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Forbidden conductors of L-functions and continued fractions of particular form

by

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In memory of Professor Andrzej Schinzel

1. Introduction. It is expected that the conductor q of an L-function from the extended Selberg class $\mathcal{S}^\sharp$ (see Section 2 for definitions) can attain only certain special values. For example, it is expected that the L-functions in $\mathcal{S}^\sharp$ with degree 2 cannot have conductor $q < 1$, and that the L-functions in the Selberg class $\mathcal{S}$ always have $q \in \mathbb{N}$. Both these expectations are far from being proved at present. In particular, no absolute lower bound for the conductor is known. It is possible, however, to estimate it in terms of other invariants of the L-function involved, as was shown in [6].

In this paper we focus on L-functions of degree 2 in $\mathcal{S}^\sharp$ and investigate the admissible values of their conductor q via an unexpected link with certain continued fractions $c(q,\mathbf{m})$, which we now define. Let $q > 0$ be given. For a vector $\mathbf{m} = (m_0, \dots, m_k) \in \mathbb{Z}^{k+1}$ with some $k \geq 0$ we set

$$c(q,\mathbf{m}) = m_k + \cfrac{1}{qm_{k-1} + \cfrac{q}{qm_{k-2} + \cfrac{q}{\ddots + \cfrac{q}{qm_0}}}}. \tag{1.1}$$

Here we assume that *all denominators in* (1.1) *are non-zero*. Such a vector $\mathbf{m}$ is called a *path* for q, or simply a path.

Of course, (1.1) can be translated to the standard continued fraction notation where all numerators are 1 and indices are in increasing order,

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$$[a_0,\dots,a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}, \qquad a_j \in \begin{cases} \mathbb{Z}, & 2 \mid j, \\ q\mathbb{Z}, & 2 \nmid j. \end{cases}$$

In this paper we use the notation $c(q,\mathbf{m})$, as it is a better fit to our transformation formula for L-functions.

The fraction $c(q,\mathbf{m})$ and the path $\mathbf{m}$ are called *proper* if all m_j, $j = 0,\dots,k-1$, are non-zero. The proper fractions are those arising naturally in connection with L-functions. The integer k is the *length* of the path, and clearly $c(q,\mathbf{m}) = m_0$ for a path of length 0. The *weight* $w_q(\mathbf{m})$ of $c(q,\mathbf{m})$ is defined for a path of length $k \geq 1$ as

$$\begin{aligned} (1.2) \qquad w_q(\mathbf{m}) &= q^{k/2} \prod_{j=0}^{k-1} |c(q,\mathbf{m}_j)| \\ &= q^{k/2} \left| m_{k-1} + \cfrac{1}{qm_{k-2} + \cfrac{q}{\ddots + \cfrac{q}{qm_0}}} \right| \left| m_{k-2} + \cfrac{1}{\ddots + \cfrac{q}{qm_0}} \right| \cdots |m_0|, \end{aligned}$$

where

$$\mathbf{m}_j = (m_0,\dots,m_j) \quad \text{for } 0 \leq j \leq k.$$

If $k = 0$ we simply write $w_q(\mathbf{m}) = 1$. Note that the weight $w_q(\mathbf{m})$ does not depend on the last entry m_k, and that always $w_q(\mathbf{m}) > 0$. Moreover, we say that *the weight w_q is unique* if $w_q(\mathbf{m}) = w_q(\mathbf{n})$ whenever $c(q,\mathbf{m}) = c(q,\mathbf{n})$.

The main result of this paper reads as follows.

THEOREM 1. *If there exists $F \in \mathcal{S}^\sharp$ of degree 2 and conductor q, then the weight w_q is unique.*

The proof of Theorem 1 is based on the properties of certain nonlinear twists of L-functions and is given in Section 3.

We shall also prove (see Lemma 4 in Section 3) that the weight w_q is unique if and only if $w(q,\mathbf{m}) = 1$ for all proper fractions of type (1.1) representing 0, i.e. such that $c(q,\mathbf{m}) = 0$. A fraction $c(q,\mathbf{m})$ representing 0 is called a *loop*; the path $\mathbf{m}$ is then also called a loop. The loop $c(q,(0))$ is *trivial.*

EXAMPLES. 1. Let $q = 2/3$. Then one easily checks that the fraction $c(2/3,(1,-1,-3))$, which has $k = 2$, satisfies $c(2/3,(1,-1,-3)) = 0$. Moreover, using the definition (1.2) we see that $w_{2/3}((1,-1,-3)) = 1/3 \neq 1$.

Hence, in view of Theorem 1, there are no functions of degree 2 in $\mathcal{S}^\sharp$ with conductor $q = 2/3$.

2. Sometimes loops can be quite long. For $q = 7/2$ the sequence

$$\mathbf{m} = (2, -5, -1, 1, -1, 1, -1, 1, -1, 1, 2)$$

is a loop with $k = 10$ and $w_{7/2}(\mathbf{m}) = 8$. By solving the Diophantine equation

$$c(7/2, (m_0, \dots, m_k)) = 0$$

for $k < 10$ we can see that there are no shorter loops of weight $\neq 1$ for $q = 7/2$. As before, we conclude that there are no functions of degree 2 in $\mathcal{S}^\sharp$ with conductor $q = 7/2$.

3. Choose now $q = \sqrt{3}$ and $\mathbf{m} = (1, 1, 1, -1, 1)$. A simple computation shows that

$$c(\sqrt{3}, \mathbf{m}) = 0 \quad \text{and} \quad w_{\sqrt{3}}(\mathbf{m}) = \sqrt{3} + 2 \neq 1,$$

hence again there are no functions of degree 2 in $\mathcal{S}^\sharp$ with conductor $q = \sqrt{3}$.

4. Finally, let $q = 2$ and $\mathbf{m} = (1, -1, 1)$. In this case we have

$$c(2, \mathbf{m}) = 0 \quad \text{and} \quad w_2(\mathbf{m}) = 1.$$

The last equality is not surprising, as it is well known that there exist L-functions in $\mathcal{S}^\sharp$ of degree 2 and conductor $q = 2$ (see Lemma 6 with $m = 4$). In fact, Theorem 1 tells us that $w_2(\mathbf{m}) = 1$ not only for $\mathbf{m} = (1, -1, 1)$ but for *every* loop $\mathbf{m} = (m_0, \dots, m_k) \in \mathbb{Z}^{k+1}$. ∎

As is clear from the above examples, Theorem 1 and Lemma 4 enable one to prove non-existence of L-functions of degree 2 with a given conductor q by producing a proper loop $c(q, \mathbf{m})$ with weight $w_q(\mathbf{m}) \neq 1$. This problem is suitable for computations and, for example, in that way we obtain the following result (see Section 4).

COROLLARY 1. *There exist no L-functions of degree 2 in $\mathcal{S}^\sharp$ with conductor of the form*

$$q = \frac{a}{nb} \quad \textit{with} \quad (a, b) = 1,\ 2 \le b \le 300,\ n \ge 1$$

if at least one of the following conditions holds:

- $b \le 300$ *and* $a/b < 1$,
- $b \le 150$ *and* $a/b < 3/2$,
- $b \le 100$ *and* $a/b < 2$,
- $b \le 30$ *and* $a/b < 3$,
- $b \le 9$ *and* $a/b < 4$,
- $a \le 25$ *and* $a/b < 4$.

