Abstract. The aim of this article is to study the $L^p$-boundedness of pseudo-differential operators on a homogeneous tree $\mathcal{X}$. For $p \in (1, 2)$, we establish a connection between the $L^p$-boundedness of the pseudo-differential operators on $\mathcal{X}$ and that on the group of integers $\mathbb{Z}$. We also prove an analogue of the Calderón–Vaillancourt theorem in the setting of homogeneous trees for $p \in (1, \infty) \setminus \{2\}$.

1. Introduction. The systematic study of pseudo-differential operators has drawn lots of motivation from partial differential equations, quantum mechanics and signal analysis. Indeed, the pioneering works on this subject in the 1960s, as explored for example by Hörmander [H65, H66] and Kohn–Nirenberg [KN65], were guided by a deep connection with elliptic and hypoelliptic equations. The boundedness results on the classical spaces of harmonic analysis play a special role due to their implications in the regularity of the solutions for appropriate equations. Recently, the calculus of pseudo-differential operators on discrete spaces has gained popularity (see for e.g. [BGR20, L14]) due to their natural connection with various problems in quantum ergodicity and in the discretization of continuous problems.

The aim of this article is to study the theory of pseudo-differential operators on homogeneous trees. A homogeneous tree $\mathcal{X}$ of degree $q + 1$ is a connected graph with no loops, in which every vertex is connected to $q + 1$ other vertices. Let us first discuss the theory of pseudo-differential operators on homogeneous trees of degree 2, that is, the group of integers $\mathbb{Z}$. Given a bounded measurable function $\psi$ on $\mathbb{Z} \times \mathbb{T}$, one considers the pseudo-
differential operator $T_\psi$ defined by

$$T_\psi f(l) = \frac{1}{\tau} \int_\mathbb{T} \psi(l, s) \mathcal{F}f(s) q^{ils} \, ds \quad \text{for all } l \in \mathbb{Z},$$

for a finitely supported function $f$ on $\mathbb{Z}$, where $\tau = \frac{2\pi}{\log q}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathcal{F}f$ denotes the Fourier transform of $f$ defined by

$$\mathcal{F}f(s) = \sum_{d \in \mathbb{Z}} f(d) q^{-ids} \quad \text{for all } s \in \mathbb{T}.$$

Such a function $\psi$ is said to be the symbol of the pseudo-differential operator $T_\psi$. Alternatively, using (1.2) one can also write (1.1) as follows:

$$T_\psi f(l) = \sum_{d \in \mathbb{Z}} f(d) \kappa(l, l - d),$$

where

$$\kappa(l, l - d) = \frac{1}{\tau} \int_\mathbb{T} \psi(l, s) q^{is(l-d)}. $$

If $T_\psi$ is a bounded operator from $L^p(\mathbb{Z})$ to itself, we will denote its operator norm by $\|\psi(\cdot, \cdot)\|_p$; it is defined as

$$\|\psi(\cdot, \cdot)\|_p = \sup_{\|f\|_{L^p(\mathbb{Z})}=1} \|T_\psi f\|_{L^p(\mathbb{Z})}.$$}

When $\psi$ is independent of the space variable, we get the Fourier multiplier operator. We shall denote such a Fourier multiplier operator on $\mathbb{Z}$ by $T_m$; it is defined by

$$T_m f(l) = \frac{1}{\tau} \int_\mathbb{T} m(s) \mathcal{F}f(s) q^{ils} \, ds.$$

Molahajloo [M10] studied the pseudo-differential operators on $\mathbb{Z}$; among other interesting results, he established sufficient conditions on the Fourier transform of the symbol with respect to the variable on $\mathbb{T}$ to ensure the $L^p$-boundedness. For higher-dimensional analogues of the above result and a comprehensive pseudo-differential calculus on $\mathbb{Z}^n$, we refer to [BGR20, C11] and the references therein. On the other hand, the authors of [RR04] studied pseudo-differential operators on $\mathbb{Z}^n$ with symbols which are bounded on $\mathbb{Z}^n \times \mathbb{T}^n$ together with their derivatives with respect to the second variable. More precisely, they proved the following analogue of the Calderón–Vaillancourt theorem on $\mathbb{Z}$ (see [RR04, Theorem 2.8]):

$$\|\psi(\cdot, \cdot)\|_p \leq C \sup_{(l,s) \in \mathbb{Z} \times \mathbb{T}} \left| \frac{d^k}{ds^k} \psi(l, s) \right| \quad \text{for all } 1 \leq p \leq \infty.$$
involves polynomial growth, and that of others, which involve exponential volume growth. We begin by discussing the case of multipliers on $\mathcal{X}$. For unexplained notations, we refer to Section 2. Corresponding to a bounded measurable function $m$ on $\mathbb{T}$, the associated Fourier multiplier operator on $\mathcal{X}$ is defined by

\begin{equation}
T_m f(h) = c_G \int_{\mathbb{T}} \int_{\Omega} m(s) \tilde{f}(s, \omega) p^{1/2 - is} (h, \omega) |c(s)|^{-2} d\nu(\omega) ds,
\end{equation}

where $c_G = q \log q/(4\pi(q + 1))$, $p$ denotes the Poisson kernel, which is precisely defined in formula (2.6), and $c(\cdot)$ is given by (2.8).

The $L^p$-boundedness of the multiplier operator on homogeneous trees has been studied by many authors, see for instance [CMW19, CMS98, CMS99]. A celebrated result of Clerc and Stein [CS74] in the context of homogeneous trees states that if $T_m$ is a bounded operator on $L^p(\mathcal{X})$, then $m$ necessarily extends to a bounded holomorphic function on the interior of the strip $S_p$ (which will be denoted by $S_p^\circ$), where

\begin{equation}
S_p = \{ z \in \mathbb{C} : |\Im z| \leq \delta_p \} \quad \text{and} \quad \delta_p = \left| \frac{1}{p} - \frac{1}{2} \right| \quad \text{for} \quad p \in [1, \infty).
\end{equation}

This necessary condition was sharpened by Cowling, Meda and Setti [CMS99, Theorem 2.1], who proved that if $T_m$ is a bounded operator on $L^p(\mathcal{X})$, then the function $s \mapsto m(s + i\delta_p)$ defines a bounded multiplier operator of the form (1.4) on $L^p(\mathbb{Z})$. They also proved the converse by using some additional condition on the multiplier $m$. However, the most important result in this direction, which is relevant for us, is the following one by Celotto, Meda and Wróbel:

**Theorem 1.1** ([CMW19, p. 176]). For $p \in [1, \infty) \setminus \{2\}$, the following are equivalent:

1. The operator $T_m$ defined by (1.6) is bounded on $L^p(\mathcal{X})$.
2. The multiplier $m$ extends to a Weyl-invariant function on the strip $S_p^\circ$, and the function $s \mapsto m(s + i\delta_p)$ defines a bounded multiplier operator of the form (1.4) on $L^p(\mathbb{Z})$.

Here and subsequently, a function $\eta$ defined on $S_p^\circ$ is said to be Weyl-invariant if $\eta$ is holomorphic on $S_p^\circ$, $\eta(z) = \eta(-z)$ and $\eta(z) = \eta(z + \tau)$ for all $z$ in $S_p^\circ$.

Now that the characterization of $L^p$ Fourier multipliers is completely settled, it is natural to inquire: what happens if one replaces the multiplier $m(s)$ by a symbol $\Psi(x, s)$? We find some natural conditions on the symbol of a pseudo-differential operator on $\mathcal{X}$ which imply the continuity of the corresponding operator. Explicitly, we contribute in two ways:
Firstly, we establish a connection between the \( L^p \)-boundedness of pseudo-differential operators on \( \mathfrak{X} \) and on the group of integers \( \mathbb{Z} \).

Secondly, we provide a sufficient condition on the symbol \( \Psi \) which guarantees the \( L^p \)-boundedness of the associated pseudo-differential operator on homogeneous trees.

Let \( \Psi : \mathfrak{X} \times \mathbb{T} \to \mathbb{C} \) be a measurable function. We define the pseudo-differential operator associated with the symbol \( \Psi \) (via the inversion formula, see Theorem 2.2) by

\[
T_\Psi f(h) = c_G \int \int_{\mathfrak{T} \Omega} \Psi(h, s) \widetilde{f}(s, \omega) p^{1/2 - is(h, \omega)} |c(s)|^{-2} d\nu(\omega) ds.
\]

In what follows, \( N \) denotes the group defined in formula (2.4) below, and \( \sigma \) is a translation of step 1 along a fixed doubly infinite chain \( \omega_0 \) (see Section 2 for details). Given a function \( F \) on \( \mathfrak{X} \times \mathbb{R} \), for every \( v \in [-\delta_p, \delta_p] \), we denote by \( F_v(\cdot, \cdot) \) the function on \( \mathfrak{X} \times \mathbb{R} \) defined by \( F_v(x, s) = F(x, s + iv) \).

