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ON WEIGHTED ESTIMATES FOR THE STREAM FUNCTION OF AXIALLY SYMMETRIC SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN A BOUNDED CYLINDER

Abstract. Higher-order estimates in weighted Sobolev spaces for solutions to a singular elliptic equation for the stream function in an axially symmetric cylinder are provided. These estimates are essential for the proof of the global existence of regular axially symmetric solutions to incompressible Navier–Stokes equations in axially symmetric cylinders. In order to derive the estimates, the technique of weighted Sobolev spaces developed by Kondrat'ev is applied. The weight is a power function of the distance to the axis of symmetry.

1. Introduction. In this note we derive estimates for solutions to the following problem:

$$\begin{cases} -\Delta\psi + \dfrac{\psi}{r^2} = \omega & \text{in } \Omega, \\ \psi = 0 & \text{on } S := \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded cylinder with boundary S. Before we go into any geometrical details (see (1.6)), we briefly justify why this problem is highly important in mathematical fluid mechanics.

Our ultimate goal is to study the regularity of weak solutions to an initial-boundary value problem for the three-dimensional axi-symmetric Navier–Stokes equations with non-vanishing *swirl*. In order to define this quantity we need to introduce cylindrical coordinates. If $x = (x_1, x_2, x_3)$

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in Cartesian coordinates, then the cylindrical coordinates (r,φ,z) are introduced by the relation $x=\mathbf{\Phi}(r,\varphi,z)$, where

$$\begin{aligned} x_1 &= r\cos\varphi,\\ x_2 &= r\sin\varphi,\\ x_3 &= z. \end{aligned}$$

Thus, the standard basis vectors are

$$\begin{aligned} \bar e_r &= \partial_r\mathbf{\Phi} = (\cos\varphi,\sin\varphi,0),\\ \bar e_\varphi &= \partial_\varphi\mathbf{\Phi} = (-\sin\varphi,\cos\varphi,0),\\ \bar e_z &= \partial_z\mathbf{\Phi} = (0,0,1). \end{aligned}$$

Let $\mathbf{w}=\mathbf{w}(x,t)$ be any vector-valued function of x and t. Then in cylindrical coordinates, $\mathbf{w}$ is expressed in the standard basis as follows:

$$\mathbf{w} = w_r(r,\varphi,z,t)\bar e_r + w_\varphi(r,\varphi,z,t)\bar e_\varphi + w_z(r,\varphi,z,t)\bar e_z. \tag{1.2}$$

We call $\mathbf{w}$ *axially-symmetric* if

$$w_{r,\varphi} = w_{\varphi,\varphi} = w_{z,\varphi} = 0.$$

Let $\mathbf{v}$ and p denote the velocity field of an incompressible fluid and the pressure, respectively. Let $\operatorname{rot}\mathbf{v}$ be the vorticity vector. Then the Navier–Stokes equations read

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Omega^T = \Omega\times(0,T),\\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega^T,\\ \mathbf{v}\cdot\bar n = 0 & \text{on } S^T = S\times(0,T),\\ \mathbf{v}\cdot\bar e_\varphi = 0 & \text{on } S^T,\\ \operatorname{rot}\mathbf{v}\cdot\bar e_\varphi = 0 & \text{on } S^T,\\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\mathbf{f}$ is the external force field and $\bar n$ is the unit outward vector normal to S, and S and Ω are the same as in (1.1).

In the mathematical theory of fluid mechanics we call the function rv_φ the *swirl*.

The problem of regularity of axially-symmetric solutions to (1.3) is in general open. Since 1968 (see [3] and [10]) it has been known that the Navier–Stokes equations have regular axially-symmetric solutions in $\mathbb{R}^3$ provided that $v_\varphi=0$ and $f_\varphi=0$ (hence the swirl is zero). In the case of non-vanishing swirl there are some partial results, e.g. [1, 4–7, 11, 13] though this list is far from complete.

However, a long list of papers concerning regularity criterions for the axially-symmetric Navier–Stokes equations can be found in [8].