REMARK. Actually, our computations were performed for $q = a/b$ with a, b satisfying one of the conditions specified. The result in Corollary 1 follows from these computations observing that if q is a forbidden conductor then q/n is also forbidden for every integer $n \geq 1$. Suppose indeed that there exists $F \in \mathcal{S}^\sharp$ of degree 2 with conductor q/n. Then FG has degree 2 and conductor q for any $G \in \mathcal{S}^\sharp$ of degree 0 and conductor n, a contradiction since such functions G actually exist (see [3]). Alternatively, Proposition 1 in Section 3 shows the link between properties of fractions $c(q, \mathbf{m})$ and $c(q/n, \mathbf{m})$ without reference to L-functions. The second assertion follows from the same computations and properties of loops described in Section 4. ∎

Theorem 1 justifies a closer study of continued fractions of type (1.1). There are some natural questions to ask about them. For instance, we would like to know for which values of q the representation of a real number a in the form $a = c(q, \mathbf{m})$, with $c(q, \mathbf{m})$ proper, is unique. For such, our method of detecting forbidden conductors does not work, because, as Lemma 4 shows, the weight is necessarily unique. So an even more interesting problem is to find all q without the above uniqueness property. Among them, there are q's such that $w(q)$ is not unique, so Theorem 1 applies. Hence the basic open question in this direction is to describe the set of such q's explicitly.

THEOREM 2. *If $q > 0$ is transcendental or $q \geq 4$, then every real number a can be represented as $a = c(q, \mathbf{m})$ with a proper fraction $c(q, \mathbf{m})$ at most in one way. In particular, in this case the weight w_q is unique.*

Let $L(q)$ denote the set of loops for a given q.

COROLLARY 2. *If $q > 0$ and $L(q)$ contains a loop of odd length, then $1/q$ is an algebraic integer.*

The proofs of these results do not lie particularly deep. In particular, they do not depend on the theory of L-functions. Moreover, it shows that the problem of the uniqueness of the weight w_q is non-trivial for algebraic $q < 4$ only. The latter case is far more subtle. In contrast to the proof of Theorem 2, our proof of the following result heavily depends on L-functions, in particular on Theorem 1 and the Hecke theory of modular forms for the triangle groups $G(\lambda)$. In passing we remark that Hecke's theory shows the existence of L-functions of degree 2 for every conductor $q \geq 4$ (see Lemma 6). Thus, although our method cannot detect forbidden conductors among the values q in Theorem 2, actually there are no forbidden conductors $q \geq 4$.

THEOREM 3. *Let $q \in \mathbb{R}$ be a positive algebraic number. The weight w_q is unique in each of the following two cases:*

(i) *q has a Galois conjugate which is greater than or equal to 4; in other words, for a certain $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have $q^\sigma \geq 4$;*
(ii) *q is a totally positive algebraic integer.*

REMARKS. 1. As remarked before, our proof of Theorem 3 depends on L-functions, but its formulation does not. The case of $q^\sigma \geq 4$ also follows from Theorem 2. A natural problem is to give a proof of the second case, independent of the theory of L-functions.

2. From Theorem 3, we know that the weight w_q is unique for the pair $q_\pm = (3 \pm \sqrt{5})/2$ of Galois conjugate algebraic integers. From Lemma 6 applied with $m = 5$ we know that there exists an L-function of degree 2 in $\mathcal{S}^\sharp$ with conductor q_+. So, in that case, the uniqueness of w_{q_+} follows from Theorem 1. In contrast, no L-function $F \in \mathcal{S}^\sharp$ of degree 2 with $q_F = q_-$ is known at present, and it is not clear if it exists at all. Analyzing the proof of Theorem 1, we see that the uniqueness of w_q for $q = q_F$ follows from consistency conditions imposed by the basic transformation formula (see Lemma 2), and hence implicitly by the functional equation of F. Thus the uniqueness of w_{q_-} can be interpreted as the lack of obstacles for the existence of $F \in \mathcal{S}^\sharp$ of degree 2 with $q_F = q_-$.

3. All algebraic integers of the form

$$q = 4\cos^2(\pi\ell/m) \quad (m \geq 3,\ 1 \leq \ell < m,\ (\ell, m) = 1)$$

are totally positive. Thus for such q's the weight w_q is unique. In particular, this shows that the set of algebraic q's for which w_q is unique is dense in the interval $(0, 4)$. ∎

In the opposite direction we have the following theorem.

Theorem 4. *The weight w_q is not unique for*

$$q = \frac{4}{n}\cos^2(\pi\ell/(2k+1)), \qquad k \geq 1,\ 1 \leq \ell < 2k+1,\ (\ell, 2k+1) = 1,\ n \geq 2.$$

In particular, there are no functions of degree 2 in $\mathcal{S}^\sharp$ with such conductors.

We conclude with some open problems.

1. Construct an L-function $F \in \mathcal{S}^\sharp$ of degree 2 with conductor $q_F = (3 - \sqrt{5})/2$ or show that it does not exist. Show that there exists a real $q > 0$ such that $w(q)$ is unique but there is no $F \in \mathcal{S}^\sharp$ of degree 2 with conductor q.

2. Show that the set of q for which the weight w_q is not unique is also dense in the interval $(0, 4)$.

3. It follows from Theorems 2 and 3 that for every $q > 0$ there exists a positive integer n such that w_{nq} is unique. The last question is whether for every algebraic $q > 0$ there exists a positive integer n such that $w_{q/n}$ is not unique.

2. Definitions and basic requisites. Throughout the paper we write $s = \sigma + it$, and $\overline{f}(s)$ for $\overline{f(\overline{s})}$. The extended Selberg class $\mathcal{S}^\sharp$ consists of non-identically-vanishing Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

absolutely convergent for $\sigma > 1$, such that $(s-1)^m F(s)$ is entire of finite order for some integer $m \geq 0$, and satisfying a functional equation of the type

$$F(s)\gamma(s) = \omega\overline{\gamma}(1-s)\overline{F}(1-s),$$

where $|\omega| = 1$ and the γ-factor

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

has $Q > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re(\mu_j) \geq 0$. The Selberg class $\mathcal{S}$ is, roughly, the subclass of $\mathcal{S}^\sharp$ of the functions having, in addition, an Euler product representation and satisfying the Ramanujan conjecture. Note that the conjugate function $\overline{F}$ has conjugate coefficients $\overline{a(n)}$, and clearly $\overline{F} \in \mathcal{S}^\sharp$. We refer to the survey papers [2, 4, 10–13] for further definitions, examples and the basic theory of the classes $\mathcal{S}^\sharp$ and $\mathcal{S}$.

The degree d, the conductor q and the ξ-invariant ξ_F of $F \in \mathcal{S}^\sharp$ are defined as

$$d = 2\sum_{j=1}^{r} \lambda_j, \quad q = (2\pi)^d Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \quad \xi_F = 2\sum_{j=1}^{r} (\mu_j - 1/2) =: \eta_F + id\theta_F$$

with $\eta_F, \theta_F \in \mathbb{R}$. In this paper we deal mainly with functions in $\mathcal{S}^\sharp$ of degree $d = 2$; the subclass of such functions is denoted by $\mathcal{S}_2^\sharp$.