Then our main result for \( 1 < p < 2 \) is the following:

**Theorem 1.2.** Let \( 1 < p < 2 \). Suppose that \( \Psi : \mathfrak{X} \times S_p \to \mathbb{C} \) is a function satisfying the following properties:

1. For each \( x \) in \( \mathfrak{X} \), \( z \mapsto \Psi(x, z) \) is Weyl-invariant on the strip \( S_p^0 \) and continuous on \( S_p \).
2. The symbol \( \Psi \) is uniformly bounded on \( \mathfrak{X} \times S_p \), that is,
   \[
   \sup_{x \in \mathfrak{X}, z \in S_p} |\Psi(x, z)| < \infty.
   \]
3. For each \( n \) in \( N \), \( (l, s) \mapsto \Psi(n\sigma^l, s - i\delta_p) \) defines a bounded pseudo-differential operator from \( L^p(\mathbb{Z}) \) to itself and
   \[
   \sup_{n \in N} \|\Psi_{-\delta_p}(n\cdot, \cdot)\|_p < \infty.
   \]

Then \( T_\Psi \) defines a bounded operator from \( L^p(\mathfrak{X}) \) to itself. Moreover, there exists a constant \( C_p > 0 \) such that

\[
\|T_\Psi f\|_{L^p(\mathfrak{X})} \leq C_p \left( \|\Psi\|_{L^\infty(\mathfrak{X} \times S_p)} + \sup_{n \in N} \|\Psi_{-\delta_p}(n\cdot, \cdot)\|_p \right) \|f\|_{L^p(\mathfrak{X})}.
\]

We next turn to the case \( 2 < p < \infty \). The \( L^p \)-boundedness of the multiplier operator (1.6) (for \( p > 2 \)) follows from a straightforward duality argument, using the fact that the kernel of the operator is radial. However, the duality argument does not work for pseudo-differential operators. We overcome this obstacle by imposing a rather stronger condition on the symbol \( \Psi \). More precisely, we prove the following analogue of the Calderón–Vaillancourt theorem [RR04, Theorem 2.8] in the setting of homogeneous trees.

**Theorem 1.3.** Let \( 2 < p < \infty \). Suppose that \( \Psi : \mathfrak{X} \times S_p \to \mathbb{C} \) is a function satisfying the following properties:
(1) For each $x$ in $\mathcal{X}$, $z \mapsto \Psi(x, z)$ is Weyl-invariant on the strip $S_p^\circ$.  

(2) The functions $d^k\Psi/dz^k$ extend continuously on $S_p$ and satisfy the differential inequalities

\begin{equation}
\left| \frac{d^k}{dz^k}\Psi(x, z) \right| \leq C_k \quad \text{for all } k = 0, 1, 2 \text{ and for all } x \in \mathcal{X}, \ z \in S_p.
\end{equation}

Then $T_{\Psi}$ defines a bounded operator from $L^p(\mathcal{X})$ to itself. Moreover, there exists a constant $C_p > 0$ such that

$$
\| T_{\Psi}f \|_{L^p(\mathcal{X})} \leq C_p \left( \sup_{(x, z) \in \mathcal{X} \times S_p, k=0,1,2} \left| \frac{d^k}{dz^k}\Psi(x, z) \right| \right) \| f \|_{L^p(\mathcal{X})}
$$

for all $f \in L^p(\mathcal{X})$.

Remark 1.4. (1) When $\Psi$ is a multiplier, one can write $T_{\Psi}$ as a convolution operator with a $K$-biinvariant kernel. This plays a crucial role in the multiplier case. However, in the case of pseudo-differential operators, one cannot use this method due to the extra variable “$x$” in the symbol $\Psi(x, s)$. This is the fundamental difference between the multiplier case and our situation.

(2) In comparison to the results for the lattice $\mathbb{Z}$, the holomorphic extension property of the symbol $\Psi$ is a new and necessary condition for the pseudo-differential operator $T_{\Psi}$ to be bounded on $L^p(\mathcal{X})$, as in the case of multipliers on $\mathcal{X}$.

(3) Next, we compare Theorems 1.2 and 1.3 with the corresponding results on rank 1 symmetric spaces of non-compact type. In [PR22, Theorem 1.6], the authors proved the $L^p$-boundedness (for $p \in (1, \infty) \setminus \{2\}$) of pseudo-differential operators on symmetric spaces by assuming (among other things) that the mixed partial derivatives of the corresponding symbols with respect to the spatial and the frequency variables up to a prescribed order (depending on the dimension of the symmetric space) have polynomial decay in terms of the frequency variable. In our case, due to the discrete nature of the spatial variable (that is, $\mathcal{X}$) and the compactness of the torus $\mathbb{T}$ (which is the frequency variable), one immediately sees that the corresponding decay condition on the symbols boils down to hypothesis (2) of Theorem 1.3 (see expression (1.9)). Consequently, Theorem 1.3 on homogeneous trees is a discrete analogue of [PR22, Theorem 1.6] on symmetric spaces. However, if $\Psi : \mathcal{X} \times \mathbb{T} \to \mathbb{C}$ is a measurable function satisfying the hypothesis of Theorem 1.3 for $1 < p < 2$, then by using (1.5), it follows that $\Psi$ satisfies the hypothesis of Theorem 1.2 and hence $T_{\Psi}$ is a bounded operator on $L^p(\mathcal{X})$. In this sense, for $p \in (1, 2)$, Theorem 1.2 on $\mathcal{X}$ can be seen as an improvement of [PR22, Theorem 1.6] on symmetric spaces.

(4) The novelty of Theorem 1.2 lies in the fact that the hypothesis of Theorem 1.2 covers a wider range of symbols in comparison to that of Theorem 1.3 on homogeneous trees.
This can be easily seen by considering the following multiplier (and hence symbol) on \( X \): For \( 1 < p < 2 \), let us define

\[
m(z) = (\gamma(z) - \gamma(i\delta_p))^\alpha \quad \text{for all } z \in S_p,
\]

where

\[
\gamma(z) = 1 - \frac{q^{1/2+iz} + q^{1/2-iz}}{q+1}
\]

and \( 1 < \alpha < 2 \) is fixed. Observe that \( s \mapsto m(s - i\delta_p) \) is a continuously differentiable function on \( \mathbb{T} \) and hence its inverse Fourier transform in \( \mathbb{Z} \) is absolutely summable. Consequently, \( m \) satisfies all the hypotheses of Theorem \ref{thm:1.2}, and so the associated pseudo-differential operator \( T_m \) is bounded on \( L^p(X) \) for \( 1 < p < 2 \). However, the second derivative of \( z \mapsto m(z) \) does not exist at \( z = i\delta_p \) and hence the \( L^p \)-boundedness (when \( 2 < p < \infty \)) of the operator \( T_m \) does not follow from Theorem \ref{thm:1.3}.

(5) In \cite{L14}, Le Masson introduced pseudo-differential operators on \( X \) associated with a more general symbol class and proved their \( L^2 \)-boundedness. The key ingredient of the proof is that the kernels of the operators have a rapid decay property. The proof of that property is essentially a paraphrase of the \( L^2 \)-Schwartz space isomorphism theorem given in \cite[Theorem 2]{CS99}. We remark that such an isomorphism theorem is not known for \( p \neq 2 \).

The article is organized as follows: Section 2 sets the necessary background from harmonic analysis on homogeneous trees. In Section 3, we express \( T_\Psi \) as an integral operator and prove Theorem \ref{thm:1.2}. Finally, in Section 4 we prove Theorem \ref{thm:1.3}.

2. Notation and preliminaries

2.1. Generalities. We denote the set of all non-negative integers by \( \mathbb{Z}_+ \), and \( \mathbb{N} = \{1, 2, \ldots\} \). For \( z \in \mathbb{C} \), we use the notations \( \Re z \) and \( \Im z \) for the real and imaginary parts of \( z \) respectively. We shall follow the standard practice of using the letters \( C, C_1, C_2 \) etc. for positive constants whose value may change from one line to another. Occasionally the constants will be suffixed to show their dependence on important parameters. For any Lebesgue exponent \( p \in (1, \infty) \), let \( p' \) denote the conjugate exponent \( p/(p - 1) \). From \eqref{eq:1.7} it is evident that \( \delta_p = \delta_{p'} \) and \( S_p = S_{p'} \) for all \( p \in (1, \infty) \). We shall henceforth write \( S^c_p \) and \( \partial S_p \) to denote the usual topological interior and the boundary of \( S_p \) respectively.

2.2. Homogeneous trees. Here we review some general facts about homogeneous trees, most of which are already known. Details regarding harmonic analysis on trees can be found in the books \cite{FNP91, FP83}.

As described earlier, a homogeneous tree \( \mathcal{X} \) of degree \( q + 1 \) is a connected and acyclic graph in which every vertex has \( q + 1 \) neighbours. We identify \( \mathcal{X} \)
with the set of all vertices where the natural distance \( d(x, y) \) between any two vertices \( x \) and \( y \) is defined as the number of edges between them. Let \( o \) be a fixed but arbitrary reference point in \( \mathcal{X} \). We shall write \( |x| \) for \( d(o, x) \). The tree \( \mathcal{X} \) being discrete is naturally equipped with the counting measure. Let \( G \) be the group of isometries of the metric space \((\mathcal{X}, d)\) and \( K \) be the stabilizer of \( o \) in \( G \). It is known that \( K \) is a maximal compact subgroup of \( G \).

