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# COMPACT MINIMALIZATION OF INVERSE SYSTEMS

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The aim of this note is to prove two theorems on inverse systems of sets.

THEOREM 1. *Let* $\{p_b^a\colon X_a \to X_b;\ a, b \in A\}$ *be a continuous inverse system of sets over a set* $A$. *If the set* $A$ *is totally directed, then the system has a compact minimalization.*

THEOREM 2. *Let* $A$ *be a directed set such that each continuous inverse system* $\{p_b^a\colon X_a \to X_b;\ a, b \in A\}$ *of sets, over the set* $A$, *has a compact minimalization. Then the set* $A$ *must be totally directed.*

Let us explain the terminology used in the theorems.

An inverse system $\{p_b^a\colon X_a \to X_b;\ a, b \in A\}$ has a *compact minimalization* iff there exist a directed subset $B \subset A$ cofinal in $A$ and a family $\{\mathcal{T}_b\colon b \in B\}$ of compact Hausdorff topologies such that the maps

$$p_b^a\colon X_a \to X_b; \quad a \geqslant b,\ a, b \in B,$$

between the topological spaces $(X_a, \mathcal{T}_a)$ and $(X_b, \mathcal{T}_b)$ are continuous. If $B = A$, then we say that the system has a *full compact minimalization*.

An ordered set $(A, \geqslant)$ is *totally directed* iff it contains a subset $B \subset A$ cofinal in $A$ and such that the order $\geqslant$ restricted to $B$ well orders $B$. Define a *directness number* $d(A)$ of an ordered set $A$ as

$$d(A) = \sup\{\tau\colon \forall B \subset A,\ |B| < \tau \Rightarrow \exists a \in A,\ a \geqslant B\},$$

where $a \geqslant B \Leftrightarrow \forall b \in B,\ a \geqslant b$. Observe that if a set $A$ is totally directed, then it contains a well-ordered cofinal subset of cardinality equal to $d(A)$.

An inverse system $\{p_b^a\colon X_a \to X_b;\ a, b \in A\}$ is *continuous* iff for each directed subset $B \subset A$, $|B| < d(A)$, and for an $a \in A$ such that $a \geqslant B$ the map

$$p_a\colon X_a \to \varprojlim\{p_c^b\colon X_b \to X_c;\ b, c \in B\}$$

induced by the bonding maps $p_b^a\colon X_a \to X_b$, $b \in B$, is onto. Notice that continuity of an inverse system ($B = \{b\}$) implies that the bonding maps must be onto.

We can infer Theorem 1 straight out from the following

THEOREM A. *Let $\gamma$ be an ordinal number and let*

$$\{p^\alpha_\beta\colon X_\alpha \to X_\beta;\ \alpha, \beta < \gamma\}$$

*be an inverse system of Hausdorff spaces X having for each $\alpha < \gamma$ a closed subset $Y_{\alpha+1} \subset X_{\alpha+1}$ and satisfying the following conditions:*

(1) *the subspace $X_{\alpha+1} \setminus Y_{\alpha+1}$ is locally compact;*

(2) $p^{\alpha+1}_\alpha(Y_{\alpha+1}) = X_\alpha$, *and* $p^{\alpha+1}_\alpha | Y_{\alpha+1}$ *is one-to-one;*

(3) *if $\alpha < \gamma$ is a limit ordinal, then*

$$X_\alpha = \varprojlim \{p^\beta_\delta\colon X_\beta \to X_\delta;\ \beta, \delta < \alpha\};$$

(4) *there exists a continuous one-to-one map $f\colon X_0 \to Y_0$ onto a compact Hausdorff space $Y_0$.*

*Then the inverse system has a full compact minimalization.*

*More precisely, on each space $X_\alpha$, $\alpha < \gamma$, it is possible to define a coarser topology such that the obtained new spaces $X^*_\alpha$ become compact Hausdorff and the bonding maps $p^\alpha_\beta\colon X^*_\alpha \to X^*_\beta$ are still continuous.*

Proof. Let $X^*_0$ be a topological space with topology given on the set $X_0$; $W$ is *open* in $X$ iff $W = f^{-1}(U)$, where $U$ is open in $Y_0$.