For $\sigma > 1$ and $F \in \mathcal{S}^\sharp$ with degree 2 and conductor q we consider the nonlinear twist

$$F(s,\alpha,\beta) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n - \beta\sqrt{n}), \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}$ and $e(x) = e^{2\pi i x}$. Note that, according to our notation above, we have

$$\overline{F}(s,\alpha,\beta) = \overline{F(\overline{s},\alpha,\beta)} = \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^s} e(\alpha n + \beta\sqrt{n}).$$

To avoid ambiguities, we also use the following notation when we consider a nonlinear twist of the conjugate function $\overline{F}$:

$$(\overline{F})(s,\alpha,\beta) = \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^s} e(-\alpha n - \beta\sqrt{n}).$$

Thanks to the periodicity of the complex exponential, for $\alpha \in \mathbb{Z}$, the twist in (2.1) reduces to the standard twist

$$F(s,\beta) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\beta\sqrt{n}),$$

and for $m \in \mathbb{Z}$, we have

$$F(s,\alpha+m,\beta) = F(s,\alpha,\beta). \tag{2.2}$$

Writing

$$n_\beta = q\beta^2/4 \quad \text{and} \quad a(n_\beta) = 0 \text{ if } n_\beta \notin \mathbb{N},$$

the *spectrum* of F is defined as

$$\begin{aligned} \text{(2.3)} \qquad \operatorname{Spec}(F) &:= \{\beta > 0 : a(n_\beta) \neq 0\} \\ &= \bigl\{2\sqrt{m/q} : m \in \mathbb{N} \text{ with } a(m) \neq 0\bigr\}. \end{aligned}$$

Moreover, for $\ell = 0, 1, \ldots$ we write

$$s_\ell = \frac{3}{4} - \frac{\ell}{2} \quad \text{and} \quad s_\ell^* = s_\ell - i\theta_F.$$

Lemma 1. *Let $\beta \neq 0$. Then the standard twist $F(s,\beta)$ is entire if $|\beta| \notin \operatorname{Spec}(F)$, while for $|\beta| \in \operatorname{Spec}(F)$ it is meromorphic on $\mathbb{C}$ with at most simple poles at the points s_ℓ^*. Moreover, when $|\beta| \in \operatorname{Spec}(F)$ the residue of $F(s,\beta)$ at $s = s_0^*$ does not vanish.*

We refer to [5, 7] for this and other results on the standard twist. Clearly

$$\operatorname{Spec}(\overline{F}) = \operatorname{Spec}(F),$$

and since $\theta_{\overline{F}} = -\theta_F$, the possible poles of $(\overline{F})(s,\beta)$ are at the points $\overline{s_\ell^*} = s_\ell + i\theta_F$, and $\overline{s_0^*}$ is again a simple pole.

Lemma 2. *Let $F \in \mathcal{S}^\sharp$ be of degree 2 and conductor q, and let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then*

$$\text{(2.4)} \qquad F(s,\alpha,\beta) = e^{as+b}\overline{F}\biggl(s + 2i\theta_F, \frac{1}{q\alpha}, -\frac{\beta}{\sqrt{q}\,\alpha}\biggr) + h(s)$$

with certain $a \in \mathbb{R}$ and $b \in \mathbb{C}$, where $h(s)$ is holomorphic for $\sigma > 1/2$.

Since the explicit values of a and b are not specified, this is a less precise form of [8, Lemma], in the case where F is suitably normalized. Moreover, Lemma 2 follows by similar but more straightforward arguments in the more general case, where θ_F is not necessarily vanishing. We will also need an analogous expression for negative values of the first parameter in $F(s,\alpha,\beta)$, thus for $\alpha > 0$, we consider the twist $F(s,-\alpha,\beta)$ and note that

$$F(s,-\alpha,\beta) = \overline{(\overline{F})(\overline{s},\alpha,-\beta)}.$$

Since the conductors of $\overline{F}$ and F are equal, from Lemma 2 we finally deduce that for $\alpha > 0$,

$$\text{(2.5)} \qquad F(s,-\alpha,\beta) = e^{as+b}(\overline{F})\biggl(s - 2i\theta_F, \frac{1}{q\alpha}, \frac{\beta}{\sqrt{q}\,\alpha}\biggr) + h(s)$$

with certain $a \in \mathbb{R}$ and $b \in \mathbb{C}$ and a function $h(s)$ holomorphic for $\sigma > 1/2$.

In the next section, we shall use an argument based on repeated applications of (2.4) and (2.5). Since what really matters in such an argument is only the value of $1/(q\alpha)$ and $|\beta/(\sqrt{q}\,\alpha)|$, to simplify notation we denote by

$$\widetilde{F}\Big(s \pm 2i\theta_F, \frac{1}{q\alpha}, \pm\Big|\frac{\beta}{\sqrt{q}\,\alpha}\Big|\Big) \tag{2.6}$$

the right hand side of both (2.4) and (2.5). Clearly, for $\sigma > 1/2$ the function in (2.6) has the same singularities as the functions $\overline{F}$ or $(\overline{F})$ on the right hand side of (2.4) or (2.5).

3. Proofs of the theorems

3.1. Some properties of fractions and weights. We start with some initial properties of the fractions and weights in (1.1) and (1.2), and we refer to Section 4 for a further discussion. Directly from the definitions, we see that

$$c(q,\mathbf{m}) = m_k + \frac{1}{qc(q,\mathbf{m}_{k-1})}, \qquad w_q(\mathbf{m}) = \sqrt{q}\,|c(q,\mathbf{m}_{k-1})|w_q(\mathbf{m}_{k-1}). \tag{3.1}$$

Moreover, it is easy to check that for $k \geq 2$, we also have

$$\begin{aligned} c(q,(m_0,\dots,m_{j-1},0,m_{j+1},\dots,m_k))& \\ = c(q,(m_0,\dots,m_{j-2},m_{j-1}+m_{j+1},m_{j+2},\dots,m_k))&. \end{aligned} \tag{3.2}$$

Thus, by repeated applications of (3.2) we can transform a path $\mathbf{m}$ to a proper path $\mathbf{m}^*$ in such a way that $c(q,\mathbf{m}) = c(q,\mathbf{m}^*)$. This process is called *zero-skipping*, and the notation $\mathbf{m}^*$ will also be used later on. For example,

$$c(q,(1,0,1)) = c(q,(2)) = 2 \quad \text{and} \quad c(q,(1,0,-1)) = c(q,(0)) = 0.$$

We also note that the zero-skipping process preserves the weight, namely $w_q(\mathbf{m}) = w_q(\mathbf{m}^*)$ if $\mathbf{m}$ and $\mathbf{m}^*$ are as above, since

$$\begin{aligned} q^{3/2}\Big|m_{j-1} + \frac{1}{0+\frac{q}{qm_{j+1}+\frac{q}{*}}}\Big|\,\Big|0+\frac{1}{qm_{j+1}+\frac{q}{*}}\Big|\,\Big|m_{j+1}+\frac{1}{*}\Big|& \\ = \sqrt{q}\,\Big|m_{j-1}+m_{j+1}+\frac{1}{*}\Big|&. \end{aligned}$$

Further, given two loops $\mathbf{m} = (m_0,\dots,m_k)$ and $\mathbf{n} = (n_0,\dots,n_\ell)$ we define the *composition* of $\mathbf{m}$ and $\mathbf{n}$ as

$$\mathbf{mn} = (m_0,\dots,m_k+n_0,\dots,n_\ell).$$

Note that $\mathbf{mn}$ is a path and moreover

$$c(q,(m_0,\dots,m_k+n_0,\dots,n_j)) = c(q,\mathbf{n_j}) \quad \text{for } j = 0,\dots,\ell. \tag{3.3}$$

Indeed, since $\mathbf{m}$ is a loop, we have $c(q,(m_0,\dots,m_k+n_0)) = c(q,(n_0))$, hence (3.3) follows. In particular, we have

$$c(q,\mathbf{mn}) = c(q,\mathbf{n}). \tag{3.4}$$

As a consequence,

(3.5) if $c(q,\mathbf{m})$ and $c(q,\mathbf{n})$ are loops then so is $c(q,\mathbf{mn})$.

