FUNDAMENTA
MATHEMATICAE
Online First version

Applications of infinity-Borel codes to definability and definable cardinals

by

William Chan (Wien) and **Stephen Jackson** (Denton, TX)

Abstract. Woodin introduced an extension of the axiom of determinacy, AD, called AD^+ which includes an assertion that all sets of reals have an ∞-Borel code. An ∞-Borel code is a pair (φ, S) where φ is a formula and S is a set of ordinals which provides a highly absolute definition for a set of reals. This paper will use AD^+ and ∞-Borel codes to establish a property of ordinal definability analogous to a property for Σ^1_1 shown by Harrington–Shore–Slaman (2017). Under AD^+, the paper will also use ∞-Borel codes to explore the cardinality of sets below $\mathscr{P}(\omega_1)$ which Woodin (2006) began investigating under $\mathsf{AD}_\mathbb{R}$ and DC. The following summarizes the main results.

Assume $\mathsf{ZF}+\mathsf{AD}^+ + \mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$. If $H \subseteq \mathbb{R}$ has the property that there is a nonempty OD set of reals K such that H is OD_z for any $z \in K$, then H is OD.

Assume $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R} + \mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$. Then there is a cardinal strictly between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}| = |\mathscr{P}(\omega_1)|$.

Assume $\mathsf{ZF} + \mathsf{AD}^+$. Then $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$ does not inject into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals. This implies $|\mathbb{R}| < |S_1|$ and $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

Assume $\mathsf{ZF} + \mathsf{AD}^+$. Let X be a surjective image of $\mathbb{R}$ and let $\mathscr{P}_{\omega_1}(X) = \{A \subseteq X : |A| < \omega_1\}$. If $\omega_1 \leq |\mathscr{P}_{\omega_1}(X)|$, then $\omega_1 \leq |X|$. If $|\mathscr{P}(\omega_1)| = |[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$, then $|\mathbb{R} \sqcup \omega_1| \leq |X|$.

$\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ implies that the uncountable cardinals below $|\mathbb{R} \times \omega_1|$ are ω_1, $|\mathbb{R}|$, $|\mathbb{R} \sqcup \omega_1|$, and $|\mathbb{R} \times \omega_1|$. An elaborate structure of cardinals below $|\mathbb{R} \times \omega_1|$ is described under the assumption of $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R} + \mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$.

1. Introduction. An ∞-*Borel code* is simply a pair (S, φ) where S is a set of ordinals and φ is a formula of set theory. The set of reals defined by (S, φ) is $\mathfrak{B}^1_{(S,\varphi)} = \{x \in \mathbb{R} : L[S, x] \models \varphi(S, x)\}$. If A is a set of reals, then one says that (S, φ) is an ∞-Borel code *for* A if $\mathfrak{B}^1_{(S,\varphi)} = A$. An ∞-Borel code for A is a highly absolute definition for A in the sense that to

2020 *Mathematics Subject Classification*: Primary 03E60; Secondary 03E45, 03E10.
Key words and phrases: determinacy, infinity-Borel codes, ordinal definability, cardinalities.
Received 1 October 2021; revised 20 January 2024.
Published online 27 March 2024.

determine membership of $x \in A$, one simply needs to go into $L[S,x]$, which is the minimal model of ZFC containing the code S and x, and ask whether $L[S,x] \models \varphi(S,x)$. Note that for any inner model $M \models \mathsf{ZF}$ with $S \in M$, $(\mathfrak{B}^1_{(S,\varphi)})^M = \mathfrak{B}^1_{(S,\varphi)} \cap M$.

The axiom of determinacy, AD, states that certain two player games have a winning strategy for one of the two players. Mathematics under AD is often regarded as being more effective, uniform, or definable. Woodin [21] isolated an extension of AD called AD^+ which includes $\mathsf{DC}_{\mathbb{R}}$, a technical statement called ordinal determinacy, and the statement that all sets of reals have an ∞-Borel code. The existence of ∞-Borel codes strengthens the claim that AD^+ captures definable combinatorics.

It is not known if AD can prove any of the three statements in AD^+. Kechris [12] and Woodin showed that if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models \mathsf{AD}^+$. Moreover, Woodin showed that in natural models of AD^+, i.e. those models which satisfy $\mathsf{ZF} + \mathsf{AD} + \mathsf{V{=}L}(\mathscr{P}(\mathbb{R}))$, more is known about the structure of ∞-Borel codes. In particular, in models of the form $L(J,\mathbb{R}) \models \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ where J is a set of ordinals, Woodin's result that $L(J,\mathbb{R})$ is a symmetric collapse extension of $\mathrm{HOD}_J^{L(J,\mathbb{R})}$ outlines a procedure to obtain ∞-Borel codes from definitions witnessing ordinal definability.

Under AD^+, the Vopěnka forcing of nonempty OD subsets of $\mathbb{R}$ ordered by $\subseteq$ becomes a very powerful tool. In the presence of strongly absolute definitions provided by the ∞-Borel codes, the method of the Vopěnka forcing in local models of the form $\mathrm{HOD}_S^{L[S,X]}$, where S is a fixed set of ordinals and X varies over the Turing degrees, combined with the ultraproduct $\prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$ where μ is the Martin measure on Turing degrees is especially useful for combinatorics under AD^+.

For instance, similar techniques were used by Woodin to prove the perfect set dichotomy [2] which generalized Silver's Π^1_1 equivalence relation dichotomy [18], and by Hjorth [10] to prove the more general E_0-dichotomy which generalizes the E_0-dichotomy of Harrington–Kechris–Louveau [8]. Woodin's result also uses the fact that countable section uniformization for relations on $\mathbb{R} \times \mathbb{R}$ holds under AD^+ (see [15, 2]). Such techniques are also used in [3] to answer a question of Foreman that there are no Suslin lines in $L(\mathbb{R}) \models \mathsf{AD}$. In [4], the ∞-Borel codes, Vopěnka forcing, and the ultraproduct are used to show that if $\langle E_\alpha : \alpha < \omega_1 \rangle$ is a sequence of equivalence relations on $\mathbb{R}$ with all classes countable such that $|\mathbb{R}/E_\alpha| = |\mathbb{R}|$, then the disjoint union $\bigsqcup_{\alpha<\omega_1} \mathbb{R}/E_\alpha$ is in bijection with $\mathbb{R} \times \omega_1$.

This article provides some new applications of ∞-Borel codes and the Vopěnka forcing to questions about ordinal definability and definable cardinals assuming AD^+ or specifically in natural models of AD^+.

Harrington, Shore, and Slaman [9] showed that if $H \subseteq \mathbb{R}$ has the property that there is a nonempty Σ^1_1 $K \subseteq \mathbb{R}$ such that H is $\Sigma^1_1(z)$ for any $z \in K$, then H is Σ^1_1. In other words, if H is Σ^1_1 in any parameter z from a nonempty Σ^1_1 set K, then H is actually Σ^1_1 with no parameters.

One can ask if a similar phenomenon holds for other notions of lightface definability. Ordinal definability is a strong notion of definability which is closed under nearly any operation which does not introduce nonordinal parameters. One can ask if $H \subseteq \mathbb{R}$ is OD_z in any parameter z from a nonempty OD set of reals K, then is H ordinal definable with no parameters.

The answer is generally not positive under ZF since Fact 3.2 shows that in the Sacks generic extension of the constructible universe L, the Sacks generic real is OD_z from any nonconstructible z but the Sacks generic real is not OD. However, in natural models of AD^+ the answer is positive:

THEOREM 3.1. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V{=}L}(\mathscr{P}(\mathbb{R}))$. *Let J be a set of ordinals. Let $H \subseteq \mathbb{R}$. Let $K \subseteq \mathbb{R}$ be nonempty and* OD_J. *If H is* $\mathrm{OD}_{J,z}$ *for all $z \in K$, then H is* OD_J.

Using the arguments of Woodin in the proof that $L(J,\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ is a symmetric collapse extension of $\mathrm{HOD}_J^{L(J,\mathbb{R})}$, one can show that in $L(J,\mathbb{R})$, there is a set of ordinals $\mathbb{X}$ which "absorbs" functions of various types. As an example, this means that for any function $\Phi : [\omega_1]^{\omega_1} \to [\omega_1]^{<\omega_1}$ (or $\Phi : \mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$), there is a real e such that for all z with $e \leq_{\mathbb{X}} z$ and $f \in [\omega_1]^{\omega_1} \cap L[\mathbb{X}, z]$, $\Phi(f) \in L[\mathbb{X}, z]$ and $\Phi \cap L[\mathbb{X}, z] \in L[\mathbb{X}, z]$. This function absorption idea is especially useful for studying definable cardinality under AD^+ and for producing intermediate cardinalities in natural models of AD^+.

It is shown in [5] that $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}| = |\mathscr{P}(\omega_1)|$ by establishing an almost everywhere continuity phenomenon for functions of the form $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. Section 4 gives a more set-theoretic argument as well as other conditions on cardinals κ which imply that $|[\kappa]^{<\kappa}| < |[\kappa]^{\kappa}|$. This section also shows that in models of the form $L(J,\mathbb{R})$, where J is a set of ordinals, there is a cardinal intermediate between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}|$:

THEOREM 4.10. *Assume* $\mathsf{ZF} + \mathsf{AD}^+$. *Let $J \subseteq \mathrm{ON}$ be a set of ordinals such that $V = L(J,\mathbb{R})$. Let $\mathbb{X} = (J, {}_{\omega}\mathbb{O}_J)$ (see Section 2 for more details). Define N_1^J by*

$$N_1^J = \bigsqcup_{r \in \mathbb{R}} ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} = \{(r,\alpha) : \alpha < ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}\}.$$

Then the following cardinal relations hold: $\neg(|N_1^J| \leq [\omega_1]^{<\omega_1})$, $|\mathbb{R} \times \omega_1| < |N_1^J| < |\mathbb{R} \times \omega_2|$, $|N_1^J| < |[\omega_1]^{\omega_1}|$, $\neg(|[\omega_1]^{\omega}| \leq |N_1^J|)$, *and* $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{<\omega_1} \sqcup N_1^J| < |[\omega_1]^{\omega_1}|$.

Intuitively, $[\omega_1]^{\omega}$ and $[\omega_1]^{<\omega_1}$ appear to be distinct subsets of $\mathscr{P}(\omega_1)$ in terms of cardinality. It is implicit in [20] that under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$,

$|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. It appears that these cardinal distinctions are obtained through an analysis of the set $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$, defined by Woodin. Section 5 will study S_1 using ∞-Borel codes and the function absorption idea under AD^+.

In just AD, one can show that $|\mathbb{R}| \le |S_1|$ and $\neg(\omega_1 \le |S_1|)$. However, all other interesting properties of S_1 appear to be only known under the existence of ∞-Borel codes. The main property of S_1 is that it does not inject into the class of ω-sequences of ordinals.

Theorem 5.7. *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_\mathbb{R}$ *and all sets of reals have* ∞*-Borel codes. Then there is no injection of* S_1 *into* ${}^\omega\mathrm{ON}$*, the class of* ω*-sequences of ordinals.*

This result can then be used to give the following cardinality computation under AD^+:

Theorem 5.8. *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_\mathbb{R}$ *and all sets of reals have* ∞*-Borel codes. Then* $|\mathbb{R}| < |S_1|$ *and* $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

The proof of Theorem 5.7 involves finding a filter which is generic over a model of ZFC for a forcing in this model which is countable in the real world satisfying AD. If one would like to imitate this argument to establish similar results on ω_2, then the naturally associated forcing in a model of ZFC would be uncountable even in the real world and hence one may not have generics for this forcing. Thus the AD^+ methods in Theorem 5.7 are not suitable for generalization to ω_2.

By its definition, S_1 involves notions of constructibility which makes ∞-Borel definition quite useful for studying properties of its cardinality. However, $[\omega_1]^\omega$ and $[\omega_1]^{<\omega_1}$ are purely combinatorial objects whose cardinal distinctions should be obtainable under AD alone. By establishing an almost everywhere continuity result for functions of the form $\Phi : [\omega_1]^\epsilon \to \omega_1$, where $\epsilon < \omega_1$, [7] shows in just AD that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$. This argument provides the suitable template for studying combinatorics on ω_2. By establishing an almost everywhere continuity result for functions of the form $\Phi : [\omega_2]^\epsilon \to \omega_2$, where $\epsilon < \omega_2$, [7] shows in AD that $|[\omega_2]^\omega| < |[\omega_2]^{<\omega_1}| < |[\omega_2]^{\omega_1}| < |[\omega_2]^{<\omega_2}|$. More recently, [6] established these almost everywhere continuity properties purely from combinatorial principles. Thus [6] showed that if κ is a weak partition cardinal (i.e. it satisfies $\kappa \to (\kappa)^{<\kappa}_2$), then for all $\chi < \kappa$, $[\kappa]^{<\kappa}$ does not inject into ${}^\chi\mathrm{ON}$, the class of length-χ sequences of ordinals, and so $|[\kappa]^\chi| < |[\kappa]^{<\kappa}|$. Hence these cardinality results apply to all the familiar weak and strong partition cardinals of determinacy such as $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ and $\boldsymbol{\delta}^2_1$.

Using the properties of S_1, one can answer an interesting question of Zapletal: If X is a set, let $\mathscr{P}_{\omega_1}(X) = \{A \subseteq X : |A| < \omega_1\}$ and let $\mathscr{P}_{\mathrm{WO}}(X)$ be the collection of $A \subseteq X$ which are wellorderable. Zapletal asked that if $\mathscr{P}_{\omega_1}(X)$ has certain cardinality properties, then what can be said about the cardinality properties of X. A concrete question would be: If ω_1 injects into $\mathscr{P}_{\omega_1}(X)$, does ω_1 already inject into X? The following gives a positive answer:

THEOREM 6.6. *Assuming* $\mathsf{ZF} + \mathsf{AD}^+$, *for all cardinals* $\kappa < \Theta$ *and all sets* X *which are surjective images of* $\mathbb{R}$, $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies* $\kappa \leq |X|$. *In particular,* $\kappa \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\kappa \leq |X|$.

COROLLARY 6.7. *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ *and all sets of reals have* ∞-*Borel codes. Let* X *be a set which is a surjective image of* $\mathbb{R}$. *Then* $\omega_1 \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies* $\omega_1 \leq |X|$. *In particular,* $\omega_1 \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\omega_1 \leq |X|$.

One can ask what other sets Y have the property that if Y injects into $\mathscr{P}_{\omega_1}(X)$, then X already has a copy of Y. Note that $\mathscr{P}_{\omega_1}(\omega_1) = [\omega_1]^{<\omega_1}$. Thus for any uncountable $Y \subseteq [\omega_1]^{<\omega_1}$ such that $|Y| \neq \omega_1$, Y injects into $\mathscr{P}_{\omega_1}(\omega_1)$, but Y does not inject into ω_1. This reflection property fails for any $Y \subseteq [\omega_1]^{<\omega_1}$ with $|Y| \neq \omega_1$. Naturally, one can ask: If $[\omega_1]^{\omega_1}$ injects into $\mathscr{P}_{\omega_1}(X)$, what can be said about the cardinality of X? The following results shows that X must contain a copy of ω_1 and $\mathbb{R}$:

THEOREM 6.10. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have* ∞-*Borel codes. Let* X *be a set which is a surjective image of* $\mathbb{R}$. *If* $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$, *then* $|\mathbb{R} \sqcup \omega_1| \leq |X|$.

A natural conjecture would be that if $[\omega_1]^{\omega_1}$ injects into $\mathscr{P}_{\omega_1}(X)$, then $[\omega_1]^{\omega_1}$ already injects into X. However, an easier question may be: If $[\omega_1]^{\omega_1}$ injects into $\mathscr{P}_{\omega_1}(X)$, does $\mathbb{R} \times \omega_1$ inject into X?

Woodin [20] showed using elaborate AD^+ techniques that under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$, the uncountable cardinals below $[\omega_1]^\omega$ are ω_1, $|\mathbb{R}|$, $|\mathbb{R} \sqcup \omega_1|$, $|\mathbb{R} \times \omega_1|$, and $[\omega_1]^\omega$. Using a simple uniformization argument, Corollary 7.6 shows that under $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, the uncountable cardinals below $|\mathbb{R} \times \omega_1|$ are ω_1, $|\mathbb{R}|$, $|\mathbb{R} \sqcup \omega_1|$, and $|\mathbb{R} \times \omega_1|$. Woodin showed that if $\mathsf{AD}_{\mathbb{R}}$ fails, then there may be other cardinalities below $|\mathbb{R} \times \omega_1|$.

The final section studies the uncountable cardinalities below $|\mathbb{R} \times \omega_1|$ in natural models of $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}}$ such as $L(J, \mathbb{R})$ where J is a set of ordinals which “absorbs” all functions from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$. Let $\mathfrak{V}$ denote all the cardinals $\mathcal{X}$ below $|\mathbb{R} \times \omega_1|$ such that $\neg(\omega_1 \leq \mathcal{X})$. Fact 7.4 shows that every cardinal $\mathcal{Z} \leq |\mathbb{R} \times \omega_1|$ either is in $\mathfrak{V}$ or is the disjoint union of ω_1 with some cardinality in $\mathfrak{V}$. Thus a complete understanding of $\mathfrak{V}$ would elucidate the structure of the cardinals below $|\mathbb{R} \times \omega_1|$.

Let $\mathcal{D}_J$ and μ_J denote the J-constructible degrees and the Martin measures on J-degrees, respectively. For any $F : \mathbb{R} \to \omega_1$ which is J-invariant, let $W_F^J = \bigsqcup_{r \in \mathbb{R}} \omega_{F(r)}^{L[J,r]}$. For any $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1 / \mu_J$, there exists an everywhere increasing J-invariant $F : \mathbb{R} \to \omega_1$ which represents $\mathcal{F}$. Let $Y_{\mathcal{F}}^J = |W_F^J|$ for any everywhere increasing J-invariant $F : \mathbb{R} \to \omega_1$ which represents $\mathcal{F}$. (It can be shown that $Y^J_{\mathcal{F}}$ is independent of the choice of F.)

Woodin showed that $\prod_{X\in\mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J = \omega_1$ for any set J of ordinals and $\prod_{X\in\mathcal{D}_J} \omega_2^{L[J,X]} = \Theta$ if J is an "ultimate ∞-Borel code" in $V = L(J,\mathbb{R})$. For $\alpha < \omega_1$, let $F^\alpha : \mathbb{R} \to \omega_1$ be the constant function taking value α. It can be shown that F^α represents the ordinal α in $\prod_{\mathcal{D}_J} \omega_1/\mu_J$. Thus $Y^J_\alpha = |W^J_{F^\alpha}|$ for each $\alpha < \omega_1$.

Let $\mathfrak{Y} = \{Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J\}$. Then $\mathfrak{Y} \subseteq \mathfrak{V}$. It can be shown that $Y^J_0 = Y^J_1 = |\mathbb{R}|$. If $\mathcal{F}_1 < \mathcal{F}_2$ in the ultrapower ordering, then $Y^J_{\mathcal{F}_1} < Y^J_{\mathcal{F}_2}$. Also for any $\mathcal{Y} \in \mathfrak{V}$, there is some $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$ such that $\mathcal{Y} \le Y^J_{\mathcal{F}}$. By analyzing the behavior of $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$ which are successor ordinals and limit ordinals of cofinality ω, one can show that $\langle Y^J_\alpha : \alpha < \omega_1\rangle$ is the length-ω_1 initial segment of $\mathfrak{V}$. The following summarizes the results of Section 7.

Theorem. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and* $V = L(J,\mathbb{R})$ *where* J *is a set of ordinals which absorbs functions from* $\mathbb{R} \times \omega_1$ *to* $\mathbb{R} \times \omega_1$.

- $\langle Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}\rangle$ *is an order preserving injection of the ultraproduct ordering into* $\mathfrak{Y}$ *with the injection ordering.*
- $\mathfrak{Y}$ *is cofinal in* $\mathfrak{V}$*: For all* $\mathcal{X} \in \mathfrak{V}$*, there is an* $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}$ *such that* $\mathcal{X} \le Y^J_{\mathcal{F}}$.
- *For any* $\mathcal{X} \in \mathfrak{V}$ *and* $F \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}$*, either* $\mathcal{X} \le Y^J_{\mathcal{F}}$ *or* $Y^J_{\mathcal{F}} \le \mathcal{X}$.
- $\{Y^J_\alpha : \alpha < \omega_1\}$ *is the length-*ω_1 *initial segment of* $\mathfrak{V}$*: for any cardinality* $\mathcal{X}$ *below* $|\mathbb{R} \times \omega_1|$ *such that* $\neg(\omega_1 \le \mathcal{X})$*, either there exists an* $\alpha < \omega_1$ *such that* $\mathcal{X} = Y^J_\alpha$ *or for all* $\alpha < \omega_1$, $Y^J_\alpha \le \mathcal{X}$.

A very appealing conjecture given these results is that $\mathfrak{V} = \mathfrak{Y}$. Let $F^{\omega_1} : \mathbb{R} \to \omega_1$ be defined by $F^{\omega_1}(x) = \omega_1^{L[J,x]}$. It can be shown that F^{ω_1} represents ω_1 in $\prod_{\mathcal{D}_J} \omega_1/\mu_J$. Is $Y^J_{\omega_1} = |W^J_{F^{\omega_1}}|$ the ω_1th cardinality in $\mathfrak{V}$ in the sense that for all $\mathcal{X} \in \mathfrak{V}$ such that $\mathcal{X} \le Y^J_{\omega_1}$, there is an $\alpha \le \omega_1$ such that $\mathcal{X} = Y^J_\alpha$? The difficulty is that the behavior of cardinalities below $Y^J_{\mathcal{F}}$ when $\mathcal{F}$ has uncountable cofinality is not well understood.

2. Basics. This section summarizes some properties of ∞-Borel codes, Vopěnka forcing, and the Martin measure that will be needed throughout the paper. The reader can refer to [2] for a detailed exposition of these ideas at least in the $L(\mathbb{R}) \models \mathsf{AD}$ setting.

Definition 2.1. Let $S \subseteq \mathrm{ON}$ be a set of ordinals and φ be a formula of set theory. The pair (S,φ) is called an ∞-*Borel code*. For any $n \in \omega$, define $\mathfrak{B}^n_{(S,\varphi)} = \{x \in \mathbb{R}^n : L[S,x] \models \varphi(S,x)\}$.

If $A \subseteq \mathbb{R}^n$, then (S, φ) is an ∞-*Borel code for* A if $\mathfrak{B}^n_{(S,\varphi)} = A$. A set $A \subseteq \mathbb{R}^n$ is said to be ∞-*Borel* if it has an ∞-Borel code.

Note that an ∞-Borel set of reals has a very absolute definition in the following sense: If $A \subseteq \mathbb{R}$ is ∞-Borel with ∞-Borel code (S, φ), then $x \in A$ is completely determined by whether $\varphi(S, x)$ holds in the minimal model of ZFC, $L[S, x]$, containing the code (S, φ) and the real x.

Definition 2.2. Let $n > 0$ and $S \subseteq \mathrm{ON}$ be a set of ordinals. Let ${}_n\mathbb{O}_S$ denote the forcing of nonempty OD_S subsets of $\mathbb{R}^n$ ordered by $\subseteq$ with largest element $1_{{}_n\mathbb{O}_S} = \mathbb{R}^n$. We will write $\mathbb{O}_S$ for ${}_1\mathbb{O}_S$.

Since there is an S-definable bijection of OD_S with ON, one can transfer ${}_n\mathbb{O}_S$ onto the ordinals. In this way, ${}_n\mathbb{O}_S$ is a forcing in HOD_S.

