For what follows, an estimate of the growth of the sequence (z_n) is also required. From Theorems 3.4 and 3.6, using (3.15), we have

$$\begin{aligned}\log z_n &= \log u_{n-1} + 2\log(u_{n-1} \mp 1) + \log \rho_{n-1} \\ &= 3\Lambda_{n-1} + \log m + \sum_{k=1}^{n-2} \log \alpha_k + o(1).\end{aligned}$$

Thus we see from (4.2) and (4.8) that

$$\log z_n = D'\mu^n(1+o(1)), \qquad D' > 0, \tag{4.9}$$

where the precise form of D' is unimportant.

In order to evaluate the limit in (4.1), we now consider the three-term recurrence relation for q_n encoded in (2.3), which is $q_{n+1} = a_{n+1}q_n + q_{n-1}$, so

$$L_{n+1} - L_n = \log a_{n+1} + \log\left(1 + \frac{q_{n-1}}{a_{n+1}q_n}\right),$$

where we set $L_n = \log q_n$. Performing the telescopic sum of the latter identity, with the initial value $L_0 = \log q_0 = 0$, and noting that the last term on the right is at most $\log 2$, since (q_n) is an increasing sequence of positive integers and $a_n \geq 1$, we obtain

$$L_n = \sum_{j=1}^{n} \log a_j + \delta_n, \qquad 0 < \delta_n < n\log 2. \tag{4.10}$$

From the discussion before Corollary 2.4, it is clear that the only possible values of the coefficients appearing in the folded continued fraction (3.14) (or its counterpart as described in Theorem 3.6) are 1, u_1, $u_1 - 1$, $u_1 - 2$ and $z_j - 1$ for $j \geq 2$. In an initial block of length $\ell_n = 3\cdot 2^{n-1} - 2$, as in (2.15), the coefficient $z_n - 1$ appears once, $z_{n-1} - 1$ appears twice, and in general $z_j - 1$ appears 2^{n-j} times. This accounts for $2^{n-1} - 1$ coefficients out of ℓ_n, while δ_{ℓ_n} and the sum of the logarithms of the remaining coefficients are both $O(2^n)$, so from (4.10) we have

$$\begin{aligned}L_{\ell_n} &= \sum_{j=2}^{n} 2^{n-j}\log(z_j - 1) + O(2^n) = 2^n D' \sum_{j=2}^{n} (\mu/2)^j(1+o(1)) + O(2^n) \\ &= D'(1-2/\mu)^{-1}\mu^n(1+o(1)),\end{aligned}$$

since $\mu > 2$ by (4.7). Now if $\epsilon_{n+1} = +1$ then folding requires an application of $\varphi^{(+1)}_{z_{n+1}}$, which gives $a_{\ell_n+1} = z_{n+1} - 1$ and so

$$L_{\ell_n+1} = L_{\ell_n} + \log(z_{n+1} - 1) + \delta_{\ell_n+1} - \delta_{\ell_n} = L_{\ell_n} + D'\mu^{n+1}(1+o(1)),$$

which gives

$$\frac{L_{\ell_n+1}}{L_{\ell_n}} = 1 + \frac{\mu}{(1-2/\mu)^{-1}} + o(1) = \mu - 1 + o(1). \tag{4.11}$$

Otherwise, if $\epsilon_{n+1} = -1$ then an application of $\varphi_{z_{n+1}}^{(-1)}$ gives $a_{\ell_n+1} = 1$, so $L_{\ell_n+1}/L_{\ell_n} = 1 + o(1)$, but $a_{\ell_n+2} = z_{n+1} - 1$, and so instead $L_{\ell_n+2}/L_{\ell_n+1} = \mu - 1 + o(1)$. Then, by considering the sequence of coefficients until the next folding happens at length ℓ_{n+1}, we may write

$$L_{\ell_n+j} = D'\mu^n(\mu + \tilde{\Delta}_{n,j})(1 + o(1))$$

for $j \geq 1$ when $\epsilon_{n+1} = +1$, or for $j \geq 2$ when $\epsilon_{n+1} = -1$, where $0 < \tilde{\Delta}_{n,j} = O(1)$ increases by an amount $\mu^{k-n} \leq 1$ each time the coefficient a_{ℓ_n+j} is equal to $z_k - 1$, and otherwise remains the same. So there is an initial step where $L_{\ell_n+j+1}/L_{\ell_n+j} = \mu - 1 + o(1)$ for $j = 0$ or 1 depending on whether $\epsilon_{n+1} = \pm 1$, and at all subsequent steps until the next folding we have $L_{\ell_n+j+1} - L_{\ell_n+j} = D'\mu^n\big(\tilde{\Delta}_{n,j+1} - \tilde{\Delta}_{n,j} + o(1)\big)$ where $\tilde{\Delta}_{n,j+1} - \tilde{\Delta}_{n,j} \leq \mu^{k-n}$ for $2 \leq k \leq n$, so $\tilde{\Delta}_{n,j+1} - \tilde{\Delta}_{n,j} \leq 1$. Hence, for these subsequent steps,