Assume that for each $\beta < \alpha$, $\alpha < \gamma$, the topological spaces $X^*_\beta$ are defined. If $\alpha$ is a limit ordinal, then let us put

$$X^*_\alpha = \varprojlim \{p^\beta_\delta\colon X^*_\beta \to X^*_\delta;\ \beta, \delta < \alpha\}.$$

If $\alpha = \beta + 1$ is a successor ordinal, then let us define a new topology on the set $X_\alpha$ generated by neighbourhood systems:

(a) If $x \in Y_\alpha = Y_{\beta+1}$, then open basic neighbourhoods of the point $x$ are sets of the form $(p^\alpha_\beta)^{-1}(V_x) \setminus A$, where $V_x$ is an open subset of $X^*_\beta$ such that $p^\alpha_\beta(x) \in V_x$ and $A \subset X_\alpha \setminus Y_\alpha$ is a compact subset of $X$.

(b) If $x \in X_\alpha \setminus Y_\alpha$, then open basic neighbourhoods of $x$ are open subsets $U_x \subset X_\alpha \setminus Y_\alpha$, $x \in U_x$.

One can verify that the new topology described on the set $X_\alpha$ is compact Hausdorff whenever the space $X^*_\beta$ is compact Hausdorff. It is clear that the map $p^\alpha_\beta\colon X^*_\alpha \to X^*_\beta$ is continuous.

COROLLARY. *A continuous inverse system*

$$\{p^\alpha_\beta\colon X^\alpha_\beta \to X_\beta;\ \alpha, \beta < \gamma\}$$

*of sets over a set of ordinals $\gamma$, $\gamma$ being regular, has a full compact minimalization.*

Since the limit of an inverse system of compact Hausdorff spaces is non-empty (see [1], Theorem 3.2.13, p. 188), Theorem 2 can be concluded from the following

THEOREM B. *If a directed set $A$ is not totally directed, then there exists a continuous inverse system*

$$\{p^a_b\colon X_a \to X_b;\ a, b \in A\}$$

*of sets, over the set $A$, with the empty limit.*

Proof. Let $\gamma$ be an ordinal number. We shall use the following notation:

$$\gamma = \{\alpha: \alpha < \gamma\} = [0, \gamma), \quad \gamma+1 = \{\alpha: \alpha \leqslant \gamma\} = [0, \gamma].$$

Each ordinal $\alpha$ can be uniquely represented as $\alpha = \beta + n$, where $\beta$ is a limit ordinal and $n \in \omega$ is a natural number. If $n = 2k+1$, $k \in \omega$, then $\alpha$ is said to be an *odd ordinal*, the other ordinals will be called *even ordinals*.

For each $a \in A$ let $X_a$ be the set of all maps $f$: $[0, \gamma] \to A$, $a = f(\gamma)$, where $\gamma < d(A)$ is an odd ordinal, and such that each map $f \in X_a$ satisfies the following conditions:

(1) If $\alpha \in \operatorname{dom} f$ is an even ordinal, then
  (i) $f(\beta) < f(\alpha)$ for each even ordinal $\beta < \alpha$;
  (j) $f(\beta) \geqslant f(\alpha)$ does not hold for any $\beta < \alpha$;
  (k) $f(\alpha+1) \leqslant f(\alpha)$.

(2) If $\alpha \in \operatorname{dom} f$ is an odd ordinal, then
  (i) $f(\beta) \geqslant f(\alpha)$ does not hold for any $\beta < \alpha - 1$;
  (j) $f(\alpha-1) \geqslant f(\alpha)$ (this follows from (1k)).

Now, let us define a bonding map

$$p_b^a: X_a \to X_b, \quad a \geqslant b, \quad a, b \in A.$$

For each $f \in X_a$, $f$: $[0, \gamma] \to A$, let us put

$$p_b^a(f) = g, \quad g: [0, \lambda] \to A,$$

where

(3) $$\lambda = 1 + \min\{\alpha < \gamma: \alpha \text{ is an even ordinal and } f(\alpha) \geqslant b\},$$

(4) $$g(\alpha) = \begin{cases} b & \text{if } \alpha = \lambda, \\ f(\alpha) & \text{if } \alpha < \lambda. \end{cases}$$

It is clear that

(5) $$g|[0, \lambda) = f|[0, \lambda).$$

The map $g$ is well defined, i.e., $g \in X_b$. To see this it suffices to check that $\lambda$ satisfies the condition (2i). Indeed, suppose that, for some $\beta < \lambda - 1$, $f(\beta) \geqslant f(\lambda) = b$. But if $\beta$ is an even cardinal, then this contradicts the definition of $\lambda$; if $\beta$ is an odd ordinal, then, by (2j), $f(\beta-1) \geqslant f(\beta) \geqslant b$, and again we get a contradiction with the definition of $\lambda$. One can verify that if $a \geqslant b \geqslant c$, then

$$p_c^b \circ p_b^a = p_c^a.$$

We shall prove that the limit of the inverse system

$$\{p_b^a: X_a \to X_b;\ a, b \in A\}$$

is empty and that the system is continuous.