Definition 2.3. Let S be a set of ordinals. For each $k \in \omega$, let $b_k = \{x \in \mathbb{R} : x(k) = 1\}$. Note that $b_k \in \mathbb{O}_S$. Let $\dot{x}_{\mathrm{gen}} = \{(\check{k}, b_k) : k \in \omega\}$. Note that $\dot{x}_{\mathrm{gen}}$ is an $\mathbb{O}_S$-name which adds a real.

One can formulate the analogous ${}_n\mathbb{O}_S$-name $\dot{x}^n_{\mathrm{gen}}$ for adding an element of $\mathbb{R}^n$ for all $n \geq 1$.

Fact 2.4. *Let S be a set of ordinals. For each $x \in \mathbb{R}^n$, $G^n_x = \{p \in {}_n\mathbb{O}_S : x \in p\}$ is a* HOD_S*-generic filter such that* $\dot{x}^n_{\mathrm{gen}}[G^n_x] = x$ *and* $\mathrm{HOD}_S[G^n_x] = \mathrm{HOD}_S[x]$.

Fact 2.5 ([10, Theorem 2.4], [2, Fact 8.1]). *Let M be a transitive inner model of* ZF. *Let $S \in M$ be a set of ordinals. Suppose* $K \in \mathrm{HOD}^M_S$ *is a set of ordinals and φ is a formula. Let N be a transitive inner model with* $\mathrm{HOD}^M_S \subseteq N$. *Let $p = \{x \in \mathbb{R} : L[K, x] \models \varphi(K, x)\}$, so p is a condition of $\mathbb{O}^M_S$. Then $N \models p \Vdash_{\mathbb{O}^M_S} L[\check{K}, \dot{x}_{\mathrm{gen}}] \models \varphi(\check{K}, \dot{x}_{\mathrm{gen}})$.*

Definition 2.6 (Woodin, [21, Section 9.1]). AD^+ consists of the following:

(1) $\mathsf{DC}_{\mathbb{R}}$.
(2) Every $A \subseteq \mathbb{R}$ is ∞-Borel.
(3) For all $\lambda < \Theta$, $A \subseteq \mathbb{R}$, and continuous $\pi : {}^{\omega}\lambda \to \mathbb{R}$, $\pi^{-1}[A]$ is determined.

Models satisfying $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$ are called *natural models* of AD^+. Woodin showed that these either are models of $\mathsf{AD}_{\mathbb{R}}$ or take the form $V = L(J, \mathbb{R})$ for a set J of ordinals:

Fact 2.7 (Woodin, [1, Corollary 3.2]). *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + \mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Then there is a set J of ordinals such that $V = L(J, \mathbb{R})$.*

Many results about $L(\mathbb{R})$ proved by Vopěnka forcing can be relativized to analogous statements about models of the form $L(J, \mathbb{R})$.

Fact 2.8 (Woodin, [1, Theorem 3.4]). *Assume* $\mathsf{ZF}+\mathsf{AD}^{+}+\mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let* J *be a set of ordinals and* $A \subseteq \mathbb{R}$. *If* A *is* OD_J, *then* A *has an* OD_J ∞*-Borel code.*

Fact 2.9 (Woodin, [1, Theorem 2.8]). *Assume* $\mathsf{ZF}+\mathsf{AD}^{+}+\mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let* J *be a set of ordinals. There is a set* $\mathbb{X}$ *of ordinals such that* $\mathrm{HOD}_J = L[\mathbb{X}]$.

Proof. See [2, Corollary 7.21] for a proof of this under $\mathsf{AD}+\mathsf{V}{=}\mathsf{L}(\mathbb{R})$. ∎

Woodin's work showing that $L(J,\mathbb{R}) \models \mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ is a symmetric collapse extension of $\mathrm{HOD}_J^{L(J,\mathbb{R})}$ gives additional information about ∞-Borel codes in such models. In particular, it shows the existence of an ultimate ∞-Borel code mentioned above, which will be particularly useful in this article for "absorbing fragments of functions" in relevant models of ZFC.

Assume $V = L(J,\mathbb{R}) \models \mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$. For each $m \le n < \omega$, let $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$ be the projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$. One can induce a map $\pi_{n,m} : {}_n\mathbb{O}_J \to {}_m\mathbb{O}_J$ by $\pi_{n,m}(p) = \pi_{n,m}[p]$, where the latter $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$ is the projection map. These maps $\pi_{n,m} : {}_n\mathbb{O}_J \to {}_m\mathbb{O}_J$ are forcing projections. Let ${}_\omega\mathbb{O}_J$ denote the finite support direct limit induced by $\langle {}_n\mathbb{O}_J, \pi_{n,m} : 1 \le m \le n < \omega\rangle$. Let $\pi_{\omega,n} : {}_\omega\mathbb{O}_J \to {}_n\mathbb{O}_J$ be the natural associated projection map.

Each $s \in \mathbb{R}^n$ canonically induces an ${}_n\mathbb{O}_J$-generic filter over $\mathrm{HOD}_J^{L(J,\mathbb{R})}$ denoted by G_s^n. Using $\pi_{\omega,n}$, let ${}_\omega\mathbb{O}_J/G_s^n$ refer to the associated remainder forcing. Moreover, every $G \subseteq {}_n\mathbb{O}_J$ which is ${}_n\mathbb{O}_J$-generic over HOD_J adds a generic element of $\mathbb{R}^n$. For each n, let τ_n be the ${}_\omega\mathbb{O}_J$-name for the real in the last coordinate of the generic n-tuple of reals added by the ${}_n\mathbb{O}_J$-generic filter induced from an ${}_\omega\mathbb{O}_J$-generic filter. Let $\dot{\mathbb{R}}_{\mathrm{sym}}$ be the ${}_\omega\mathbb{O}_J$-name for the set $\{\tau_n : n \in \omega\}$. Let $\dot{x}^n_{\mathrm{gen}}$ be a name denoting $\langle \tau_i : i < n\rangle$.

Fact 2.10 (Woodin). *Suppose* $L(J,\mathbb{R}) \models \mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$. *Let* $s \in \mathbb{R}^n$, $z \in L[J, {}_\omega\mathbb{O}_J, s]$, *and* φ *be a formula. Then* $L(J,\mathbb{R}) \models \varphi(J,s,z)$ *if and only if*

$$L[J, {}_\omega\mathbb{O}_J, s] \models 1_{{}_\omega\mathbb{O}_J/G_s^n} \Vdash_{{}_\omega\mathbb{O}_J/G_s^n} L(\check{J}, \dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(\check{J}, \dot{x}^n_{\mathrm{gen}}, \check{z}).$$

Fact 2.10 can be used to show that in $L(J,\mathbb{R}) \models \mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$, for any $A \subseteq \mathbb{R}$, there is an $r \in \mathbb{R}$ and a formula φ such that $(J \oplus {}_\omega\mathbb{O}_J \oplus r, \varphi)$ forms an $\mathrm{OD}_{J,s}$ ∞-Borel code for A, where $J \oplus {}_\omega\mathbb{O}_J \oplus r$ is a set of ordinals that codes these three objects in some fixed way. It also gives the following result.

Fact 2.11 (Woodin). *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ *and there is a set* $J \subseteq \mathrm{ON}$ *such that* $V = L(J,\mathbb{R})$. *For each* $x \in \mathbb{R}$, $\mathrm{HOD}_{J,x}^{L(J,\mathbb{R})} = L[J, {}_\omega\mathbb{O}_J, x]$.

A more detailed exposition of the above results can be found in [2] in the $L(\mathbb{R})$ case. It should be noted that here these results are stated for the Vopěnka forcing $\mathbb{O}$. These results were initially proved using $\mathbb{A}$, which is the forcing of nonempty sets of reals with OD ∞-Borel codes. It was then shown that $\mathbb{O}$ and $\mathbb{A}$ are the same.

DEFINITION 2.12. Let $x \leq_{\mathsf{Turing}} y$ indicate that x is Turing reducible to y. Let $x \equiv_{\mathsf{Turing}} y$ indicate $x \leq_{\mathsf{Turing}} y$ and $y \leq_{\mathsf{Turing}} y$. Let $\mathcal{D} = \mathbb{R}/{\equiv_{\mathsf{Turing}}}$ denote the collection of Turing degrees. For $X, Y \in \mathcal{D}$, let $X \leq Y$ indicate that there are $x \in X$ and $y \in Y$ such that $x \leq_{\mathsf{Turing}} y$. If $X \in \mathcal{D}$, then the *Turing cone* above X is the set $\{Y \in \mathcal{D} : X \leq Y\}$. The *Martin measure* μ on $\mathcal{D}$ is the collection of subsets of $\mathcal{D}$ which contain a Turing cone.

Let $J \subseteq \mathrm{ON}$ be a set of ordinals. On $\mathbb{R}$, define $x \leq_J y$ if and only if $x \in L[J, y]$. Let $x \equiv_J y$ if and only if $x \leq_J y$ and $y \leq_J x$. Let $\mathcal{D}_J = \mathbb{R}/{\equiv_J}$ denote the collection of J-constructibility degrees. If $X, Y \in \mathcal{D}_J$, then let $X \leq Y$ indicate that there exist $x \in X$ and $y \in Y$ such that $x \leq_J y$. If $X \in \mathcal{D}_J$, then the *J-cone* above X is the set $\{Y \in \mathcal{D}_J : X \leq Y\}$. Let μ_J be the collection of subsets of $\mathcal{D}_J$ which contain a J-cone.

FACT 2.13 (Martin). *Assume $\mathsf{ZF} + \mathsf{AD}$. Then μ is a countably complete ultrafilter. For any $J \subseteq \mathrm{ON}$, μ_J is a countably complete ultrafilter.*

FACT 2.14 (Woodin, [1, Section 2.2]). *Assume $\mathsf{ZF} + \mathsf{AD}^+$. Then the ultrapower of the ordinals by Martin's Turing cone measure, $\prod_{X \in \mathcal{D}} \mathrm{ON}/\mu$, is a wellordering. So also is $\prod_{X \in \mathcal{D}_J} \mathrm{ON}/\mu_J$, the ultrapower of the ordinals by the J-constructibility cone measure.*

COROLLARY 2.15. *Assume $\mathsf{ZF} + \mathsf{AD}^+$. Let $S \subseteq \mathrm{ON}$ be a set of ordinals. Then $\prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$ is wellfounded.*

Proof. Suppose $F \in \prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$. Let f be a function on $\mathcal{D}$ such that $[f]_\mu = F$. Define $\phi(f)$ by $\phi(f)(X) = \mathrm{rk}^{\mathrm{HOD}_S^{L[S,X]}}(f(X))$. Let $\Phi : \prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu \to \prod_{X \in \mathcal{D}} \mathrm{ON}/\mu$ be defined by $\Phi([f]_\mu) = [\phi(f)]_\mu$. Then Φ is a well-defined function. Moreover, it has the property that if $F \in G$, then $\Phi(F) < \Phi(G)$. Fact 2.14 implies that $\prod_{X \in \mathcal{D}} \mathrm{HOD}_S^{L[S,X]}/\mu$ is wellfounded. ∎

3. OD **in** OD **is** OD. We will write $\mathbb{R}$ for ${}^\omega 2$, which is the collection of functions $f : \omega \to 2$.

THEOREM 3.1. *Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V{=}L}(\mathscr{P}(\mathbb{R}))$. Let J be a set of ordinals. Let $H \subseteq \mathbb{R}$. Let $K \subseteq \mathbb{R}$ be nonempty and OD_J. If H is $\mathrm{OD}_{J,z}$ for all $z \in K$, then H is OD_J.*

Proof. For simplicity, assume $J = \emptyset$. By Fact 2.9, let $\mathbb{X} \in \mathrm{HOD}^V$ be such that $\mathrm{HOD}^V = L[\mathbb{X}]$. Using the constructibility ordering of $L[\mathbb{X}]$, let $\langle (S_\alpha, \varphi_\alpha) : \alpha \in \mathrm{ON} \rangle$ enumerate all the ∞-Borel codes in $L[\mathbb{X}] = \mathrm{HOD}^V$. (This is merely the canonical constructibility enumeration of all pairs (S, φ)

in $\mathrm{HOD}^V = L[\mathbb{X}]$ where S is a set of ordinals and φ is a formula.) The main observation is that for any $X \in \mathcal{D}$, $\mathrm{HOD}^V = L[\mathbb{X}] \subseteq \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and therefore the sequence $\langle (S_\alpha, \varphi_\alpha) : \alpha \in \mathrm{ON}\rangle$ is definable in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ uniformly (by the same formula using $\mathbb{X}$ as a parameter for all $X \in \mathcal{D}$). In particular, every OD^V ∞-Borel code belongs to $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$.

Claim 1. *For any $R \subseteq \mathbb{R}$, R is OD_z^V for some $z \in \mathbb{R}$ if and only if there is some OD^V ∞-Borel code (S,φ) such that*

$$\begin{aligned} R = (\mathfrak{B}^2_{(S,\varphi)})_z &= \{x \in \mathbb{R} : (z,x) \in \mathfrak{B}^2_{(S,\varphi)}\} \\ &= \{x \in \mathbb{R} : L[S,z,x] \models \varphi(S,z,x)\}. \end{aligned}$$

Proof. ($\Rightarrow$) Suppose R is OD_z^V. There is some formula ς such that $x \in R \Leftrightarrow V \models \varsigma(z,x,\bar\alpha)$ where $\bar\alpha$ is a tuple of ordinals. Let $R' = \{(a,b) \in \mathbb{R}^2 : \varsigma(a,b,\bar\alpha)\}$. Then R' is an OD^V subset of $\mathbb{R}^2$. By Fact 2.8, there is some $(S,\varphi) \in \mathrm{HOD}^V$ such that $\mathfrak{B}^2_{(S,\varphi)} = R'$. Then $R = (\mathfrak{B}^2_{(S,\varphi)})_z$. ($\Leftarrow$) is clear. ∎

Since $K \subseteq \mathbb{R}$ is OD^V, K has an ∞-Borel code $(U,\psi) \in \mathrm{HOD}^V$ by Fact 2.8. Since $K \neq \emptyset$, let $z^* \in K$. Let $Z^* = [z^*]_{\equiv_{\mathsf{Turing}}}$. Throughout this argument, we will only work on the Turing cone above Z^*.

For all $X \in \mathcal{D}$, since $(U,\psi) \in \mathrm{HOD}^V = L[\mathbb{X}] \subseteq L[\mathbb{X},X]$, $(U,\psi) \in \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. For any $X \geq Z^*$, let $q^X = \{x \in \mathbb{R}^{L[\mathbb{X},X]} : L[U,x] \models \psi(U,x)\}$. Note that q^X is $\mathrm{OD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. Since $z^* \in \mathbb{R}^{L[\mathbb{X},X]}$ and $z^* \in K$, and (U,ψ) is the ∞-Borel code for K, one has $V \models L[U,z^*] \models \psi(U,z^*)$. Thus $L[\mathbb{X},X] \models L[U,z^*] \models \psi(U,z^*)$. Thus $z^* \in q^X$ and $q^X \neq \emptyset$. Therefore, $q^X \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$.

Case I. There is a cone of $X \in \mathcal{D}$ such that there are no atoms in

$$\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X = \{p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} : p \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} q^X\}.$$

Let $Z^{**} \in \mathcal{D}$ with $Z^{**} \geq Z^*$ be a base of a cone for which the Case I assumption holds. Now suppose $X \in \mathcal{D}$ with $X \geq Z^{**}$.

Claim 2. *There is a sequence $\mathcal{J} = \langle J_n : n \in \omega\rangle$ of dense open subsets of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$ and a sequence $\langle \epsilon_n : n \in \omega\rangle$ of ordinals such that for all $h \in \mathbb{R}$ which are $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$-generic with respect to $\mathcal{J}$, the following holds:*

(1) *$h \in K$.*
(2) *h is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$.*
(3) *There is some $m \in \omega$ such that $H = (\mathfrak{B}^2_{(S_{\epsilon_m},\varphi_{\epsilon_m})})_h$.*

Proof. Since $L[\mathbb{X},X] \models \mathsf{ZFC}$ and $V \models \mathsf{AD}$, we see that ω_1^V is inaccessible in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. This can be used to show that $\mathbb{O}_X^{L[\mathbb{X},X]} \restriction q^X$ is a countable atomless forcing. In V, fix a forcing isomorphism $\Phi : \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X \to \mathbb{C}$, where $\mathbb{C}$ is the Cohen forcing. (Specifically, $\mathbb{C} = ({}^{<\omega}2, \leq_{\mathbb{C}})$ is the forcing of finite binary strings ordered by $\leq_{\mathbb{C}}$ which is reverse string inclusion. Note there is generally no way to uniformly choose Φ depending on the degree X.) Let $\mathcal{E}$ be the collection of all dense open subsets of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$ which belong to $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for some $y \in \mathbb{R}^{L[\mathbb{X},X]}$. Since $V \models \mathsf{AD}$, $L[\mathbb{X},X] \models \mathsf{ZFC}$, and $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models \mathsf{ZFC}$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$, it follows that $\mathcal{E}$ is countable in V. Let $\mathcal{F} = \{\Phi[D] : D \in \mathcal{E}\}$. Then $\mathcal{F}$ is a countable collection of dense open subsets of Cohen forcing $\mathbb{C}$.

For each $g \in \mathbb{R}$, let $G_g^{\mathbb{C}} \subseteq \mathbb{C}$ be the derived $\mathbb{C}$-filter defined by $G_g^{\mathbb{C}} = \{g\restriction n : n \in \omega\}$. One says that g is $\mathbb{C}$-*generic* with respect to $\mathcal{F}$ if $G_g^{\mathbb{C}}$ intersects each dense open set in $\mathcal{F}$. Similarly, if $\mathcal{J}$ is a collection of dense open subsets of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$, one says that a real $x \in \mathbb{R}$ is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-*generic* with respect to $\mathcal{J}$ if there is an $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic filter $G \subseteq \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$ such that G meets each dense open set in $\mathcal{J}$ and $\dot{x}_{\mathrm{gen}}[G] = x$.

Since $\mathcal{F}$ is countable in V, let $C \subseteq \mathbb{R}$ be the comeager set of reals which are $\mathbb{C}$-generic with respect to $\mathcal{F}$. Let B be the collection of reals which are $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$. By the definition of Φ, $\mathcal{E}$, and $\mathcal{F}$, the forcing isomorphism Φ induces a bijection $\tilde{\Phi} : B \to C$.

For each $g \in C$, let $G_{\tilde{\Phi}^{-1}(g)} = \Phi^{-1}[G_g^{\mathbb{C}}]$. Then $G_{\tilde{\Phi}^{-1}(g)}$ is an $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic filter over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$. Note that $\dot{x}_{\mathrm{gen}}[G_{\tilde{\Phi}^{-1}(g)}] = \tilde{\Phi}^{-1}(g)$. Since $q^X \in G_{\tilde{\Phi}^{-1}(g)}$ and q^X is a condition of the form mentioned in Fact 2.5,

$$\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[G_{\tilde{\Phi}^{-1}(g)}] \models L[U, \tilde{\Phi}^{-1}(g)] \models \psi(U, \tilde{\Phi}^{-1}(g)).$$

Thus

$$V \models L[U, \tilde{\Phi}^{-1}(g)] \models \psi(U, \tilde{\Phi}^{-1}(g)).$$

Since (U, ψ) is the ∞-Borel code for K, $\tilde{\Phi}^{-1}(g) \in K$. Therefore, for $g \in C$, $\tilde{\Phi}^{-1}(g) \in K$.

By assumption, H is OD_x for all $x \in K$. In particular, for each $g \in C$, H is $\mathrm{OD}_{\tilde{\Phi}^{-1}(g)}$. By Claim 1, there is some $\epsilon \in \mathrm{ON}$ such that $H = (\mathfrak{B}^2_{(S_\epsilon,\varphi_\epsilon)})_{\tilde{\Phi}^{-1}(g)}$. Define $\Psi : C \to \mathrm{ON}$ by letting $\Psi(g)$ be the least such ϵ.

Under AD, a wellordered union of meager sets is meager, therefore, there must be some $\epsilon \in \mathrm{ON}$ such that $\Psi^{-1}[\{\epsilon\}]$ is nonmeager. Let $\delta_0 \in \mathrm{ON}$ be the least ordinal such that $\Psi^{-1}[\{\delta_0\}]$ is nonmeager. Suppose $\delta_\xi \in \mathrm{ON}$ has been defined. If $\bigcup_{\alpha \geq \delta_\xi} \Psi^{-1}[\{\alpha\}]$ is meager, the construction is finished. Otherwise, again using the fact that wellordered unions of meager sets are meager under AD, there is a least ordinal $\delta_{\xi+1}$ greater than δ_ξ such that $\Phi^{-1}[\{\delta_{\xi+1}\}]$ is nonmeager. If ξ is a limit ordinal and δ_ζ has been defined for all $\zeta < \xi$, then let $\delta_\xi = \sup\{\delta_\zeta : \zeta < \xi\}$. Since all sets of reals have the Baire property under AD and the topology on $\mathbb{R}$ has the countable chain condition, there must be a countable $\lambda \in \mathrm{ON}$ such that the construction is finished at stage λ.

As λ is countable, one can enumerate $\langle \delta_\xi : \xi < \lambda\rangle$ by $\langle \epsilon_n : n \in \omega\rangle$. Let $D = \bigcup_{n\in\omega} \Psi^{-1}[\{\epsilon_n\}]$, which is comeager by definition of λ being the ordinal at which the construction finished.

Since D is comeager, there is a sequence $\langle I_n : n \in \omega\rangle$ of topologically dense open subsets of $\mathbb{R}$ such that $\bigcap_{n\in\omega} I_n \subseteq D$. Let $J_n = \{\Phi^{-1}(\sigma) : \sigma \in \mathbb{C} \wedge N_\sigma \subseteq I_n\}$, where $N_\sigma = \{f \in \mathbb{R} : \sigma \subseteq f\}$ is the basic open subset of $\mathbb{R}$ determined by σ and recall that $\mathbb{C} = {}^{<\omega}2$. Define $\mathcal{J} = \langle J_n : n \in \omega\rangle$ to be is a sequence of dense open subsets of $\mathbb{O}_{\mathbb{S}}^{L[\mathbb{X},X]}\restriction q^X$. Note that if x is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic with respect to $\mathcal{J} = \langle J_n : n \in \omega\rangle$, then $\tilde{\Phi}(x) \in D$. Since $D \subseteq C$ and by the observation above, $G_x = G_{\tilde{\Phi}^{-1}(\tilde{\Phi}(x))}$ is $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$ and $x = \dot{x}_{\mathrm{gen}}[G_x]$. This completes the proof of Claim 2. ∎

We will construct a sequence of conditions in $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$ for as long as possible:

Let $p_{-1} = q^X$. Suppose p_k has been constructed.

Subcase i. There is some $y \in \mathbb{R}^{L[\mathbb{X},X]}$ and some $u \leq_{\mathbb{O}_X^{L[\mathbb{X},X]}\restriction q^X} p_k$ such that

$$y \notin H \wedge \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models u \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \varphi_{\epsilon_{k+1}}(\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y})$$

or

$$y \in H \wedge \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models u \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \neg\varphi_{\epsilon_{k+1}}(\check{S}_{\epsilon_{k+1}}, \dot{x}_{\mathrm{gen}}, \check{y}).$$

In this case, let $p_{k+1} \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$ be the least $u \in J_{k+1}$ according to the canonical wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$.

Subcase ii. Otherwise, declare that the construction has failed at stage $k+1$.