$$\frac{L_{\ell_n+j+1}}{L_{\ell_n+j}} = 1 + \frac{\tilde{\Delta}_{n,j+1} - \tilde{\Delta}_{n,j} + o(1)}{(\mu + \tilde{\Delta}_{n,j})\big(1 + o(1)\big)},$$

which in the limit is at most $1 + \mu^{-1}$, until the next folding happens and there is a term with limit $\mu - 1$, obtained from the ratio of terms like (4.11). Now $1 + \mu^{-1} \leq \mu - 1$ for $\mu \geq 1 + \sqrt{2}$, which holds by (4.7). Thus from (4.1) we find

$$\mu(\xi) = 1 + \limsup_{n\to\infty} \frac{L_{n+1}}{L_n} = 1 + \mu - 1 = \max(\nu, \lambda),$$

as required.

For the second part of the theorem, where (u_n) is subject to (3.12) or (3.17), for a series of Lüroth/alternating Lüroth type with signs, as appropriate, (4.5) is modified to

$$\Lambda_{n+2} - (s+2)\Lambda_{n+1} - (1+r)\Lambda_n = \Delta_n, \qquad \Delta_n = C\nu^n + o(1),$$

and the largest characteristic root is

$$\lambda = \tfrac{1}{2}\big(s + 2 + \sqrt{(s+2)^2 + 4(r+1)}\,\big) \geq 1 + \sqrt{2},$$

so $\mu = \max(\nu, \lambda) \geq 1 + \sqrt{2}$ still holds, and the rest of the proof is the same. ∎

Remark 4.2. A suitable modification of the preceding argument should show that the number (3.20) defined in Example 3.9 has irrationality exponent 2.

As is well known, the set of irrational numbers with irrationality exponent greater than 2 has measure zero. If $\mu(\xi) = 2$ then there is no simple criterion to decide whether ξ is transcendental or not. Nevertheless, we have reason to expect that none of the ξ defined by strong Engel series with signs are algebraic.

CONJECTURE 4.3. *All of the real numbers ξ defined by a series of the form (2.12), for arbitrary $p/q \in \mathbb{Q}$ and positive integer parameters $z_2, z_3, \ldots,$ are transcendental.*

To explain why the above conjecture is plausible, we return to the case considered in [13], that is, $p/q = 1$ with all $\epsilon_j = +1$, when the strong Engel series for ξ has the form

$$S = 1 + \sum_{j=2}^{\infty} \frac{1}{z_2^{2^{j-2}} z_3^{2^{j-3}} \cdots z_j} \tag{4.12}$$

(it is necessary to assume that $z_j > 1$ for at least one j to ensure convergence). Suppose that we replace the first n of the parameters by variables, so $z_{j+1} = \zeta_j^{-1}$ for $j = 1, \ldots, n$, and regard all the other z_j as fixed. Then (4.12) becomes a power series

$$S(\zeta_1, \ldots, \zeta_n) = 1 + \sum_{j=1}^{\infty} c_j \prod_{i=1}^{\min(j,n)} \zeta_i^{2^{j-i}} \tag{4.13}$$

for suitable coefficients c_j defined in terms of $z_{n+2}, z_{n+3}, \ldots,$ with $c_j = 1$ for $1 \le j \le n$. Then in principle, the series (4.13) should be amenable to the techniques of Loxton and van der Poorten [19], who proved that, subject to some recursive systems of functional equations being satisfied, certain power series in several variables, with algebraic coefficients, take only transcendental values at algebraic points.

The result of [19] is a very broad generalization of a result of Mahler [21], who showed that the series

$$f(\zeta) = \sum_{n=0}^{\infty} \zeta^{2^n},$$

which satisfies the functional equation

$$f(\zeta^2) = f(\zeta) - \zeta,$$

takes transcendental values at algebraic points α with $0 < |\alpha| < 1$. In particular, this includes the transcendence of the Kempner number and the other values of the series (1.11) for integers $u \ge 2$.

The analysis of the series (4.13) by the methods of Loxton and van der Poorten, and a proof of the above conjecture, is an interesting challenge for the future.

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