One way to investigate the existence of solutions to (1.3) is to start with the following observation: if $\mathbf{v}$ is an axially symmetric solution to (1.3), then in light of (1.2) we have

$$\mathbf{v} = v_r(r,z,t)\bar{e}_r + v_\varphi(r,z,t)\bar{e}_\varphi + v_z(r,z,t)\bar{e}_z$$

and

$$\operatorname{rot}\mathbf{v} = -v_{\varphi,z}(r,z,t)\bar{e}_r + \omega(r,z,t)\bar{e}_\varphi + \frac{1}{r}(rv_\varphi)_{,r}(r,z,t)\bar{e}_z,$$

where

$$\omega = v_{r,z} - v_{z,r}. \tag{1.4}$$

Expressing $(1.3)_2$ in cylindrical coordinates yields

$$(rv_r)_{,r} + (rv_z)_{,z} = 0$$

and combining this equation with (1.4) suggests introducing a stream function ψ such that

$$v_r = -\psi_{,z}, \qquad v_z = \frac{1}{r}(r\psi)_{,r}. \tag{1.5}$$

Since

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$$

we see that this stream function satisfies (1.1). Note that $(1.1)_2$ follows from $(1.3)_3$. This explains why (1.1) is of primary interest. Solutions to this problem are essential for establishing global, regular and axially-symmetric solutions to the Navier–Stokes equations with non-vanishing swirl (see [12]). We demonstrate this idea for the case of small swirl in [8]. Having proper estimates for solutions to (1.1), the proof in [12] works.

There is a challenge in investigating (1.1), which we shall now discuss. Let $a > 0$ and $R > 0$. In cylindrical coordinates, the bounded cylinder Ω is given by

$$\Omega = \{x \in \mathbb{R}^3 \colon r < R,\ |z| < a\}, \tag{1.6}$$

where $S = \partial\Omega = S_1 \cup S_2$ and

$$S_1 = \{x \in \mathbb{R}^3 \colon r = R,\ |z| < a\},$$
$$S_2 = \{x \in \mathbb{R}^3 \colon r < R,\ z \in \{-a, a\}\}.$$

It follows that the terms $\frac{1}{r^2}\psi$ and $\frac{1}{r}\psi_r$ might be undefined for $r = 0$. There are a few possibilities of overcoming this issue: one could

- remove the ϵ-neighborhood of $r = 0$, derive necessary estimates and let $\epsilon \to 0^+$ (see e.g. [3]),
- consider $\frac{1}{r^{1-\epsilon}}\psi$, derive necessary estimates and let $\epsilon \to 0^+$ (see e.g. [4]),
- use weighted Sobolev spaces.

We take the third approach. The classical results for the Poisson equation tell us that if $\omega \in H^1$, then $\psi \in H^3$. We would expect a similar outcome here but we need to handle $\frac{1}{r}$ and similar terms carefully.

If we were interested in basic energy estimates we could proceed the standard way: multiply (1.1) by ψ, integrate by parts, use the Hölder and Cauchy inequalities. This would be justified because in light of [5] and [7, Remark 2.4] we have

$$\psi = O(r) \quad \text{as } r \to 0^+ \tag{1.7}$$

provided that ψ is introduced through (1.5) and $\mathbf{v}$ is an axially symmetric vector field of class $\mathcal{C}^1(0,R)$. Moreover, if $\mathbf{v} \in \mathcal{C}^3(0,R)$, then

$$\psi = a_1(z,t)r + a_3(z,t)r^3 + o(r^5) \quad \text{as } r \to 0^+, \tag{1.8}$$

where a_1 and a_3 are smooth functions. Since basic energy estimates are not enough in our case, more sophisticated tools and techniques are needed. Weighted Sobolev spaces seem to be the right choice.

To conduct our analysis we introduce the quantity $\psi_1 = \psi/r$. We see that it satisfies

$$\begin{cases} -\Delta\psi_1 - \dfrac{2}{r}\psi_{1,r} = \dfrac{\omega}{r} \equiv \omega_1 & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } S. \end{cases} \tag{1.9}$$

Since $\psi = \frac{\psi}{r} r = \psi_1 r$ and r is bounded by R, we see that any estimates for ψ_1 are immediately applicable to ψ. In fact, in [12] we need estimates for ψ_1 because this function appears naturally in some auxiliary problems.

To examine problem (1.9) in weighted Sobolev spaces we have to derive estimates with respect to r and z separately. To derive an estimate with respect to r we have to examine solutions to (1.9) independently both in a neighborhood of the axis of symmetry and in a neighborhood at a positive distance from it. To perform such analysis we treat z as a parameter and we introduce a partition of unity $\{\zeta^{(1)}(r), \zeta^{(2)}(r)\}$ such that

$$\sum_{i=1}^{2} \zeta^{(i)}(r) = 1$$

and

$$\zeta^{(1)}(r) = \begin{cases} 1 & \text{for } r \le r_0, \\ 0 & \text{for } r \ge 2r_0, \end{cases} \qquad \zeta^{(2)}(r) = \begin{cases} 0 & \text{for } r \le r_0, \\ 1 & \text{for } r \ge 2r_0, \end{cases}$$

where $r_0 > 0$ is fixed in such a way that $2r_0 < R$.