Claim 3. *The construction must fail at some stage.*

Proof. Suppose the construction never fails. Then one would successfully produce a sequence $\langle p_k : k \in \omega\rangle$ in $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$ with the properties specified above. Let $\hat{G}$ be the $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic filter produced by $\leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X}$-upward closing $\{p_k : k \in \omega\}$. By construction, $p_k \in J_k$. Hence $\hat{G}$ is an $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic filter with respect to $\mathcal{J}$. Let $h = \dot{x}_{\mathrm{gen}}[\hat{G}]$ be the associated $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}\restriction q^X$-generic real. By Claim 2, $h \in K$, h is $\mathbb{O}_X^{L[\mathbb{X},X]}\restriction q^X$-generic over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$, and there is some $m \in \omega$ such that

$H = (\mathfrak{B}^2_{(S_{\epsilon_m}, \varphi_{\epsilon_m})})_h$. However, the construction did not fail at stage m. Without loss of generality (the other case being similar), there is a p_m with the property that there is some $y \in \mathbb{R}^{L[\mathbb{X},X]}$ such that

$$y \notin H \wedge \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models p_m \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\epsilon_m}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \varphi_{\epsilon_m}(\check{S}_{\epsilon_m}, \dot{x}_{\mathrm{gen}}, \check{y}).$$

Thus

$$\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y][h] \models L[S_{\epsilon_m}, h, y] \models \varphi_{\epsilon_m}(S_{\epsilon_m}, h, y),$$

and so

$$V \models L[S_{\epsilon_m}, h, y] \models \varphi_{\epsilon_m}(S_{\epsilon_k}, h, y).$$

Since $H = (\mathfrak{B}^2_{(S_{\epsilon_m}, \varphi_{\epsilon_m})})_h$, this implies that $y \in H$. However, it was also assumed that $y \notin H$, a contradiction. This completes the proof of Claim 3. ■

Claim 4. *For all $X \geq Z^{**}$, there is some $p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and some ordinal ϵ such that for all $y \in \mathbb{R}^{L[\mathbb{X},X]}$, $y \in H$ if and only if*

$$\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models p \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\epsilon}, \dot{x}_{\mathrm{gen}}, \check{y}] \models \varphi(\check{S}_{\epsilon}, \dot{x}_{\mathrm{gen}}, \check{y}).$$

Proof. By Claim 3, the construction described above must fail at some stage k. This means that the forcing relation written above in $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$ for p_{k-1} and the ∞-Borel code $(S_{\epsilon_k}, \varphi_{\epsilon_k})$ can be used to determine membership of $y \in H$ for any $y \in \mathbb{R}^{L[\mathbb{X},X]}$. This completes the proof of Claim 4.

As mentioned in the proof of Claim 2, we nonuniformly selected a forcing isomorphism Φ. The choice of Φ is irrelevant, however, since we will only need the existence of any condition p with the above property in Claim 4. ■

For $X \geq Z^{**}$, using the canonical wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$, let $\langle p_\alpha^X : \alpha < \delta^X \rangle$, where $\delta^X \in \mathrm{ON}$, be the canonical enumeration of $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$.

We established thus far that for any $y \in \mathbb{R}$, if one drops into a local model $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y]$, where X is a sufficiently strong Turing degree (i.e. $X \geq Z^{**} \oplus [y]_{\mathrm{Turing}}$), then one can determine membership of y in H by merely two pieces of information: a condition $p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and an ordinal ϵ. Note that p is coded by an ordinal, since one can identify p with the least ordinal $\alpha < \delta^X$ such that $p = p_\alpha^X$. Next, we will show that roughly all this local information can be captured by just two ordinals by taking an ultrapower by μ.

Using Claim 4, let $\Sigma_{\alpha^*} : \mathcal{D} \to \mathrm{ON}$ be defined by letting $\Sigma_{\alpha^*}(X)$ be the least α such that p_α^X satisfies Claim 4 for some ϵ whenever $X \geq Z^{**}$. Otherwise, let $\Sigma_{\alpha^*}(X) = 0$. Define $\Sigma_{\epsilon^*} : \mathcal{D} \to \mathrm{ON}$ by letting $\Sigma_{\epsilon^*}(X)$ be the least ϵ satisfying Claim 4 with respect to $p_{\Sigma_{\alpha^*}(X)}$ whenever $X \geq Z^{**}$. Otherwise, let $\Sigma_{\epsilon^*}(X) = 0$.

$[\Sigma_{\alpha^*}]_\mu$ and $[\Sigma_{\epsilon^*}]_\mu$ are ordinals since $\prod_{X \in \mathcal{D}} \mathrm{ON}/\mu$ is a wellordering by Fact 2.14. Let $\alpha^* = [\Sigma_{\alpha^*}]_\mu$ and $\epsilon^* = [\Sigma_{\epsilon^*}]_\mu$.

Claim 5. *H is* OD.

Proof. Note that for $y \in \mathbb{R}$, $y \in H$ if and only if for any $\Sigma_0, \Sigma_1 : \mathcal{D} \to \mathrm{ON}$ such that $[\Sigma_0]_\mu = \alpha^*$ and $[\Sigma_1]_\mu = \epsilon^*$, for a cone of $X \in \mathcal{D}$,

$$\begin{aligned}\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}[y] \models p^X_{\Sigma_0(X)} \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} L[\check{S}_{\Sigma_1(X)}, \dot{x}_{\mathrm{gen}}, \check{y}] \\ \models \varphi_{\Sigma_1(X)}(\check{S}_{\Sigma_1(X)}, \dot{x}_{\mathrm{gen}}, \check{y}).\end{aligned}$$

The latter is ordinal definable (using the two ordinals α^* and ϵ^*). The expression successfully defines H by the definition of $\alpha^* = [\Sigma_{\alpha^*}]_\mu$ and $\epsilon^* = [\Sigma_{\epsilon^*}]_\mu$ as well as Claim 4. ∎

The proof is complete in the setting of Case I.

Case II. There is a cone of $X \in \mathcal{D}$ such that there is an atom in $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X$.

Let $Z^{**} \geq Z^*$ be the base of a cone satisfying the Case II assumption. Fix an $X \geq Z^{**}$. Let $p \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} \restriction q^X} q^X$ be an atom.

Claim 6. *There is some $r \in K \cap \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. Note that $r \in K$ implies there is an ordinal ϵ such that $H = (\mathfrak{B}^2_{(S_\epsilon,\varphi_\epsilon)})_r$.*

Proof. Since $p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$, we know that $p \neq \emptyset$. Let $r \in p$. Let $G^1_r = \{p \in \mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]} : r \in p\}$. By Fact 2.4, G^1_r is an $\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}$-generic filter over $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$ and $\dot{x}_{\mathrm{gen}}[G^1_r] = r$. Also, $p \in G^1_r$. Therefore, thinking of reals as subsets of ω, for each $n \in \omega$, $n \in r$ if and only if $p \Vdash_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} \check{n} \in \dot{x}_{\mathrm{gen}}$ since p was assumed to be an atom and hence has no nontrivial extensions. The latter is $\mathrm{OD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. This shows that $r \in \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. (Since $r \in p$ was arbitrary, this argument actually shows that $p = \{r\}$.) Since $p \leq_{\mathbb{O}_{\mathbb{X}}^{L[\mathbb{X},X]}} q^X$ and $p \in G^1_r$, one sees that $r \in q^X$. By definition of q^X, one finds that $L[U, r] \models \Psi(U, r)$. Since (U, ψ) is the ∞-Borel code for K, $V \models r \in K$. Therefore, $r \in K \cap \mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. ∎

Let $\langle r^X_\alpha : \alpha < \delta^X \rangle$, where $\delta^X \in \mathrm{ON}$, be the enumeration of $\mathbb{R}^{\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}}$ according to the canonical wellordering of $\mathrm{HOD}_{\mathbb{X}}^{L[\mathbb{X},X]}$. Define $\Sigma_{\alpha^*} : \mathcal{D} \to \mathrm{ON}$ by letting $\Sigma_{\alpha^*}(X)$ be the least ordinal α such that r^X_α satisfies Claim 6 whenever $X \geq Z^{**}$. Otherwise, let $\Sigma_{\alpha^*}(X) = 0$. Let $\Sigma_{\epsilon^*} : \mathcal{D} \to \mathrm{ON}$ be defined by letting $\Sigma_{\epsilon^*}(X)$ be the least $\epsilon \in \mathrm{ON}$ such that $H = (\mathfrak{B}^2_{(S_\epsilon,\varphi_\epsilon)})_{r^X_{\Sigma_{\alpha^*}(X)}}$ whenever $X \geq Z^{**}$. Otherwise, let $\Sigma_{\epsilon^*}(X) = 0$.

Again, since $\prod_{X \in \mathcal{D}} \mathrm{ON}/\mu$ is a wellordering by Fact 2.14, $[\Sigma_{\alpha^*}]_\mu$ and $[\Sigma_{\epsilon^*}]_\mu$ are ordinals. Let $\alpha^* = [\Sigma_{\alpha^*}]_\mu$ and $\epsilon^* = [\Sigma_{\epsilon^*}]_\mu$.

Claim 7. *H is* OD.

Proof. Note that for all $y \in \mathbb{R}$, $y \in H$ if and only for all $\Sigma_0, \Sigma_1 : \mathcal{D} \to \mathrm{ON}$ such that $[\Sigma_0]_\mu = \alpha^*$ and $[\Sigma_1]_\mu = \epsilon^*$, for a cone of $X \in \mathcal{D}$,

$$L[S_{\Sigma_1(X)}, r^X_{\Sigma_0(X)}, y] \models \varphi_{\Sigma_1(X)}(S_{\Sigma_1(X)}, r^X_{\Sigma_0(X)}, y).$$

This equivalence is true by Claim 6 and the definitions of Σ_{α^*} and Σ_{ϵ^*}. The latter is ordinal definable (using the ordinals α^* and ϵ^*). ∎

The theorem has been shown in Case II as well. The entire argument is complete. ∎

Some assumptions beyond ZF or ZFC are necessary to prove the conclusion of Theorem 3.1. The next result shows that in a Sacks forcing extension of the constructible universe L, there is a nonempty OD set K and a real g such that g is OD_z for any $z \in K$ but g is not OD.

FACT 3.2. *Let $\mathbb{S}$ denote the Sacks forcing of perfect trees. Let $G \subseteq \mathbb{S}$ be an $\mathbb{S}$-generic filter over L.*

In $L[G]$, let $K = \mathbb{R}^{L[G]} \setminus \mathbb{R}^L$ be the collection of nonconstructible reals. This is an OD *set of reals. Let $g \in \mathbb{R}^{L[G]}$ be the $\mathbb{S}$-generic real over L derived from G. Then g is OD_z for any $z \in K$, but g is not* OD.

Proof. A perfect tree is a subset p of ${}^{<\omega}2$ with the property that for all $\sigma, \tau \in {}^{<\omega}2$, if $\sigma \subseteq \tau$ and $\tau \in p$, then $\sigma \in p$, and for all $\sigma \in p$, there exists a $\tau \supseteq \sigma$ such that $\tau\hat{}0, \tau\hat{}1 \in p$. Let $\mathbb{S}$ consist of the collection of perfect trees. Define $p \leq_{\mathbb{S}} q$ if and only if $p \subseteq q$. The largest element is $1_{\mathbb{S}} = {}^{<\omega}2$. Sacks forcing is $\mathbb{S} = (\mathbb{S}, \leq_{\mathbb{S}}, 1_{\mathbb{S}})$. If $p \in \mathbb{S}$, then set $[p] = \{f \in {}^\omega 2 : (\forall n)(f \restriction n \in p)\}$. If $r \in \mathbb{R}$, then let $G^{\mathbb{S}}_r = \{p \in \mathbb{S} : r \in [p]\}$. If $G^{\mathbb{S}}_r$ is an $\mathbb{S}$-generic filter over L, then one says that r is an $\mathbb{S}$-*generic real over* L. See [11, Chapter 15] for the basic facts about the Sacks forcing $\mathbb{S}$.

Fix a Sacks generic filter $G \subseteq \mathbb{S}$ over L. Work in $L[G]$. Let g be the Sacks generic real derived from G, i.e. $\{g\} = \bigcap_{p \in G}[p]$.

Let $K = \mathbb{R}^{L[G]} \setminus \mathbb{R}^L$ be the collection of nonconstructible reals. The set K is OD. Using a fusion argument, one can reconstruct g from any nonconstructible real z (that is, $z \in K$) using only parameters in L. (This is the argument used in [11, Theorem 15.34] to show that g is a real of minimal constructibility degree. It also shows that every element of K is itself an $\mathbb{S}$-generic real for some $\mathbb{S}$-generic filter over L.) So g is OD_z for any $z \in K$.

However, g is not OD. Suppose otherwise. Then there must be some formula φ and some ordinal ϵ such that g is the unique solution $v \in L[G]$ to $L[G] \models \varphi(v, \epsilon)$. Therefore, there is some $q \in G$ such that $L \models q \Vdash_{\mathbb{S}} \varphi(\dot{x}_{\mathrm{gen}}, \check{\epsilon})$ where $\dot{x}_{\mathrm{gen}}$ is the canonical $\mathbb{S}$-name for the generic real added by an $\mathbb{S}$-generic filter. Since q is still a perfect tree in $L[G]$, $[q]^{L[G]}$ must contain a nonconstructible real h with $h \neq g$. As mentioned above, by the fusion argument of [11, Theorem 15.34], h is also $\mathbb{S}$-generic over L. Let $G^{\mathbb{S}}_h =$

$\{p \in \mathbb{S} : h \in [p]\}$ be the $\mathbb{S}$-generic filter over L derived from h such that $\dot{x}_{\mathrm{gen}}[G_h^{\mathbb{S}}] = h$. Note that $G_h^{\mathbb{S}} \in L[G]$ and $q \in G_h^{\mathbb{S}}$. Thus $L[G_h^{\mathbb{S}}] \models \varphi(h, \epsilon)$. Since [11, Theorem 15.34] implies every nonconstructible real in $L[G]$ has minimal constructibility degree, $L[G] = L[G_h^{\mathbb{S}}]$. Hence $L[G] \models \varphi(h, \epsilon)$ and $h \neq g$. This contradicts g being the unique solution in $L[G]$ to $\varphi(v, \epsilon)$. ∎

4. Cardinals below $[\omega_1]^{\omega_1}$**.** This section will show under AD^+ that $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}| = |\mathscr{P}(\omega_1)|$. In $L(\mathbb{R})$, a cardinality intermediate between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}|$ will be isolated.

The argument for Theorem 4.5 showing that $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$, presented below using Fact 4.1, was suggested by Neeman and is simpler than the original argument. The original argument will be presented later and is required in other settings involved, absorbing a fragment of an arbitrary injection into a suitable ZFC model. This idea is a powerful technique for studying cardinalities under AD^+ and especially for producing an intermediate cardinality under $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}}$.

FACT 4.1. *Assume* ZF. *Suppose κ is a cardinal which is inaccessible in any inner model of* ZFC. *Then* $|[\kappa]^{<\kappa}| < |[\kappa]^\kappa|$.

Proof. Suppose there was an injection $\Phi : [\kappa]^\kappa \to [\kappa]^{<\kappa}$. Consider $\hat{\Phi} \subseteq [\kappa]^\kappa \times \kappa$ defined by $(f, \alpha) \in \hat{\Phi} \Leftrightarrow \alpha \in \Phi(f)$. (Here $[\kappa]^{<\kappa}$ is identified as a subset of κ.) Note that if $f \in L[\hat{\Phi}]$, then $\Phi(f) \in L[\hat{\Phi}]$.

Identify the predicate Φ with $\hat{\Phi}$. Then $L[\Phi] \models \mathsf{ZFC}$ and $L[\Phi] \models$ "Φ is an injection". By Cantor's theorem, $L[\Phi] \models |[\kappa]^\kappa| = 2^\kappa > \kappa$. However, since $L[\Phi]$ thinks κ is inaccessible, $L[\Phi] \models |[\kappa]^{<\kappa}| = |2^{<\kappa}| = \kappa$. Then within $L[\Phi]$, Φ induces an injection of 2^κ into κ, which is not possible. ∎

FACT 4.2. *Assume* ZF. *Suppose κ is a cardinal such that there is a κ-complete nonprincipal ultrafilter on κ. Let M be any inner model of* ZFC. *Then κ is inaccessible in M.*

Proof. Let μ be a κ-complete measure on κ. It is clear that κ is regular in M.

Suppose κ is not a strong limit cardinal in M. Then there is a $\delta < \kappa$ such that $M \models |\mathscr{P}(\delta))| \geq \kappa$. Since $M \models \mathsf{ZFC}$, one can find a length-κ sequence of distinct subsets of δ, $\langle A_\alpha : \alpha < \kappa \rangle$.

For each $\beta < \delta$, let $C_\beta^0 = \{\alpha < \kappa : \beta \notin A_\alpha\}$ and $C_\beta^1 = \{\alpha < \kappa : \beta \in A_\alpha\}$. As $C_\beta^0 \cup C_\beta^1 = \kappa$ and μ is a measure, there is some $i_\beta \in 2$ such that $C_\beta^{i_\beta} \in \mu$. Let $A = \{\beta : i_\beta = 1\}$. Since μ is κ-complete and $\delta < \kappa$, $C = \bigcap_{\beta<\delta} C_\beta^{i_\beta} \in \mu$. Since μ is nonprincipal, let $\alpha_0, \alpha_1 \in C$ with $\alpha_0 \neq \alpha_1$. Then $A_{\alpha_0} = A_{\alpha_1} = A$. This contradicts $\langle A_\alpha : \alpha < \kappa \rangle$ being a sequence of distinct subsets of δ. ∎

FACT 4.3 ([16, Theorem 3.2]). *Assume* ZF. *Let κ be a cardinal. Let $\eta < \kappa$ be a limit ordinal. The partition relation $\kappa \to (\kappa)_2^{\eta+\eta}$ implies that the η-club filter on κ, W_κ^η, is a normal κ-complete ultrafilter on κ.*

FACT 4.4. *Assume* $\mathsf{ZF} + \mathsf{AD}$.

(Solovay) $\omega_1 \to (\omega_1)_2^{\omega_1}$ *and therefore ω_1 is measurable.*
(Martin) $\omega_2 \to (\omega_2)_2^{\alpha}$, *for each $\alpha < \omega_2$, and therefore ω_2 is measurable.*
([13]) *Suppose $A \subseteq \mathbb{R}$. Let δ_A be the least ordinal such that $L_\delta(A, \mathbb{R}) \prec_1 L(A, \mathbb{R})$. Then $\delta_A \to (\delta_A)_2^{\delta_A}$ and hence δ_A is measurable.*

THEOREM 4.5. *Assume* $\mathsf{ZF} + \mathsf{AD}$.

- $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$.
- $|[\omega_2]^{<\omega_2}| < |[\omega_2]^{\omega_2}|$.
- *For any set $A \subseteq \mathbb{R}$, $|[\delta_A]^{<\delta_A}| < |[\delta_A]^{\delta_A}|$.*
- *More generally, for any cardinal κ satisfying the partition relation $\kappa \to (\kappa)_2^{\omega+\omega}$, one has $|[\kappa]^{<\kappa}| < |[\kappa]^{\kappa}|$.*

Proof. Under AD, ω_1, ω_2, and δ_A, for any $A \subseteq \mathbb{R}$, satisfy the $\omega + \omega$ exponent partition relation by Fact 4.4 and are thus measurable cardinals by Fact 4.3. Each result now follows from Facts 4.2 and 4.1. ∎

FACT 4.6. *Assume $V = L(J, \mathbb{R}) \models \mathsf{AD}{+}\mathsf{DC}_{\mathbb{R}}$, where J is a set of ordinals. Suppose $\Phi : [\kappa]^\kappa \to [\kappa]^{<\kappa}$. Then there is an $e \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ with $e \leq_{J,{}_\omega\mathbb{O}_J} x$ (which refers to the $(J, {}_\omega\mathbb{O}_J)$-constructibility reduction), one has the following properties:*

(i) *For all $f \in [\kappa]^\kappa \cap L[J, {}_\omega\mathbb{O}_J, x]$, $\Phi(f) \in L[J, {}_\omega\mathbb{O}_J, x]$.*
(ii) *$\Phi \cap L[J, {}_\omega\mathbb{O}_J, x] \in L[J, {}_\omega\mathbb{O}_J, x]$.*

(i) *and* (ii) *together imply that $\Phi \cap L[J, {}_\omega\mathbb{O}_J, x]$ is a function, which is even a set in $L[J, {}_\omega\mathbb{O}_J, x]$.*

Proof. In $L(J, \mathbb{R})$, every set is $\mathrm{OD}_{J,e}$ for some real e. Let φ be a formula and let $\bar{\alpha}$ be a tuple of ordinals such that

$$(f, \sigma) \in \Phi \Leftrightarrow L(J, \mathbb{R}) \models \varphi(J, e, \bar{\alpha}, f, \sigma).$$

Now fix $x \in \mathbb{R}$ such that $e \in L[J, {}_\omega\mathbb{O}_J, x]$. By Fact 2.10 and the above, for all $(f, \sigma) \in ([\kappa]^\kappa \times [\kappa]^{<\kappa}) \cap L[J, {}_\omega\mathbb{O}_J, x]$,

$$(f, \sigma) \in \Phi \Leftrightarrow L[J, {}_\omega\mathbb{O}_J, x] \models 1_{{}_\omega\mathbb{O}_J/G_x^n} \Vdash_{{}_\omega\mathbb{O}_J/G_x^1} L(\check{J}, \dot{\mathbb{R}}_{\mathrm{sym}}) \models \varphi(J, e, \bar{\alpha}, f, \sigma).$$

By comprehension in $L[J, {}_\omega\mathbb{O}_J, x]$, one sees that (ii) follows.

Note that for each $f \in [\kappa]^\kappa$ and $\beta \in \kappa$, one has

$$\beta \in \Phi(f) \Leftrightarrow L(J, \mathbb{R}) \models (\exists\sigma)(\varphi(J, e, \bar{\alpha}, f, \sigma) \wedge \beta \in \sigma).$$

(Here $\sigma \in [\kappa]^{<\kappa}$ is construed as a subset of κ.)

So for each $x \in \mathbb{R}$ such that $e \in L[J, {}_\omega\mathbb{O}_J, x]$, if $f \in L[J, {}_\omega\mathbb{O}_J, x]$, one has

$$\beta \in \Phi(f) \Leftrightarrow L[J, {}_\omega\mathbb{O}_J, x] \models 1_{{}_\omega\mathbb{O}_J/G^1_x} \Vdash_{{}_\omega\mathbb{O}_J/G^1_x} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists\sigma)(\varphi(J, e, \bar{\alpha}, f, \sigma) \wedge \beta \in \sigma).$$

Again by comprehension in $L[J, {}_\omega\mathbb{O}_J, x]$, one finds that $\Phi(f) \in L[J, {}_\omega\mathbb{O}_J, x]$ and thus (i) holds. ∎

The following result due to Steel is proved by inner model-theoretic techniques:

Fact 4.7 (Steel, [19, Theorem 8.27]). *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$. *If* κ *is regular, then for all* $x \in \mathbb{R}$, $\mathrm{HOD}_x \models$ "*κ is measurable*".

Theorem 4.8. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{V}{=}\mathsf{L}(\mathbb{R})$. *Suppose* $\kappa < \Theta$ *is regular. Then* $|[\kappa]^{<\kappa}| < |[\kappa]^\kappa|$.

Proof. If $\kappa < \Theta$ is regular, then Fact 4.7 implies that $\mathrm{HOD}_x^{L(\mathbb{R})} \models$ "κ is measurable" for any $x \in \mathbb{R}$. Let $\mathbb{X} = {}_\omega\mathbb{O}$. By Fact 2.11, $\mathrm{HOD}_x^{L(\mathbb{R})} = L[\mathbb{X}, x]$.

