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A NEW CONSTRUCTION OF THE BLOCK SYSTEMS $B(4,1,25)$ AND $B(4,1,28)$

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Let $B(v)$ denote a family of 4-element subsets, called *blocks*, of a v-element set E such that each 2-element subset of E is contained in exactly one block $B(v)$.

Constructions of $B(13)$ and $B(16)$ are well known: there exists exactly one $B(13)$ (projective plane PG [2], [3]) and exactly one $B(16)$ (Euclidean plane EG [2], [4]).

In 1939, Bose [1] constructed $B(25)$, $B(28)$, and $B(37)$. His results were then used by Hanani [3] who proved that the relation

$$v \equiv 1 \text{ or } 4 \pmod{12}$$

is a sufficient and necessary condition for the existence of $B(v)$. Pukanow [4] showed that there exist at least two non-isomorphic $B(v)$ for each $v \geqslant 181$. Wojtas [5] proved, using essentially results of [4], that there exist at least $2[(v-1)/720]$ non-isomorphic $B(v)$ for each $v \geqslant 781$. His method, however, cannot be applied for small v.

In this paper we give a construction of $B(25)$ and $B(28)$ which are non-isomorphic to those in [1]; and, therefore, there are at least two $B(25)$ and $B(28)$.

In the bibliography of Steiner systems [2], compiled by Doyen and Rosa, there is a table providing numbers of non-isomorphic Steiner systems $B(k, \lambda, v)$ for $v \leqslant 26$ and the case of $B(4,1,25)$ is there open. Thus the present paper yields an answer: the number is at least 2.

$B(25)$. In the set

$$E = \{0, (i,j): i = 1, \ldots, 6,\ j = 1, \ldots, 4\}$$

we construct the following blocks:

$\{0, (1,j), (2,j), (3,j)\}$, $\{0, (4,j), (5,j), (6,j)\}$ for $j = 1, 2, 3, 4$;

$\{(i,j), (i+3,j), (i,j+k), (i+3,j+k)\}$ for $i = 1, j = 1, 3, k = 1$, or $i = 2, j = 1, 2, k = 2$, or $i = 3, j = 2, k = 1$;

$\{(i,j),(i+4,j),(i,j+k),(i+4,j+k)\}$ for $i=1, j=1, k=3$, or $i=1, j=2, k=1$, or $i=2, j=1,3, k=1$, or $i=3, j=1,2, k=2, i+4=4$;

$\{(i,j),(i+5,j),(i,j+k),(i+5,j+k)\}$ for $i=1, j=1,2, k=2$, or $i=2, j=1, k=3, i+5=4$, or $i=2, j=2, k=1$, or $i=3, j=1,3, k=1, i+5=5$;

$\{(i,1),(i+1,2),(i+2,3),(2i+2,4)\}$;

$\{(i,1),(i+2,2),(i+1,4),(2i+3,3)\}$;

$\{(i,1),(i+1,3),(i+2,4),(2i+4,2)\}$;

$\{(i,1),(2i+3,2),(2i+2,3),(2i+4,4)\}$, $(2i+k)\in\{4,5,6\}$ and $2i+k$ is taken mod 3.

In order to show that this block system is non-isomorphic to that in [1] it is sufficient to notice that it has the following property:

For each pair of blocks that contain 0, say

$$\{0, x_1, x_2, x_3\} \quad \text{and} \quad \{0, y_1, y_2, y_3\},$$

one can choose two other blocks containing 0, say

$$\{0, z_1, z_2, z_3\} \quad \text{and} \quad \{0, t_1, t_2, t_3\},$$

such that blocks

$$\{x_1, y_1, z_1, t_1\}, \quad \{x_2, y_2, z_2, t_2\}, \quad \{x_3, y_3, z_3, t_3\}$$

belong to $B(25)$.

It is easy to see that none of the elements in the construction from [1] has that property.

$B(28)$. We take

$$E = \{(i,j): i=1,\ldots,7,\ j=1,\ldots,4\}.$$

The set E can be written in the form of a (7×4)-matrix $A=(i,j)$ for $i=1,\ldots,7$, and $j=1,\ldots,4$.

Let us form a (7×3)-matrix B consisting of pairs of $I=\{1,\ldots,7\}$ according to the following procedure.

A pair (x,y) belongs to the j-th column if $x+y\equiv j \pmod 7$, a pair (x,y) belongs to the i-th row if $|x-y|\equiv i \pmod 7$. Next, let C be the (6×3)-matrix:

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 3 & 4 & 1 \end{pmatrix}.$$

Blocks: Consider all pairs (a,b) of elements of C such that a and b belong to the same column and $a\neq 0$, $b\neq 0$, $a\neq b$. Let a be in the i-th row and in the j-th column, and b in the k-th row and in the j-th column. If (x,y) is in the i-th row and in the n-th column of B, and (z,t) in the k-th row and in the n-th column of B, then $\{x,y,z,t\}\in B(28)$, where

(x, y) are elements of the a-th row of the matrix A, and (z, t) are elements of the b-th row of the matrix A, and $i \neq k, i = 1, 2, 3, n = 1, \ldots, 7$.

Moreover, in $B(28)$ we have

$\{(i, 1), (i, 2), (i, 3), (i, 4)\}$;

$\{(i, 1), (i+1, 2), (i+3, 4), (i+5, 3)\}$;

$\{(i, 1), (i+6, 2), (i+2, 3), (i+5, 4)\}$, $i = 1, \ldots, 7$, and $i+k$ is taken mod 7.

It is easy to see that this construction has only 21 blocks that form 3 groups of 7 mutually disjoint blocks each, and the remaining blocks have not this property. In the construction described in [1] all blocks form 9 groups of 7 mutually disjoint blocks each.

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