Clearly, by zero-skipping, we may transform $c(q,\mathbf{mn})$ to a proper loop with the same weight. Finally, weight is multiplicative with respect to composition of loops: if $c(q,\mathbf{m})$ and $c(q,\mathbf{n})$ are loops, then

$$w_q(\mathbf{mn}) = w_q(\mathbf{m})w_q(\mathbf{n}). \tag{3.6}$$

Indeed, thanks to (1.2) and (3.3) we have

$$\begin{aligned} w_q(\mathbf{mn}) &= q^{(k+\ell)/2}\prod_{j=0}^{k-1}|c(q,\mathbf{m}_j)|\prod_{j=0}^{\ell-1}|c(q,(m_0,\dots,m_k+n_0,\dots,n_j))| \\ &= q^{k/2}\prod_{j=0}^{k-1}|c(q,\mathbf{m}_j)|q^{\ell/2}\prod_{j=0}^{\ell-1}|c(q,\mathbf{n}_j)| = w_q(\mathbf{m})w_q(\mathbf{n}). \end{aligned}$$

It can be shown that the proper loops form a group under composition (with zero-skipping) and thus w_q is a group homomorphism. The inverse of a loop $\mathbf{m}=(m_0,\dots,m_k)$ is $(-m_k,\dots,-m_0)$.

Lemma 3. *Let $q>0$. If two proper fractions satisfy $c(q,\mathbf{m})=c(q,\mathbf{n})$, then there is a proper loop $\mathbf{u}$ such that $\mathbf{m}=(\mathbf{un})^*$. The loop $\mathbf{u}$ is non-zero if and only if $\mathbf{m}\neq\mathbf{n}$.*

Proof. Let $\mathbf{m}=(m_0,\dots,m_k)$ and $\mathbf{n}=(n_0,\dots,n_\ell)$ be such that

$$c(q,\mathbf{m})=c(q,\mathbf{n}).$$

If $\mathbf{m}=\mathbf{n}$, the assertions are clear. Otherwise, $\mathbf{u}'=(m_0,\dots,m_k-n_\ell,-n_{\ell-1},\dots,-n_0)$ is a loop. Indeed,

$$c(q,(m_0,\dots,m_{k-1},m_k-n_\ell,-n_{\ell-1},\dots,-n_{j+1}))=c(q,\mathbf{n}_j),\quad j=\ell-1,\dots,0,$$

by induction, and finally $c(q,(m_0,\dots,m_{k-1},m_k-n_\ell,-n_{\ell-1},\dots,-n_0))=0$. Let $\mathbf{u}=\mathbf{u}'^*$, where $*$ denotes zero-skipping, and let j be the number of times (3.2) was applied in this operation. i.e. the largest integer such that

$$m_{k-i}=n_{\ell-i},\qquad i=0,\dots,j-1.$$

We have $j<\min(k+1,\ell+1)$, otherwise $\mathbf{u}$ would be a loop with the first or last entry zero, which is impossible. Hence

$$\begin{aligned} (\mathbf{un})^* &= ((m_0,\dots,m_{k-j-1},m_{k-j}-n_{\ell-j},-n_{\ell-j-1},\dots,-n_0)(n_0,\dots,n_\ell))^* \\ &= ((m_0,\dots,m_{k-j-1},m_{k-j},n_{\ell-j+1},\dots,n_\ell))^* \\ &= (\mathbf{m})^* = \mathbf{m}. \ \blacksquare \end{aligned}$$

Lemma 4. *Let $q>0$. The following statements are equivalent:*

(i) *the weight w_q is unique;*
(ii) *$w_q(\mathbf{m})=1$ for every proper loop $\mathbf{m}$;*
(iii) *$w_q(\mathbf{m})=w_q(\mathbf{n})$ for any non-trivial proper loops $\mathbf{m}$ and $\mathbf{n}$.*

Proof. The implication (i)⇒(ii) follows from the convention that $w_q((0)) = 1$ and $c(q,\mathbf{m}) = 0 = c(q,(0))$ for every loop $\mathbf{m}$. The implication (ii)⇒(iii) is trivial. Now we assume (iii) and prove first (ii) and then (i). Let $c(q,\mathbf{m})$ be a non-trivial proper loop. Consider the composition $\mathbf{mm}$ and recall that, thanks to (3.5), $c(q,\mathbf{mm})$ is also a loop. Therefore by (3.4) and (3.6) applied with $\mathbf{n} = \mathbf{m}$ we get

$$w_q(\mathbf{m}) = w_q(\mathbf{mm}) = w_q(\mathbf{m})^2,$$

thus $w_q(\mathbf{m}) = 1$. This implies (ii). The equality of fractions $c(q,\mathbf{m}) = c(q,\mathbf{n})$ implies $\mathbf{m} = \mathbf{un}$ for some proper loop $\mathbf{u}$, by Lemma 3. Hence

$$w_q(\mathbf{m}) = w_q(\mathbf{u})w_q(\mathbf{n}) = w_q(\mathbf{n})$$

by (3.6) and (ii). This proves (i). ∎

Recall that $L(q)$ denotes the set of loops for a given q.

Proposition 1. *Let $q > 0$. Let $\mathbf{m} = (m_0,\ldots,m_k) \in L(q)$ and r, r' be rational numbers with $rr' > 0$ such that*

$$m'_j = \begin{cases} rm_j, & j \equiv k \pmod 2,\\ r'm_j, & j \not\equiv k \pmod 2,\end{cases}$$

are all integers. Moreover, let $\mathbf{m}' = (m'_0,\ldots,m'_k)$. Then for $q' = \frac{q}{rr'}$ we have $\mathbf{m}' \in L(q')$ and

$$w_{q'}(\mathbf{m}') = \begin{cases} w_q(\mathbf{m}), & 2 \mid k,\\ \sqrt{r/r'}\, w_q(\mathbf{m}), & 2 \nmid k.\end{cases}$$

Proof. This follows from the relation

$$c(q',\mathbf{m}'_j) = \begin{cases} rc(q,\mathbf{m}_j), & j \equiv k \pmod 2,\\ r'c(q,\mathbf{m}_j), & j \not\equiv k \pmod 2,\end{cases}$$

for $j = 0,\ldots,k$. ∎

Corollary 3. *If $L(q)$ contains a loop $\mathbf{m}$ of odd length, then the weight $w_{q/n}$ is not unique for any integer $n \geq 2$.*

Proof. Let $q' = q/n$. By Proposition 1 there exist $\mathbf{m}', \mathbf{m}'' \in L(q')$ with

$$w_{q'}(\mathbf{m}') = \sqrt{n}\, w_q(\mathbf{m}) \quad \text{and} \quad w_{q'}(\mathbf{m}'') = \sqrt{1/n}\, w_q(\mathbf{m}),$$

so at least one of $w_{q'}(\mathbf{m}')$, $w_{q'}(\mathbf{m}'')$ is different from 1. The corollary follows from Lemma 4. ∎

3.2. Proof of Theorem 1. Theorem 1 is an immediate consequence of Lemma 4 and the following lemma.

Lemma 5. *Let $F \in \mathcal{S}^\sharp$ be of degree 2 and conductor q. Then $w_q(\mathbf{m}) = 1$ for every proper loop $c(q,\mathbf{m})$.*

Proof. Let $\beta \in \mathrm{Spec}(F)$; we use the notation in (1.1), (1.2) and (2.6). By repeated applications of (2.2), (2.4) and (2.5) we obtain

$$\begin{aligned}
F(s,\beta) &= F(s,m_0,\beta)\\
&= \widetilde{F}\left(s \pm 2i\theta_F, \frac{1}{qm_0}, \pm\frac{\beta}{\sqrt{q}\,|m_0|}\right)\\
&= \widetilde{F}\left(s \pm 2i\theta_F, m_1 + \frac{1}{qm_0}, \pm\frac{\beta}{\sqrt{q}\,|m_0|}\right)\\
&= \widetilde{F}\left(s + 2n_1 i\theta_F, \frac{1}{qm_1 + \frac{q}{qm_0}}, \pm\frac{\beta}{q\big|(m_1 + \frac{1}{qm_0})m_0\big|}\right)\\
&= \widetilde{F}\left(s + 2n_1 i\theta_F, m_2 + \frac{1}{qm_1 + \frac{q}{qm_0}}, \pm\frac{\beta}{q\big|(m_1 + \frac{1}{qm_0})m_0\big|}\right)\\
&= \cdots\\
&= \widetilde{F}\left(s + 2n_{k-1} i\theta_F, c(q,\mathbf{m}), \pm\frac{\beta}{w_q(\mathbf{m})}\right)
\end{aligned}$$

where $\mathbf{m} = (m_0,\dots,m_k)$ and the n_j's, $j = 1,\dots,k-1$, are certain integers.