Now suppose that there is an injection $\Phi : [\kappa]^\kappa \to [\kappa]^{<\kappa}$. By Fact 4.6, there is an $e \in \mathbb{R}$ such that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and this set is a function in $L[\mathbb{X}, e]$. Let $\Psi = \Phi \cap L[\mathbb{X}, e]$. By absoluteness, $L[\mathbb{X}, e] \models$ "$\Psi : [\kappa]^\kappa \to [\kappa]^{<\kappa}$ is an injection". However, since κ is measurable in $\mathrm{HOD}_e = L[\mathbb{X}, e]$, one has $L[\mathbb{X}, e] \models |[\kappa]^{<\kappa}| = \kappa$. By Cantor's theorem applied in $L[\mathbb{X}, e]$, such an injection cannot exist. ∎

By Theorem 4.5, $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{\omega_1}|$. A natural question at this point would be whether it is possible under $\mathsf{ZF} + \mathsf{AD}$ that there exists a set K such that $|[\omega_1]^{<\omega_1}| < |K| < |[\omega_1]^{\omega_1}|$. Next, it will be shown that such a set exists under $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R} + \mathsf{V}{=}\mathsf{L}(\mathscr{P}(\mathbb{R}))$. Recall that under this assumption, there is a set J of ordinals such that $V = L(J, \mathbb{R})$.

Definition 4.9. Assume $\mathsf{ZF} + \mathsf{AD}^+$. Let $J \subseteq \mathrm{ON}$ be a set of ordinals such that $V = L(J, \mathbb{R})$. Let $\mathbb{X} = (J, {}_\omega\mathbb{O}_J)$.

$$N_1^J = \bigsqcup_{r \in \mathbb{R}} ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} = \{(r, \alpha) : \alpha < ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}\}.$$

In other words, this is a disjoint union over $r \in \mathbb{R}$ of the successors of $\omega_1^{L(J,\mathbb{R})}$ as computed in $L[\mathbb{X}, r]$.

Theorem 4.10. *Assume* $\mathsf{ZF} + \mathsf{AD}^+$ *and there is a set* $J \subseteq \mathrm{ON}$ *such that* $V = L(J, \mathbb{R})$. *Then:*

(1) $\neg(|N_1^J| \leq [\omega_1]^{<\omega_1})$.
(2) $|\mathbb{R} \times \omega_1| < |N_1^J| < |\mathbb{R} \times \omega_2|$.
(3) $|N_1^J| < |[\omega_1]^{\omega_1}|$.
(4) $\neg(|[\omega_1]^\omega| \leq |N_1^J|)$.
(5) $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{<\omega_1} \sqcup N_1^J| < |[\omega_1]^{\omega_1}|$.

Proof. Let $\mathbb{X} = (J, {}_\omega\mathbb{O}_J)$. Suppose there is an injection $\Phi : N_1^J \to [\omega_1]^{<\omega_1}$. By the idea of Fact 4.6, there is an $e \in \mathbb{R}$ such that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and $L[\mathbb{X}, e]$ thinks that $\tilde{\Phi} = \Phi \cap L[\mathbb{X}, e]$ is an injective function with domain $N_1^J \cap L[\mathbb{X}, e]$. Thus with the model $L[\mathbb{X}, e]$, the restriction of $\tilde{\Phi}$ to $\{e\} \times ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},e]}$ is an injection into $([\omega_1^{L(J,\mathbb{R})}]^{<\omega_1^{L(J,\mathbb{R})}}) \cap L[\mathbb{X}, e]$. This is impossible since the inaccessibility of $\omega_1^{L(J,\mathbb{R})}$ in the model $L[\mathbb{X}, e]$ implies that $L[\mathbb{X}, e] \models |[\omega_1^{L(J,\mathbb{R})}]^{<\omega_1^{L(J,\mathbb{R})}}| = \omega_1^{L(J,\mathbb{R})}$. This shows that $\neg(N_1^J \leq [\omega_1]^{<\omega_1})$. This also implies $|[\omega_1]^{<\omega_1}| < |[\omega_1]^{<\omega_1} \sqcup N_1^J|$.

Suppose there is an injection $\Phi : N_1^J \to \mathbb{R} \times \omega_1$. Using the same idea as applied in the proof of Fact 4.6, there is an e such that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and $L[\mathbb{X}, e]$ thinks that $\Phi \cap L[\mathbb{X}, e]$ is an injective function with domain $N_1^J \cap L[\mathbb{X}, e]$. Let $\tilde{\Phi} = \Phi \cap L[\mathbb{X}, e]$. Then $L[\mathbb{X}, e] \models$ "$\tilde{\Phi}$ restricted to $\{e\} \times ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},e]} = \{e\} \times (\omega_1^{L(J,\mathbb{R})})^+$ is an injection of $\{e\} \times (\omega_1^{L(J,\mathbb{R})})^+$ into $\mathbb{R} \times \omega_1^{L(J,\mathbb{R})}$". Note that $L[\mathbb{X}, e] \models |\mathbb{R}| < \omega_1^{L(J,\mathbb{R})}$ since $\omega_1^{L(J,\mathbb{R})}$ is inaccessible in $L[\mathbb{X}, e]$. Thus $L[\mathbb{X}, e] \models |\mathbb{R} \times \omega_1^{L(J,\mathbb{R})}| = \omega_1^{L(J,\mathbb{R})}$. It is impossible that $L[\mathbb{X}, e]$ has an injection of the successor $\omega_1^{L(J,\mathbb{R})}$ (as computed in $L[\mathbb{X}, e]$) into $\omega_1^{L(J,\mathbb{R})}$. This establishes $\neg(|N_1^J| \leq |\mathbb{R} \times \omega_1|)$.

Suppose there is an injection $\Phi : \mathbb{R} \times \omega_2 \to N_1^J$. Again using the idea applied for Fact 4.6, there is an e such that $\Phi \cap L[\mathbb{X}, e] \in L[\mathbb{X}, e]$ and $L[\mathbb{X}, e]$ thinks that $\tilde{\Phi} = \Phi \cap L[\mathbb{X}, e]$ is a function with domain $(\mathbb{R} \times \omega_2^{L(J,\mathbb{R})}) \cap L[\mathbb{X}, e]$. Since $L[\mathbb{X}, e] \models \mathsf{AC}$ and there are no uncountable wellordered sequences of distinct reals, $L[\mathbb{X}, e] \models |\mathbb{R}| < \omega_1^{L(J,\mathbb{R})}$. Since AD implies that ω_1 and ω_2 are measurable, the argument for Fact 4.2 implies that there are no uncountable wellordered sequences of distinct reals and no ω_2 length sequences of distinct subsets of ω_1. Thus $\mathbb{R}^{L[\mathbb{X},e]}$ is countable and for each $r \in \mathbb{R}$, $((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} < \omega_2^{L(J,\mathbb{R})}$. Hence $L[\mathbb{X}, e] \models |\bigsqcup_{r\in\mathbb{R}}((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}| < \omega_2^{L(J,\mathbb{R})}$. Thus it is impossible that $L[\mathbb{X}, e]$ thinks that $\tilde{\Phi}$ restricted to $\{e\} \times \omega_2^{L(J,\mathbb{R})}$ is an injection of $\{e\} \times \omega_2^{L(J,\mathbb{R})}$ into

$$L[\mathbb{X}, e] \cap N_1^J = \bigsqcup_{r\in\mathbb{R}^{L[\mathbb{X},e]}} ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}.$$

This establishes that $\neg(|\mathbb{R} \times \omega_2| \leq |N_1^J|)$.

As observed above, for each $r \in \mathbb{R}$, $((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} < \omega_2^{L(J,\mathbb{R})}$. Thus it is clear that N_1^J is a subset of $\mathbb{R} \times \omega_2$. Thus $|\mathbb{R} \times \omega_1| < |N_1^J| < |\mathbb{R} \times \omega_2|$.

For each $r \in \mathbb{R}$, define $A_r = \{f \in [\omega_1^{L(J,\mathbb{R})}]^{\omega_1^{L(J,\mathbb{R})}} : \min(f) \geq \omega\}$ in $L[\mathbb{X}, r]$. Observe that $L[\mathbb{X}, r] \models |A_r| = |[\omega_1^{L(J,\mathbb{R})}]^{\omega_1^{L(J,\mathbb{R})}}| = |2^{\omega_1^{L(J,\mathbb{R})}}| \geq (\omega_1^{L(J,\mathbb{R})})^+$. Let $\Psi_r : ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]} \to A_r$ be the least injection from $((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},r]}$ into A_r according to the constructibility order on $L[\mathbb{X}, r]$. (Note that $\langle \Psi_r :$

$r \in \mathbb{R}\rangle$ does exist as a set in $L(J, \mathbb{R})$.) Out in $L(J, \mathbb{R})$, define an injection $\Gamma : N_1^J \to [\omega_1]^{\omega_1}$ by $\Gamma(r, \alpha) = r\hat{\ }\Psi_r(\alpha)$, which is well-defined if one considers $\mathbb{R}$ as $[\omega]^\omega$, the collection of strictly increasing ω-sequences in ω, and the fact that $\min \Psi_r(\alpha) \geq \omega$ since $\Psi_r(\alpha) \in A_r$. Then Γ witnesses that $|N_1^J| \leq |[\omega_1]^{\omega_1}|$.

Let $\mathrm{add} : \omega_1 \times [\omega_1]^{<\omega_1} \to [\omega_1]^{<\omega_1}$ be defined by $\mathrm{add}(\alpha, f)(\beta) = \alpha + f(\beta)$, whenever $\beta < \mathrm{dom}(f)$. If $B \subseteq \omega_1$ is unbounded in ω_1, then let $\mathrm{enum}_B : \omega_1 \to \omega_1$ denote the increasing enumeration of B. Let

$$\Lambda(f) = \langle \sup(f) \rangle\hat{\ }\mathrm{add}(\sup(f), f)\hat{\ }\mathrm{enum}_{\omega_1 \setminus \mathrm{rang}(\mathrm{add}(\sup(f), f))}.$$

In words, $\Lambda(f)$ first outputs $\sup(f)$, then outputs the values $\sup(f) + f(\beta)$ for each $\beta < \mathrm{dom}(f)$, and then fills up the rest with an increasing enumeration of the remaining countable ordinals. Thus Λ is an injection of $[\omega_1]^{<\omega_1}$ into $[\omega_1]^{\omega_1}$.

Let $A = \{f \in [\omega_1]^{<\omega_1} : \min(f) \geq \omega\}$. Observe that $|A| = |[\omega_1]^{<\omega_1}|$. Note that $\Lambda[A]$ and $\Gamma[N_1^J]$ are disjoint subsets of $[\omega_1]^{\omega_1}$ since for any $f \in \Lambda[A]$, $\min(f) \geq \omega$, but for all $f \in \Gamma[N_1^J]$, $\min(f) < \omega$. Thus one can merge these two injections together to obtain an injection of $[\omega_1]^{<\omega_1} \sqcup N_1^J$ into $[\omega_1]^{\omega_1}$. This shows that

$$|[\omega_1]^{<\omega_1} \sqcup N_1^J| \leq |[\omega_1]^{\omega_1}|.$$

Now suppose $\Phi : [\omega_1]^\omega \to N_1^J$ is an injection. Let $\pi : \mathbb{R} \times \omega_2 \to \mathbb{R}$ denote the projection onto the first coordinate. Thinking of $N_1^J \subseteq \mathbb{R} \times \omega_2$, $\pi \circ \Phi : [\omega_1]^\omega \to \mathbb{R}$. Thinking of $\mathbb{R}$ as ${}^\omega 2$, let $\sigma_n : \mathbb{R} \to 2$ be defined to be the projection onto the nth coordinate, that is, $\sigma_n(r) = r(n)$. Thus for each $n \in \omega$, $\sigma_n \circ \pi \circ \Phi : [\omega_1]^\omega \to 2$. By the correct-type partition relation, $\omega_1 \to_* (\omega_1)^\omega_2$, there is a club C_n and $i_n \in 2$ such that for all $f \in [C_n]^\omega_*$, $\sigma_n(\pi(\Phi(f))) = i_n$, where $[C_n]^\omega_*$ is the collection of all $f \in [C_n]^\omega$ which are of the correct type. (See [2, Section 2] for the definition of functions of correct type, the correct-type partition relation, and its equivalence with the usual partition property.) By $\mathsf{AC}^{\mathbb{R}}_\omega$, let $\langle C_n : n \in \omega \rangle$ be such that C_n is a club subset of ω_1 which is homogeneous for $\sigma_n \circ \pi \circ \Phi$ in the above sense for each $n \in \omega$. Let $s \in \mathbb{R}$ be defined by $s(n) = i_n$. Let $C = \bigcap_{n \in \omega} C_n$. Then for all $f \in [C]^\omega_*$, $\pi(\Phi(f)) = s$. Thus Φ restricted to $[C]^\omega_*$ is an injection of $[C]^\omega_*$ into $\{s\} \times ((\omega_1^{L(J,\mathbb{R})})^+)^{L[\mathbb{X},e]}$. This is impossible since $[C]^\omega_*$ is not wellorderable under AD. This shows

$$\neg(|[\omega_1]^\omega| \leq |N_1^J|).$$

Now suppose $\Phi : [\omega_1]^{\omega_1} \to [\omega_1]^{<\omega_1} \sqcup N_1^J$. Define $P : [\omega_1]^{\omega_1} \to 2$ by

$$P(f) = \begin{cases} 0, & \Phi(f) \in [\omega_1]^{<\omega_1}, \\ 1, & \Phi(f) \in N_1^J. \end{cases}$$

By $\omega_1 \to (\omega_1)^{\omega_1}_2$, choose a $C \subseteq \omega_1$ with $|C| = \omega_1$ and homogeneous for P. If C is homogeneous for 0, then Φ gives an injection of $[C]^{\omega_1}$ (which is in bijection with $[\omega_1]^{\omega_1}$) into $[\omega_1]^{<\omega_1}$. This contradicts Theorem 4.5. Suppose C is homogeneous for P taking value 1. Then Φ is an injection of $[C]^{\omega_1}$ into N_1^J. From this, one obtains an injection of $[\omega_1]^\omega$ into N_1^J. But it was shown above that $\neg(|[\omega_1]^\omega| \leq |N_1^J|)$.

This completes the proof of the theorem. ∎

Note that the failure of $\mathsf{AD}_\mathbb{R}$ is important. With $\mathsf{AD}_\mathbb{R}$, one cannot have a set $\mathbb{X}$ that absorbs fragments of functions as in Fact 4.6. Moreover, the natural analogs of the N_1^J sets under $\mathsf{AD}_\mathbb{R}$ are simply in bijection with $\mathbb{R}\times\omega_1$.

FACT 4.11. *Assume* $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$. *Let* $S \subseteq \mathrm{ON}$ *be a set of ordinals. Let* $N = \bigsqcup_{r\in\mathbb{R}}((\omega_1^V)^+)^{L[S,r]}$. *Then* $|N| = |\mathbb{R} \times \omega_1|$.

Proof. Using a prewellordering on $\mathbb{R}$ of length ω_1, one can code subsets of ω_1 (and also subsets of $\omega_1 \times \omega_1$) by reals using the Moschovakis coding lemma. Define a relation $R \subseteq \mathbb{R} \times \mathbb{R}$ by $R(x, y)$ if and only if y codes a subset of $\omega_1 \times \omega_1$ which is a wellordering of ω_1 of order type $((\omega_1^V)^+)^{L[S,x]}$. By $\mathsf{AD}_\mathbb{R}$, let $F : \mathbb{R} \to \mathbb{R}$ be a uniformizing function for R. For each $x \in \mathbb{R}$, let $\Psi_x : \omega_1^V \to ((\omega_1^V)^+)^{L[S,x]}$ be the bijection induced by the wellordering on ω_1 coded by $F(x)$ according to the fixed prewellordering of length ω_1.

Define $\Phi : \mathbb{R} \times \omega_1 \to N$ by $\Phi(x, \alpha) = (x, \Psi_x(\alpha))$. Then Φ is a bijection. ∎

A natural question, under $\mathsf{AD}_\mathbb{R}$, is whether there is an intermediate cardinal between $|[\omega_1]^{<\omega_1}|$ and $|[\omega_1]^{\omega_1}|$.

5. Cardinality of S_1. Recall the definition of S_1 from the introduction.

DEFINITION 5.1 (Woodin). Let $S_1 = \{f \in [\omega_1]^{<\omega_1} : \sup(f) = \omega_1^{L[f]}\}$.

This section will establish several properties of the cardinality of S_1 under AD and $\mathsf{DC}_\mathbb{R}$, the statement that all sets of reals have ∞-Borel codes. It will be shown that S_1 does not inject into ${}^\omega\mathrm{ON}$, the class of ω-sequences of ordinals, which implies that $|[\omega_1]^\omega| < |[\omega_1]^{<\omega_1}|$.

Woodin [20] defines the set S_1 and establishes an elaborate dichotomy which asserts that S_1 has a special position among uncountable subsets of $[\omega_1]^{<\omega_1}$.

FACT 5.2 (Woodin's S_1 dichotomy [20, Theorem 19]). *Assume* $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_\mathbb{R}$. *If* $X \subseteq [\omega_1]^{<\omega_1}$ *is uncountable, then either* $|X| \leq |[\omega_1]^\omega|$ *or* $|S_1| \leq |X|$.

The proof of Woodin's S_1 dichotomy is very elaborate. This section will present some elementary arguments to establish several of the basic cardinal properties of S_1 under AD^+.

The next result shows that S_1 contains a copy of $\mathbb{R}$ but has no uncountable wellorderable subsets. These properties are mentioned in [20] without proof, but for completeness, the brief arguments given in [4] will be reproduced below.

Fact 5.3 (Woodin). *Assume* ZF. *Then* $|\mathbb{R}| \leq |S_1|$.

Assume ZF *and there are no uncountable wellorderable sets of reals. Then* $\neg(\omega_1 \leq |S_1|)$.

Proof. For this proof, consider $\mathbb{R}$ as the collection of infinite subsets of ω. For each $r \in \mathbb{R}$, let $A_r = r \cup \{\alpha : \omega \leq \alpha < \omega_1^{L[r]}\}$. Let $f_r \in [\omega_1]^{<\omega_1}$ be the increasing enumeration of A_r. Note that $\omega_1^{L[f_r]} = \omega_1^{L[r]} = \sup(f_r)$. Thus $f_r \in S_1$. The function $\Phi : \mathbb{R} \to S_1$ defined by $\Phi(r) = f_r$ is an injection.

Suppose $\Phi : \omega_1 \to S_1$ is an injection.

Claim. $\sup\{\omega_1^{L[\Phi(\alpha)]} : \alpha < \omega_1\} = \omega_1$.

Suppose not. Let $\epsilon = \sup\{\sup(\Phi(\alpha)) : \alpha < \omega_1\}$ and $\epsilon < \omega_1$. Since Φ maps into S_1, one has $\sup\{\omega_1^{L[\Phi(\alpha)]} : \alpha < \omega_1\} = \sup\{\sup(\Phi(\alpha)) : \alpha < \omega_1\} = \epsilon < \omega_1$. Then Φ would be an injection into $[\epsilon+1]^{<\epsilon+1}$ which is in bijection with $\mathbb{R}$. This is impossible since there are no uncountable wellorderable sets of reals.

Let $\varpi : \omega_1 \times \omega_1 \to \omega_1$ be a constructible bijection, for instance the Gödel pairing function. Think of $S_1 \subseteq [\omega_1]^{<\omega_1}$ as subsets of ω_1. Then let $\tilde{\Phi} = \{\varpi(\alpha,\beta) : \beta \in \Phi(\alpha)\}$. Note that $\tilde{\Phi}$ is a subset of ω_1 which codes the function Φ. That is, $\Phi \in L[\tilde{\Phi}]$. Therefore, $\Phi \in L[\Phi] \models \mathsf{ZFC}$.

Since there are no uncountable wellordered sets of reals, one sees that $\omega_1^{L[\Phi]} < \omega_1$. By the claim, there is some $\alpha < \omega_1$ such that $\omega_1^{L[\Phi(\alpha)]} > \omega_1^{L[\Phi]}$. However, since $\Phi \in L[\Phi]$, $\Phi(\alpha) \in L[\Phi]$. Thus $\omega_1^{L[\Phi(\alpha)]} \leq \omega_1^{L[\Phi]}$, a contradiction. ∎

Woodin's S_1-dichotomy (Fact 5.2) and Fact 5.3 are not sufficient to distinguish $|S_1|$ from $|\mathbb{R}|$, or $|[\omega_1]^\omega|$ from $|[\omega_1]^{<\omega_1}|$. Next, Theorem 5.7 will be shown in order to make these distinctions. (These cardinality distinctions seem to be implicit in [20].) The most interesting properties of S_1 require at least some of the properties of AD^+.

First, we will fix a simple coding for elements of ${}^{<\omega_1}\omega_1$ by reals.

Definition 5.4. Let $\rho : \omega \times \omega \to \omega$ denote a fixed recursive and bijective pairing function. Thinking of $\mathbb{R}$ as ${}^\omega 2$, one can code relations on ω by reals. That is, for each $x \in X$, let $R_x(n,m) \Leftrightarrow x(\rho(n,n)) = 1$. Recall WO is the collection of x such that R_x is a wellordering on ω.

For each $x \in \mathbb{R}$, let $x_n \in \mathbb{R}$ be defined by $x_n(k) = x(\rho(n,k))$. We say that $x \in \mathsf{BS}$ if $x_0 \in \mathrm{WO}$ and for all $n \in \omega$, $(x_1)_n \in \mathrm{WO}$. For each $x \in \mathsf{BS}$, let $\sigma_x : \mathrm{ot}(x_0) \to \omega_1$ be defined by $\sigma_x(\alpha) = \beta$ if for the unique $n \in \omega$ with rank α according to the wellordering R_{x_0}, $\mathrm{ot}((x_1)_n) = \beta$.

In this way, every $\sigma \in {}^{<\omega_1}\omega_1$ has a code $x \in \mathsf{BS}$ such that $\sigma_x = \sigma$.

Fact 5.5. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*, and all sets of reals have* ∞*-Borel codes. Suppose* $R \subseteq {}^{<\omega_1}\omega_1 \times \kappa$*, where* $\kappa < \Theta$*. Then there is a set* $S \subseteq \mathrm{ON}$

and a formula ϑ such that for all $\sigma \in {}^{<\omega_1}\omega_1$ and $\beta < \kappa$,

$$R(\sigma, \beta) \Leftrightarrow L[S, \sigma] \models \vartheta(S, \sigma, \beta).$$

If $\Phi : {}^{<\omega_1}\omega_1 \to {}^{\omega}\kappa$ is a function, then there is a set $S \subseteq \mathrm{ON}$ such that for all $\sigma \in {}^{<\omega_1}\omega_1$, $\Phi(\sigma) \in L[S, \sigma]$.

Proof. Since $\kappa < \Theta$, let $\preceq$ be a prewellordering on $\mathbb{R}$ of length κ. Let (J', ϕ') be an ∞-Borel code for $\preceq$. Let $\varphi : \mathbb{R} \to \kappa$ be the associated ranking function of $\preceq$.

Fix $R \subseteq {}^{<\omega_1}\omega_1 \times \kappa$. Let $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R}$ be defined by

$$\tilde{R}(x, y) \Leftrightarrow x \in \mathsf{BS} \wedge R(\sigma_x, \varphi(y)).$$

Let (J'', ϕ'') be an ∞-Borel code for $\tilde{R}$.

Let J be a set of ordinals coding in some fixed constructible way the two sets of ordinals J' and J''. Let ${}_{\omega}\mathbb{O}_J$ be the finite support direct limit of the Vopěnka forcing $\langle {}_n\mathbb{O}_J, \pi_{n,m} : 0 < m \leq n < \omega\rangle$. Let S be a set of ordinals that codes $(J, {}_{\omega}\mathbb{O}_J)$.