The map \( g \mapsto g \cdot o \) identifies \( \mathcal{X} \) with the coset space \( G/K \), so that functions on \( \mathcal{X} \) correspond to \( K \)-right invariant functions on \( G \). Furthermore, radial functions on \( \mathcal{X} \), that is, functions which only depend on \( |x| \), correspond to \( K \)-biinvariant functions on \( G \).

\[ \text{2.3. The height function and the boundary of } \mathcal{X}. \] An infinite geodesic ray \( \omega \) is a one-sided sequence \( \{\omega_j : j = 0, 1, \ldots\} \) where the \( \omega_j \) are in \( \mathcal{X} \) and \( d(\omega_i, \omega_j) = |i - j| \) for all non-negative integers \( i \) and \( j \). We say that an infinite geodesic \( \omega \) starts from \( x \) if \( \omega_0 = x \). For a given infinite geodesic ray \( \omega \), we define the associated height function \( h_\omega : \mathcal{X} \to \mathbb{Z} \) by
\[ h_\omega(x) = \lim_{j \to \infty} (j - d(x, \omega_j)). \]

For details, we refer to [CMS98]. The height function \( h_\omega \) is the discrete analogue of the Busemann function in Riemannian geometry. We note that for all \( x \) and \( \omega \), the sequence in (2.1) is eventually constant and hence the limit exists. Furthermore, for every \( m \in \mathbb{Z} \), we define the \( \omega \)-horocycles
\[ \mathcal{H}(\omega, m) = \{ x \in \mathcal{X} : h_\omega(x) = m \}, \]
and see that \( \mathcal{X} \) decomposes into the disjoint union
\[ \mathcal{X} = \bigcup_{m \in \mathbb{Z}} \mathcal{H}(\omega, m). \]

Two infinite geodesics \( \omega = \{\omega_j : j = 0, 1, \ldots\} \) and \( \omega' = \{\omega'_j : j = 0, 1, \ldots\} \) are said to be equivalent if they meet at infinity, that is, there exist \( k, N \in \mathbb{N} \) such that for all \( j \geq N \), \( \omega_j = \omega'_{j+k} \). This identification is an equivalence relation and partitions the set of all infinite geodesics into equivalence classes. The equivalence class of a generic geodesic ray \( \omega \) will henceforth be denoted by \([\omega]\). In every equivalence class, there exists a unique geodesic ray starting from \( o \). The boundary \( \Omega \) is defined as the set of all infinite geodesic rays starting from \( o \). It is known that \( K \) acts transitively on \( \Omega \) via the map \( k \mapsto k \cdot \omega \).

\[ \text{2.4. The groups } A \text{ and } N. \] From now on, we fix a doubly infinite geodesic \( \omega_0 \) of the form \( \{\omega^0_j : j \in \mathbb{Z}\} \) such that \( \omega^0_0 = o \) and \( d(\omega^0_i, \omega^0_j) = |i - j| \) for all integers \( i \) and \( j \). Furthermore, we will denote by \( \omega^+_0 \) the infinite geodesic \( \{\omega^+_0 : j = 0, 1, \ldots\} \). Let \( \sigma \) in \( G \) be a translation of step 1 along the doubly infinite chain \( \omega_0 \). More explicitly, let \( \sigma(\omega^0_j) = \omega^0_{j+1} \) for all \( j \) in \( \mathbb{Z} \). We
denote by $A$ the subgroup generated by $\sigma$, that is,

$$A = \{\sigma^j : j \in \mathbb{Z}\}.$$  

It is important to note that $A$ is abelian and isomorphic to the group $\mathbb{Z}$. Let $G_{\omega_0^+}$ be the stabilizer of the equivalence class $[\omega_0^+]$ in $G$. We define the group $N$ by

\begin{equation}
N = \{n \in G_{\omega_0^+} : n \cdot x = x \text{ for some } x \in \mathfrak{X}\}.
\end{equation}

It is well-known that $N$ is locally compact. We shall denote the Haar measure on $N$ by $\mu$. From [V02, Lemma 3.3] it is evident that $N$ is unimodular and the measure $\mu$ is normalized by the condition $\mu(N \cap K) = 1$. Moreover, it was proved in [V02, Corollary 3.2] that the subgroup $N$ acts transitively on every $\omega_0^+$-horocycle $\mathfrak{H}(\omega_0^+, m)$ defined by (2.2). By using this fact together with the horocyclic decomposition (2.3), Veca [V02, Theorem 3.5] proved the following Iwasawa-type decomposition of the group $G$ and arrived at the following integral formula for functions on $G$.

**Theorem 2.1.** Let $G$, $N$, $K$ and $A$ be as defined above. Then for every $g \in G$, there exist $n \in N$, $j \in \mathbb{Z}$ and $k \in K$ such that $g = n\sigma^j k$. Furthermore, if $f$ is a compactly supported function defined on $G$, then

$$\int_G f(g) \, dg = \int_N \sum_{j \in \mathbb{Z}} \int_K f(n\sigma^j k)q^{-j} \, dk \, d\mu(n).$$

It is also known that the subgroup $A$ acts on $N$ by conjugation (see [V02, Lemma 3.8]). Moreover, for a compactly supported function $f$ on $N$, we have

\begin{equation}
\int_N f(\sigma^j n\sigma^{-j}) \, d\mu(n) = q^{-j} \int_N f(n) \, d\mu(n).
\end{equation}

We endow the set $NA$ with the binary operation induced by this conjugation and refer to it as the semidirect product $NA$. By Theorem 2.1 we may identify $K$-right invariant functions on $G$ with functions on $NA$. In fact, the corresponding $L^p$-norms coincide. For the sake of simplicity, we denote the Haar measure $d\mu(n)$ by $dn$.

**2.5. The spherical functions and Fourier transform on $\mathfrak{X}$.** On the boundary $\Omega$, there exists a unique $K$-invariant, $G$-quasi-invariant probability measure $\nu$, and the Poisson kernel $p(g \cdot o, \omega)$ is defined to be the Radon–Nikodym derivative $d\nu(g^{-1} \cdot \omega)/d\nu(\omega)$. The Poisson kernel can be explicitly written as

\begin{equation}
p(x, \omega) = q^{h_\omega(x)} \text{ for all } x \in \mathfrak{X} \text{ and all } \omega \in \Omega.
\end{equation}

See [FP83, Chapter 3, Section 2] for details.
Let $C(\Omega)$ be the space of all continuous functions defined on the boundary $\Omega$. For $z \in \mathbb{C}$, we define the representations $\pi_z$ of $G$ on $C(\Omega)$ by
\[
\pi_z(g)\eta(\omega) = p^{1/2 + iz}(g \cdot o, \omega)\eta(\omega^{-1} \cdot \omega) \quad \text{for all } g \in G \text{ and all } \omega \in \Omega.
\]
It is clear that $\pi_z = \pi_{z + \tau}$, where $\tau = 2\pi/\log q$. We write $\mathbb{T}$ for the torus $\mathbb{R}/\tau\mathbb{Z}$, which we usually identify with the interval $[-\tau/2, \tau/2)$. The elementary spherical function $\phi_z$ is now defined as
\[
\phi_z(x) = \langle \pi_z(x)1, 1 \rangle = \int_\Omega p^{1/2 + iz}(x, \omega)\, d\nu(\omega), \quad \text{where } x \in \mathfrak{X}.
\]
The following explicit formula of $\phi_z$ is well-known (see [FP82, Theorem 2]):
\[
\phi_z(x) = \begin{cases}
(q^{-1} - |x| + 1)q^{-|x|/2} & \forall z \in \tau\mathbb{Z}, \\
(q^{-1} - |x| + 1)q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2 + \tau\mathbb{Z}, \\
c(z)q^{iz-1/2}|x| + c(-z)q^{-iz-1/2}|x| & \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z},
\end{cases}
\]
where $c$ is the meromorphic function given by
\[
c(z) = \frac{q^{1/2}}{q + 1} \frac{q^{1/2 + iz} - q^{-1/2 - iz}}{q^{iz} - q^{-iz}} \quad \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}.
\]
We note that for every $x$ in $\mathfrak{X}$, the map $z \mapsto \phi_z(x)$ is an entire function. From the explicit formula above, it is also clear that $\phi_z$ is a radial function which satisfies $\phi_z = \phi_{-z} = \phi_{z + \tau}$ for every $z$ in $\mathbb{C}$.

The spherical transform $\hat{f}$ of a finitely supported radial function $f$ on $\mathfrak{X}$ is defined by the formula
\[
\hat{f}(z) = \sum_{x \in \mathfrak{X}} f(x)\phi_z(x), \quad \text{where } z \in \mathbb{C}.
\]
One sees immediately that $z \mapsto \hat{f}(z)$ is an entire function. Moreover, the symmetry property and the $\tau$-periodicity of $\phi_z$ imply that $\hat{f}$ is even and $\tau$-periodic on $\mathbb{C}$. Following [CMW19], we say that a holomorphic function $\eta$ defined on $S_p^0$ is Weyl-invariant if $\eta(z) = \eta(-z)$ and $\eta(z) = \eta(z + \tau)$ for all $z$ in $S_p^0$.