If $c(q,\mathbf{m})$ is a non-trivial proper loop, then the above equation reduces, essentially, to the equality of two standard twists; more precisely, it becomes

$$F(s,\beta) = \widetilde{F}\left(s + 2n_{k-1} i\theta_F, \pm\frac{\beta}{w_q(\mathbf{m})}\right).$$

Since $\beta \in \mathrm{Spec}(F)$, both sides must have a simple pole at $s = s_0^*$ and hence by Lemma 1 we have $\beta/w_q(\mathbf{m}) \in \mathrm{Spec}(F)$ as well. Moreover, Lemma 1 implies that $s_0^* + 2n_{k-1} i\theta_F$ must be either s_0^* or $\overline{s_0^*}$. Thus the opposite implication holds as well, namely if $\beta/w_q(\mathbf{m}) \in \mathrm{Spec}(F)$ then $\beta \in \mathrm{Spec}(F)$. Therefore

$$\beta \in \mathrm{Spec}(F) \iff \beta/w_q(\mathbf{m}) \in \mathrm{Spec}(F).$$

This, in view of the shape of $\mathrm{Spec}(F)$ in (2.3), implies that $w_q(\mathbf{m}) = 1$, and the lemma follows. ■

3.3. Proof of Theorem 2 and its corollary. For a proper path $\mathbf{m} = (m_0,\dots,m_k)$ we define the rational function

$$R(x,\mathbf{m}) = m_k + \cfrac{1}{xm_{k-1} + \cfrac{x}{\ddots + \cfrac{x}{xm_0}}}. \tag{3.7}$$

Moreover, we define the polynomials $P_\ell(x,\mathbf{m})$ and $Q_\ell(x,\mathbf{m})$, $0 \le \ell \le k$, inductively as

$$P_0(x,\mathbf{m}) \equiv m_0, \qquad Q_0(x,\mathbf{m}) \equiv 1, \tag{3.8}$$

and for $1 \le \ell \le k$,

$$(3.9)\quad P_\ell(x,\mathbf{m}) = m_\ell x P_{\ell-1}(x,\mathbf{m}) + Q_{\ell-1}(x,\mathbf{m}), \quad Q_\ell(x,\mathbf{m}) = xP_{\ell-1}(x,\mathbf{m}).$$

Then we have

$$(3.10)\qquad R(x,\mathbf{m}) = \frac{P_k(x,\mathbf{m})}{Q_k(x,\mathbf{m})}.$$

By a trivial induction we show that

$$(3.11)\qquad \deg P_\ell = \deg Q_\ell, \quad 0 \le \ell < k.$$

Now we show that two rational functions of the above type, say $R(x,\mathbf{m})$ and $R(x,\mathbf{n})$ with $\mathbf{m} = (m_0,\dots,m_k)$ and $\mathbf{n} = (n_0,\dots,n_\ell)$, coincide if and only if $\mathbf{m} = \mathbf{n}$. Sufficiency is trivial, and so is necessity for $k = \ell = 0$. Suppose first that $k, \ell > 0$. Then

$$(3.12)\quad R(x,\mathbf{m}) = m_k + \frac{1}{xR(x,\mathbf{m}_{k-1})} \quad \text{and} \quad R(x,\mathbf{n}) = n_l + \frac{1}{xR(x,\mathbf{n}_{\ell-1})}.$$

By (3.11) we have $R(x,\mathbf{m}) \asymp 1$ and $R(x,\mathbf{n}) \asymp 1$ as $|x| \to \infty$. Thus (3.12) gives

$$m_k = n_\ell + O(1/|x|)$$

as $|x| \to \infty$ and hence $m_k = n_\ell$. Again by (3.12) we have $R(x,\mathbf{m}_{k-1}) = R(x,\mathbf{n}_{\ell-1})$, therefore by induction we conclude that $R(x,\mathbf{m}) = R(x,\mathbf{n})$ implies $\mathbf{m} = \mathbf{n}$. Finally, if $k > 0$ and $\ell = 0$ (or vice versa) then

$$R(x,\mathbf{m}) = m_k + \frac{1}{xR(x,\mathbf{m}_{k-1})} = n_0,$$

a contradiction proving our assertion in this case as well.

After this preparation we can conclude the proof. Suppose that a real number a has two different representations as a proper fraction with transcendental parameter q. Then

$$a = R(q,\mathbf{m}) = R(q,\mathbf{n})$$

for two different proper paths $\mathbf{m} = (m_0,\dots,m_k)$ and $\mathbf{n} = (n_0,\dots,n_\ell)$. Since the rational functions $R(x,\mathbf{m})$ and $R(x,\mathbf{n})$ are distinct, we deduce that the polynomial

$$H(x) := P_k(x,\mathbf{m})Q_l(x,\mathbf{n}) - P_\ell(x,\mathbf{n})Q_k(x,\mathbf{m}) \in \mathbb{Z}[x]$$

is not identically vanishing and moreover $H(q) = 0$. This is impossible if q is transcendental.

Suppose now $q \ge 4$ and suppose $\mathbf{m} = (m_0,\dots,m_k)$ is the shortest nonzero proper path such that $|c(q,\mathbf{m})| \le 1/2$. We have $|m_0| \ge 1$, so $k \ne 0$. Since $|c(q,\mathbf{m}_{k-1})| > 1/2$, we have

$$\left|\frac{1}{qc(q,\mathbf{m}_{k-1})}\right| < \frac{1}{2}.$$

From $m_k = c(q, \mathbf{m}) - \frac{1}{qc(q,\mathbf{m}_{k-1})}$ we obtain $|m_k| < 1$, contradicting $m_k \neq 0$. Therefore there is no non-zero proper path $\mathbf{m}$ such that $|c(q, \mathbf{m})| \leq 1/2$. In particular, there is no non-zero proper loop. The assertion follows from Lemma 3. ∎

To prove the corollary we note that, by Theorem 2, the number q is algebraic. Let

$$a_n x^n + \cdots + a_1 x + a_0$$

be its minimal polynomial, where $a_0, \ldots, a_n$ are integers with $\gcd(a_0, \ldots, a_n) = 1$. For $k = 2l + 1$ we have $P_{2l+1}(x, \mathbf{m}) = x^l(1 + xH(x))$, where $H(x)$ is some polynomial with integer coefficients depending on $\mathbf{m}$. If $L(q)$ contains a loop of length k, we have $P_{2l+1}(x, \mathbf{m}) = 0$. This implies that the minimal polynomial of q divides $1 + xH(x)$. Hence $|a_0| = 1$. ∎

3.4. Proof of Theorem 3. We need two further lemmas.

LEMMA 6. *Let $q \in \mathbb{R}$. Suppose that either $q = 4\cos^2(\pi/m)$ for some $m \geq 3$, or $q \geq 4$. Then there exists an L-function in $\mathcal{S}^\sharp$ with degree 2 and conductor q.*