Fix $\sigma \in {}^{<\omega}\omega_1$ and let $\mathbb{P}_\sigma$ denote the forcing $\mathrm{Coll}(\omega, \sup(\sigma))$. Observe that forcing with $\mathbb{P}_\sigma$ over $L[J, \sigma]$ canonically adds a surjection of ω onto $\sup(\sigma)$. From this, one can canonically obtain a bijection of ω with $\sup(f)$. Thus one can naturally produce an element of BS which codes σ in any $\mathbb{P}_\sigma$-generic extension of $L[S, \sigma]$. Let τ_σ be a $\mathbb{P}_\sigma$-name in $L[S, \sigma]$ for this naturally produced element of BS which codes σ.

Let ϑ be the following formula: $\vartheta(S, \sigma, \beta)$ if and only if

$$\begin{aligned} 1_{\mathbb{P}_\sigma} \Vdash_{\mathbb{P}_\sigma} L[J, {}_{\omega}\mathbb{O}_J, \tau_\sigma] \models 1_{{}_{\omega}\mathbb{O}_J/G^1_{\tau_\sigma}} \Vdash_{{}_{\omega}\mathbb{O}_J/G^1_{\tau_\sigma}} \\ L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma, y] \models \phi''(J'', \tau_\sigma, y)). \end{aligned}$$

In the above, "$\varphi(y) = \beta$" is an abbreviation for a statement asserting that β is the rank of y in the prewellordering defined by the ∞-Borel code (J', ϕ').

It is very important that "$\varphi(y) = \beta$" is expressed in this way. The purpose of using $L(J, \mathbb{R})$ and Woodin's results on the symmetric collapse is to express "$\varphi(y) = \beta$", which cannot be computed correctly by evaluating the prewellordering directly in an inner model of ZFC which can only contain countably many of the reals of the original universe satisfying determinacy.

Claim. *For all $\sigma \in {}^{<\omega_1}\omega_1$, $R(\sigma, \beta)$ if and only if $L[S, \sigma] \models \vartheta(S, \sigma, \beta)$.*

To see this: ($\Rightarrow$) Let $p \in \mathbb{P}_\sigma$. Since $\sup(\sigma) < \omega_1$, the powerset of $\mathbb{P}_\sigma$ computed in $L[S, \sigma]$ is countable in the real universe satisfying determinacy. Thus there is a $G \subseteq \mathbb{P}_\sigma$ containing p which is $\mathbb{P}_\sigma$-generic over $L[S, \sigma]$. In $L[S, \sigma][G]$, $\tau_\sigma[G] \in \mathsf{BS}$ is a code for σ, that is $\sigma_{\tau_\sigma[G]} = \sigma$. In $L(J, \mathbb{R})$, there is a $y \in \mathbb{R}$ such that $\varphi(y) = \beta$. Hence $\tilde{R}(\tau_\sigma[G], y)$. Thus

$$L(J, \mathbb{R}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

By Fact 2.10,

$$L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

In particular,

$$L[S,\sigma][G] \models L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

By the forcing theorem and the fact that $p \in G$, there is a $q \leq_{\mathbb{P}_\sigma} p$ such that

$$L[S,\sigma] \models q \Vdash_{\mathbb{P}_\sigma} L[J, {}_\omega\mathbb{O}_J, \tau_\sigma] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma}} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma, y] \models \phi''(J'', \tau_\sigma, y)).$$

Since $p \in \mathbb{P}_\sigma$ was arbitrary, $L[S,\sigma]$ believes that $1_{\mathbb{P}_\sigma}$ forces the statement in the forcing language above. Thus $L[S,\sigma] \models \vartheta(S,\sigma,\beta)$.

($\Leftarrow$) Since the powerset of $\mathbb{P}_\sigma$ computed in $L[S,\sigma] \models \mathsf{ZFC}$ is countable in the real world satisfying AD, there exists a $G \in V$ which is $\mathbb{P}_\sigma$-generic over $L[S,\sigma]$. Note that by the explicit definition of the coding used in BS, one has $\tau_\sigma[G] \in \mathsf{BS}$ and $\sigma_{\tau_\sigma[G]} = \sigma$ by absoluteness. Since $L[S,\sigma] \models \vartheta(S,\sigma,\beta)$, one has

$$L[S,\sigma][G] \models L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

Since G is a set in the real world V,

$$V \models L[J, {}_\omega\mathbb{O}_J, \tau_\sigma[G]] \models 1_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} \Vdash_{{}_\omega\mathbb{O}_J/G^1_{\tau_\sigma[G]}} L(J, \dot{\mathbb{R}}_{\mathrm{sym}}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

Fact 2.10 implies

$$L(J, \mathbb{R}) \models (\exists y)(\varphi(y) = \beta \wedge L[J'', \tau_\sigma[G], y] \models \phi''(J'', \tau_\sigma[G], y)).$$

Since (J'', ϕ'') is the ∞-Borel code for $\tilde{R}$, it follows that $\tilde{R}(\tau_\sigma[G], y)$ holds. By definition of $\tilde{R}$ and the fact that $\tau_\sigma[G] \in \mathsf{BS}$ is a code for σ, $R(\sigma,\beta)$ holds.

This concludes the proof of the claim and hence the first statement of the fact.

Now suppose $\Phi : {}^{<\omega_1}\omega_1 \to {}^\omega\kappa$ is a function. Let $R(\sigma, n, \beta)$ assert that $\Phi(\sigma)(n) = \beta$. By the first part, there is a set $S \subseteq \mathrm{ON}$ and a formula ϑ such that

$$R(\sigma, n, \beta) \Leftrightarrow L[S,\sigma] \models \vartheta(S, \sigma, n, \beta).$$

Then by comprehension in $L[S,\sigma]$, one finds that $\Phi(\sigma) \in L[S,\sigma]$. ∎

A consequence of Fact 5.5 is that (under $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ and all sets of reals have ∞-Borel codes) every subset A of $[\omega_1]^{<\omega_1}$ has an ∞-Borel code (S,φ) in the sense that $\sigma\in A$ if and only if $L[S,\sigma]\models\varphi(S,\sigma)$.

A key idea of the previous argument was to use ∞-Borel codes to go into a suitable $L(J,\mathbb{R})\models\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$ and then by considering the forcing language $\mathrm{Coll}(\omega,\sup(\sigma))$, one can speak of a canonical real coding σ. For $f\in{}^{\omega}\kappa$, there are various ways to code f by a real; however, it is unclear where to find or how to uniformly speak of a real coding f within the ZFC model $\mathrm{HOD}_J^{L(J,\mathbb{R})}=L[J,{}_{\omega}\mathbb{O}_J]$.

We can only prove the following weaker result which is quite similar to Fact 4.6:

Fact 5.6. *Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ and all sets of reals have ∞-Borel codes. Let $\Phi:{}^{\omega}\kappa\to{}^{<\omega_1}\omega_1$ be a partial function, where $\kappa<\Theta$. Then there is a set $S\subseteq\mathrm{ON}$ such that for all $z\in\mathbb{R}$, and all $f\in\mathrm{dom}(\Phi)\cap L[S,z]$, one has $\Phi(f)\in L[S,z]$.*

Proof. Since $\kappa<\Theta$, let $\preceq$ be a prewellordering of $\mathbb{R}$ of length κ. Let φ be the associated ranking function. Let (J',ϕ') denote the ∞-Borel code for $\preceq$.

For each $x\in\mathbb{R}$, let x_n denote the nth section of x. Define $f_x\in{}^{\omega}\kappa$ by $f_x(n)=\varphi(x_n)$. In this way, every $f\in{}^{\omega}\kappa$ has an $x\in\mathbb{R}$ such that $f_x=f$.

Define a relation $R\subseteq\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ by $R(x,v,w)$ if and only if

$$f_x\in\mathrm{dom}(\Phi)\wedge v,w\in\mathrm{WO}\wedge\mathrm{ot}(v)\in\mathrm{dom}(\Phi(f_x))\wedge\Phi(f_x)(\mathrm{ot}(v))=\mathrm{ot}(w).$$

Let (J'',ϕ'') be an ∞-Borel code for R.

Let J be a set of ordinals that codes J' and J'' in some fixed constructible manner.

Now work in $L(J,\mathbb{R})\models\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$. In $L(J,\mathbb{R})$, R is OD_J. Let ς be a formula with ordinal parameters such that $L(J,\mathbb{R})\models R(x,v,w)\Leftrightarrow L(J,\mathbb{R})\models\varsigma(J,x,v,w)$. In $L(J,\mathbb{R})$, let ${}_{\omega}\mathbb{O}_J$ denote the finite support direct limit of J-Vopěnka forcing.

Define $\vartheta(z,J,f,\alpha,\beta)$ by

$$\begin{aligned}&1_{{}_{\omega}\mathbb{O}_J/G_z^1}\Vdash_{{}_{\omega}\mathbb{O}_J/G_z^1}L(J,\dot{\mathbb{R}}_{\mathrm{sym}})\\&\quad\models(\exists x,v,w)((\forall n)(\varphi(x_n)=f(n)\wedge\alpha=\mathrm{ot}(v)\wedge\beta=\mathrm{ot}(w)\wedge\varsigma(J,x,v,w))).\end{aligned}$$

Then for any $z\in\mathbb{R}$, by Fact 2.10, one can conclude for all $f\in L[J,{}_{\omega}\mathbb{O}_J,z]$ that $L(J,\mathbb{R})\models\Phi(f)(\alpha)=\beta$ if and only if $L[J,{}_{\omega}\mathbb{O}_J,z]\models\vartheta(z,J,f,\alpha,\beta)$. By comprehension, $\Phi(f)\in L[J,{}_{\omega}\mathbb{O}_J,z]$. ∎

Theorem 5.7. *Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ and all sets of reals have ∞-Borel codes. Then there is no injection of S_1 into ${}^{\omega}\mathrm{ON}$, the class of ω-sequences of ordinals.*

Proof. Suppose $\Phi : S_1 \to {}^{\omega}\mathrm{ON}$ is an injection. Since $\mathbb{R}$ surjects onto ${}^{<\omega_1}\omega_1$ (for example, by BS and the coding from Definition 5.4), one has that $\mathbb{R}$ surjects onto $S_1 \subseteq {}^{<\omega_1}\omega_1$. Thus one can show that $A = \bigcup\{\mathrm{rang}(\Phi(\sigma)) : \sigma \in S_1\}$ is a collection of ordinals which is a surjective image of $\mathbb{R}$. Thus the Mostowski collapse of A is some ordinal $\kappa < \Theta$. Hence from Φ, one can derive an injection $\Psi : S_1 \to {}^{\omega}\kappa$. Since Ψ is an injection, $\Psi^{-1} : {}^{\omega}\kappa \to S_1$ is a partial function.

Let $S \subseteq \mathrm{ON}$ be a set of ordinals satisfying Fact 5.5 for the function Ψ and Fact 5.6 for the partial function Ψ^{-1}.

Since ω_1 is measurable in $L[S] \models \mathsf{ZFC}$, let $\zeta < \omega_1$ be an inaccessible cardinal of $L[S]$. Let $\mathrm{Coll}(\omega, {<}\zeta)$ be the Lévy collapse of ζ. Since $\zeta < \omega_1$ and $L[S] \models \mathsf{ZFC}$, the powerset of $\mathrm{Coll}(\omega, {<}\zeta)$ is countable in the real world satisfying AD. Thus in the real world, there is a $G \subseteq \mathrm{Coll}(\omega, {<}\zeta)$ which is $\mathrm{Coll}(\omega, {<}\zeta)$-generic over $L[S]$.

From G and its generic surjection of ζ onto ζ, one can find a cofinal function $g : \zeta \to \zeta$ such that $L[g] = L[G]$. Since $L[g] = L[G]$, we have $\omega_1^{L[g]} = \omega_1^{L[G]} = \zeta = \sup(g)$. Thus $g \in S_1$.

By the property of S from Fact 5.5, $\Psi(g) \in L[S, g]$. Since $\Psi(g) \in {}^{\omega}\kappa$, and by using the main property of the Lévy collapse $\mathrm{Coll}(\omega, < \zeta)$, there exists some $\xi < \zeta$ such that $\Psi(g) \in L[S][G \restriction \xi]$. By using the $\mathrm{Coll}(\omega, \xi)$-generic obtained from G, one sees that there is a real $z \in L[S][G]$ such that $L[S][G \restriction \xi] \subseteq L[S][z]$. Thus $\Psi(g) \in L[S, z]$. By the property of S from Fact 5.6 for the partial function Ψ^{-1}, one has $g = \Psi^{-1}(\Psi(g)) \in L[S, z]$. Thus $L[S][G] = L[S][g] \subseteq L[S][z] \subseteq L[S][G \restriction (\xi + 1)]$. It is impossible that $L[S][G] = L[S][G \restriction (\xi + 1)]$ for any $\xi < \zeta$.

Therefore, no such injection can exist. ∎

THEOREM 5.8. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have* ∞*-Borel codes. Then* $|\mathbb{R}| < |S_1|$ *and* $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$.

Proof. Since $|\mathbb{R}| = |{}^{\omega}\omega|$, Theorem 5.7 implies that there is no injection of S_1 into $\mathbb{R}$ or $[\omega_1]^{\omega}$. Thus $|\mathbb{R}| < |S_1|$. Since $S_1 \subseteq [\omega_1]^{<\omega_1}$ and S_1 does not inject into $[\omega_1]^{\omega}$, one has $|[\omega_1]^{\omega}| < |[\omega_1]^{<\omega_1}|$. ∎

6. Countable powerset operation

DEFINITION 6.1. Let X be a set. Let $\mathscr{P}_{\omega_1}(X) = \{A \subseteq X : |A| \leq \aleph_0\}$ be the collection of countable subsets of X.

This section will discuss the question of what cardinality properties of $\mathscr{P}_{\omega_1}(X)$ must have already been exhibited by X. For example, it will be shown that if κ is a cardinal and κ injects into $\mathscr{P}_{\omega_1}(X)$, then κ already injects into X. It will also be shown that if $\mathscr{P}(\omega_1)$ injects into $\mathscr{P}_{\omega_1}(X)$, then $\mathbb{R} \sqcup \omega_1$ already injects into X.

FACT 6.2 (Woodin's perfect set dichotomy). *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have an* ∞*-Borel code. Let* E *be an equivalence relation on* $\mathbb{R}$*. Then exactly one of the following holds:*

(1) $\mathbb{R}/E$ *is wellorderable.*
(2) $\mathbb{R}$ *injects into* $\mathbb{R}/E$*.*

Moreover, if $\mathbb{R}/E$ *is wellorderable and if* (S, φ) *is an* ∞*-Borel code for* E*, then there is a uniform procedure that takes* (S, φ) *to an* $\mathrm{OD}_S^{L(S,\mathbb{R})}$ *wellordering of* $\mathbb{R}/E$*.*

Proof. This result is attributed to Woodin by Hjorth [10]. A proof of these results can be found in [2, Section 8] and [4] which give particular attention to the uniformity aspects of (1) and (2). ■

DEFINITION 6.3. Let X be a set. Let $\mathscr{P}_{\mathrm{WO}}(X) = \{A \subseteq X : A \text{ is wellorderable}\}$. Note that $\mathscr{P}_{\omega_1}(X) \subseteq \mathscr{P}_{\mathrm{WO}}(X)$.

FACT 6.4. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have* ∞*-Borel codes. Let* $\kappa < \Theta$ *and* E *be an equivalence relation on* $\mathbb{R}$*. Suppose* $\Phi : \kappa \to \mathscr{P}_{\mathrm{WO}}(\mathbb{R}/E)$ *is a function. Then there is a sequence* $\langle <_\alpha : \alpha < \kappa \rangle$ *such that* $<_\alpha$ *is a wellordering of* $\Phi(\alpha)$ *for each* $\alpha < \kappa$*.*

Proof. Let (J_0, ϕ_0) be an ∞-Borel code for E. Let $\preceq$ be a prewellordering on $\mathbb{R}$ of length κ. Let $\varsigma : \mathbb{R} \to \kappa$ be the ranking function of $\preceq$. Let (J_1, ϕ_1) be an ∞-Borel code for $\preceq$. Define $R \subseteq \mathbb{R} \times \mathbb{R}$ by $R(x, y) \Leftrightarrow [y]_E \in \Phi(\varsigma(x))$. Let (J_2, ϕ_2) be an ∞-Borel code for R. Let J be a set of ordinals that codes J_0, J_1, and J_2.

Now work in $L(J, \mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. Note that from J, one can recover in $L(J, \mathbb{R})$ the sets E, $\preceq$, R, and Φ. In fact, all these sets are $\mathrm{OD}_J^{L(J,\mathbb{R})}$. Thus for each $\alpha < \kappa$, $\Phi(\alpha)$ is $\mathrm{OD}_J^{L(J,\mathbb{R})}$ with a witnessing definition obtained uniformly in α. Consider $\bigcup \Phi(\alpha) \subseteq \mathbb{R}$. Let $E_\alpha = E \restriction \bigcup \Phi(\alpha)$. E_α is $\mathrm{OD}_J^{L(J,\mathbb{R})}$ uniformly from the definitions witnessing E and $\Phi(\alpha)$ is $\mathrm{OD}_J^{L(J,\mathbb{R})}$. The $\mathrm{OD}_J^{L(J,\mathbb{R})}$ set E_α has an $\mathrm{OD}_J^{L(J,\mathbb{R})}$ ∞-Borel code obtained uniformly from a definition witnessing that E_α is $\mathrm{OD}_J^{L(J,\mathbb{R})}$. (This follows from an application of Fact 2.10.) If the ∞-Borel codes for each equivalence relation in $\langle E_\alpha : \alpha < \kappa \rangle$ can be obtained uniformly, then Fact 6.2 states that one can uniformly produce a sequence of wellorderings $\langle <_\alpha : \alpha < \kappa \rangle$ such that each $<_\alpha$ is a wellordering of $(\bigcup \Phi(\alpha))/E_\alpha$ which is $\Phi(\alpha)$. ■

The following is the "Boldface GCH". It was established first in $L(\mathbb{R})$ by Steel. Woodin extended this result to AD^+.

FACT 6.5 (Woodin). *Assume* $\mathsf{ZF} + \mathsf{AD}^+$*. Let* $\kappa < \Theta$ *be a cardinal. If* $X \subseteq \mathscr{P}(\kappa)$ *is wellorderable, then* $|X| \le \kappa$*.*

THEOREM 6.6. *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have* ∞*-Borel codes. Suppose* $\kappa < \Theta$ *is a cardinal with the property that for all* $\delta < \kappa$*, there is no length-*κ *sequence of distinct subsets of* $\mathscr{P}(\delta)$*. Let* X *be a set such that there is a surjection* $\pi : \mathbb{R} \to X$*. Then* $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies that* $\kappa \leq |X|$*. In particular,* $\kappa \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\kappa \leq |X|$*.*

Assuming $\mathsf{ZF}+\mathsf{AD}^+$*, for all cardinals* $\kappa < \Theta$ *and all sets* X *which are surjective images of* $\mathbb{R}$*,* $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies* $\kappa \leq |X|$*. In particular,* $\kappa \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\kappa \leq |X|$*.*

Proof. Define an equivalence relation on $\mathbb{R}$ by $x\,E\,y$ if and only if $\pi(x) = \pi(y)$. Then X is in bijection with $\mathbb{R}/E$. Thus we will work with $\mathbb{R}/E$ rather than directly with X. If $\kappa \leq |\mathscr{P}_{\mathrm{WO}}(X)|$, then one has an injection $\Phi : \kappa \to \mathscr{P}_{\mathrm{WO}}(\mathbb{R}/E)$. By Fact 6.4, let $\langle <_\alpha : \alpha < \omega_1\rangle$ be a sequence such that for each $\alpha < \kappa$, $<_\alpha$ is a wellordering of $\Phi(\alpha)$.

By using the usual wellordering on κ and the sequence of wellorderings $\langle <_\alpha : \alpha < \kappa\rangle$, one can define a wellordering of $\bigcup \Phi[\kappa] = \bigcup\{\Phi(\alpha) : \alpha < \kappa\}$. Thus $|\bigcup \Phi[\kappa]|$ is a wellordered cardinal.

The claim is that $|\bigcup \Phi[\kappa]| \geq \kappa$. To see this, suppose $|\bigcup \Phi[\kappa]| = \delta$ for some $\delta < \kappa$. Let $\Psi : \bigcup \Phi[\kappa] \to \delta$ be a bijection. Then $\Gamma(\alpha) = \Psi[\Phi(\alpha)] = \{\Psi(x) : x \in \Phi(\alpha)\}$ is an injection of κ into $\mathscr{P}(\delta)$. However, by assumption, there are no length-κ sequences of distinct subsets of $\mathscr{P}(\delta)$. The claim has been shown.

The claim immediately implies that $\kappa \leq |\mathbb{R}/E| = |X|$.

In the setting of $\mathsf{ZF}+\mathsf{AD}^+$, Fact 6.5 implies that for every cardinal $\delta < \kappa$, every wellorderable set of subsets of δ has cardinality δ. Thus κ cannot inject into $\mathscr{P}(\delta)$. The second result now follows from the first. ∎

COROLLARY 6.7. *Assume* $\mathsf{ZF}+\mathsf{DC}_{\mathbb{R}}+\mathsf{AD}$ *and all sets of reals have* ∞*-Borel codes. Let* X *be a set which is a surjective image of* $\mathbb{R}$*. Then* $\omega_1 \leq |\mathscr{P}_{\mathrm{WO}}(X)|$ *implies* $\omega_1 \leq |X|$*. In particular,* $\omega_1 \leq |\mathscr{P}_{\omega_1}(X)|$ *implies* $\omega_1 \leq |X|$*.*

To analyze the structure of the cardinality of sets X such that $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$, one needs an almost everywhere (with respect to the strong partition measure) continuity result for functions $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. The result holds in $\mathsf{ZF}+\mathsf{AD}$ and its proof is quite different from the method used in this article.

FACT 6.8 ([5]). *Assume* $\mathsf{ZF}+\mathsf{AD}$*. For every function* $\Phi : [\omega_1]^{\omega_1} \to \omega_1$*, there is a club* $C \subseteq \omega_1$ *such that* $\Phi\restriction[C]^{\omega_1}_* \to \omega_1$ *is continuous.*

If $C \subseteq \omega_1$ is club, then $[C]^{\omega_1}_*$ is the collection of $f \in [C]^{\omega_1}$ which are of the correct type, i.e. have uniform cofinality ω and are discontinuous everywhere. One can check that $|[\omega_1]^{\omega_1}| = |[C]^{\omega_1}_*|$. The function $\Phi\restriction[C]^{\omega_1}_*$ being continuous means that for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ such that for all $g \in [C]^{\omega_1}_*$, if $f\restriction\alpha = g\restriction\alpha$, then $\Phi(f) = \Phi(g)$.

Zapletal has also asked the authors whether if one partitions $[\omega_1]^{\omega_1}$ into ω_1 many sets, then must one of the pieces have cardinality $|[\omega_1]^{\omega_1}|$, under determinacy assumptions. The almost everywhere continuity property gives a positive answer.

FACT 6.9 ([5]). *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* $\langle X_\alpha : \alpha < \omega_1\rangle$ *be such that* $X_\alpha \subseteq [\omega_1]^{\omega_1}$ *for each* X *and* $\bigcup_{\alpha<\omega_1} X_\alpha = [\omega_1]^{\omega_1}$. *Then there exists* $\alpha < \omega_1$ *such that* $|X_\alpha| = |[\omega_1]^{\omega_1}$.

THEOREM 6.10. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and all sets of reals have an* ∞*-Borel code. Let* X *be a set which is a surjective image of* $\mathbb{R}$. *If* $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(X)|$, *then* $|\mathbb{R} \sqcup \omega_1| \leq |X|$.