The Helgason–Fourier transform $\tilde{f}$ of a finitely supported function $f$ on $\mathfrak{X}$ is a function on $\mathbb{C} \times \Omega$ defined by the formula
\[
\tilde{f}(z, \omega) = \sum_{x \in \mathfrak{X}} f(x)p^{1/2 + iz}(x, \omega).
\]
It is clear that $\tilde{f}(z, \omega) = \tilde{f}(z + \tau, \omega)$ for every $z$ in $\mathbb{C}$. A simple computation reveals that if $f$ is radial, its Helgason–Fourier transform $\tilde{f}$ reduces to the spherical transform $\hat{f}$. We conclude this section by stating the inversion formula for the Helgason–Fourier transform on $\mathfrak{X}$. For details, we refer
Theorem 2.2. If $f$ is a finitely supported function on $\mathfrak{X}$, then
$$f(x) = c_G \int \int_{\Omega} p^{1/2 - is(x, \omega)} \tilde{f}(s, \omega) |c(s)|^{-2} d\nu(\omega) \, ds$$ for all $x \in \mathfrak{X},$
where $c_G = q \log q / (4\pi (q + 1)).$

3. Analysis of pseudo-differential operators, $1 < p < 2$. In this section we prove our result concerning the $L^p$-boundedness of pseudo-differential operators on $\mathfrak{X}$ for $1 < p < 2.$ We begin with a couple of preparatory lemmas. Both these lemmas are essentially known (for details, see [CMW19, Lemmas 2.2 & 2.3]). However, it seems that there is a minor error in the statement of [CMW19, Lemma 2.2] which we have taken care of, and we provide a proof for the sake of completeness.

Lemma 3.1 ([CMW19, Lemma 2.2]). Let $N$ be as in (2.4). Then:

1. For every $n \in N$, $d(o, n \cdot o)$ is either zero or a positive even number.
2. For every $n \in N$ and every $j \in \mathbb{Z}$ satisfying $j \leq d(o, n \cdot o)/2$,
$$d(o, n \sigma^j \cdot o) = d(o, n \cdot o) - j.$$

Proof. Before going into the details of the proof, let us first observe that the subgroup $N$ can also be written as
$$N = \bigcup_{k \in \mathbb{Z}_+} N_{\omega_k^0},$$
where, for every $k$, $N_{\omega_k^0} = \{ n \in G_{\omega_k^0} : n \cdot \omega_k^0 = \omega_k^0 \}.$ Indeed, if $n$ is in $N$, then $n \cdot x = x$ for some $x$ in $\mathfrak{X}.$ Since $n$ also fixes the equivalence class $[\omega_0^+]$, $n$ must fix every vertex of the infinite geodesic ray $\omega_r^+$ starting from $x$, and in the direction of $\omega_0^+.$ By definition, the geodesic rays $\omega_x^+$ and $\omega_0^+$ are equivalent, and consequently there exists some $k_0$ in $\mathbb{Z}_+$ such that for all $k \geq k_0$, $n \cdot \omega_k^0 = \omega_k^0.$ Hence $n$ is in $N_{\omega_k^0}$ for all $k \geq k_0.$ By applying a similar reasoning, one sees immediately that $N_{\omega_k^0} \subset N_{\omega_{k+1}^0}$ for all $k.$ This further implies that $N$ can be written as the disjoint union
$$N = N_{\omega_0^0} \cup \bigcup_{k \in \mathbb{N}} (N_{\omega_k^0} \setminus N_{\omega_{k-1}^0}).$$

1. If $n$ is in $N$, then by using (3.1) it follows that $n \in N_{\omega_0^0}$ or $n \in N_{\omega_k^0} \setminus N_{\omega_{k-1}^0}$ for some $k \in \mathbb{N}.$ In the former case, $d(n \cdot o, o) = 0,$ whereas in the latter, by using the facts that $n \cdot \omega_k^0 = \omega_k^0$ and $n \cdot \omega_l^0 \neq \omega_l^0$ for all $0 \leq l \leq k - 1$, we get
$$d(n \cdot o, o) = d(n \cdot o, n \cdot \omega_k^0) + d(n \cdot \omega_k^0, o) = 2d(o, \omega_k^0) = 2k.$$
(2) If \( n \in N_{\omega_0} \), then \( d(o, n \cdot o) = 0 \) and hence for all \( j \leq 0 \),
\[
d(o, n \sigma^j \cdot o) = d(n^{-1} \cdot o, \sigma^j \cdot o) = d(o, n \cdot o) - j.
\]
Now suppose that \( n \in N_{\omega_k} \setminus N_{\omega_{k-1}} \) for some \( k \in \mathbb{N} \). Then \( d(o, n \cdot o) = 2k \) and consequently, for all \( j \leq k \), we get
\[
d(o, n \sigma^j \cdot o) = d(o, n \sigma^k \cdot o) + d(n \sigma^k \cdot o, n \sigma^j \cdot o)
\]
\[
= d(o, n \cdot \omega^0_k) + d(n \cdot \omega^0_k, n \cdot \omega^0_j) = d(o, \omega^0_k) + d(\omega^0_k, \omega^0_j) = 2k - j,
\]
which establishes our claim. \( \blacksquare \)

**Remark 3.2.** As mentioned before, Lemma 3.1 does not hold for all \( n \in N \) and for all \( j > d(o, n \cdot o)/2 \). Here we give a counterexample. We choose \( n \in N_{\omega_k} \setminus N_{\omega_{k-1}} \) for some \( k \) positive. Then from the preceding arguments, we have \( d(o, n \cdot o) = 2k \) and also \( n \in N_{\omega_l} \) for all \( l \geq k \). Consequently, for all \( j > k \), \( d(o, n \sigma^j \cdot o) = j > 2k - j = d(o, n \cdot o) - j \).

**Lemma 3.3** ([CMW19, Lemma 2.3]). For \( p \in [1, 2) \), define \( Q_p : N \to \mathbb{R} \) by \( Q_p(n) = q^{-|n \cdot o|/p} \). Then the function \( n \mapsto |n \cdot o|^l Q_p(n) \) belongs to \( L^1(N) \) for each non-negative integer \( l \).

### 3.1. Decomposition of the operator \( T_\Psi \)

We now express \( T_\Psi \) as an integral operator on \( G \), and thereafter we shall decompose its kernel into two parts. Let \( f \) be a finitely supported function on \( \mathcal{X} \). Then \( (1.8) \) gives us
\[
T_\Psi f(h) = c_G \int_{\mathcal{T}} \int_{\Omega} \Psi(h, s) \tilde{f}(s, \omega) p^{1/2 - is}(h, \omega)|c(s)|^{-2} \, d\nu(\omega) \, ds, \quad h \in G.
\]
Substituting the expression of \( \tilde{f}(s, \omega) \), applying Fubini’s Theorem and using the expression of \( \phi_s(g^{-1}h) \) from [FN91, p. 55], we obtain
\[
T_\Psi f(h) = c_G \int_{G} f(g) \int_{\mathcal{T}} \Psi(h, s)|c(s)|^{-2} \cdot \left( \int_{\Omega} p^{1/2 - is}(h, \omega)p^{1/2 + is}(g, \omega) \, d\nu(\omega) \right) \, ds \, dg
\]
\[
= c_G \int_{G} f(g) \left( \int_{\mathcal{T}} \Psi(h, s)\phi_s(g^{-1}h)|c(s)|^{-2} \, ds \right) \, dg
\]
\[
= \int_{G} f(g) K(h, g) \, dg,
\]
where
\[
K(h, g) = c_G \int_{\mathcal{T}} \Psi(h, s)\phi_s(g^{-1}h)|c(s)|^{-2} \, ds.
\]
Using Theorem 2.1 with \( g = n\sigma^j k_1 \) and \( h = m\sigma^l k_2 \), we can write
\[
T_\Psi f(m\sigma^l) = \int \sum_{N, j \in \mathbb{Z}} f(n\sigma^j) K(m\sigma^l, n\sigma^j) q^{-j} \ dn
\]
\[
= T^+_\Psi f(m\sigma^l) + T^-_\Psi f(m\sigma^l) \quad \text{(say)},
\]
where the operators \( T^\pm_\Psi f \) are defined by the formulae
\[
T^\pm_\Psi f(m\sigma^l) = \int \sum_{N, j \in \mathbb{Z}} f(n\sigma^j) K(m\sigma^l, n\sigma^j) \chi^\pm (l - j) q^{-j} \ dn,
\]
and \( \chi^\pm \) are functions on \( \mathbb{Z} \) given by
\[
\chi^+(l - j) = \chi_{[0,\infty)}(l - j) \quad \text{and} \quad \chi^-(l - j) = \chi_{(-\infty,-1]}(l - j).
\]
Consequently, the \( L^p \)-boundedness of \( T_\Psi \) follows from that of the operators \( T^\pm_\Psi \), which we shall take up separately in the following theorems.

In the case of multipliers on \( \mathfrak{X} \), Meda et al. [CMW19] decomposed \( T_m \) (defined by (1.6)) into the operators \( T^\pm_m \), which are of convolution-type with \( K \)-biinvariant functions \( K^\pm_m \) (say). In order to get the boundedness of \( T^-_m \), they first proved a general transference result for convolution operators (see [CMW19, Theorem 3.3]) and then used the estimate of the kernel \( K^-_m \). On the other hand, the boundedness of \( T^+_m \) was an outcome of a basic convolution-type inequality given in [HR79, Corollary 20.14(ii,iv)]. However, in the case of pseudo-differential operators, we do not have the privilege to use the above methods directly. This is the crucial difference between the multiplier case and our situation. We prove the boundedness of \( T^-_\Psi \) by establishing a connection with pseudo-differential operators on \( \mathbb{Z} \). In the case of \( T^+_\Psi \), we shall broadly follow the approach of Ionescu [I02] for non-compact symmetric spaces.