Proof. From the classical Hecke theory we know that there are non-trivial automorphic forms f for the Hecke trangle group $G(\lambda)$ if $\lambda \geq 2$ or $\lambda = 2\cos(\pi/m)$ with integer $m \geq 3$ (see e.g. [1]). The corresponding normalized L-function L_f satisfies the functional equation

$$\left(\frac{\lambda}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L_f(s) = \omega \left(\frac{\lambda}{2\pi}\right)^{1-s} \Gamma\left(1 - s + \frac{k-1}{2}\right) L_f(1-s),$$

where $\omega = \pm 1$. We cannot claim that L_f belongs to $\mathcal{S}^\sharp$ because conjugation of $L_f(1-s)$ is missing in the above functional equation. This however can easily be repaired. Without loss of generality we may assume that $L_f(s)$ has at least one coefficient with non-zero real part, otherwise we consider $iL_f(s)$. Then $F(s) := L_f(s) + \overline{L_f}(s)$ has real coefficients, satisfies a functional equation of the right type and is not identically zero. Thus it belongs to $\mathcal{S}^\sharp$, has degree 2 and its conductor equals λ^2; therefore the lemma follows. ∎

LEMMA 7. *Let α be a totally positive algebraic integer with all conjugates smaller than 4. Then there exist positive integers m and ℓ satisfying $m \geq 3$, $1 \leq \ell < m$, $(\ell, m) = 1$ and $\alpha = 4\cos^2(\pi\ell/m)$.*

Proof. Let $\beta := \sqrt{\alpha}$. Then β is a totally real algebraic integer with all Galois conjugates smaller than 2 in absolute value. By the Kronecker theorem, $\beta = 2\cos(\pi\ell/m)$ for certain positive coprime integers ℓ and m (see [9, Theorem 2.5]). Thus $\alpha = 4\cos^2(\pi\ell/m)$. Since $0 < \alpha < 4$, we have $m \geq 3$. Moreover, by the periodicity of $\cos^2 x$, we can assume that $1 \leq \ell < m$, and the lemma follows. ∎

Let $q \in \overline{\mathbb{Q}}$ be a positive algebraic number and suppose $\mathbf{m} = (m_0, \ldots, m_k) \in \mathbb{Z}^{k+1}$ is a non-trivial loop for q, namely

$$m_k + \cfrac{1}{qm_{k-1} + \cfrac{q}{\ddots + \cfrac{q}{qm_0}}} = 0.$$

Then for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have

$$m_k + \cfrac{1}{q^\sigma m_{k-1} + \cfrac{q^\sigma}{\ddots + \cfrac{q^\sigma}{q^\sigma m_0}}} = 0$$

so that $\mathbf{m}$ is a loop for q^σ as well. Moreover, it is easy to check that $w_q(\mathbf{m}) = 1$ if and only if $w_{q^\sigma}(\mathbf{m}) = 1$. Hence we conclude that the weights w_q and w_{q^σ} are simultaneously unique or not.

We can now complete the proof of Theorem 3. If $q^\sigma \geq 4$ for a certain $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ then by Lemma 6 there exists an L-function in $\mathcal{S}_2^\sharp$ with conductor q^σ, and the weight w_{q^σ} is unique by Theorem 1. Consequently, w_q is unique as well, thus proving (i). To show (ii) we assume that q is a totally positive algebraic integer with all conjugates in the interval $(0, 4)$. By Lemma 7 this means that $q = 4\cos^2(\pi\ell/m)$ for certain $m \geq 3$ and $1 \leq \ell < m$, $(m, \ell) = 1$. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ map $\exp(2\pi i\ell/m)$ to $\exp(2\pi i/m)$. Then $q^\sigma = 4\cos^2(\pi/m)$. According to Lemma 6 there exists an L-function in $\mathcal{S}_2^\sharp$ with such a conductor, thus w_{q^σ} is unique according to Theorem 1. Consequently, w_q is unique as well, and the proof is complete. ∎

3.5. Proof of Theorem 4. We need the following explicit expression for the polynomials $P_\ell(x, \mathbf{m})$ defined in the proof of Theorem 2. The proof of this expression is by a straightforward induction, which we omit.

Lemma 8. *Let $k \geq 1$ and $\mathbf{m} = (m_0, \ldots, m_{2k})$. Then*

$$P_{2k-1}(x, \mathbf{m}) = x^{k-1} \sum_{j=0}^{k} \Big(\sum_{A \in I(2k-1, 2j-1)} \prod_{i \in A} m_i \Big) x^j,$$

$$P_{2k}(x, \mathbf{m}) = x^{k} \sum_{j=0}^{k} \Big(\sum_{A \in I(2k, 2j)} \prod_{i \in A} m_i \Big) x^j,$$

where $I(h, j)$ denotes the set of subsets $\{a_0, \ldots, a_j\}$ of $\{0, \ldots, h\}$ such that $a_0 < \cdots < a_j$ and $a_i \equiv i \pmod 2$ for every $i = 0, \ldots, j$.

Let k and ℓ be as in Theorem 4 and

$$q = 4\cos^2(\pi\ell/(2k+1)) = \left(e\left(\frac{\ell}{4k+2}\right) + e\left(-\frac{\ell}{4k+2}\right)\right)^2. \tag{3.13}$$

By Corollary 3 it suffices to show that $L(q)$ contains a loop of odd length. Let

$$\mathbf{m} = (m_0, \ldots, m_{2k-1}), \qquad m_j = (-1)^j,\ j = 0, \ldots, 2k-1.$$

By Lemma 8 we have

$$\begin{aligned}
&q^{-k+1}P_{2k-1}(q,\mathbf{m}) \\
&\qquad = \sum_{j=0}^{k}\Big(\sum_{A\in I(2k-1,2j-1)}\prod_{i\in A}(-1)^i\Big)\left(e\left(\frac{\ell}{4k+2}\right) + e\left(-\frac{\ell}{4k+2}\right)\right)^{2j} \\
&\qquad = \sum_{j=0}^{k}(-1)^j|I(2k-1,2j-1)|\sum_{m=-j}^{j}\binom{2j}{j+m}e\left(\frac{\ell m}{2k+1}\right) \\
&\qquad = \sum_{m=-k}^{k}e\left(\frac{\ell m}{2k+1}\right)\sum_{j=|m|}^{k}(-1)^j|I(2k-1,2j-1)|\binom{2j}{j+m}.
\end{aligned}$$

The subsets $\{a_0, \ldots, a_{2j-1}\} \in I(2k-1, 2j-1)$ correspond one-to-one to the subsets

$$\{b_0, \ldots, b_{2j-1}\} \subseteq \{0, \ldots, k+j-1\}$$

by the mapping $b_i = (a_i + i)/2$, so $|I(2k-1,2j-1)| = \binom{k+j}{2j}$. Hence from the identity

$$\sum_{i=0}^{m}(-1)^i\binom{n-i}{n-2i}\binom{n-2i}{m-i} = 1, \qquad n \ge 0,\ 0 \le 2m \le n,$$

which can be shown by induction, it follows that

$$\begin{aligned}
q^{-k+1}P_{2k-1}(q,\mathbf{m}) &= \sum_{m=-k}^{k}e\left(\frac{\ell m}{2k+1}\right)\sum_{j=|m|}^{k}(-1)^j\binom{k+j}{2j}\binom{2j}{j+m} \\
&= \sum_{m=-k}^{k}e\left(\frac{\ell m}{2k+1}\right)\sum_{j=0}^{k-|m|}(-1)^{k-j}\binom{2k-j}{2k-2j}\binom{2k-2j}{k-m-j} \\
&= (-1)^k\sum_{m=-k}^{k}e\left(\frac{\ell m}{2k+1}\right) = 0.
\end{aligned}$$

Thus for q as in (3.13) there exists an integer vector $\mathbf{m}$ of odd length $2k-1$ with $P_{2k-1}(q,\mathbf{m}) = 0$; however, $\mathbf{m}$ may not be a path for q.