Let

$$\tilde\psi_1^{(i)} = \psi_1\zeta^{(i)}, \qquad \tilde\omega_1^{(i)} = \omega_1\zeta^{(i)}, \qquad i = 1,2,$$

and $\dot\zeta = \frac{d}{dr}\zeta$, $\ddot\zeta = \frac{d^2}{dr^2}\zeta$. Then from (1.9) we obtain two problems:

$$\begin{cases} -\Delta\tilde\psi_1^{(1)} - \dfrac{2}{r}\tilde\psi_{1,r}^{(1)} = \tilde\omega_1^{(1)} - 2\psi_{1,r}\dot\zeta^{(1)} - \psi_1\ddot\zeta^{(1)} - \dfrac{2}{r}\psi_1\dot\zeta^{(1)} & \text{in } \Omega^{(1)}, \\ \tilde\psi_1^{(1)} = 0 & \text{on } \partial\Omega^{(1)}, \end{cases} \tag{1.10}$$

where

$$\Omega^{(1)} = \{(r,z)\colon r>0,\ z\in(-a,a)\}, \qquad \partial\Omega^{(1)} = \{(r,z)\colon z\in\{-a,a\},\ r>0\},$$

and

$$\begin{cases} -\Delta\tilde\psi_1^{(2)} - \dfrac{2}{r}\tilde\psi_{1,r}^{(2)} = \tilde\omega_1^{(2)} - 2\psi_{1,r}\dot\zeta^{(2)} - \psi_1\ddot\zeta^{(2)} - \dfrac{2}{r}\psi_1\dot\zeta^{(2)} & \text{in } \Omega^{(2)},\\ \tilde\psi^{(2)} = 0 & \text{on } \partial\Omega^{(2)}, \end{cases} \tag{1.11}$$

where

$$\Omega^{(2)} = \{(r,z)\colon r_0<r<R,\ z\in(-a,a)\}, \quad \partial\Omega^{(2)} = \partial\Omega_1^{(2)} \cup \partial\Omega_2^{(2)} \tag{1.12}$$

and

$$\begin{aligned} \partial\Omega_1^{(2)} &= \{(r,z)\colon z\in\{-a,a\},\ r_0<r<R\},\\ \partial\Omega_2^{(2)} &= \{(r,z)\colon z\in(-a,a),\ r=R\}. \end{aligned}$$

We temporarily simplify the notation using

$$\begin{aligned} &u = \tilde\psi_1^{(1)}, \qquad w = \tilde\psi_1^{(2)},\\ &f = \tilde\omega_1^{(1)} - 2\psi_{1,r}\dot\zeta^{(1)} - \psi_1\ddot\zeta^{(1)} - \frac{2}{r}\psi_1\dot\zeta^{(1)},\\ &g = \tilde\omega_1^{(2)} - 2\psi_{1,r}\dot\zeta^{(2)} - \psi_1\ddot\zeta^{(2)} - \frac{2}{r}\psi_1\dot\zeta^{(2)}. \end{aligned} \tag{1.13}$$

Then (1.10) and (1.11) become

$$\begin{cases} -\Delta u - \dfrac{2}{r}u_{,r} = f & \text{in } \Omega^{(1)},\\ u = 0 & \text{on } \partial\Omega^{(1)}, \end{cases} \tag{1.14}$$

and

$$\begin{cases} -\Delta w - \dfrac{2}{r}w_{,r} = g & \text{in } \Omega^{(2)},\\ w = 0 & \text{on } \partial\Omega^{(2)}. \end{cases} \tag{1.15}$$

As we can see, the above two problems are similar; they only differ in the domain. In the case of $\Omega^{(2)}$ we can safely use the classical theory for the Poisson equation.

Since $r_0 > 0$ we instantly deduce that problem (1.15) can be solved classically.