**Theorem 3.4.** Let \( 1 < p < 2 \). Suppose that \( \Psi \) satisfies the hypothesis of Theorem 1.2 and \( T^-_\Psi \) is as in (3.4). Then \( T^-_\Psi \) is bounded from \( L^p(NA) \) to itself. Moreover, there exists a constant \( C_p > 0 \) such that
\[
\|T^-_\Psi f\|_{L^p(NA)} \leq C_p \left( \|\Psi\|_{L^\infty(\mathfrak{X} \times \mathbb{S}^p)} + \sup_{m \in \mathbb{N}} \|\Psi_{-\delta_p}(m\cdot, \cdot)\|_p \right) \|f\|_{L^p(NA)}
\]
for all \( f \in L^p(NA) \).

**Proof.** We will prove this theorem in several steps:

**Step I: Analysis of the kernel.** We recall from (3.2) that
\[
K(h, g) = c_G \int_T \Psi(h, s) \phi_s(g^{-1} h) |\mathcal{C}(s)|^{-2} \ ds.
\]
Putting in the explicit expression of \( \phi_s \) from (2.7) and using the fact that
\(|c(s)|^2 = c(s)c(-s)|\) for all \(s \in \mathbb{T}\), we obtain
\[
K(h, g) = c_G \int_{\mathbb{T}} \Psi(h, s)q^{(is-1/2)}|g^{-1}h|o|c(-s)^{-1}\, ds
+ c_G \int_{\mathbb{T}} \Psi(h, s)q^{(-is-1/2)}|g^{-1}h|o|c(s)^{-1}\, ds.
\]

After the change of variable \(s \mapsto -s\) in the second integral and using the Weyl invariance of the function \(\Psi(h, \cdot)\), we get
\[
(3.5) \quad K(h, g) = C \int_{\mathbb{T}} \Psi(h, s)q^{(is-1/2)}|g^{-1}h|o|c(-s)^{-1}\, ds.
\]

We note that the above integrand is a holomorphic function on \(S^0_p\). Therefore applying Cauchy’s integral theorem on the closed rectangle
\[
\Gamma(z) = \{ z \in \mathbb{C} : \Im z = 0, -\tau/2 \leq \Re z \leq \tau/2 \}
\cup \{ z \in \mathbb{C} : \Re z = \tau/2, 0 \leq \Im z \leq \delta_r \}
\cup \{ z \in \mathbb{C} : \Im z = \delta_r, \tau/2 \leq \Re z \leq -\tau/2 \}
\cup \{ z \in \mathbb{C} : \Re z = -\tau/2, \delta_r \leq \Im z \leq 0 \}
\]
shows that for every \(r \in (p, 2)\),
\[
K(h, g) = C \int_{\mathbb{T}} \Psi(h, s + i\delta_r)q^{(is-1/r)}|g^{-1}h|o|c(-s - i\delta_r)^{-1}\, ds.
\]

Using the dominated convergence theorem and letting \(r \to p\), we finally get
\[
K(h, g) = C \int_{\mathbb{T}} \Psi(h, s + i\delta_p)q^{(is-1/p)}|g^{-1}h|o|c(-s - i\delta_p)^{-1}\, ds
= Cq^{-|g^{-1}h|o/p} \int_{\mathbb{T}} \Psi_{\delta_p}(h, s)q^{is|g^{-1}h|o}c_{-\delta_p}(-s)^{-1}\, ds.
\]

Plugging in the above expression into \(3.4\), we obtain
\[
T_{\Psi}^\perp f(m\sigma^l) = C \int \sum_{N \in \mathbb{Z}} f(n\sigma^j)q^{-|\sigma^{-j}n^{-1}m\sigma^l|_o/p}
\cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(m\sigma^l, s)q^{is|\sigma^{-j}n^{-1}m\sigma^l|o}c_{-\delta_p}(-s)^{-1}\, ds \right) \chi_-(l - j)q^{-j}\, dn.
\]

The change of variable \(n \mapsto m^{-1}n\) implies that
\[
T_{\Psi}^\perp f(m\sigma^l) = C \int \sum_{N \in \mathbb{Z}} f(mn\sigma^j)q^{-|\sigma^{-j}n^{-1}m\sigma^l|/o/p}
\cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(m\sigma^l, s)q^{is|\sigma^{-j}n^{-1}m\sigma^l|o}c_{-\delta_p}(-s)^{-1}\, ds \right) \chi_-(l - j)q^{-j}\, dn.
\]
After the change of variable \( n \mapsto \sigma^{-j} n \sigma^j \) and using (2.5), we have

\[
T^{\Psi}_f (m \sigma^l) = C \int_N \sum_{j \in \mathbb{Z}} f(m \sigma^j n) q^{-|n^{-1} \sigma^{-j} \cdot o|/p} \\
\cdot \left( \int_{\mathbb{T}} \psi_{\delta_p}(m \sigma^l, s) q^{i \cdot q s |n^{-1} \sigma^{-j} \cdot o|} c_{-\delta_p}(-s)^{-1} ds \right) \chi_-(l-j) \, dn.
\]

Since \( \chi_-(l-j) = 0 \) whenever \( l \geq j \), using Lemma 3.1(2) we get

\[
T^{\Psi}_f f(m \sigma^l) = C \int_N \sum_{j \in \mathbb{Z}} f(m \sigma^j n) q^{-|n^{-1} \cdot o|/p} q^{(l-j)/p} \\
\cdot \left( \int_{\mathbb{T}} \psi_{\delta_p}(m \sigma^l, s) q^{i s |n^{-1} \cdot o| - (l-j)} c_{-\delta_p}(-s)^{-1} ds \right) \chi_-(l-j) \, dn.
\]

For fixed \( m, n \in N \), let us introduce the notation

\[
\kappa_m(l, |n^{-1} \cdot o| - (l-j)) = \int_{\mathbb{T}} \psi_{\delta_p}(m \sigma^l, s) q^{i s |n^{-1} \cdot o| - (l-j)} c_{-\delta_p}(-s)^{-1} ds = \kappa_{m,n}(l, l-j).
\]

We shall use both the conventions \( \kappa_m(\cdot, \cdot) \) and \( \kappa_{m,n}(\cdot, \cdot) \) for the above integral as and when necessary. Consequently, the expression of \( T^{\Psi}_f f \) in (3.6) takes the form

\[
T^{\Psi}_f f(m \sigma^l) = C \int_N \sum_{j \in \mathbb{Z}} f(m \sigma^j n) q^{-|n^{-1} \cdot o|/p} q^{(l-j)/p} \kappa_{m,n}(l, l-j) \chi_-(l-j) \, dn.
\]

We recall from Lemma 3.3 that \( Q_p(n) = q^{-|n\cdot o|/p} \). Consequently,

\[
\|T^{\Psi}_f f\|_{L^p(NA)} = C \left\| \int_N Q_p(n^{-1}) \sum_{j \in \mathbb{Z}} f(m \sigma^j n) q^{-j/p} \kappa_{m,n}(l, l-j) \\
\cdot \chi(-\infty, -1](l-j) \, dn \right\|_{L^p(\mathbb{Z}, l)} \right\|_{L^p(N, dm)},
\]

and using Minkowski’s inequality, we deduce that

\[
\|T^{\Psi}_f f\|_{L^p(NA)} \leq C \int_N Q_p(n^{-1}) \left\| \sum_{j \in \mathbb{Z}} f(m \sigma^j n) q^{-j/p} \kappa_{m,n}(l, l-j) \\
\cdot \chi(-\infty, -1](l-j) \right\|_{L^p(\mathbb{Z}, l)} \left\|_{L^p(N, dm)} \right. \, dn.
\]

For a fixed \( m \in N \), let us define the operator

\[
B_n \theta(l) = \sum_{j \in \mathbb{Z}} \theta(j) \kappa_{m,n}(l, l-j) \chi(-\infty, -1](l-j) \quad \text{for all } l \in \mathbb{Z}.
\]
We claim that $\mathcal{B}_n$ is a bounded pseudo-differential operator from $L^p(\mathbb{Z})$ to itself. Moreover, we will show there exists a constant $C_p > 0$, independent of $m$ and $n$, such that

$$\|\mathcal{B}_n \theta\|_{L^p(\mathbb{Z})} \leq C_p (1 + |n^{-1} \cdot o|) \|\theta\|_{L^p(\mathbb{Z})} \quad \text{for all } \theta \in L^p(\mathbb{Z}). \tag{3.9}$$

Assuming the claim above, we complete the proof of Theorem 3.4. Plugging in the estimate (3.9) with $\theta(j) = f(m\sigma^j n)q^{-j/p}$ into (3.8), we derive

$$\|T_{\Psi} f\|_{L^p(NA)} \leq C_p \left( \sup_{|l| \leq n^{-1} \cdot o} \left( \sum_{j \in \mathbb{Z}} |\theta(j)\kappa_m(l, |n^{-1} \cdot o| - (l - j))\chi_{(-\infty,-1]}(l - j)|^p \right)^{1/p} \right)$$

$$= C_p \left( \sup_{|l| \leq n^{-1} \cdot o} \left( \sum_{j \in \mathbb{Z}} |\theta(j)\kappa_m(l, j - l)\chi_{(-\infty,-1]}(j - l)|^p \right)^{1/p} \right)$$

$$\leq C_p \|f\|_{L^p(NA)},$$

where in the last inequality we have used Theorem 2.1 and Lemma 3.3. This proves Theorem 3.4 modulo the claim in (3.9).