Let $B \subset A$ be a directed subset of $A$ (we do not exclude that $B = A$). Assume that $x = \{f_b : b \in B\}$ belongs to

$$\lim_{\leftarrow} \{p_b^a : X_a \to X_b;\ a, b \in B\}.$$

Define a map $F_x : [0, \gamma_x] \to A$ by

$$F_x(\alpha) = f_b(\alpha), \quad \alpha < \gamma_b,\ b \in B, \tag{6}$$

where $\operatorname{dom} f_b = [0, \gamma_b]$, $f_b \in x$, and

$$\gamma_x = \sup\{\operatorname{dom} f_b : b \in B,\ f_b \in x\}. \tag{7}$$

To see that the map $F_x$ is well defined let us verify that for each pair $a, b \in B$ and $\alpha \in \gamma_a \cap \gamma_b$ we have $f_a(\alpha) = f_b(\alpha)$. Indeed, choose a $c \in B$ such that $c \geqslant a$ and $c \geqslant b$. Since $p_a^c(f_c) = f_a$ and $p_b^c(f_c) = f_b$, from (5) we get

$$f_c(\alpha) = f_a(\alpha) \quad \text{and} \quad f_c(\alpha) = f_b(\alpha) \quad \text{if } \alpha \in \gamma_a \cap \gamma_b.$$

Thus the map $F_x$ does not depend on the choice of a point $b \in B$. From the condition (2j) it follows that for each $b \in B$

$$F_x(\gamma_b - 1) = f_b(\gamma_b - 1) \geqslant b \tag{8}$$

and we conclude that $F_x$ has the following property:

The set $\{f(\alpha) : \alpha \text{ is an even ordinal, } \alpha < \gamma_x\}$ is a well-ordered subset of $A$ cofinal with $B$.

Thus, when $B = A$, we infer that the limit of the inverse system must be empty.

Now, we must show that the system is continuous. Assume that $|B| < d(A)$ and let $a \in A$ be such that $a \geqslant B$. We shall define a map $f_a^x \in X_a$ satisfying

$$p_b^a \circ f_a^x = f_b \tag{9}$$

for each $b \in B$, $f_b \in x$.

Consider two cases:

(a) $\gamma_x$ is a limit ordinal,

(b) $\gamma_x$ is a successor ordinal.

If $\gamma_x$ is a limit ordinal, choose a $c \in A$ such that $c > a$ and $c > F_x(\alpha)$ for each $\alpha < \gamma_x$, and put

$$f_a^x(\alpha) = \begin{cases} F_x(\alpha) & \text{if } \alpha < \gamma_x, \\ c & \text{if } \alpha = \gamma_x, \\ a & \text{if } \alpha = \gamma_x + 1, \end{cases} \tag{10}$$

where $\operatorname{dom} f_a^x = [0, \gamma_x + 1]$.

If $\gamma_x$ is a successor ordinal, then by (7) there is a $b_0 \in B$ such that $\gamma_x = \gamma_{b_0}$ ($\gamma_{b_0} = \operatorname{dom} f_{b_0}$, $f_{b_0} \in x$). Choose a $c \in A$ such that $c > a$ and $c > f_{b_0}(\gamma_{b_0} - 1)$.