To study the existence and properties of solutions to (1.14) we need weighted Sobolev spaces. They are defined at the beginning of Section 2. In addition we will be using Kondrat'ev's technique (see [2]). It offers a way to deal with expressions of the form $\frac{u}{r^\alpha}$ when $\alpha > 0$. We saw in (1.7) that ψ_1 is well defined at $r = 0$ but in the case of the weighted Sobolev space H_0^3 we need to handle $\frac{\psi_1}{r^3}$ in L_2. The function ψ_1 does not vanish sufficiently fast when $r \to 0^+$, thus it has to be modified in a certain way. The modifications appear in the formulations of Theorems 1.1–1.4. They depend on the weighted Sobolev spaces applied. These kinds of modifications form the essence of this note.

The very first theorem we prove is the following:

THEOREM 1.1. *Suppose that* ψ_1 *is a solution to* (1.9). *Assume that* $\omega_1 \in L_{2,\mu}(\Omega)$, $\mu \in (0,1)$. *Then*

$$\|\psi_1 - \psi_1(0)\|^2_{L_2(-a,a;H^2_\mu(0,R))} + \|\psi_{1,zr}\|^2_{L_{2,\mu}(\Omega)} + \|\psi_{1,zz}\|^2_{L_{2,\mu}(\Omega)} \\ + 2\mu(2-2\mu)\|\psi_{1,z}\|^2_{L_{2,\mu-1}(\Omega)} \le c\|\omega_1\|^2_{L_{2,\mu}(\Omega)},$$

where $\psi_1(0) = \psi_1|_{r=0}$.

In light of (1.8) we cannot expect $\psi_1 \in H^2_\mu(0,R)$ for almost all z. However, this should be the case for the difference $\psi_1 - \psi_1|_{r=0}$.

In a similar manner we obtain higher order regularity.

THEOREM 1.2. *Let* ψ_1 *be a solution to* (1.9). *Let* $\omega_1 \in H^1_\mu(\Omega)$, $\mu \in (0,1)$. *Then*

$$\|\psi_1 - \psi_1(0)\|^2_{L_2(-a,a;H^3_\mu(0,R))} + \|\psi_{1,zzz}\|^2_{L_{2,\mu}(\Omega)} + \|\psi_{1,zzr}\|^2_{L_{2,\mu}(\Omega)} \\ + 2\mu(2-2\mu)\|\psi_{1,zz}\|^2_{L_{2,\mu-1}(\Omega)} \le c\|\omega_1\|^2_{H^1_\mu(\Omega)}.$$

The above theorems are useful but we need estimates when $\mu = 0$. We cannot simply let $\mu \to 0$ because $\psi_1 - \psi_1(0)$ is neither in H^2_0 nor in H^3_0. Instead we construct two auxiliary functions χ and η that we subtract from ψ_1 (this construction is presented in Lemmas 3.6 and 3.7). This allows us to derive necessary estimates in H^3_0. We emphasize that H^3_0 denotes the weighted Sobolev space with weight $\mu = 0$ (see Section 2). To show that ψ_1 satisfying the assertions of either Theorem 1.1 or Theorem 1.2 belongs to either H^2_0 or H^3_0, respectively, we need additional modifications of ψ_1 near the axis of symmetry. The modifications are described in Theorems 1.3 and 1.4, respectively.

In the theorems below we assume that ψ_1 is a weak solution to (1.9). Basic energy estimates and the existence of weak solutions are discussed in Section 2.

THEOREM 1.3. *Suppose that* ψ_1 *is a weak solution to* (1.9). *Let* $\omega_1 \in L_2(\Omega)$ *and introduce*

$$\chi(r,z) = \int_0^r \psi_{1,\tau}(1+K(\tau))\,d\tau,$$

where $K(\tau)$ *is a smooth function with compact support such that*

$$\lim_{r\to 0^+} \frac{K(r)}{r^2} = c_0 < \infty.$$

Then

$$\|\psi_1-\psi_1(0)-\chi\|^2_{L_2(-a,a;H^2_0(0,R))}+\|\psi_{1,zr}\|^2_{L_2(\Omega)}+\|\psi_{1,zz}\|^2_{L_2(\Omega)}\le c\|\omega_1\|^2_{L_2(\Omega)},$$

In the case of H^3_0 we have

THEOREM 1.4. *Let ψ_1 be a weak solution to* (1.9). *Let $\omega_1\in H^1(\Omega)$. Then*

$$\int_{-a}^{a}\|\psi_1-\psi_1(0)-\eta\|^2_{H^3_0(0,R)}\,dz + \int_\Omega(|\psi_{1,zzz}|^2+|\psi_{1,zzr}|^2+|\psi_{1,zz}|^2)\,r\,dr\,dz\le c\|\omega_1\|^2_{H^1(\Omega)},$$

where

$$\eta(r,z)=-\int_0^r(r-\tau)\bigg(\frac{3}{r}\psi_{1,\tau}+\psi_{1,zz}+\omega_1\bigg)(1+K(\tau))\,d\tau$$

and K is as in Theorem 1.3.