Proof. Let $\pi : \mathbb{R} \to X$ be a surjection. Again define an equivalence relation on $\mathbb{R}$ by $x\,E\,y$ if and only if $\pi(x) = \pi(y)$. Since $|X| = |\mathbb{R}/E|$, we will work with the quotient by E. Now suppose $\Phi : [\omega_1]^{\omega_1} \to \mathscr{P}_{\omega_1}(\mathbb{R}/E)$ is an injection.

Note that $|[\omega_1]^{\omega_1}| \leq |\mathscr{P}_{\omega_1}(\mathbb{R}/E)|$ implies, in particular, that $\omega_1 \leq |\mathbb{R}/E|$ by Corollary 6.7. Suppose $\neg(|\mathbb{R}| \leq |\mathbb{R}/E|)$. Then the Woodin perfect set dichotomy (Fact 6.2) implies that $\mathbb{R}/E$ is wellorderable and hence there is some cardinal κ such that $|\mathbb{R}/E| = \kappa$. Let $\Lambda : \mathbb{R}/E \to \kappa$ be a bijection.

Let $\Gamma : [\omega_1]^{\omega_1} \to [\kappa]^{<\omega_1}$ be defined by $\Gamma(f) = \Lambda[\Phi(f)]$. Since $\Phi(f) \in \mathscr{P}_{\omega_1}(\mathbb{R}/E)$, $\Phi(f)$ is a countable subset of $\mathbb{R}/E$. Thus $\Lambda[\Phi(f)] = \{\Lambda(x) : x \in \Phi(f)\}$ is a countable subset of κ.

Let $\mathrm{ot}(\Lambda[\Phi(f)])$ be the ordertype of this countable subset of κ in the usual ordering on κ, which of course is a countable ordinal. Note that $\mathrm{ot}\circ\Gamma : [\omega_1]^{\omega_1} \to \omega_1$.

By letting $X_\alpha = (\mathrm{ot}\circ\Gamma)^{-1}(\{\alpha\})$, one has $[\omega_1]^{\omega_1} = \bigcup_{\alpha<\omega_1} X_\alpha$. By Fact 6.9, there is some $\alpha < \omega_1$ such that $|X_\alpha| = |[\omega_1]^{\omega_1}|$. Let $\Xi : [\omega_1]^{\omega_1} \to X_\alpha$ be a bijection.

Since $\alpha < \omega_1$, let $B : \omega \to \alpha$ be a bijection. For each $f \in [\kappa]^\alpha$, define $\Sigma(f) \in [\kappa]^\omega$ by recursion as follows: $\Sigma(f)(0) = f(B(0))$ and $\Sigma(f)(n+1) = \Sigma(f)(n) + f(B(n+1))$. The map $\Sigma : [\kappa]^\alpha \to [\kappa]^\omega$ is an injection. Then $\Sigma \circ \Gamma \circ \Xi : [\omega_1]^{\omega_1} \to [\kappa]^\omega$ is an injection. Since $|S_1| \leq |[\omega_1]^{\omega_1}|$, one can derive an injection of S_1 into $[\kappa]^\omega$. This violates Theorem 5.7.

Therefore, $|\mathbb{R}| \leq |\mathbb{R}/E| = |X|$. Thus $|\mathbb{R} \sqcup \omega_1| \leq |\mathbb{R}/E| = |X|$. ∎

7. The cardinalities below $\mathbb{R} \times \omega_1$. This section will investigate the cardinalities below $\mathbb{R} \times \omega_1$. Assuming $\mathsf{AD}_{\mathbb{R}}$, a uniformization argument will show there are only four uncountable cardinalities below $|\mathbb{R} \times \omega_1|$. In models of the form AD^+, $\neg\mathsf{AD}_{\mathbb{R}}$, and $V = L(\mathscr{P}(\mathbb{R}))$, this section will show that there are many intermediate cardinalities below $\mathbb{R} \times \omega_1$. This large family of cardinalities will correspond to the ultrapower of ω_1 by the J-constructibility degree measure for a certain set J of ordinals.

Definition 7.1. Let $\Phi : \mathbb{R} \to \omega_1$. Define $\bigsqcup \Phi = \{(r, \alpha) : \alpha < \Phi(r)\}$, which is an $\mathbb{R}$-index disjoint union of countable ordinals given by the function Φ.

Fact 7.2. *Assume* AD. *Then for every* $\Phi : \mathbb{R} \to \omega_1$, ω_1 *does not inject into* $\bigsqcup \Phi$. *If* $\{r : \Phi(r) > 0\}$ *is uncountable, then* $|\mathbb{R}| \leq |\bigsqcup \Phi|$.

Proof. Let $\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ denote the projection onto the first coordinate. Suppose $\Psi : \omega_1 \to \bigsqcup \Phi$ is an injection. Since for all $r \in \mathbb{R}$, $\Phi(r) < \omega_1$, the set of α such that $\pi_1(\Psi(\alpha)) = r$ is countable. Thus $X = \{r : (\exists \alpha < \omega_1)(\pi_1(\Psi(\alpha)) = r)\}$ is an uncountable set of reals. X is wellorderable by setting $x \sqsubset y$ if and only if the least α such that $\pi_1(\Psi(\alpha)) = x$ is less than the least α such that $\pi_1(\Psi(\alpha)) = y$. This is a contradiction since there are no uncountable wellorderable sequences of reals.

Suppose $Y = \{r : \Phi(r) > 0\}$ is uncountable. By the perfect set property, let $\Lambda' : \mathbb{R} \to Y$ be an injection. Then $\Lambda : \mathbb{R} \to \bigsqcup \Phi$ defined by $\Lambda(r) = (\Lambda'(r), 0)$ is an injection. ∎

Fact 7.3. *For all* $X \subseteq \mathbb{R} \times \omega_1$ *such that* $\neg(\omega_1 \leq |X|)$, *there exists a* $\Phi : \mathbb{R} \to \omega_1$ *such that* $X \approx \bigsqcup \Phi$.

Proof. For each $r \in \mathbb{R}$, let $X_r = \{\alpha : (r, \alpha) \in X\}$. Since ω_1 does not inject into X, X_r is countable. Let δ_r be the order type of X_r. Let $\varpi_r : X_r \to \delta_r$ be the associated collapse map. Let $\Phi : \mathbb{R} \to \omega_1$ be defined by $\Phi(r) = \delta_r$. Define $\Lambda : X \to \bigsqcup \Phi$ by $\Lambda(x) = (\pi_1(x), \varpi_{\pi_1(x)}(\pi_2(x)))$, where $\pi_1 : \mathbb{R} \times \omega_1 \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times \omega_1 \to \omega_1$ are the projections onto the first and second coordinate, respectively. Λ is a bijection. ∎

Fact 7.4. *Assume* AD. *For every* $X \subseteq \mathbb{R} \times \omega_1$, *one of the following holds:*

(1) $|X| = |\mathbb{R} \times \omega_1|$.
(2) $|X| = \aleph_1$.
(3) X *is an uncountable set such that* $\neg(\omega_1 \leq |X|)$.
(4) *There is an uncountable* Y *such that* $\neg(\omega_1 \leq |Y|)$ *and* $|X| = |Y \sqcup \omega_1|$.
(5) $|X| \leq \aleph_0$.

Proof. Let $X \subseteq \mathbb{R} \times \omega_1$. For each $r \in \mathbb{R}$, let $X_r = \{\alpha : (r, \alpha) \in X\}$. Let $\delta_r = \mathrm{ot}(X_r)$. For each $r \in \mathbb{R}$, let $\varpi_r : X_r \to \delta_r$ denote the collapse map.

Let $A = \{r : |X_r| = \aleph_1\}$. Suppose A is uncountable. Let $\Psi : \mathbb{R} \to A$ be a bijection which exists by the perfect set property and the Cantor–Schröder–Bernstein theorem. Define $\Lambda : \mathbb{R} \times \omega_1 \to X$ by $\Lambda(r, \alpha) = (\Psi(r), \varpi^{-1}_{\Psi(r)}(\alpha))$. As Λ is a bijection, we have $|X| = |\mathbb{R} \times \omega_1|$. This gives possibility (1).

From now on, assume A is countable. Then $\mathbb{R} \setminus A$ is uncountable. Let $\Phi : \mathbb{R} \to \omega_1$ be defined by

$$\Phi(r) = \begin{cases} \delta_r, & r \notin A, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Lambda : \bigsqcup \Phi \to X$ be defined by $\Lambda(r,\alpha) = (r, \varpi_r^{-1}(\alpha))$. Then Λ is an injection. In fact, it is a bijection onto $X \cap (\mathbb{R} \setminus A \times \omega_1)$. Thus $X \cap (\mathbb{R} \setminus A \times \omega_1)$ does not contain a copy of ω_1 by Fact 7.2. If $B = \{r \in \mathbb{R} \setminus A : \Phi(r) > 0\}$ is uncountable, then $X \cap (\mathbb{R} \setminus A \times \omega_1)$ is an uncountable set without a copy of ω_1. If B is countable, then since a countable union of countable ordinals is countable, $X \cap (\mathbb{R} \setminus A \times \omega_1)$ is a countable set.

Suppose A is nonempty. One can show that a countable union of sets in bijection with ω_1 is in bijection with ω_1. Thus $X \cap (A \times \omega_1) \approx \omega_1$.

Note that $X = X \cap (A \times \omega_1) \sqcup X \cap ((\mathbb{R} \setminus A) \times \omega_1)$. If A is empty and B is countable, then $|X| \le \aleph_0$, which gives case (5). If A is empty and B is uncountable, then X is an uncountable set without a copy of ω_1, which gives case (3). If A is nonempty and B is countable, then $|X| = \aleph_1$, which gives case (2). If A is nonempty and B is uncountable, then X is a union of two sets: one set which is in bijection with ω_1 and another set which is an uncountable set without a copy of ω_1, which gives case (4). ∎

FACT 7.5. *Assume* $\mathsf{AD}_{\mathbb{R}}$. *Every* $X \subseteq \mathbb{R} \times \omega_1$ *such that* $\neg(\omega_1 \le |X|)$ *injects into* $\mathbb{R}$.

Proof. Let WO be the set of reals coding wellorderings with underlying domain ω. Let $X_r = \{\alpha : (r,\alpha) \in X\}$, let $\delta_r = \mathrm{ot}(X_r)$ and let $\varpi_r : X_r \to \delta_r$ be the collapse map of X_r.

Define $R \subseteq \mathbb{R} \times \mathbb{R}$ by $R(x,w)$ if and only if $w \in \mathrm{WO}$ and $\mathrm{ot}(w) = \delta_x$. By $\mathsf{AD}_{\mathbb{R}}$, let $\Sigma : \mathbb{R} \to \mathbb{R}$ be a uniformization for R. For each $w \in \mathrm{WO}$ and for each $\alpha < \mathrm{ot}(w)$, let α^w denote the element of ω with rank α according to w. (If w codes a finite ordinal, then let $n^w = n$.)

Define $\Lambda : X \to \mathbb{R} \times \omega$ by $\Lambda(x) = (\pi_1(x), \varpi_{\pi_1(x)}(\pi_2(x))^{\Sigma(\pi_1(x))})$. Then Λ is an injection. Since $|\mathbb{R} \times \omega| = |\mathbb{R}|$, the proof is complete. ∎

COROLLARY 7.6. *Assume* $\mathsf{AD}_{\mathbb{R}}$. *The uncountable cardinals below* $|\mathbb{R} \times \omega_1|$ *are* $|\mathbb{R}|$, $\aleph_1$, $|\mathbb{R} \sqcup \omega_1|$, *and* $|\mathbb{R} \times \omega_1|$.

Proof. This follows from Facts 7.4 and 7.5.

This is also a consequence of Woodin's dichtomy below $|[\omega_1]^\omega|$ [20, Theorem 18] which is proved under $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}$. However, the proof above under $\mathsf{AD}_{\mathbb{R}}$ uses an elementary uniformization argument while Woodin's stronger result uses sophisticated AD^+ techniques. ∎

We will need several facts about J-constructibility degrees and J-pointed perfect trees:

DEFINITION 7.7. Let J be a set of ordinals. A perfect tree $p \subseteq {}^{<\omega}2$ is *J-pointed* if for all $x \in [p]$, $p \le_J x$.

DEFINITION 7.8. Let p be a perfect tree on 2. Then $s \in p$ is a *split node* of p if $s\hat{\ }0, s\hat{\ }1 \in p$.

By recursion, define $\Xi^p : {}^{<\omega}2 \to {}^{<\omega}2$ by: $\Xi^p(\emptyset)$ is the least split node of p, and if $\Xi^p(s)$ has been defined, then let $\Xi^p(s\hat{\ }i)$ be the least split node of p extending $\Xi^p(s)\hat{\ }i$.

Define $\Upsilon^p : {}^{\omega}2 \to [p]$ by letting $\Upsilon^p(r) = \bigcup_{n\in\omega} \Xi^p(r\restriction n)$. The map Υ^p is called the *canonical homeomorphism* between ${}^{\omega}2$ and $[p]$.

FACT 7.9 (Martin). *Assume* AD. *For all $A \subseteq \mathbb{R}$, A or $\mathbb{R} \setminus A$ contains the body of a Turing pointed tree. Hence for any set J of ordinals, A or $\mathbb{R} \setminus A$ contains the body of a J-pointed tree.*

The Martin Turing degree measure μ, and the J-degree measure μ_J, are countably complete ultrafilters.

Proof. Let $A \subseteq \mathbb{R}$. Let G_A denote the game

$$\begin{array}{cccccccccc} & \mathrm{I} & x_0 & & x_2 & & x_4 & & \cdots & \\ G_A & & & & & & & & & x \\ & \mathrm{II} & & x_1 & & x_3 & & x_5 & \cdots & \end{array}$$

where Player 1 wins if and only if $x \in A$.

Suppose Player 1 has a winning strategy σ. For any $r \in \mathbb{R}$, let $\sigma(r)$ be Player 1's response using σ when Player 2 plays r. Similarly, if $t \in {}^{<\omega}2$, then $\sigma(t)$ is Player 1's response using σ when Player 2 plays t in the finite partial run of G_A.

Thinking of σ as an element of ${}^{\omega}2$, let σ_n denote the nth bit of σ. Let $Z = \{x \in {}^{\omega}2 : (\forall n)(x(2n) = \sigma_n)\}$. Note that Z is the body of a perfect tree.

Let p be the $\subseteq$-downward closure of $\{\sigma(x\restriction n) \oplus (x\restriction n) : n \in \omega \wedge x \in Z\}$. (Recall that if $s, t \in {}^{<\omega}\omega$ are of the same length k, then $s \oplus t$ has length $2k$ where $(s \oplus t)(2j) = s(j)$ and $(s \oplus t)(2j + 1) = t(j)$ whenever $j < k$. If $x, y \in {}^{\omega}\omega$, one can similarly define $x \oplus y$.) Observe that p is a perfect tree and p is Turing reducible to σ. Suppose $f \in [p]$. There is an $x \in Z$ such that $f = \sigma(x) \oplus x$. Since σ is a Player 1 winning strategy, $f = \sigma(x) \oplus x \in A$. This shows that $[p] \subseteq A$. Note that p is Turing reducible to f since $\sigma_n = f(4n+1)$ for all n. Thus, p is a Turing pointed tree. Every Turing pointed tree is a J-pointed tree.

If Player 2 has a winning strategy τ, then a similar argument shows that ${}^{\omega}2 \setminus A$ contains the body of a Turing pointed tree.

Suppose $C \subseteq \mathcal{D}_J$. Let $\tilde{C} = \{x \in {}^{\omega}2 : [x]_J \in C\}$. By the above, $\tilde{C}$ or $\mathbb{R} \setminus \tilde{C}$ contains the body of a J-pointed tree p. Without loss of generality, suppose $[p] \subseteq \tilde{C}$. Suppose $x \in \mathbb{R}$ is such that $p \leq_J x$. Note $\Upsilon^p(x) \leq_J p \oplus x \leq_J x$. Since $\Upsilon^p(x) \in [p]$ and p is J-pointed, $p \leq_J \Upsilon^p(x)$. With knowledge of p, $x = (\Upsilon^p)^{-1}(\Upsilon^p(x)) \leq_J \Upsilon^p(x)$. Thus $\Upsilon^p(x)$ has the same J-degree as x. It has been shown that for any $x \geq_J p$, there is a $y \in [p] \subseteq \tilde{C}$ with the same J-degree as x. Thus C contains the J-cone above the J-degree of p. If

$\mathbb{R} \setminus \tilde{C}$ contains a J-pointed tree, then the same argument shows that $\mathcal{D}_J \setminus C$ contains a J-cone. This shows that μ_J is an ultrafilter.

Suppose $\langle A_n : n \in \omega \rangle$ is a countable sequence from μ_J. Using $\mathsf{AC}_\omega^\mathbb{R}$, let $\langle a_n : n \in \omega \rangle$ be a sequence of reals such that for all $n \in \omega$, $[a_n]_{\equiv_J}$ is the base of J-cone inside A_n. Let $a = \bigoplus a_n$, where $\bigoplus$ is some recursion coding of sequences of reals by a real. Then $[a]_{\equiv_J}$ is a base of a J-cone within $\bigcap_{n\in\omega} A_n$. This shows that μ_J is countably complete (in fact, AD alone implies that every ultrafilter is countably complete). ∎

Lemma 7.10. *Let J be a set of ordinals. Suppose $\Sigma : {}^\omega 2 \to {}^\omega 2$ is a Lipschitz continuous function. Suppose p is a J-pointed tree such that $\Sigma \leq_J p$. Assume that Σ is not constant on any basic neighborhood of $[p]$. Then there is a J-pointed subtree $q \subseteq p$ such that for all $r \in [q]$, $\Sigma(r) \oplus q \equiv_J r$.*

Proof. Since Σ is a Lipschitz continuous function, Σ can be considered as a Player 2 stategy in a game where both players make moves from $\{0, 1\}$. In this way, one will consider Σ as a real. Since Σ is Lipschitz, for each $u \in {}^{<\omega}2$ let $\Sigma(u) \in {}^{|u|}2$ be the string t such that for every $x \in {}^\omega 2$ with $u \subseteq x$, $t \subseteq \Sigma(x)$. If one considers Σ as a Player 2 winning strategy, then $\Sigma(u)$ is just the response of Player 2 using Σ when Player 1 plays u.

Fix a J-pointed tree p. We will construct a sequence $\langle u_s : s \in {}^{<\omega}2 \rangle$ in the tree p and a sequence $\langle n_s : s \in {}^{<\omega}2 \rangle$ of natural numbers with the following properties:

(1) For all $s \in {}^{<\omega}2$, $u_s \subseteq u_{s\hat{\ }i}$ for both $i \in 2$.
(2) For all $s \in {}^{<\omega}2$, if $t \subsetneq s$, then $n_t < n_s$.
(3) For all $s \in {}^{<\omega}2$ and $i \in 2$, $\Sigma(u_{s\hat{\ }i})(n_s) = i$.
(4) Both $\langle u_s : s \in {}^{<\omega}2 \rangle$ and $\langle n_s : s \in {}^{<\omega}2 \rangle$ are Turing computable from $p \oplus \Sigma$. Since $\Sigma \leq_J p$, both sequences belong to $L[J, p]$.

First suppose that such sequences exist. Let q be the $\subseteq$-downward closure of $\{u_s : s \in {}^{<\omega}2\}$. Then q is a perfect subtree of p. We know that q is Turing computable from $p \oplus \Sigma$ and therefore, $q \leq_J p$. Suppose $r \in [q]$. Then $r \in [p]$. Since p is J-pointed, $p \leq_J r$. Thus $q \leq_J r$. This shows that q is also a J-pointed tree.

Let f be the left-most branch of q, i.e. $\Upsilon^q(\bar{0})$ where $\bar{0} \in {}^\omega 2$ is the constant 0 sequence. Note that $f \leq_J q$. Since $f \in [p]$, $p \leq_J f$. Thus $p \leq_J q$ and as a result $p \equiv_J q$. Hence Σ, $\langle u_s : s \in {}^{<\omega}2 \rangle$, and $\langle n_s : s \in {}^{<\omega}2 \rangle$ belong to $L[J, q]$.

Now suppose $r \in [q]$. As observed above, $p \leq_J r$. We seek to define a sequence $\langle v_n : n \in \omega \rangle \leq_J q \oplus \Sigma(r)$ in ${}^{<\omega}2$ such that for all $n \in \omega$, $v_n \subseteq v_{n+1}$, $|v_n| = n$, and $u_{v_n} \subseteq r$.

Let $v_0 = \emptyset$. By construction of q, $u_{v_0} = u_\emptyset \subseteq r$. Suppose v_n has been defined. Let $v_{n+1} = v_n\hat{\ }(\Sigma(r)(n_{v_n}))$. By the induction hypothesis, $u_{v_n} \subseteq r$. If

$r \in [q]$, then $u_{v_n\hat{}0}$ or $u_{v_n\hat{}1}$ is an initial segment of r. By construction, one can determine which of the two is an initial segment of r by determining the value of $\Sigma(r)(n_{v_s})$. This shows that $u_{v_{n+1}} \subseteq r$. This completes the construction of the sequence $\langle v_n : n \in \omega\rangle$ which is Turing computable from $\langle u_s : s \in {}^{<\omega}2\rangle$, $\langle n_s : s \in {}^{<\omega}2\rangle$, and $\Sigma(r)$. Thus $\langle v_n : n \in \omega\rangle \leq_J q \oplus \Sigma(r)$.

Note that $r = \bigcup_{n\in\omega} u_{v_n}$. Thus $r \in L[J, q, \Sigma(r)]$, i.e. $r \leq_J q \oplus \Sigma(r)$.

Also, since $r \in [q]$ and q is J-pointed, $\Sigma \leq_J q \leq_J r$. Thus $q \oplus \Sigma(r) \leq_J r$. It has been shown that $r \equiv_J q \oplus \Sigma(r)$.

Therefore, it remains to show that one can construct the sequences $\langle u_s : s \in {}^{<\omega}2\rangle$ and $\langle n_s : s \in {}^{<\omega}2\rangle$.

Let $u_\emptyset = \emptyset$. Since Σ is not constant, find the least triple (u_0, u_1, m) such that $u_0 \in p$, $u_1 \in p$, $u_0(m) = 0$ and $u_1(m) = 1$. Let $n_\emptyset = m$, $u_{\langle 0\rangle} = u_0$, and $u_{\langle 1\rangle} = u_1$.

Let $s \in {}^{<\omega}2$ and $|s| > 0$. Suppose u_s and $n_{s\restriction|s|-1}$ have been defined. Since Σ is not constant on N_{u_s}, find the least triple (u_0, u_1, m) such that $u_0 \in p$, $u_1 \in p$, $u_s \subseteq u_0$, $u_s \subseteq u_1$, $m > n_{s\restriction|s|-1}$, $|u_0| > m$, $|u_1| > m$, $\Sigma(u_0)(m) = 0$, and $\Sigma(u_1)(m) = 1$. Let $u_{s\hat{}0} = u_0$, $u_{s\hat{}1} = u_1$, and $n_s = m$. This produces the sequences $\langle u_s : s \in {}^{<\omega}2\rangle$ and $\langle n_s : s \in {}^{<\omega}2\rangle$ with the desired property. ■

Definition 7.11. A function $F : \mathbb{R} \to \omega_1$ is *J-invariant* if for all $x, y \in \mathbb{R}$, $x \equiv_J y$ implies $F(x) = F(y)$.

If $F : \mathbb{R} \to \omega_1$ is a J-invariant function, then let $\tilde{F} : \mathcal{D}_J \to \omega_1$ be the induced function on $\mathcal{D}_J$. That is, $\tilde{F}(X) = F(x)$, where $x \in X$.