**Step II: Connection with pseudo-differential operators on $\mathbb{Z}$**. Here we prove (3.9). We recall from (3.7) an alternative definition of $\kappa_{m,n}^\prime(\cdot, \cdot)$, and observe that the $L^p(\mathbb{Z})$ operator norm of $\mathcal{B}_n$ is equal to that of $\mathcal{B}_n^\prime$, where

$$\mathcal{B}_n^\prime \theta(l) := \sum_{j \in \mathbb{Z}} \theta(j)\kappa_m(-l, l - j)\chi_{[1+|n^{-1} \cdot o|, \infty)}(l - j) \quad \text{for all } l \in \mathbb{Z}.$$ 

In fact, an explicit calculation yields

$$\|\mathcal{B}_n\|_p = \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} \theta(j)\kappa_m(l, |n^{-1} \cdot o| - (l - j))\chi_{(-\infty,-1]}(l - j) \right|^p \right)^{1/p}$$

$$= \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} \theta(j - |n^{-1} \cdot o|)\kappa_m(l, j - l)\chi_{(-\infty,-1]}(j - l) \right|^p \right)^{1/p}$$

$$= \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} \theta(j - |n^{-1} \cdot o|)\kappa_m(-l, j - l)\chi_{[1+|n^{-1} \cdot o|, \infty)}(j - l) \right|^p \right)^{1/p}$$

$$= \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} \theta(-j - |n^{-1} \cdot o|)\kappa_m(-l, j - l)\chi_{[1+|n^{-1} \cdot o|, \infty)}(j - l) \right|^p \right)^{1/p}$$

$$= \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left| \sum_{j \in \mathbb{Z}} \theta(j)\kappa_m(-l, j - l)\chi_{[1+|n^{-1} \cdot o|, \infty)}(j - l) \right|^p \right)^{1/p}$$

$$= \|\mathcal{B}_n^\prime\|_p.$$
Thus, it is enough to prove that $B'_n$ satisfies (3.9). We can write
\begin{equation}
B'_n \theta(l) = I_+ \theta(l) - I_- \theta(l) - I_n \theta(l) \quad \text{for all } l \in \mathbb{Z},
\end{equation}
where $I_+$, $I_-$ and $I_n$ are pseudo-differential operators on $\mathbb{Z}$ defined by
\begin{align*}
I_+ \theta(l) &= \sum_{j \in \mathbb{Z}} \theta(j) \kappa_m(-l, l - j), \\
I_- \theta(l) &= \sum_{j \in \mathbb{Z}} \theta(j) \kappa_m(-l, l - j) \chi(-\infty, -1](l - j), \\
I_n \theta(l) &= \sum_{j \in \mathbb{Z}} \theta(j) \kappa_m(-l, l - j) \chi[0, |n^{-1}a|](l - j).
\end{align*}

First we shall estimate $I_+$. By using the explicit expression of $\kappa_m(\cdot, \cdot)$ from (3.7), it follows that
\[
I_+ \theta(l) = \sum_{j \in \mathbb{Z}} \theta(j) \left( \int_{\mathbb{T}} \Psi_{\delta_p}(m\sigma^{-l}, s) q^{is(l-j)} c_{-\delta_p}(-s)^{-1} ds \right)
= \int_{\mathbb{T}} \Psi_{\delta_p}(m\sigma^{-l}, s) F\theta(s) c_{-\delta_p}(-s)^{-1} q^{isl} ds.
\]
The change of variable $s \mapsto -s$ and the Weyl invariance of $\Psi$ yield
\[
I_+ \theta(l) = \int_{\mathbb{T}} \Psi_{-\delta_p}(m\sigma^{-l}, s) F\theta(-s) c_{-\delta_p}(s)^{-1} q^{-isl} ds.
\]
Hence the operator norm of $I_+$ on $L^p(\mathbb{Z})$ becomes
\[
\|I_+\|_p = \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left| \int_{\mathbb{T}} \Psi_{-\delta_p}(m\sigma^{-l}, s) F\theta(-s) c_{-\delta_p}(s)^{-1} q^{-isl} ds \right|^p \right)^{1/p}
= \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left| \int_{\mathbb{T}} \Psi_{-\delta_p}(m\sigma^l, s) F\theta(-s) c_{-\delta_p}(s)^{-1} q^{isl} ds \right|^p \right)^{1/p}.
\]
We observe that the inner integral above defines a pseudo-differential operator on $\mathbb{Z}$ as in (1.1) with symbol $\psi(l, s) = \Psi(m\sigma^l, s - i\delta_p)$. Since $\Psi_{-\delta_p}(m\cdot, \cdot)$ defines a bounded pseudo-differential operator on $L^p(\mathbb{Z})$, the above expression gives
\begin{equation}
\|I_+\|_p \leq \|\Psi_{-\delta_p}(m\cdot, \cdot)\|_p \left( \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \|F^{-1}(F\theta(-\cdot) c_{-\delta_p}(\cdot)^{-1})\|_{L^p(\mathbb{Z})} \right).
\end{equation}
Define $\theta^#(j) = \theta(-j)$, where $j \in \mathbb{Z}$. Then a simple calculation shows that $F(\theta^#)(s) = F\theta(-s)$. Implementing this fact and using Young’s inequality
in (3.11) we obtain
\[
\|I_+\|_p \leq \|\Psi_{-\delta_p}(m\cdot, \cdot)\|_p \left( \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \|\theta^\#_{L^p(\mathbb{Z})} \mathcal{F}^{-1}(c_{-\delta_p}(\cdot)^{-1})\|_{L^p(\mathbb{Z})} \right)
\]
\leq \|\Psi_{-\delta_p}(m\cdot, \cdot)\|_p \left( \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \|\theta^\#_{L^p(\mathbb{Z})} \mathcal{F}^{-1}(c_{-\delta_p}(\cdot)^{-1})\|_{L^1(\mathbb{Z})} \right).
\]

Since \( s \mapsto c(-s - i\delta_p)^{-1} \) is a smooth function on \( \mathbb{T} \), we finally get
\[
(3.12) \quad \|I_+\|_p \leq C_p \left( \sup_{m \in \mathbb{N}} \|\Psi_{-\delta_p}(m\cdot, \cdot)\|_p \right).
\]

Next, we consider the operator
\[
I_- \theta(l) = \sum_{j \in \mathbb{Z}} \theta(j) \kappa_m(-l, l - j) \chi_{(-\infty, -1]}(l - j).
\]

Recalling the expression of \( \kappa_m(\cdot, \cdot) \) from (3.7), it follows that
\[
\kappa_m(-l, l - j) = \int_{\mathbb{T}} \Psi_{\delta_p}(m\sigma^{-l}, s) q^{is(l-j)} c_{-\delta_p}(-s)^{-1} ds.
\]

We observe that the integrand above is holomorphic on the strip
\[
\mathbb{S}_{2\delta_p} := \{ z \in \mathbb{C} : -2\delta_p < \Im z < 0 \}.
\]

Thus, using Cauchy’s integral theorem and the dominated convergence theorem, we get
\[
\kappa_m(-l, l - j) = q^{2\delta_p(l-j)} \int_{\mathbb{T}} \Psi_{-\delta_p}(m\sigma^{-l}, s) q^{is(l-j)} c_{-\delta_p}(-s)^{-1} ds.
\]

Taking modulus on both sides of the above expression, we get the pointwise estimate
\[
|\kappa_m(-l, l - j)| \leq C_p q^{2\delta_p(l-j)} \|\Psi\|_{L^\infty(\mathbb{R} \times S_p)} \quad \text{for all } l, j \in \mathbb{Z}.
\]

This in turn implies that
\[
(3.13) \quad \|I_-\|_p \leq \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} |\theta(j)| \left| \kappa_m(-l, l - j) \chi_{(-\infty, -1]}(l - j) \right|^p \right)^{1/p} \right)^1/p
\]
\[
\leq C_p \|\Psi\|_{L^\infty(\mathbb{R} \times S_p)} \cdot \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} |\theta(j)| q^{2\delta_p(l-j)} \chi_{(-\infty, -1]}(l - j) \right)^p \right)^{1/p}
\]
\[
= C_p \|\Psi\|_{L^\infty(\mathbb{R} \times S_p)} \cdot \sup_{\|\theta\|_{L^p(\mathbb{Z})} = 1} \left( \sum_{l \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} |\theta(l - j)| q^{2\delta_p(l-j)} \chi_{(-\infty, -1]}(j) \right)^p \right)^{1/p}
\]
\[ \leq C_p \|\Psi\|_{L^\infty(X \times S_p)} \sup_{\|\theta\|_{L^p(Z)}=1} \left( \sum_{j \in \mathbb{Z}} \left( q^{2j\delta_p} \left( \sum_{l \in \mathbb{Z}} |\theta(l - j)|^p \right)^{1/p} \chi_{(-\infty,-1]}(j) \right) \right) \]

\[ \leq C_p \|\Psi\|_{L^\infty(X \times S_p)}. \]

Finally, we investigate the operator norm of

\[ I_n \theta(l) = \sum_{j \in \mathbb{Z}} \theta(j) \kappa_m(-l, l - j) \chi_{[0,|n-1|]}(l - j). \]

The expression of \( \kappa_m(\cdot, \cdot) \) from (3.7) gives us the trivial estimate

\[ |\kappa_m(-l, l - j)| \leq C_p \|\Psi\|_{L^\infty(X \times S_p)} \text{ for all } l, j \in \mathbb{Z}. \]

Consequently, as in (3.13) we have

\[ (3.14) \quad \|I_n\|_p \leq C_p (1 + |n^{-1} \cdot o|) \|\Psi\|_{L^\infty(X \times S_p)}. \]

Plugging in the estimates (3.12), (3.13) and (3.14) of the operators \( I_+ \), \( I_- \), and \( I_n \) respectively into (3.10), we get the desired claim (3.9). This concludes the proof of Theorem 3.4.