At this point the estimates from Theorems 1.3 and 1.4 may look surprising. In [8] we show how to eliminate $\psi_1(0)$, χ and η by using the data.

Using some properties of ψ_1 presented in this paper we prove in [12] the following global estimate for axially symmetric solutions to the Navier–Stokes equations:

$$\|\omega_r/r\|_{V(\Omega^t)}+\|\omega/r\|_{V(\Omega^t)}\le\phi(\mathrm{data(t)}),\tag{1.16}$$

where ω_r and ω are the radial and angular coordinates of the vorticity, and the energy norm of $V(\Omega^t)$ is defined by

$$\|u\|_{V(\Omega^t)}=\sup_{t'\le t}\|u(t')\|_{L_2(\Omega)}+\|\nabla u\|_{L_2(\Omega^t)}.$$

However, to prove (1.16) we need that $\psi_1=0$ on the axis of symmetry.

To end this introduction it is worth mentioning that we could continue the process of deriving higher-order estimates for ψ_1. In light of (1.8) it would require more subtractions from ψ_1 when $r=0$. However, we do not see any potential gain or immediate applications for such estimates.

2. Notation and auxiliary results

Notation. By c we denote a generic constant which may vary from line to line.

We use $\mathbb{N}=\{1,2,\ldots\}$ and $\mathbb{N}_0=\{0,1,2,\ldots\}$.

The set $\{(r,z)\colon r>0,\ z\in\mathbb{R}\}$ is denoted by $\mathbb{R}^2_+$.

By ϕ we always denote an increasing positive function.

Function spaces

Definition 2.1. Let Ω be either a cylindrical domain $(0,R)\times(-a,a)$ or $\Omega=\mathbb{R}^2_+$. We introduce the following norms:

$$\|u\|^2_{L_{2,\mu}(\Omega)}=\int\limits_\Omega |u(r,z)|^2 r^{2\mu} r\,dr\,dz,\quad \mu\in\mathbb{R},$$
$$\|u\|^2_{H^k_\mu(\Omega)}=\sum_{|\alpha|\le k}\int\limits_\Omega |D^\alpha_{r,z}u(r,z)|^2 r^{2(\mu+|\alpha|-k)} r\,dr\,dz,$$

where $D^\alpha_{r,z}=\partial_r^{\alpha_1}\partial_z^{\alpha_2}$, $|\alpha|=\alpha_1+\alpha_2$, $|\alpha|\le k$, $\alpha_i\in\mathbb{N}_0$, $i=1,2$, $k\in\mathbb{N}_0$ and $\mu\in\mathbb{R}$.

Then we have the compatibility condition

$$L_{2,\mu}(\Omega)=H^0_\mu(\Omega).$$

Fourier transform. Let $f\in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}$. Then the Fourier transform and its inverse are defined by

$$\hat f(\lambda)=\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} e^{-i\lambda\tau}f(\tau)\,d\tau,\qquad \check{\hat f}(\tau)=\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} e^{i\lambda\tau}\hat f(\lambda)\,d\lambda \tag{2.1}$$

and $\check{\hat f}=\hat{\check f}=f$.

Remark 2.2. For smooth functions with respect to z we introduce the weighted norms

$$\|u\|^2_{H^k_\mu(\mathbb{R}_+)}=\sum_{i=0}^k\int\limits_{\mathbb{R}_+}|\partial_r^i u|^2 r^{2(\mu-k+i)} r\,dr \tag{2.2}$$

where $\mu\in\mathbb{R}$ and $k\in\mathbb{N}_0$.

With the transformation $\tau=-\ln r$, $r=e^{-\tau}$, $dr=-e^{-\tau}\,d\tau$ we will prove the equivalence

$$\sum_{i=0}^k\int\limits_{\mathbb{R}_+}|\partial_r^i u|^2 r^{2(\mu-k+i)} r\,dr\sim\sum_{i=0}^k\int\limits_{\mathbb{R}}|\partial_\tau^i u'|^2 e^{2h\tau}\,d\tau \tag{2.3}$$

for $u'(\tau)=u'(-\ln r)=u(r)$, $h=k-1-\mu$.