A J-invariant function F is *everywhere increasing* if for all $x, y \in \mathbb{R}$, $x \leq_J y$ implies $F(x) \leq F(y)$.

A J-invariant function F is *increasing μ_J-almost everywhere* if there is an $a \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ with $a \leq_J x$ and $a \leq_J y$, $x \leq_J y$ implies that $F(x) \leq F(y)$.

Definition 7.12. Let J be a set of ordinals. For each $\mathfrak{F}, \mathfrak{G} \in \prod_{X\in\mathcal{D}_J} \mathrm{ON}$, define $\mathfrak{F} =_{\mu_J} \mathfrak{G}$ if $\{X \in \mathcal{D}_J : \mathfrak{F}(X) = \mathfrak{G}(X)\} \in \mu_J$. Let $\mathfrak{F} <_{\mu_J} \mathfrak{G}$ if $\{X \in \mathcal{D}_J : \mathfrak{F}(X) < \mathfrak{G}(X)\} \in \mu_J$.

The ultraproduct $\prod_{X\in\mathcal{D}_J} \mathrm{ON}/\mu_J$ consists of the equivalence classes of $\prod_{X\in\mathcal{D}_J} \mathrm{ON}$ under $=_{\mu_J}$. For two elements $\mathcal{F}, \mathcal{G} \in \prod_{X\in\mathcal{D}_J} \mathrm{ON}/\mu_J$, we let $\mathcal{F} < \mathcal{G}$ if for all $\mathfrak{F} \in \mathcal{F}$ and $\mathfrak{G} \in \mathcal{G}$, $\mathfrak{F} <_{\mu_J} \mathfrak{G}$.

Let $\prod_{\mathcal{D}_J} \omega_1/\mu_J$ consist of the equivalence classes having a representative which is a function $\mathfrak{F} : \mathcal{D}_J \to \omega_1$.

Fact 7.13 (Woodin). *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let J be a set of ordinals. Then* $\prod_{X\in\mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J = \omega_1$.

Proof. For each $\alpha < \omega_1$, let $F_\alpha : \mathbb{R} \to \omega_1$ be the constant function taking value α. Note that $\tilde{F}_\alpha \in \prod_{X\in\mathcal{D}_J} \omega_1^{L[J,X]}$. By the countable additivity of μ_J, $[\tilde{F}_\alpha]_{\mu_J} = \alpha$. Thus $\omega_1 \subseteq \prod_{X\in\mathcal{D}_J} \omega_1^{L[J,X]}$.

Let $\mathcal{F} \in \prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J$. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function such that $\tilde{F}$ is a representative of $\mathcal{F}$. onsider the following game from [14, Lemma 3.3]:

$$G_F \qquad \begin{array}{lcccccccccc} \mathrm{I} & x_0 & & x_1 & & x_2 & & \cdots & & x & \\ \mathrm{II} & & y_0, z_0 & & y_1, z_1 & & y_2, z_2 & & \cdots & & y, z \end{array}$$

Player 2 wins if and only if $x \leq_J y$, $z \in \mathrm{WO}^{L[J,y]}$, and $\mathrm{ot}(z) = F(y)$.

Claim 1. *Player* 2 *has a winning strategy in this game.*

Suppose otherwise that Player 1 has a winning strategy σ. Consider σ as both a real and as a strategy. Since $\tilde{F} \in \prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}$, pick a $y \geq_J \sigma$ such that $F(y) < \omega_1^{L[J,y]}$. Pick a $z \in \mathrm{WO}^{L[J,y]}$ such that $\mathrm{ot}(z) = F(y)$. Note that $\sigma(y, z) \leq_J y$ since $\sigma, y, z \leq_J y$. Thus Player 2 has won, which contradicts σ being a Player 1 winning strategy. This proves Claim 1.

Thus suppose τ is a Player 2 winning strategy. Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projections onto the first and second coordinate, respectively. Since τ is a winning strategy for Player 2, $\pi_2[\tau[\mathbb{R}]]$ is a Σ^1_1 subset of WO. By boundedness, there is a $\delta < \omega_1$ such that for all $v \in \pi_2[\tau[\mathbb{R}]]$, $\mathrm{ot}(v) < \delta$. Now take $x \geq_J \tau$. Then $\tau(x) \leq_J x$ and therefore $\pi_1(\tau(x)) \leq_J x$. Since τ is a winning strategy for Player 2, $x \leq_J \pi_1(\tau(x))$. So $x \equiv_J \pi_1(\tau(x))$. Since F is J-invariant, $F(x) = F(\pi_1(\tau(x))) = \mathrm{ot}(\pi_2(\tau(x))) < \delta$. Then by the countable additivity of μ_J, there is an $\alpha < \delta$ such that for μ_J-almost all x, $F(x) = \alpha$. Hence $[\tilde{F}]_{\mu_J} = \alpha$.

This shows that $\prod_{X \in \mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J \subseteq \omega_1$, which completes the proof. ∎

Fact 7.14. *Assume* $\mathsf{ZF} + \mathsf{DC}_\mathbb{R} + \mathsf{AD}$. *Let J be a set of ordinals. Then every J-invariant function is increasing μ_J-almost everywhere.*

Proof. Consider the set $A = \{x \in \mathbb{R} : (\forall y)(x \leq_J y \Rightarrow F(x) \leq F(y))\}$. Since F is a J-invariant function, A is a J-invariant set. Let $\tilde{A} = A/{\equiv_J}$ be the corresponding set of J-degrees. By Fact 7.9, $\tilde{A} \in \mu_J$ or $\mathcal{D}_J \setminus \tilde{A} \in \mu_J$.

Case 1. Suppose $\mathcal{D}_J \setminus \tilde{A} \in \mu_J$. There is some $\iota \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ with $\iota \leq x$, $x \notin A$. Let $C_\iota = \{x \in \mathbb{R} : \iota \leq x\}$. Thus for all $x \in C_\iota$, there is a $y \in \mathbb{R}$ with $x \leq_J y$ and $F(y) < F(x)$. Since $\iota \leq_J x \leq_J y$, in fact for all $x \in C_\iota$ there is some $y \in C_\iota$ such that $F(y) < F(x)$. Define a binary relation R on C_ι by $y \mathrel{R} x$ if and only if $F(y) < F(x)$. By $\mathsf{DC}_\mathbb{R}$, there is a sequence $\langle x_n : n \in \omega \rangle$ such that $F(x_{n+1}) < F(x_n)$. This contradicts the wellfoundedness of ON. Thus Case 1 cannot occur.

Case 2. Suppose $A \in \mu_J$. There is some $\iota \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ with $\iota \leq_J x$, $x \in A$. Suppose $x, y \in \mathbb{R}$ is such that $\iota \leq_J x \leq_J y$. By definition of $x \in A$, $F(x) \leq F(y)$, so F is increasing on the cone above ι.

Since only Case 2 can occur, F must be increasing μ_J-almost everywhere. ■

FACT 7.15. *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$. *Let J be a set of ordinals. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function. Then there is a $G : \mathbb{R} \to \omega_1$ which is a J-invariant everywhere increasing function such that $\tilde{F} \sim_{\mu_J} \tilde{G}$.*

Proof. By Fact 7.14, there is an $\iota \in \mathbb{R}$ such that F is increasing above the J-cone of ι. Define $G(x) = \sup\{F(z) : \iota \leq_J z \leq_J x\}$. (If this set is empty, then $G(x) = 0$.) Then G is J-invariant.

If $x \leq_J y$, then $\{z : \iota \leq_J z \leq_J x\} \subseteq \{z : \iota \leq_J z \leq_J y\}$. Thus $G(x) \leq G(y)$. Therefore G is everywhere increasing.

If $x \in \mathbb{R}$ is such that $\iota \leq_J x$, then $G(x) = \sup\{F(z) : \iota \leq_J z \leq x\} = F(x)$ since F is increasing on the cone above ι. ■

FACT 7.16 (Woodin, [17, Theorem 5.9]). *Assume* AD. *Let J be a set of ordinals. For μ_J-almost all $x \in \mathbb{R}$, $L[J, x] \models \mathsf{CH}$.*

FACT 7.17. *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ *and $V = L(J, \mathbb{R})$ for some set J of ordinals. Then there is a set $\mathbb{X}_J$ of ordinals that absorbs every function on $\mathbb{R} \times \omega_1$ in the following sense: for every partial function $\Lambda : \mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$, there is a real z, a formula φ, and an ordinal ξ such that for all $(r, \alpha) \in \mathrm{dom}(f)$, $\Lambda(r, \alpha) \in L[\mathbb{X}_J, z, r]$ and $\Lambda(r, \alpha) = (s, \beta) \Leftrightarrow L[\mathbb{X}_J, z, r, s] \models \varphi(\mathbb{X}_J, z, \xi, r, \alpha, s, \beta)$. In this context, z is said to code Λ.*

Proof. The proof is quite similar to those of Facts 4.6 and 5.6. As in those arguments, one can take $\mathbb{X}_J$ to be $J \oplus {}_\omega\mathbb{O}_J$. ■

REMARK 7.18. Next, we will study the cardinals below $\mathbb{R} \times \omega_1$ under the failure of $\mathsf{AD}_{\mathbb{R}}$. By Fact 2.7, if one is working in the theory $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V{=}L}(\mathscr{P}(\mathbb{R})) + \neg\mathsf{AD}_{\mathbb{R}}$, then there is set J of ordinals such that $V = L(J, \mathbb{R})$. In the rest of this section, we will work with models of the form $L(J, \mathbb{R}) \models \mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. By Fact 7.17, there is an associated set of ordinals $\mathbb{X}_J \in L(J, \mathbb{R})$ which absorbs all functions $\Lambda : \mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$ in $L(J, \mathbb{R})$. Without loss of generality, by replacing J with $\mathbb{X}_J$, we can assume that J is a set of ordinals that absorbs all functions from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$.

DEFINITION 7.19. Let J be a set of ordinals. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function. Define $\Phi_F : \mathbb{R} \to \omega_1$ by $\Phi_F(x) = \omega_{F(x)}^{L[J,x]}$. Let $W_F^J = \bigsqcup \Phi_F$.

FACT 7.20. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let $F_1, F_2 : \mathbb{R} \to \omega_1$ be two everywhere increasing J-invariant functions such that $\tilde{F}_1 =_{\mu_J} \tilde{F}_2$. Then $W_{F_1}^J \approx W_{F_2}^J$.*

Proof. Let $\ell \in \mathbb{R}$ be such that for all $x \geq_J \ell$, $F_1(x) = F_2(x)$. By Fact 7.9, let p be a J-pointed tree such that $[p] \subseteq \{x \in \mathbb{R} : \ell \leq_J x\}$.

Define $\Lambda : W_{F_1}^J \to W_{F_2}^J$ by letting $\Lambda(x, \alpha) = (\Upsilon^p(x), \alpha)$. Since p is J-pointed, $p \leq_J \Upsilon^p(x)$. Hence $p \in L[J, \Upsilon^p(x)]$. Using p and $\Upsilon^p(x)$, one can

Turing compute x. Thus $x \leq_J \Upsilon^p(x)$. Since $\Upsilon^p(x) \in [p]$, we have $F_1(\Upsilon^p(x)) = F_2(\Upsilon^p(x))$. Thus $\alpha < \omega_{F_1(x)}^{L[J,x]} \leq \omega_{F_1(x)}^{L[J,\Upsilon^p(x)]} \leq \omega_{F_1(\Upsilon^p(x))}^{L[J,\Upsilon^p(x)]} = \omega_{F_2(\Upsilon^p(x))}^{L[J,\Upsilon^p(x)]}$ since $x \leq_J \Upsilon^p(x)$, F_1 is everywhere increasing, and F_1 and F_2 are equal on $[p]$. This shows that Λ is well-defined. It is an injection. Thus $|W_{F_1}^J| \leq |W_{F_2}^J|$.

By reversing the roles of F_1 and F_2 in this argument, one sees that $|W_{F_2}^J| \leq |W_{F_1}^J|$. Hence $W_{F_1}^J \approx W_{F_2}^J$. ∎

Definition 7.21. Assume $\mathsf{ZF}+\mathsf{DC}_{\mathbb{R}}+\mathsf{AD}$ and there is a set J of ordinals such that $V = L(J,\mathbb{R})$. For each $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$, define the cardinality $Y_{\mathcal{F}}^J$ to be $|W_F^J|$, where $F : \mathbb{R} \to \omega_1$ is any J-invariant everywhere increasing function such that $\tilde{F} \in \mathcal{F}$. (Note that such an F exists by Fact 7.15 and this definition is correct by Fact 7.20.)

Fact 7.22. *Let J be a set of ordinals. For every $\Phi : \mathbb{R} \to \omega_1$, there is an everywhere increasing J-invariant function F such that $|\bigsqcup \Phi| \leq |W_F^J|$.*

Thus every subset of $\mathbb{R} \times \omega_1$ without a copy of ω_1 injects into W_F^J for some everywhere increasing J-invariant function F. Of course, W_F^J does not contain a copy of ω_1 either, since it is of the form $\bigsqcup \Phi$ for some function Φ.

Proof. Let $F' : \mathbb{R} \to \omega_1$ be defined by letting $F'(x)$ to be the ordinal such that $L[J,x] \models |\Phi(x)| = \aleph_{F'(x)}$.

For each $x \in \mathbb{R}$, let $\Gamma^x : \Phi(x) \to \omega_{F'(x)}^{L[J,x]}$ be the $L[J,x]$-least bijection. Then $\Lambda' : \bigsqcup \Phi \to W_{F'}^J$ defined by $\Lambda'(x,\alpha) = (x, \Gamma^x(\alpha))$ is a bijection.

Let $F(x) = \sup\{F'(z) : z \leq_J x\}$. Then F' is everywhere increasing and $W_{F'}^J$ injects into W_F^J.

The last statement follows from Fact 7.3. ∎

Example 7.23. Let J be a set of ordinals. Let $H_0, H_1 : \mathbb{R} \to \omega_1$ denote the constant 0 and constant 1 function, respectively. Then $|W_{H_0}^J| = |W_{H_1}^J| = |\mathbb{R}|$.

Proof. Note $W_{H_0}^J = \bigsqcup \omega_0^{L[J,x]} \approx \mathbb{R} \times \omega \approx \mathbb{R}$.

For each $x \in \mathbb{R}$, let $\Gamma^x : \omega_1^{L[J,x]} \to \mathbb{R}$ denote the $L[J,x]$-least injection of $\omega_1^{L[J,x]}$ into $\mathbb{R}^{L[J,x]}$. Define $\Lambda : W_{H_0}^J \to \mathbb{R} \times \mathbb{R}$ by $\Lambda(x,\alpha) = (x, \Gamma^x(\alpha))$. Λ is an injection witnessing $|W_{H_1}^J| \leq |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. Thus $W_{H_1}^J \approx \mathbb{R}$. ∎

Fact 7.24. *Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ and $V = L(J,\mathbb{R})$ where J is a set of ordinals that absorbs all functions from $\mathbb{R}\times\omega_1$ into $\mathbb{R}\times\omega_1$ as in Fact 7.17 and Remark 7.18. Suppose $F_1, F_2 : \mathbb{R} \to \omega_1$ are everywhere increasing J-invariant functions such that $\tilde{F}_1 <_{\mu_J} \tilde{F}_2$ and F_1 is not μ_J-almost everywhere equal to 0. Then $|W_{F_1}^J| < |W_{F_2}^J|$.*

Proof. Since F_1 is not μ_J-almost everywhere 0 and $F_1 <_{\mu_J} F_2$, let $\ell \in \mathbb{R}$ be such that for all $x \in \mathbb{R}$ with $\ell \leq_J x$, $1 \leq F_1(x) < F_2(x)$. Let p be a

J-pointed tree such that $[p] \subseteq \{x \in \mathbb{R} : \ell \leq_J x\}$. Define $\Lambda : W^J_{F_1} \to W^J_{F_2}$ by $\Lambda(x,\alpha) = (\Upsilon^p(x),\alpha)$. For all $(x,\alpha) \in W^J_{F_1}$, $\alpha < \omega^{L[J,x]}_{F_1(x)} \leq \omega^{L[J,\Upsilon^p(x)]}_{F_1(\Upsilon^p(x))} < \omega^{L[J,\Upsilon^p(x)]}_{F_2(\Upsilon^p(x))}$ since $x \leq_J \Upsilon^p(x)$, F_1 is everywhere increasing, and $\ell \leq_J \Upsilon^p(x)$. Thus Λ is a well-defined injection witnessing $|W^J_{F_1}| \leq |W^J_{F_2}|$.

Suppose there was an injection $\Lambda : W^J_{F_2} \to W^J_{F_1}$. Since J absorbs all functions, let $z \in \mathbb{R}$ and φ be some formulas such that within $L[J,z]$, Λ is correctly defined in the sense of Fact 7.17. That is, for all $(r,\alpha) \in W^J_{F_2}$, $\Lambda(r,\alpha) \in L[J,z,r]$ and $\Lambda(r,\alpha) = (s,\beta) \Leftrightarrow L[J,z,r] \models \varphi(J,z,r,\alpha,s,\beta)$. By Fact 7.16, let $e \in \mathbb{R}$ be such that for all $x \in \mathbb{R}$, $e \leq_J x$ implies that $L[J,x] \models \mathsf{CH}$.

Let $w = z \oplus \ell \oplus e$. Within $L[J,w]$, Λ as defined by φ is a injection of $W^J_{F_2} \cap L[J,w]$ into $W^J_{F_1} \cap L[J,w]$. In particular, within $L[J,w]$, there is an injection of $\{w\} \times \omega^{L[J,w]}_{F_2(w)}$ into $W^J_{F_1} \cap L[J,w] \subseteq \mathbb{R}^{L[J,w]} \times \omega^{L[J,w]}_{F_1(w)}$ since F_1 is an everywhere increasing function. Since $L[J,w] \models \mathsf{CH}$, $|\mathbb{R}|^{L[J,w]} = \omega_1^{L[J,w]}$. By the definition of ℓ, for all x such that $\ell \leq_J x$, $F_1(x) \geq 1$. Thus $L[J,w] \models |\mathbb{R} \times \omega_{F_1(w)}| = \omega_{F_1(w)}$. Thus within $L[J,w]$, one has an injection of $\omega^{L[J,w]}_{F_2(w)}$ into $\omega^{L[J,w]}_{F_1(w)}$. Since $\ell \leq_J w$, we have $F_2(w) > F_1(w)$. Such an injection cannot exist in $L[J,w]$, a contradiction. This shows $|W^J_{F_1}| < |W^J_{F_2}|$. ∎

Corollary 7.25 (Woodin). *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + \mathsf{V{=}L}(\mathscr{P}(\mathbb{R}))$. *There is a set* $X \subseteq \mathbb{R} \times \omega_1$ *such that* $|\mathbb{R}| < |X|$ *and* $\neg(\omega_1 \leq |X|)$,

Proof. By Fact 2.7, there is a set J of ordinals such that $V = L(J,\mathbb{R})$ and J absorbs functions. Let $F^1, F^2 : \mathbb{R} \to \omega_1$ be the constant function taking values 1 and 2, respectively. By Example 7.23, $W^J_{F^1} \approx \mathbb{R}$. Then by Fact 7.24, $|\mathbb{R}| = |W^J_{F^1}| < |W^J_{F^2}|$.

The set $W^J_{F^2}$ is essentially the example in [20, Theorem 25]. ∎

Theorem 7.26. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and* $V = L(J,\mathbb{R})$ *for some set* J *of ordinals which absorbs functions from* $\mathbb{R} \times \omega_1$ *into* $\mathbb{R} \times \omega_1$. *Let* $\mathfrak{V}$ *be the collection of* $|X|$ *such that* $X \subseteq \mathbb{R} \times \omega_1$ *and* $\neg(\omega_1 \leq |X|)$*; that is,* $\mathfrak{V}$ *is the collection of cardinalities of sets below* $\mathbb{R} \times \omega_1$ *that do not possess a copy of* ω_1.

The sequence $\{Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}\}$ *is an order-preserving injection of the wellordering* $\prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}$ *with the ultrapower ordering into* $\mathfrak{V}$ *with the natural cardinality ordering induced by injections. Moreover, this sequence is cofinal in* $\mathfrak{V}$ *in the sense that if* $Y \in \mathfrak{V}$*, then there is an* $\mathcal{F} \in \prod_{\mathcal{D}} \omega_1/\mu \setminus \{0\}$ *such that* $Y \leq Y^J_{\mathcal{F}}$.

Proof. This is clear from Facts 7.22 and 7.24. Also note that it is necessary to remove 0, for otherwise the sequence would not be injective since $Y^J_0 = |\mathbb{R}| = Y^J_1$ by Example 7.23. ∎

FACT 7.27 (Woodin). *Assume* $\mathsf{ZF}+\mathsf{DC}_{\mathbb{R}}+\mathsf{AD}$ *and* $V = L(J,\mathbb{R})$ *for some set* J *of ordinals. Let* $\mathbb{X}_J = J \oplus_\omega \mathbb{O}_J$. *Then* $\prod_{\mathcal{D}_{\mathbb{X}_J}} \omega_2^{L[\mathbb{X}_J,X]}/\mu_{\mathbb{X}_J} = \Theta^{L(J,\mathbb{R})}$.

Proof. This is shown in [14, Theorem 5.16]. ∎

As in Remark 7.18, if one has that $V = L(J,\mathbb{R})$, one could have always chosen the set of ordinals which absorbed functions to be $J \oplus_\omega \mathbb{O}_J$. Moreover $L(J,\mathbb{R}) = L(J \oplus_\omega \mathbb{O}_J, \mathbb{R})$. Thus $\{Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J\}$ is quite long.

Let $\mathfrak{Y} = \{Y^J_{\mathcal{F}} : \mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}\}$. A natural question would be whether $\mathfrak{V}$, the collection of uncountable cardinals below $\mathbb{R}\times\omega_1$ which does not contain a copy of ω_1, is the same as $\mathfrak{Y}$. Certainly, $\mathfrak{Y} \subseteq \mathfrak{V}$ and $\mathfrak{Y}$ is cofinal in $\mathfrak{V}$. Moreover, for all $\mathcal{Y} \in \mathfrak{Y}$ and $\mathcal{X} \in \mathfrak{V}$, either $\mathcal{X} \leq \mathcal{Y}$ or $\mathcal{Y} \leq \mathcal{X}$. This will follow from the next result. Moreover, the game in the proof is important for later results.

THEOREM 7.28. *Assume* $\mathsf{ZF}+\mathsf{AD}$. *Let* J *be a set of ordinals. Let* $F : \mathbb{R} \to \omega_1$ *be an everywhere increasing* J*-invariant function such that for all* $x \in \mathbb{R}$, $F(x) \geq 1$. *Let* $\Phi : \mathbb{R} \to \omega_1$ *be any function. Consider the following game* S^Φ_F:

$$S^\Phi_F \quad \begin{array}{lcccccccccc} I & r_0 & & r_1 & & r_2 & & r_3 & & \cdots & r \\ II & & x_0 & & x_1 & & x_2 & & x_3 & \cdots & x \end{array}$$

where Players 1 *and* 2 *separately play natural numbers to produce reals* r *and* x. *Player* 2 *wins* S^Φ_F *if and only if* $L[J,r,x] \models \Phi(r) < \omega_{F(r\oplus x)}$. *If Player* 2 *has a winning strategy in* S^Φ_F, *then* $|\bigsqcup\Phi| \leq |W^J_F|$. *If Player* 1 *has a winning strategy in* S^Φ_F, *then* $|W^J_F| \leq |\bigsqcup\Phi|$.

Thus either $|\bigsqcup\Phi| \leq |W^J_F|$ *or* $|W^J_F| \leq |\bigsqcup\Phi|$.

Proof. Suppose Player 2 has a winning strategy τ. For each $r \in \mathbb{R}$, let $\tau(r)$ denote the real that Player 2 produces using τ when Player 1 plays r.