The remainder of this section will be devoted to the proof of \( L^p(NA) \)-boundedness of the operator \( T_\Psi^+ \) for \( 1 < p < 2 \). Consequently, we will be able to conclude that \( T_\Psi \) is bounded from \( L^p(\mathcal{X}) \) to itself.

**Theorem 3.5.** Let \( 1 < p < 2 \). Suppose that \( \Psi \) satisfies the hypotheses (1) and (2) of Theorem 1.2 and let \( T_\Psi^+ \) be as in (3.4). Then \( T_\Psi^+ \) is bounded from \( L^p(NA) \) to itself. Moreover, there exists a constant \( C_p > 0 \) such that

\[ \|T_\Psi^+ f\|_{L^p(NA)} \leq C_p \|\Psi\|_{L^\infty(X \times S_p)} \|f\|_{L^p(NA)} \quad \text{for all } f \in L^p(NA). \]

**Proof.** Recalling the definition of \( T_\Psi^+ \) from (3.4), we have

\[ T_\Psi^+ f(m\sigma^l) = \int \sum_{N,j \in \mathbb{Z}} f(n\sigma^j) K(m\sigma^l, n\sigma^j) \chi_+(l - j) q^{-j} \, dn, \]

which after substituting the expression of \( K(m\sigma^l, n\sigma^j) \) from (3.5) becomes

\[ T_\Psi^+ f(m\sigma^l) = C \int \sum_{N,j \in \mathbb{Z}} f(n\sigma^j) \]

\[ \cdot \left( \int_{\mathbb{T}} \Psi(m\sigma^l, s) c(-s)^{-1} q^{(i(s-1)/2)|\sigma^{-j} n^{-1} m l | - o} \, ds \right) \chi_+(l - j) q^{-j} \, dn. \]

Using the same argument as in Theorem 3.4, we move the contour of inte-
gration in the inner integral from $\mathbb{T}$ to $\mathbb{T} + i \delta_p$ and obtain

$$
(3.15) \quad T^+_{\psi} f(m^l) = C \int \sum_{N} \sum_{j \in \mathbb{Z}} f(n \sigma^j) q^{-|\sigma^{-j} n^{-1} m \sigma^l \cdot o|/p} \cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(m \sigma^l, s) c_{-\delta_p}(-s)^{-1} q^{i s |\sigma^{-j} n^{-1} m \sigma^l \cdot o|} ds \right) \chi_+ (l - j) q^{-j} dn.
$$

To get the desired result, it suffices to prove that for any compactly supported functions $f, \varphi : NA \to \mathbb{C}$, one has

$$
|\langle T^+_{\psi} f, \varphi \rangle| = \left| \int \sum_{N} \sum_{l \in \mathbb{Z}} T^+_{\psi} f(m \sigma^l) \varphi(m \sigma^l) q^{-l} dm \right| \leq C_p \| f \|_{L^p(NA)} \| \varphi \|_{L^{p'}(NA)}.
$$

Plugging in the expression of $T^+_{\psi} f$ from (3.15), we get

$$
\langle T^+_{\psi} f, \varphi \rangle = C \int \sum_{N} \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f(n \sigma^j) \varphi(n m \sigma^l) \chi_+ (l - j) q^{-|\sigma^{-j} n^{-1} m \sigma^l \cdot o|/p} q^{-j - l} \cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(n m \sigma^l, s) c_{-\delta_p}(-s)^{-1} q^{i s |\sigma^{-j} m \sigma^l \cdot o|} ds \right) dm dn,
$$

which after the change of variables $m \mapsto n^{-1} m$ yields

$$
\langle T^+_{\psi} f, \varphi \rangle = C \int \sum_{N} \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f(n \sigma^j) \varphi(n \sigma^j m \sigma^{l-j}) \chi_+ (l - j) q^{-|\sigma^{-j} m \sigma^{l-j} \cdot o|/p} q^{-j - l - \sigma^{-j} m \sigma^{l-j} \cdot o} \cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(n \sigma^j m \sigma^{l-j}, s) c_{-\delta_p}(-s)^{-1} q^{i s |m \sigma^{l-j} \cdot o|} ds \right) dm dn.
$$

We recall that the map $m \mapsto \sigma^{-j} m \sigma^j$ is a dilation of $N$. Hence, by (2.5), we get

$$
\langle T^+_{\psi} f, \varphi \rangle = C \int \sum_{N} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} f(n \sigma^j) \varphi(n \sigma^j m \sigma^{l-j}) \chi_+ (l - j) q^{-|m \sigma^{l-j} \cdot o|/p} q^{-j - l - \sigma^{-j} m \sigma^{l-j} \cdot o} \cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(n \sigma^j m \sigma^{l-j}, s) c_{-\delta_p}(-s)^{-1} q^{i s |m \sigma^{l-j} \cdot o|} ds \right) dm dn.
$$

Next, by using Fubini’s theorem and the change of variables $l \mapsto l - j$, it follows that

$$
\langle T^+_{\psi} f, \varphi \rangle = C \int \sum_{N} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}_+} f(n \sigma^j) \varphi(n \sigma^j m \sigma^l) q^{-|m \sigma^l \cdot o|/p} q^{-l - j} \cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(n \sigma^j m \sigma^l, s) c_{-\delta_p}(-s)^{-1} q^{i s |m \sigma^l \cdot o|} ds \right) dm dn.
$$

Taking modulus of both sides of the above expression and using the bound-
edness of $\Psi$ and $c_{-\delta_p}(-s)^{-1}$ on $\mathcal{X} \times S_p$ and $T$ respectively, we deduce that

\begin{equation}
|\langle T^+_\Psi f, \varphi \rangle| \leq C_p \|\Psi\|_{L^\infty(\mathcal{X} \times S_p)} \cdot \int \int \sum_{N} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^+} |f(n^{\sigma_j})| |\varphi(n^{\sigma_j}m^{\sigma_l})| q^{-|m^{\sigma_l} \cdot o|/p} q^{-l-j} \, dm \, dn.
\end{equation}

Now let us define

\begin{equation}
F(\sigma^j) = \left[ \int_{N} |f(n^{\sigma_j})|^p \, dn \right]^{1/p}, \quad \Phi(\sigma^j) = \left[ \int_{N} |\varphi(n^{\sigma_j})|^{p'} \, dn \right]^{1/p'}.
\end{equation}

Applying Hölder’s inequality in \eqref{3.16} and using \eqref{3.17}, we get

\begin{equation}
|\langle T^+_\Psi f, \varphi \rangle| \leq C_p \|\Psi\|_{L^\infty(\mathcal{X} \times S_p)} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^+} F(\sigma^j) \Phi(\sigma^{l+j}) q^{-|m^{\sigma_l} \cdot o|/p} q^{-l-j} \, dm,
\end{equation}

which after using Fubini’s theorem yields

\begin{equation}
|\langle T^+_\Psi f, \varphi \rangle| \leq C_p \|\Psi\|_{L^\infty(\mathcal{X} \times S_p)} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^+} F(\sigma^j) \Phi(\sigma^{l+j}) q^{-l-j} \int_{N} q^{-|m^{\sigma_l} \cdot o|/p} \, dm.
\end{equation}

Since the Abel transform of a radial function is even (see [CMS98 Theorem 2.5]), it follows that

\[ \int_{N} q^{-|m^{\sigma_l} \cdot o|/p} \, dm = q^l \int_{N} q^{-|m^{\sigma_l} \cdot o|/p} \, dm \quad \text{for all } l \in \mathbb{Z}^+. \]

Putting the formula above in \eqref{3.18}, we obtain

\begin{equation}
|\langle T^+_\Psi f, \varphi \rangle| \leq C_p \|\Psi\|_{L^\infty(\mathcal{X} \times S_p)} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^+} F(\sigma^j) \Phi(\sigma^{l+j}) q^{-j} \int_{N} q^{-|m^{\sigma_l} \cdot o|/p} \, dm.
\end{equation}

Using Lemma 3.1(2) in the inequality above shows that

\begin{equation}
|\langle T^+_\Psi f, \varphi \rangle| \leq C_p \|\Psi\|_{L^\infty(\mathcal{X} \times S_p)} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^+} F(\sigma^j) \Phi(\sigma^{l+j}) q^{-j-l/p} \int_{N} q^{-|m \cdot o|/p} \, dm.
\end{equation}

Finally, using Lemma 3.3 and applying Hölder’s inequality, we conclude that

\begin{equation}
|\langle T^+_\Psi f, \varphi \rangle| \leq C_p \|\Psi\|_{L^\infty(\mathcal{X} \times S_p)} \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^+} F(\sigma^j) q^{-j/p} \Phi(\sigma^{j+l}) q^{-(j+l)/p'} q^{l(1/p' - 1/p)}
\leq C_p \|\Psi\|_{L^\infty(\mathcal{X} \times S_p)} \|f\|_{L^p(NA)} \|\varphi\|_{L^{p'}(NA)}.
\end{equation}

This completes the proof of Theorem 3.5.