To show (2.3), we first take $k = 2$. Then $h = 1-\mu$ and $\partial_\tau^2 = -r\partial_r(-r\partial_r) = r^2\partial_r^2 + r\partial_r$. We have

$$\begin{aligned}\int_{\mathbb{R}} (|\partial_\tau^2 u'|^2 + |\partial_\tau u'|^2 + |u'|^2)e^{2(1-\mu)\tau}\,d\tau\\ &= \int_{\mathbb{R}_+} \big(|r\partial_r(r\partial_r u)|^2 + |r\partial_r u|^2 + |u|^2\big) r^{2(\mu-1)}\frac{1}{r}\,dr\\ &\le 2\int_{\mathbb{R}_+}\bigg(|\partial_r^2 u|^2 + \frac{|\partial_r u|^2}{r^2} + \frac{|u|^2}{r^4}\bigg) r^{2\mu} r\,dr,\end{aligned}$$

and conversely

$$\begin{aligned}\int_{\mathbb{R}_+}\bigg(|\partial_r^2 u|^2 + \frac{|\partial_r u|^2}{r^2} + \frac{|u|^2}{r^4}\bigg) r^{2\mu\tau} r\,dr\\ &= \int_{\mathbb{R}_+} (r^4|\partial_r^2 u|^2 + r^2|\partial_r u|^2 + |u|^2) r^{2\mu-4} r\,dr\\ &= \int_{\mathbb{R}_+} (|r\partial_r(r\partial_r u) - r\partial_r u|^2 + |r\partial_r u|^2 + |u|^2) r^{2\mu-4} r\,dr\\ &\le 2\int_{\mathbb{R}_+} (|r\partial_r(r\partial_r u)|^2 + |r\partial_r u|^2 + |u|^2) r^{2\mu-4} r\,dr\\ &\le 2\int_{\mathbb{R}} (|\partial_\tau^2 u'|^2 + |\partial_\tau u'|^2 + |u'|^2) e^{2(1-\mu)\tau}\,d\tau.\end{aligned}$$

The above considerations imply (2.3) for $k = 2$. Similarly we can prove it for $k \ge 3$.

Using the Fourier transform we introduce norms equivalent to (2.2) and convenient for examining solutions of differential equations. Hence, by the Parseval identity we have

$$\int_{-\infty+ih}^{+\infty+ih} \sum_{j=0}^{k} |\lambda|^{2j}|\hat{u}(\lambda)|^2\,d\lambda = \int_{\mathbb{R}}\sum_{j=0}^{k} |\partial_\tau^j u|^2 e^{2h\tau}\,d\tau, \tag{2.4}$$

where the r.h.s. norm is equivalent to (2.2) under the equivalence (2.3). This ends Remark 2.2.

Energy estimates and weak solutions

LEMMA 2.3. *Assume that $\omega_1 \in L_2(\Omega)$. Then there exists a weak solution to problem* (1.9) *such that $\psi_1 \in H^1(\Omega)$ and we have*

$$\|\psi_1\|^2_{H^1(\Omega)} + \int_{-a}^{a} \psi_1^2(0)\,dz \le c\|\omega_1\|^2_{L_2(\Omega)}, \tag{2.5}$$

where $\psi_1(0) = \psi_1|_{r=0}$.

Proof. Multiplying (1.9) by ψ_1, integrating over Ω and using the boundary condition and the Poincaré inequality we derive (2.5). Then the existence follows from the Fredholm alternative. ∎

REMARK 2.4. We deduce from (1.8) that

$$\psi_1 = a_1(z,t) + a_2(z,t)r^2 + o(r^4) \quad \text{when } r \to 0^+. \tag{2.6}$$

In particular, $\psi(0) = \psi|_{r=0} = 0$ but $\psi_1(0) = \psi_1|_{r=0} \neq 0$.