Now let $\mathbf{n} = (n_0, \ldots, n_{2j-1}) \in \mathbb{Z}^{2j}$ be such that $P_{2j-1}(q,\mathbf{n}) = 0$ with the smallest possible j. In view of (3.7) and (3.10) (see also Proposition 2 in the next section), if $\mathbf{n}$ is not a path for q we have $P_i(q,\mathbf{n}) = 0$ for some $i < 2j-1$, and i is even by the minimality of j. It follows from (3.8) and (3.9) that $Q_i(q,\mathbf{n}) \neq 0$, and moreover $P_{i+1}(q,\mathbf{n}) = Q_i(q,\mathbf{n})$. Hence $i+1 < 2j-1$, so $i < 2j-3$. Further, we have

$$\begin{aligned} &Q_{i+1}(q,\mathbf{n}) = 0, \quad P_{i+2}(q,\mathbf{n}) = n_{i+2}qQ_i(q,\mathbf{n}) = cP_0(q,\mathbf{n}'), \\ &Q_{i+2}(q,\mathbf{n}) = qQ_i(q,\mathbf{n}) = cQ_0(q,\mathbf{n}'), \end{aligned}$$

where $\mathbf{n}' = (n_{i+2},\dots,n_{2j-1})$ and $c = qQ_i(q,\mathbf{n}) \neq 0$. Consequently, using (3.9) again, we have

$$P_{i+2+h}(q,\mathbf{n}) = cP_h(q,\mathbf{n}') \quad \text{for } h = 1,\dots,2j-i-3,$$

and in particular

$$P_{2j-1}(q,\mathbf{n}) = cP_{2j-i-3}(q,\mathbf{n}').$$

Hence $P_{2j-i-3}(q,\mathbf{n}') = 0$, contrary to the minimality of j. Therefore $\mathbf{n}$ is a path, and also a loop, of odd length; the theorem now follows. ∎

4. Computations. With the aid of machine computations we have been able to find loops of weight $\neq 1$ for rational $q = a/b$, $(a,b) = 1$, $0 < q < 4$, in each of the following cases:

- $a \leq 25$, arbitrary b;
- $b \leq 300$ and $q < 1$;
- $b \leq 150$ and $q < 3/2$;
- $b \leq 100$ and $q < 2$;
- $b \leq 30$ and $q < 3$;
- $b \leq 9$.

An excerpt from the results is shown in Tables 1 and 2. Complete results and the Python scripts needed to reproduce them are available online at https://maciejr.web.amu.edu.pl/computations/conductors.

In our computations we make use of some observations that we state here without complete proofs.

Proposition 2. *A vector $\mathbf{m} \in \mathbb{Z}^{k+1}$ is a path for a given q if and only if $P_l(q,\mathbf{m}) \neq 0$ for all $0 \leq l < k$. In that case $\mathbf{m}$ is a loop if and only if $P_k(q,\mathbf{m}) = 0$. Moreover, $w_q(\mathbf{m}) = |q^{-k/2}Q_k(q,\mathbf{m})|$.*

Corollary 4. *For $q > 0$ the set $L(q)$ contains a loop of length 1 if and only if $q = 1/b$ for some positive integer b. In that case the weight w_q is not unique unless $q = 1$.*

Proof. Loops of length 1 are solutions of $P_1(q,(m_0,m_1)) = 0$, i.e. $m_0m_1q + 1 = 0$, so q needs to be of the above form. In that case $(b,-1)$ is a loop of weight $\sqrt{b}$. ∎

Table 1. Examples of loops for $q = a/b$ with $b \le 4$

q	$\mathbf{m}$	$w_q(\mathbf{m})$
$\frac{1}{2}$	$(1, -2)$	$\sqrt{\frac{1}{2}}$
$\frac{3}{2}$	$(-1, 1, -2)$	$\frac{1}{2}$
$\frac{5}{2}$	$(-1, 1, -1, 1, 2)$	$\frac{1}{4}$
$\frac{7}{2}$	$(2, 1, -1, 1, -1, 1, -1, 1, -1, -5, 2)$	$\frac{1}{8}$
$\frac{1}{3}$	$(1, -3)$	$\sqrt{\frac{1}{3}}$
$\frac{2}{3}$	$(1, -1, -3)$	$\frac{1}{3}$
$\frac{4}{3}$	$(-1, 1, -3)$	$\frac{1}{3}$
$\frac{5}{3}$	$(-1, 1, -1, -1, -3)$	$\frac{1}{9}$
$\frac{7}{3}$	$(-1, 1, -1, 2, -1, -1, 3)$	$\frac{1}{27}$
$\frac{8}{3}$	$(1, -1, 1, -1, 6)$	$\frac{1}{9}$
$\frac{10}{3}$	$(2, -1, 1, -1, 1, -1, 1, -5, 15)$	$\frac{1}{81}$
$\frac{11}{3}$	$(-1, -1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -30, 1, -8)$	$\frac{1}{243}$
$\frac{1}{4}$	$(1, -4)$	$\sqrt{\frac{1}{4}}$
$\frac{3}{4}$	$(-1, 1, 4)$	$\frac{1}{4}$
$\frac{5}{4}$	$(-1, 1, -4)$	$\frac{1}{4}$
$\frac{7}{4}$	$(1, -1, 1, 2, -2)$	$\frac{1}{8}$
$\frac{9}{4}$	$(-1, 1, -1, 2, 2)$	$\frac{1}{8}$
$\frac{11}{4}$	$(-1, 1, -1, 1, -2, -1, 4)$	$\frac{1}{32}$
$\frac{13}{4}$	$(1, -1, 1, -1, 1, -1, 36)$	$\frac{1}{64}$
$\frac{15}{4}$	$(-2, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -6, 11, -1, 8, -1)$	$\frac{1}{4096}$

Corollary 5. *For $q > 0$ the set $L(q)$ contains a loop of length 2 if and only if $q = 1/u + 1/v$ for some non-zero integers u, v, with $u \neq -v$. In that case the weight w_q is not unique unless $q = 1$ or $q = 2$. In particular, the weight is not unique whenever $q = 2/b$ for some integer $b \geq 3$.*

Proof. Loops of length 2 are solutions of

$$P_2(q, (m_0, m_1, m_2)) = m_0 m_1 m_2 q^2 + (m_0 + m_2)q = 0,$$

which implies that q is of the required form. Conversely, for $q = 1/u + 1/v$ the sequence $\mathbf{m} = (u, -1, v)$ is always a loop and $w_q(\mathbf{m}) = |u/v|$, which is $\neq 1$ unless $v = u$. Suppose $v = u$, so $q = 2/u$. If $2 \mid u$, the assertion follows from Corollary 4. Otherwise, unless $q = 2$, we have $u = 2l + 1$ for some positive integer l and there is a loop $\mathbf{m}' = (1, -l, -2l - 1)$ of weight $\frac{1}{2l+1}$. ∎

Proposition 3. *Let $\mathbf{m} = (m_0, \ldots, m_k)$ be a path for some $q = a/b$, where a and b are coprime positive integers. Let*

$$u_j/v_j = ac(q, \mathbf{m}_j), \qquad j = 0, \ldots, k,$$

be reduced fractions, in particular $u_k = 0$ *and* $v_k = 1$ *if* $\mathbf{m} \in L(q)$. *Then for arbitrary* $\varepsilon_j = \pm 1$, $j = 0, \dots, k$, *and for every positive* b' *satisfying* $b' \equiv \varepsilon_j \varepsilon_{j+1} b \pmod{u_j}$, $j = 0, \dots, k-1$, *the sequence* $\mathbf{m}' = (m'_0, \dots, m'_k)$, *where* $m'_0 = \varepsilon_0 m_0$ *and*

$$m'_{j+1} = \varepsilon_{j+1} m_{j+1} + \frac{\varepsilon_{j+1} b - \varepsilon_j b'}{u_j} v_j, \qquad j = 0, \dots, k-1,$$

satisfies $c(q', \mathbf{m}') = c(q, \mathbf{m})$ *and* $w_{q'}(\mathbf{m}') = (b/b')^{k/2} w_q(\mathbf{m})$, *where* $q' = a/b'$.