In this case let us put

$$(11) \qquad f_a^x(\alpha) = \begin{cases} F_x(\alpha) & \text{if } \alpha < \gamma_x, \\ F_x(\gamma_x - 1) & \text{if } \alpha = \gamma_x, \\ c & \text{if } \alpha = \gamma_x + 1, \\ a & \text{if } \alpha = \gamma_x + 2. \end{cases}$$

It is easy to see that the condition (9) is fulfilled but before that we ought to check that $f_a^x \in X_a$. We shall verify that the map $f_a^x$ is well defined when $\gamma_x$ is a limit ordinal. It suffices to check that $\alpha = \gamma_x$ satisfies the condition (1) and $\alpha = \gamma_x + 1$ satisfies the condition (2). The choice of $c$ implies that the conditions (1i), (1j), (1k) and (2j) are fulfilled. Suppose that (2i) does not hold for $\alpha = \gamma_x + 1$, i.e., suppose that there is a $\beta < \gamma_x$ such that

$$f_a^x(\beta) \geqslant f_a^x(\gamma_x + 1) = a.$$

Since $\gamma_x$ is a limit ordinal, we may assume that, for some $b \in B$, $f_a^x(\beta) = f_b(\beta)$ and $\beta < \gamma_b - 1$. We get

$$f_b(\beta) \geqslant a \geqslant b = f_b(\gamma_b),$$

a contradiction with (2i). When $\gamma_x$ is a successor ordinal, the proof that $f_a^x \in X_a$ is similar. This completes the proof of the theorem.

Let $\boldsymbol{R}$ be the set of real numbers. Let us define

$$\boldsymbol{R}^* = (\boldsymbol{R}^*, \geqslant),$$

where

$$\boldsymbol{R}^* = \{a \subset \boldsymbol{R}: |a| \leqslant \omega\} \quad \text{and} \quad a \geqslant b \Leftrightarrow a \supset b.$$

Notice that $d(\boldsymbol{R}^*) = \omega_1$. It is easy to observe that $|\boldsymbol{R}| = \omega_1$ iff the set $\boldsymbol{R}^*$ is totally directed.

Thus Theorems 1 and 2 imply

COROLLARY. *The Continuum Hypothesis is equivalent to the following statement:*

*Each continuous inverse system* $\{p_b^a: X_a \to X_b;\ a, b \in \boldsymbol{R}^*\}$ *of sets, over the set* $\boldsymbol{R}^*$*, has a compact minimalization.*

Remarks. The assumption in Theorem 1 that the inverse system is continuous is necessary. For example, the existence of the Suslin line implies that there exists an inverse system

$$\{p_\beta^\alpha: X_\alpha \xrightarrow{\text{onto}} X_\beta;\ \alpha, \beta < \omega_1\}$$

of sets $X_\alpha$, $|X_\alpha| = \omega$, $\alpha < \omega_1$, which has the empty limit (see [5], p. 73). Hence the system has no compact minimalization. Another reason for the assumption of continuity is the observation that if we know that an inverse system the bonding maps of which are onto has a full compact minimalization, then the system must be continuous (this follows from Theorem 3.2.12 in [1], p. 189).

The condition (4) of Theorem A is fulfilled when $X_0$ is a locally compact Hausdorff space. Indeed, let $Y_0$ be a space obtained by introducing a new topology on the set $X_0$ in the following way: Choose a point $x_0 \in X_0$. Define a set $U \subset X_0$ to be open in the space $Y_0$ iff $U$ is open in the space $X_0$ and $X_0 \setminus U$ is compact whenever $x_0 \in U$. The space $Y_0$ obtained in such a way is compact Hausdorff and the identity map $f: X_0 \to Y_0$, $f(x) = x$, is continuous. In general, the question what kind of spaces have continuous one-to-one maps onto compact Hausdorff spaces is difficult. The question was raised independently by Banach (see Problem 1 in [6]) and Russian mathematicians (see [3]).

Observe that each inverse sequence

$$\{p_m^n: X_n \xrightarrow{\text{onto}} X_m;\ n, m \in \omega\}$$

of discrete spaces satisfies the assumptions of Theorem A. On the other hand, each 0-dimensional complete metric space is an inverse sequence of discrete spaces with the bonding maps being into. Thus from the above remark and from Theorem A we get immediately

COROLLARY. *For each 0-dimensional complete metric space $X$ there exists a continuous one-to-one map $f: X \to Y$ onto a compact Hausdorff space $Y$.*

The result was proved by the author in [4] and in a very complicated way in [7]. Theorem A gives a simple proof of the fact.

Theorem B is a generalization of a construction of Henkin [2], who has proved

THEOREM. *If a directed set $A$ does not contain a countable cofinal subset, then there exists an inverse system*

$$\{p_b^a: X_a \xrightarrow{\text{onto}} X_b;\ a, b \in A\}$$

*of sets, over the set $A$, with the empty limit.*

Of course, the inverse system constructed by Henkin is not, in general, continuous.

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