4. Boundedness of pseudo-differential operators, $2 < p < \infty$. In order to prove Theorem 1.3, we will broadly follow the approach of the proof of Theorem 3.5. To avoid repetition, we shall only highlight the crucial steps.
Proof of Theorem 1.3. Our strategy is to prove that for any compactly supported functions $f, \varphi : NA \to \mathbb{C}$,

\[ |\langle T_{\Psi} f, \varphi \rangle| = \left| \sum_{N \in \mathbb{Z}} T_{\Psi}^+ f(m\sigma^l) \varphi(m\sigma^l) q^{-l} \, dm \right| \]

\[ \leq C_p \| f \|_{L^p(NA)} \| \varphi \|_{L^{p'}(NA)}. \]

As in the case $1 < p < 2$, we decompose $T_{\Psi} f$ into two parts, $T_{\Psi}^+ f$ and $T_{\Psi}^- f$, where

\[ T_{\Psi}^\pm f(m\sigma^l) = C \sum_{N \in \mathbb{Z}} f(n\sigma^j) \]

\[ \cdot \left( \int_{\mathbb{T}} \Psi(m\sigma^l, s) c(-s)^{-1} q^{(is - 1/2)|\sigma^{-j} n^{-1} m\sigma^l \cdot o|} \, ds \right) \chi_{\pm}(l - j) q^{-j} \, dn. \]

We first prove that $T_{\Psi}^+$ satisfies (4.1). Sending the contour of integration in the inner integral from $\mathbb{T}$ to $\mathbb{T} + i\delta_p$ and noting that $\delta_p = \delta_{p'} = 1/p' - 1/2$ (as $p > 2$), we get (3.15) with $p$ being replaced by $p'$. Then applying the same change of variables as in the proof of Theorem 3.5, we obtain

\[ \langle T_{\Psi}^+ f, \varphi \rangle = C \sum_{N \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} f(n\sigma^j) \varphi(n\sigma^j m\sigma^l) q^{-|m\sigma^l \cdot o|/p'} q^{-j - l} \]

\[ \cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(n\sigma^j m\sigma^l, s) c_{\delta_p}(-s)^{-1} q^{is|m\sigma^l \cdot o|} \, ds \right) \, dm \, dn. \]

Observe that $h_{\omega^0_+}(m\sigma^l \cdot o)$ and hence $|m\sigma^l \cdot o|$ is strictly positive for all $l > 0$. Consequently, integrating by parts the inner integrals of (4.2) for all $l \in \mathbb{N}$, we deduce that

\[ \langle T_{\Psi}^+ f, \varphi \rangle = C \sum_{N \in \mathbb{Z}} \sum_{l \in \mathbb{N}} f(n\sigma^j) \varphi(n\sigma^j m\sigma^l) q^{-|m\sigma^l \cdot o|/p'} q^{-j - l} \frac{1}{|m\sigma^l \cdot o|^2} \]

\[ \cdot \left( \int_{\mathbb{T}} \frac{d^2}{ds^2} (\Psi_{\delta_p}(n\sigma^j m\sigma^l, s) c_{\delta_p}(-s)^{-1}) q^{is|m\sigma^l \cdot o|} \, ds \right) \, dm \, dn \]

\[ + C \sum_{N \in \mathbb{Z}} \sum_{l \in \mathbb{N}} f(n\sigma^j) \varphi(n\sigma^j m) q^{-j - |m \cdot o|/p'} \]

\[ \cdot \left( \int_{\mathbb{T}} \Psi_{\delta_p}(n\sigma^j m, s) c_{\delta_p}(-s)^{-1} q^{is|m \cdot o|} \, ds \right) \, dm \, dn. \]

Taking modulus of both sides, using hypothesis (1.9) and the smoothness of
\[ \langle T^+ f, \varphi \rangle \leq C_p \left( \sup_{(x, z) \in \mathbb{R} \times S_p, k = 0, 1, 2} \left| \frac{d^k}{dz^k} \Psi(x, z) \right| \right) \]
\[ \times \left[ \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} F(\sigma^j) \Phi(\sigma^{l+j}) q^{-j-l} \int_N \frac{q^{-|m\sigma^l|/p'}}{|m\sigma^l|} \ dm \right. \]
\[ \left. + \sum_{j \in \mathbb{Z}} F(\sigma^j) \Phi(\sigma^j) q^{-j} \int_N q^{-|m\sigma^l|/p'} \ dm \right]. \]

Now the evenness of the Abel transform of a radial function shows that
\[ \langle T^+ f, \varphi \rangle \leq C_p \left( \sup_{(x, z) \in \mathbb{R} \times S_p, k = 0, 1, 2} \left| \frac{d^k}{dz^k} \Psi(x, z) \right| \right) \]
\[ \times \left[ \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}} F(\sigma^j) \Phi(\sigma^{l+j}) q^{-j-l} \int_N \frac{q^{-|m\sigma^l|/p'}}{|m\sigma^l|} \ dm \right. \]
\[ \left. + \sum_{j \in \mathbb{Z}} F(\sigma^j) \Phi(\sigma^j) q^{-j} \int_N q^{-|m\sigma^l|/p'} \ dm \right]. \]

Implementing the above fact in (4.3) and following the same line of arguments as in the proof of Theorem 3.5, we finally obtain
\[ \langle T^+ f, \varphi \rangle \leq C_p \left( \sup_{(x, z) \in \mathbb{R} \times S_p, k = 0, 1, 2} \left| \frac{d^k}{dz^k} \Psi(x, z) \right| \right) \| f \|_{L^p(NA)} \| \varphi \|_{L^{p'}(NA)}. \]

Next, we shall estimate \( T^+_f \). Proceeding in a similar way to the proof of \( T^+_f \), we get
\[ \langle T^- f, \varphi \rangle = C \int \int \sum_{N} \sum_{\sigma^j} f(n\sigma^j) \varphi(n\sigma^j m\sigma^l) q^{-|m\sigma^l|/p'} q^{-j-l} \]
\[ \cdot \left( \int_{\mathbb{T}} \Psi_{\delta^p}(n\sigma^j m\sigma^l, s) c_{-\delta^p} (-s)^{-1} q^{is|m\sigma^l|} ds \right) dm \ dn. \]

Since \( l < 0 \) in the equation above, using Lemma 3.1(2), we have
\[ \langle T^- f, \varphi \rangle = C \int \int \sum_{N} \sum_{\sigma^j} f(n\sigma^j) \varphi(n\sigma^j m\sigma^l) q^{(l-|m\sigma^l|)/p'} q^{-j-l} \]
\[ \cdot \left( \int_{\mathbb{T}} \Psi_{\delta^p}(n\sigma^j m\sigma^l, s) c_{-\delta^p} (-s)^{-1} q^{is|m\sigma^l|} ds \right) dm \ dn. \]

An application of Hölder’s inequality and (3.17) finally gives us
\[ \| \langle T^- f, \varphi \rangle \| \leq C_p \| \Psi \|_{L^{\infty}(\mathbb{R} \times S_p)} \left( \sum_{N} \sum_{\sigma^j} F(\sigma^j) \Phi(\sigma^{l+j}) q^{(l-|m\sigma^l|)/p'} q^{-j-l} \right) dm \]
\[ \leq C_p \| \Psi \|_{L^{\infty}(\mathbb{R} \times S_p)} \| f \|_{L^p(NA)} \| \varphi \|_{L^{p'}(NA)}, \]
which is the desired conclusion. ■

Acknowledgements. The authors would like to thank Prof. Pratyooosh Kumar and Prof. Sanjoy Pusti for many useful discussions in the course of this work. The authors are also thankful to the anonymous referee for the
careful reading of the manuscript and for many insightful comments and suggestions.

The first author was supported by a research fellowship from CSIR (India). The second author gratefully acknowledges the support provided by the NBHM (National Board of Higher Mathematics) post-doctoral fellowship (Number: 0204/3/2021/R&D-II/7386) from the Department of Atomic Energy (DAE), Government of India.

References


Tapendu Rana
Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076, Maharashtra, India
E-mail: tapendurana@gmail.com

Sumit Kumar Rano
Stat-Math Unit
Indian Statistical Institute
Kolkata 700108, India
E-mail: sumitrano1992@gmail.com