Since τ is a Player 2 winning strategy, for all $r \in \mathbb{R}$, $L[J,r,\tau(r)] \models \Phi(r) < \omega_{F(r\oplus\tau(r))}$. Define $\Lambda : \bigsqcup\Phi \to W^J_F$ by $\Lambda(r,\alpha) = (r\oplus\tau(r),\alpha)$. Λ is an injection witnessing $|\bigsqcup\Phi| \leq |W^J_F|$.

Suppose now Player 1 has a winning strategy σ. For each $x \in \mathbb{R}$, let $\sigma(x)$ be the response by Player 1 using σ when Player 2 plays x.

Since σ is a Player 1 winning strategy, for all $x \in \mathbb{R}$, $L[J,\sigma(x),x] \models \omega_{F(\sigma(x)\oplus x)} \leq \Phi(\sigma(x))$. Note that if $x_0, x_1 \in \mathbb{R}$ are such that $\sigma(x_0) = \sigma(x_1)$ and $\sigma(x_0)\oplus x_0 \equiv_J \sigma(x_1)\oplus x_1$, then $\omega^{L[J,\sigma(x_0),x_0]}_{F(\sigma(x_0)\oplus x_0)} = \omega^{L[J,\sigma(x_1),x_1]}_{F(\sigma(x_1)\oplus x_1)}$.

By Fact 7.16, let $e \in \mathbb{R}$ be such that for all $x \in \mathbb{R}$ with $e \leq_J x$, we have $L[J,x] \models \mathsf{CH}$. By Fact 7.9, let p be a J-pointed perfect tree such that $e \oplus \sigma \leq_J p$, i.e. $[p]$ is inside the cone above $e\oplus\sigma$.

Note that when one considers $\sigma : \mathbb{R} \to \mathbb{R}$ as a Lipschitz function, it cannot be constant on any neighborhood of $[p]$ since $\omega_{F(\sigma(x)\oplus x)}^{L[J,\sigma(x),x]} \leq \Phi(\sigma(x))$ and $F(x) \geq 1$ for all $x \in \mathbb{R}$. Thus by Lemma 7.10, there is a J-pointed perfect subtree $q \subseteq p$ with the property that for all $x \in [q]$, $\sigma(x) \oplus q \equiv_J x$.

Before proceeding, we should give intuition for the next function: σ as a Lipschitz function is not an injection; however, for any $r \in \sigma[[q]]$, one knows where the possible preimages of r come from. Precisely, for any $r \in \sigma[[q]]$, $\sigma^{-1}[\{r\}] \subseteq \mathbb{R}^{L[r\oplus q]}$. Thus there are at most $|\mathbb{R}|^{L[J,r\oplus q]}$ many $x \in \mathbb{R}$ such that $\sigma(x) = r$. Since $L[J, r \oplus q] \models \mathsf{CH}$, we have $L[J, r \oplus q] \models |\mathbb{R}| = \omega_1$. In anticipation of many possible x sharing the same r as its image, we will split $\omega_{F(r\oplus q)}^{L[J,r\oplus q]}$ into $\mathbb{R}^{L[J,r\oplus q]}$ many disjoint pieces of size $\omega_{F(r\oplus q)}^{L[J,r\oplus q]}$. This makes room for each of the possible x such that $\sigma(x) = r$. The details are as follows:

For each $r \in \sigma[[q]]$, let $\Pi^r : \mathbb{R}^{L[J,r\oplus q]} \times \omega_{F(r\oplus q)}^{L[J,r\oplus q]} \to \omega_{F(r\oplus q)}^{L[J,r\oplus q]}$ be the $L[J, r \oplus q]$-least injection which exists since $L[J, r \oplus q] \models \mathsf{CH}$ and $F(x) \geq 1$ for all $x \in \mathbb{R}$. Define $\Lambda' : \bigsqcup_{x\in[q]} \omega_{F(x)}^{L[J,x]} \to \bigsqcup \Phi$ by

$$\Lambda'(x, \alpha) = (\sigma(x), \Pi^{\sigma(x)}(x, \alpha)).$$

Note this is well-defined since for all $x \in [q]$, $\sigma \leq_J q \leq_J x$ and thus $\sigma(x) \oplus x \equiv_J x \equiv_J \sigma(x)\oplus q$. If $x \in [q]$ and $\alpha < \omega_{F(x)}^{L[J,x]}$, then $x \in \mathbb{R}^{L[J,x]} = \mathbb{R}^{L[J,\sigma(x)\oplus q]}$ and $\alpha < \omega_{F(x)}^{L[J,x]} = \omega_{F(\sigma(x)\oplus q)}^{L[J,\sigma(x)\oplus q]}$. Thus (x, α) is in the domain of $\Pi^{\sigma(x)}$. Also, $\Pi^{\sigma(x)}$ maps into $\omega_{F(\sigma(x)\oplus q)}^{L[J,\sigma(x)\oplus q]} = \omega_{F(\sigma(x)\oplus x)}^{L[J,\sigma(x)\oplus x]} \leq \Phi(\sigma(x))$.

Suppose $(x_0, \alpha_0) \neq (x_1, \alpha_1)$ belong to $\bigsqcup_{x\in[q]} \omega_{F(x)}^{L[J,x]}$. If $\sigma(x_0) \neq \sigma(x_1)$, then it is clear that $\Lambda'(x_0, \alpha_0) \neq \Lambda'(x_1, \alpha_1)$. Suppose $\sigma(x_0) = \sigma(x_1)$, and let r be this common value. As noted above, since $x_0, x_1 \in [q]$, one has $x_0 \equiv_J \sigma(x_0) \oplus q \equiv_J r \oplus q \equiv_J \sigma(x_1) \oplus q \equiv_J x_1$. Thus $x_0, x_1 \in \mathbb{R}^{L[J,r\oplus q]}$. Since $x_0 \neq x_1$, we have $\Pi^r(x_0, \alpha_0) \neq \Pi^r(x_1, \alpha_1)$ since Π^r is an injection. By definition of Λ', $\Lambda'(x_0, \alpha_0) \neq \Lambda'(x_1, \alpha_1)$. Thus Λ' is an injection.

Finally, define $\Lambda'' : W_F^J \to \bigsqcup_{x\in[q]} \omega_{F(x)}^{L[J,x]}$ by $\Lambda''(x, \alpha) = (\Upsilon^q(x), \alpha)$. Note $x \leq_J \Upsilon^q(x)$ since q is J-pointed. Therefore $\omega_{F(x)}^{L[J,x]} \leq \omega_{F(x)}^{L[J,\Upsilon^q(x)]} \leq \omega_{F(\Upsilon^q(x))}^{L[J,\Upsilon^q(x)]}$ since F is everywhere increasing. Thus Λ'' is a well-defined injection.

Thus $|W_F^J| \leq |\bigsqcup_{x\in[q]} \omega_{F(x)}^{L[J,x]}| \leq |\bigsqcup \Phi|$. ■

Corollary 7.29. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let J be a set of ordinals. Let $F : \mathbb{R} \to \omega_1$ be a J-invariant function such that $F(x) \geq 1$ for all $x \in \mathbb{R}$. Suppose $X \subseteq \mathbb{R} \times \omega_1$ and $\neg(\omega_1 \leq |X|)$. Then either $|X| \leq Y_F^J$ or $Y_F^J \leq |X|$.*

In other words, for all $\mathcal{X} \in \mathfrak{V}$ and $\mathcal{Y} \in \mathfrak{Y}$, $\mathcal{X} \leq \mathcal{Y}$ or $\mathcal{Y} \leq \mathcal{X}$.

Proof. By Fact 7.3, there is some $\Phi : \mathbb{R} \to \omega_1$ such that $|X| = |\bigsqcup \Phi|$. The result now follows from Theorem 7.28. ■

THEOREM 7.30. *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Let* J *be a set of ordinals. Let* $F : \mathbb{R} \to \omega_1$ *be an everywhere increasing* J*-invariant function. Let* $X \subseteq W^J_{F+1}$, *where* $(F+1)(x) = F(x) + 1$. *Then either* $|X| \le |W^J_F|$ *or* $|W^J_{F+1}| = |X|$.

Proof. By Fact 7.3, there is a $\Phi : \mathbb{R} \to \omega_1$ such that $|X| = |\bigsqcup \Phi|$.

Consider the game S^{Φ}_{F+1} from Theorem 7.28:

$$\begin{array}{llcccccccccc} & \mathrm{I} & r_0 & & r_1 & & r_2 & & r_3 & & \cdots & r \\ S^X_F & & & & & & & & & & & \\ & \mathrm{II} & & x_0 & & x_1 & & x_2 & & x_3 & \cdots & x \end{array}$$

where Player 1 and Player 2 separately play natural numbers to produce reals r and x. Player 2 wins S^X_F if and only if $L[J, r, x] \models \Phi(r) < \omega_{F(r \oplus x)+1}$. By AD, one of the two players has a winning strategy.

By Theorem 7.28, if Player 1 has a winning strategy then $|W^J_{F+1}| \le |\bigsqcup \Phi| = |X| \le |W^J_{F+1}|$. Thus $|X| = |W^J_{F+1}|$.

Suppose now Player 2 has a winning strategy τ. We will need a more careful look at the proof of statement 1 in Theorem 7.28.

For each $r \in \mathbb{R}$, let $\tau(r)$ denote the real that Player 2 produces using τ when Player 1 plays r. Since τ is a Player 2 winning strategy, for all $r \in \mathbb{R}$, $L[J, r, \tau(r)] \models \Phi(r) < \omega_{F(r \oplus \tau(r))+1}$. That is, $L[J, r, \tau(r)] \models |\Phi(r)| \le \omega_{F(r \oplus \tau(r))}$. Let $\Gamma^r : \Phi(r) \to \omega^{L[J,r,\tau(r)]}_{F(r \oplus \tau(r))}$ denote the $L[J, r, \tau(r)]$-least injection of $\Phi(r)$ into $\omega^{L[J,r,\tau(r)]}_{F(r \oplus \tau(r))}$.

Define $\Lambda : \bigsqcup \Phi \to W^J_F$ by $\Lambda(r, \alpha) = (r \oplus \tau(r), \Gamma^r(\alpha))$. Then Λ is an injection witnessing $|\bigsqcup \Phi| \le |W^J_F|$. ∎

Note that the assumption for Theorems 7.28 and 7.30 is just $\mathsf{ZF} + \mathsf{AD}$ and J is any set of ordinals (with no assumption about function absorption, although the two cardinalities may degenerate without these assumptions).

COROLLARY 7.31. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *and* $V = L(J, \mathbb{R})$ *where* J *is a set of ordinals which absorbs functions from* $\mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$. *Then for all* $n \in \omega \setminus \{0\}$, *there are no cardinalities between* Y^J_n *and* Y^J_{n+1}. *In particular, there are no cardinalities between* $|\mathbb{R}| = Y^J_1$ *and* Y^J_2.

THEOREM 7.32. *Assume* $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \mathsf{AD}$ *and* $V = L(J, \mathbb{R})$, *where* J *is a set of ordinals. Let* $\mathcal{F} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J \setminus \{0\}$ *be such that* $\mathrm{cof}(\mathcal{F}) = \omega$. *Let* $\langle \mathcal{F}_n : n \in \omega \rangle$ *be any* ω*-cofinal sequence through* $\mathcal{F}$. *Then there exist everywhere increasing* J*-invariant functions from* $\mathbb{R}$ *into* ω_1, F *and* $\langle F_n : n \in \omega \rangle$, *such that* $[\tilde{F}]_{\mu_J} = \mathcal{F}$ *and for all* $n \in \omega$, $[\tilde{F}_n]_{\mu_J} = \mathcal{F}_n$.

Furthermore, assume J *is a set of ordinals which absorbs functions from* $\mathbb{R} \times \omega_1$ *to* $\mathbb{R} \times \omega_1$. *Then for any* $X \subseteq W^J_F$, *either* $|X| = |W^J_F|$ *or there exists an* $n \in \omega$ *such that* $|X| \le |W^J_{F_n}|$.

Proof. By Fact 7.15, every $\mathcal{G} \in \prod_{\mathcal{D}_J} \omega_1/\mu_J$ has an everywhere increasing J-invariant $G : \mathbb{R} \to \omega_1$ such that $\mathcal{G} = [\tilde{G}]_{\mu_J}$. Since AD implies $\mathsf{AC}_\omega^\mathbb{R}$ and every set in $L(J,\mathbb{R})$ is ordinal definable from J and a real, one finds that $L(J,\mathbb{R})$ satisfies AC_ω, the full axiom of countable choice. Thus one can obtain F and $\langle F_n : n \in \omega\rangle$ as in the first statement of the theorem. We may assume that for all $n \in \omega$ and all $x \in \mathbb{R}$, $F_n(x) \geq 1$.

Now fix an $X \subseteq W_F^J$. Suppose there is no n such that $|X| \leq |W_{F_n}^J|$. Let $m \in \omega$. Suppose $\langle X_k : k < m\rangle$ is a sequence of disjoint subset of X and $X_k \approx W_{F_k}^J$ for all $k < m$. Let $Y = X \setminus (\bigcup_{k<m} X_k)$. For each $r \in \mathbb{R}$, let $\delta_r = \mathrm{ot}(Y_r)$. Let $\Phi : \mathbb{R} \to \omega_1$ be defined by $\Phi(r) = \delta_r$. Note that $Y \approx \bigsqcup \Phi$.

Consider the game $S_{F_m}^\Phi$ from Theorem 7.28.

Case 1: Suppose Player 2 has a winning strategy in $S_{F_m}^\Phi$. By Theorem 7.28, there is an injection $\Lambda : \bigsqcup \Phi \to W_{F_m}^J$. Since $Y \approx \bigsqcup \Phi$, there is an injection of Y into $W_{F_m}^J$.

Note that $W_{F_m}^J$ is in bijection with $\bigsqcup_{k\leq m} W_{F_m}^J$. Since $X_k \approx W_{F_k}^J$ and $|W_{F_k}| \leq |W_{F_m}|$ for all $k < m$, there are injections of X_k into $W_{F_m}^J$. Thus there is an injection of $X = Y \sqcup \bigsqcup_{k<m} X_k$ into $\bigsqcup_{k\leq m} W_{F_m}^J \approx W_{F_m}^J$. This contradicts the assumption that there is no $n \in \omega$ such that $|X| \leq |W_{F_n}^J|$. So Case 1 cannot occur.

Case 2: Player 1 has a winning strategy in $S_{F_m}^\Phi$. Theorem 7.28 states that there is an injection $\Lambda_m : W_m^J \to Y$. Let X_m be the image of Λ_m.

Consider the tree T of $(\Lambda_0, \dots, \Lambda_{m-1})$ such that each $\Lambda_i : W_{F_i}^J \to X$ is an injection and for all $i < j < m$, $\Lambda_i[W_i^J] \cap \Lambda_j[W_j^J] = \emptyset$. Order this tree by extension. By the analysis above, this tree has no dead branches. Since $L(J,\mathbb{R}) \models \mathsf{DC}_\mathbb{R}$ and all sets are ordinal definable from J and a real, $L(J,\mathbb{R}) \models \mathsf{DC}$. Thus let $\langle \Lambda_i : i \in \omega\rangle$ be a branch through the tree T.

Define $K : \mathbb{R} \times \omega_1 \to \mathbb{R} \times \omega_1$ by

$$K(r,\alpha) = \begin{cases} F_\alpha(r), & \alpha < \omega, \\ 0, & \text{otherwise.} \end{cases}$$

Since J absorbs functions, as in Fact 7.17, there is an $\ell_0 \in \mathbb{R}$ such that for all $x \geq_J \ell_0$ and $\alpha < \omega_1$, $K(x,\alpha) \in L[J,x]$. In particular, by absorbing K, one sees that for all x with $\ell_0 \leq x$, $\langle F_n(x) : n \in \omega\rangle \in L[J,x]$.

Since $\langle \mathcal{F}_n : n \in \omega\rangle$ is cofinal through $\mathcal{F}$, one can use the countable additivity of μ_J to find an $\ell \geq_J \ell_0$ such that for all $x \in \mathbb{R}$ with $\ell \leq_J x$, $\langle F_n(x) : n \in \omega\rangle$ is a cofinal sequence through $F(x)$. Let p be a J-pointed tree such that $\ell \leq_J p$. For each $s \in [p]$, let $\Sigma^s : \omega_{F(x)}^{L[J,s]} \to \bigsqcup_{n\in\omega} \omega_{F_n(s)}^{L[J,s]}$ be the $L[J,s]$-least injection. (Note it is important that $\langle F_n(s) : s \in \omega\rangle \in L[J,s]$ for this to make sense.)

Let $\Lambda^* : \bigsqcup_{x\in[p]} \omega_{F(x)}^{L[J,x]} \to X$ be defined by

$$\Lambda^*(x,\alpha) = \Lambda_{\pi_1(\Sigma^x(x,\alpha))}(x, \pi_2(\Sigma^x(x,\alpha))).$$

Here, we think of $\bigsqcup_{n\in\omega} \omega_{F_n(x)}^{L[J,s]} = \{(n,\alpha) : n \in \omega \wedge \alpha < \omega_{F_n(x)}^{L[J,x]}\}$ as a subset of $\omega \times \omega_1$. The functions $\pi_1 : \omega \times \omega_1 \to \omega$ and $\pi_2 : \omega \times \omega_1 \to \omega_1$ are the projections onto the first and second coordinates, respectively. Here we consider $W_{F_i}^J$ as a subset of $\mathbb{R} \times \omega_1$. Observe that Λ^* is an injection.

As usual, $\Lambda^\star : W_{F(x)}^J \to \bigsqcup_{x\in[p]} \omega_{F(x)}^{L[J,x]}$ defined by $\Lambda^\star(x,\alpha) = (\Upsilon^p(x), \alpha)$ is an injection. It has been shown that $|W_F^J| \leq |X|$ and hence $|X| = |W_F^J|$. ∎

By Fact 7.13, the first ω_1 elements of $\prod_{\mathcal{D}_J} \omega_1/\mu_J$ are the elements of $\prod_{X\in\mathcal{D}_J} \omega_1^{L[J,X]}/\mu_J$. For each $\alpha < \omega_1$, let $F^\alpha : \mathbb{R} \to \omega_1$ be the constant function α. Note that $[\tilde{F}^\alpha]_{\mu_J}$ is α in the ultrapower. Thus, $Y_\alpha^J = |W_{F^\alpha}^J|$. From the results shown so far, one can determine the ω_1-initial segment of $\mathfrak{Y}$, the collection of cardinalities below $|\mathbb{R} \times \omega_1|$ without a copy of ω_1:

THEOREM 7.33. *Assume* $\mathsf{ZF} + \mathsf{AD} + \mathsf{V}{=}\mathsf{L}(J,\mathbb{R})$ *where J is a set of ordinals which absorbs functions from $\mathbb{R} \times \omega_1$ into $\mathbb{R} \times \omega_1$. The collection of cardinalities $\{Y_\alpha^J : 1 \leq \alpha < \omega_1\}$ is closed under the injection relation, $\leq$. That is, if $\mathcal{X}$ is an uncountable cardinality and there is some $\alpha < \omega_1$ such that $\mathcal{X} \leq Y_\alpha^J$, then there is some $1 \leq \beta \leq \alpha$ such that $\mathcal{X} = Y_\beta^J$. Moreover, $\{Y_\alpha^J : 1 \leq \alpha < \omega_1\}$ is an initial segment of $\mathfrak{Y}$ under the injection relation in the sense that for all $\mathcal{X} \in \mathfrak{Y}$, either $\mathcal{X} \in \{Y_\alpha^J : 1 \leq \alpha < \omega_1\}$ or for all $\alpha < \omega_1$, $Y_\alpha^J \leq \mathcal{X}$.*

Acknowledgements. The first author was supported by NSF grant DMS-1703708. The second author was supported by NSF grant DMS-1800 323.

References

[1] A. E. Caicedo and R. Ketchersid, *A trichotomy theorem in natural models of* AD^+, in: Set Theory and its Applications, Contemp. Math. 533, Amer. Math. Soc., Providence, RI, 2011, 227–258.

[2] W. Chan, *An introduction to combinatorics of determinacy*, in: Trends in Set Theory, Contemp. Math. 752, Amer. Math. Soc., Providence, RI, 2020, 21–75.

[3] W. Chan and S. Jackson, $\mathbf{L}(\mathbb{R})$ *with determinacy satisfies the Suslin hypothesis*, Adv. Math. 346 (2019), 305–328.

[4] W. Chan and S. Jackson, *Cardinality of wellordered disjoint unions of quotients of smooth equivalence relations*, Ann. Pure Appl. Logic 172 (2021), art. 102988, 21 pp.

[5] W. Chan and S. Jackson, *Definable combinatorics at the first uncountable cardinal*, Trans. Amer. Math. Soc. 374 (2021), 2035–2056.

[6] W. Chan, S. Jackson, and N. Trang, *Almost everywhere behavior of functions according to partition measures*, Forum Math. Sigma 12 (2024), art. e1b.

[7] W. Chan, S. Jackson, and N. Trang, *More definable combinatorics around the first and second uncountable cardinals*, J. Math. Log. 23 (2023), art. 2250029, 31 pp.

[8] L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. 3 (1990), 903–928.

[9] L. A. Harrington, R. A. Shore, and T. A. Slaman, Σ^1_1 *in every real in a* Σ^1_1 *class of reals is* Σ^1_1, in: Computability and Complexity, Lecture Notes in Computer Sci. 10010, Springer, Cham, 2017, 455–466.

[10] G. Hjorth, *A dichotomy for the definable universe*, J. Symbolic Logic 60 (1995), 1199–1207.

[11] T. Jech, *Set Theory*, 3rd ed., Springer Monogr. Math., Springer, Berlin, 2003.

[12] A. S. Kechris, *The axiom of determinacy implies dependent choices in* $L(\mathbb{R})$, J. Symbolic Logic 49 (1984), 161–173.

[13] A. S. Kechris, E. M. Kleinberg, Y. N. Moschovakis, and W. H. Woodin, *The axiom of determinacy, strong partition properties and nonsingular measures*, in: Cabal Seminar 77–79, Lecture Notes in Math. 839, Springer, Berlin, 1981, 75–99.

[14] R. Ketchersid, *More structural consequences of* AD, in: Set Theory and its Applications, Contemp. Math. 533, Amer. Math. Soc., Providence, RI, 2011, 71–105.

[15] R. Ketchersid, P. Larson, and J. Zapletal, *Ramsey ultrafilters and countable-to-one uniformization*, Topology Appl. 213 (2016), 190–198.

[16] E. M. Kleinberg, *Infinitary combinatorics and the axiom of determinateness*, Lecture Notes in Math. 612, Springer, Berlin, 1977.

[17] P. Koellner and W. H. Woodin, *Large cardinals from determinacy*, in: Handbook of Set Theory, Springer, Dordrecht, 2010, 1951–2119.

[18] J. H. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Ann. Math. Logic 18 (1980), 1–28.

[19] J. R. Steel, *An outline of inner model theory*, in: Handbook of Set Theory, Springer, Dordrecht, 2010, 1595–1684.

[20] W. H. Woodin, *The cardinals below* $|[\omega_1]^{<\omega_1}|$, Ann. Pure Appl. Logic 140 (2006), 161–232.

[21] W. H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, rev. ed., De Gruyter Ser. Logic Appl. 1, de Gruyter, Berlin, 2010.

William Chan
Institute for Discrete Mathematics and Geometry
Vienna University of Technology
1040 Wien, Austria
E-mail: william.chan@tuwien.ac.at

Stephen Jackson
Department of Mathematics
University of North Texas
Denton, TX 76203, USA
E-mail: stephen.jackson@unt.edu