Proof. It suffices to show that $ac(q', \mathbf{m}'_j) = \varepsilon_j u_j / v_j$ for $j = 0, \dots, k$. Indeed, we have $u_0/v_0 = am_0 = \varepsilon_0 a m'_0$ and

$$\frac{u_{j+1}}{v_{j+1}} = \frac{a m_{j+1} u_j + abv_j}{u_j} = \pm \frac{a m'_{j+1} \varepsilon_0 u_j + ab' v_j}{\varepsilon_0 u_j}$$

for $j = 0, \dots, k-1$. ∎

Corollary 6. *Let* $\mathbf{m} = (m_0, \dots, m_k)$ *and* $\mathbf{n} = (n_0, \dots, n_l)$ *be such that*

$$c(q, \mathbf{m}) = c(q, \mathbf{n}) \quad \text{for some } q = a/b, \text{ where } (a, b) = 1.$$

Let $u_j/v_j = ac(q, \mathbf{m}_j)$ *for* $j = 0, \dots, k-1$ *and* $x_j/y_j = ac(q, \mathbf{n}_j)$ *for* $j = 0, \dots, l-1$ *be reduced fractions and let* N *be a positive integer such that*

$$u_j \mid N, \ j = 0, \dots, k-1, \quad \text{and} \quad x_j \mid N, \ j = 0, \dots, l-1.$$

If $k \neq l$, *then* $w_{q'}$ *is non-unique for every* $q' = a/b'$ *such that*

$$b' \equiv \pm b \pmod{N}, \qquad b' \neq \left(\frac{w_q(\mathbf{m})}{w_q(\mathbf{n})}\right)^{2/(k-l)} b. \tag{4.1}$$

If $k = l$ *and* $w_q(\mathbf{m}) \neq w_q(\mathbf{n})$, *then* $w_{q'}$ *is non-unique for every* $q' = a/b'$ *such that*

$$b' \equiv \pm b \pmod{N}.$$

Table 2. Examples of families of loops of weight $\neq 1$ for $q' = a/b'$ where $b' \equiv \pm b$ mod N), $b \neq b''$, and $N = na$. For each family we list the data necessary to apply Corollary 6: a, the residue b, the modulus N, and the paths (denoted $\mathbf{m}$ and $\mathbf{n}$ in the corollary). The possible exceptional value of b' in (4.1) is denoted as b'' here.

a	b	N	b''	$\mathbf{m}$	$\mathbf{n}$
3	1	3	1	$(-1, 0, 2)$	(1)
4	1	4	1	$(-1, 0, 2)$	(1)
5	1	5	1	$(-1, 0, 2)$	(1)
5	2	10	2	$(-2, 0, 3)$	(1)
5	3	10	none	$(-1, 1, -1, -1, -3)$	(0)

It also follows from Proposition 1 with $r = r' = -1$ and the fact that w_q is a group homomorphism that $\mathbf{m} = (m_0, \dots, m_k)$ is a loop with weight $\neq 1$ if and only if $(-m_0, \dots, -m_k)$ and $(m_k, \dots, m_0)$ are such loops.

The computations for $q = a/b$ employed several methods, depending on the case being considered; below is the list of the methods. In the complete table available online, each example of loop is labelled with the number of the method by which it was obtained. This number also corresponds to the script number.

1. The case $q = 1/b$ was handled using Corollary 4.

2. For $a = 3, \dots, 25$ and b relatively prime to a in all possible congruence classes mod na, starting with $n = 1$, a search for paths satisfying the assumptions of Corollary 6, with $N = an$, was performed. (The cases $a = 1$ and $a = 2$ follow from Corollaries 4 and 5 respectively.) Examples of paths and congruence classes that we have found are shown in Table 2. Exceptions (asserted in the corollary) were noted and later handled by subsequent methods, with the results stored in the full version of Table 1 (online). Whenever appropriate paths were not found, classes modulo a higher modulus had to be considered, either by incrementing n, or by splitting the current class mod an to classes mod ann' with the smallest possible n'. The decision to increment n or split the class was based on how many residue classes mod an were already successfully handled. Covering the next case, $a = 26$, with the current algorithm would probably require around two months of machine time.

3. For a given $q = a/b$ all possible loops of a given length may be found by solving the diophantine equation $P_k(q, (m_0, \dots, m_k)) = 0$ in non-zero, integer m_j. This was mainly done recursively by

- finding an upper bound M for $\min |m_j|$,
- checking all possible cases of $i = 0, \dots, k$, $|m_i| = \min |m_j| \leq M$,
- substituting possible values $|m_i| \leq M$ and then solving each case.

For example, for $q = 26/23$ and length 6 the equation to solve is

$$\begin{aligned}
&2^3 \cdot 13^3 m_0m_1m_2m_3m_4m_5m_6 + 2^2 \cdot 13^2 \cdot 23(m_0m_1m_2m_3m_4 + m_0m_1m_2m_3m_6 \\
&\quad + m_0m_1m_2m_5m_6 + m_0m_1m_4m_5m_6 + m_0m_3m_4m_5m_6 + m_2m_3m_4m_5m_6) \\
&\quad + 2 \cdot 13 \cdot 23^2(m_0m_1m_2 + m_0m_1m_4 + m_0m_1m_6 + m_0m_3m_4 + m_0m_3m_6 \\
&\quad + m_0m_5m_6 + m_2m_3m_4 + m_2m_3m_6 + m_2m_5m_6 + m_4m_5m_6) \\
&\quad + 23^3(m_0 + m_2 + m_4 + m_6) = 0.
\end{aligned}$$

This implies $\min |m_j| \leq 2$, which reduces to $m_j \in \{1, 2\}$ for at least one $j \in \{0, 1, 2, 3\}$. Each of these 8 substitutions produces an equation in 6 variables (of degree 6) which can be solved by applying the same method recursively. Ultimately we reach 3384779 subcases involving a quadratic equation in two variables, for which a dedicated algorithm was used. Searching for longer loops requires solving much more complex equations, e.g., those corresponding to length 10 would not fit on one page.

The computational complexity of this method is hard to estimate. Roughly, it behaves like $(1+q^{-1})^{k!}$, but it is much less regular. The unique feature of this method is that it also allows us to prove that loops of a given length do not exist, and thus that a loop of the next possible length (e.g., found with another method) has the smallest possible length.

This method was applied for $3 \le a \le b \le 300$ and loops of length 2, 4 and, sometimes, longer loops. Results for a given q also showed the existence of loops of weight $\neq 1$ for other q, in accordance with Propositions 1 and 3, allowing us to avoid the direct application of the method for many of the smaller q, where the computation time would be particularly long.

4. Where previous methods were unsuccessful, loops of weight $\neq 1$ were found by examining sequences constructed using the following simple heuristic:

- start with $m_0 = 1$ or $m_0 = b$;
- given $m_0, \dots, m_k$, consider several possible values of m_{k+1} such that
$$|c(q,(m_0,\dots,m_{k+1}))| < C$$
for some fixed $C > 0$;
- if more than N paths were generated (where N is around 10^7) discard those with largest numerators.

Direct application of method 3 allows us to exclude the existence of loops of a given length and weight $\neq 1$ for a given q. This way we were able to check that many of the results in the full table available online are optimal, in the sense that there are no shorter loops of weight $\neq 1$ for such q.

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