LEMMA 2.5. *Assume that $\omega \in L_2(\Omega)$. Then there exists a solution $\psi \in H^1(\Omega)$ to problem* (1.1) *which satisfies*

$$\|\psi\|^2_{H^1(\Omega)} + \int_\Omega \frac{\psi^2}{r^2}\,dx \le c\|\omega\|^2_{L_2(\Omega)} \tag{2.7}$$

and

$$\|\psi_{,rz}\|^2_{L_2(\Omega)} + \|\psi_{,zz}\|^2_{L_2(\Omega)} + \int_\Omega \frac{\psi_{,z}^2}{r^2}\,dx \le c\|\omega\|^2_{L_2(\Omega)}. \tag{2.8}$$

The proof of (2.7) is similar to the proof of (2.5). Moreover, in view of (2.6) the integral on the l.h.s. of (2.7) is finite.

Proof of Lemma 2.5. Multiplying $(1.1)_1$ by ψ, integrating over Ω, using boundary conditions and the Poincaré inequality we obtain (2.7). Multiplying (1.1) by $-\psi_{,zz}$ and integrating over Ω yields

$$\int_\Omega \psi_{,rr}\psi_{,zz}\,dx + \int_\Omega \frac{1}{r}\psi_{,r}\psi_{,zz}\,dx + \int_\Omega \psi_{,zz}^2\,dx + \int_\Omega \frac{\psi_{,z}^2}{r^2}\,dx = -\int_\Omega \omega\psi_{,zz}\,dx. \tag{2.9}$$

Integrating by parts in the first term and using the boundary conditions, we get

$$\begin{aligned}\int_\Omega &\psi_{,rr}\psi_{,zz}\,dx\\ &= \int_\Omega (\psi_{,rr}\psi_{,z})_{,z}\,dx - \int_\Omega (\psi_{,rz}\psi_{,z}r)_{,r}\,dr\,dx + \int_\Omega \psi_{,rz}\psi_{,z}\,dr\,dz + \int_\Omega \psi_{,rz}^2\,dx,\end{aligned}$$

where the first two terms on the r.h.s. vanish because $\psi_{,rr}|_{S_2} = 0$ and $\psi_{,z}|_{r=0}^{r=R} = 0$. Using the equality in (2.9) yields

$$\begin{aligned}\int_\Omega (\psi_{,rz}^2 + \psi_{,zz}^2)\,dx + \int_\Omega \frac{\psi_{,z}^2}{r^2}dx + \int_\Omega \psi_{,r}\psi_{,zz}\,dr\,dz + \int_\Omega \psi_{,rz}\psi_{,z}\,dr\,dz&\\ = \int_\Omega \omega\psi_{,zz}\,dx.&\end{aligned} \tag{2.10}$$

Integrating by parts with respect to z in the last but one term on the l.h.s.

of (2.10) and using

$$\int_0^R (\psi_{,r}\psi_{,zz})|_{S_2}\,dt = 0$$

we find that the sum of last two terms on the l.h.s. of (2.10) vanishes. Then (2.10) implies (2.8) and concludes the proof. ■

From (1.9) we derive the following problem:

$$\begin{cases} -\Delta\psi_{1,z} - \dfrac{2}{r}\psi_{1,rz} = \omega_{1,z} & \text{in } \Omega, \\ \psi_{1,z} = 0 & \text{on } \{r = R,\ z \in (-a,a)\}, \\ \psi_{1,zz} = 0 & \text{on } \{z \in \{-a,a\},\ r < R\}, \end{cases} \tag{2.11}$$

where the last boundary condition follows from (1.9) and $\omega_1|_{z\in\{-a,a\},\,r<R} = 0$.

LEMMA 2.6. *Suppose that* $\omega_{1,z} \in L_2(\Omega)$. *Then there exists a weak solution to* (2.11) *such that* $\psi_{1,z} \in H^1(\Omega)$ *and*

$$\|\psi_{1,z}\|^2_{H^1(\Omega)} + \int_{-a}^{a} \psi^2_{1,z}(0)\,dz \le c\|\omega_1\|^2_{L_2(\Omega)}, \tag{2.12}$$

where $\psi_{1,z}(0) = \psi_{1,z}|_{r=0}$.

Proof. Multiplying $(2.11)_1$ by $\psi_{1,z}$ and integrating over Ω yields

$$-\int_\Omega \psi_{1,rrz}\psi_{1,z}\,dx - \int_\Omega \psi_{1,zzz}\psi_{1,z}\,dx - 3\int_\Omega \psi_{1,zr}\psi_{1,z}\,dr\,dz = \int_\Omega \omega_{1,z}\psi_{1,z}\,dx. \tag{2.13}$$

Integrating by parts with respect to r in the first term yields

$$-\int_\Omega (\psi_{1,rz}\psi_{1,z}r)_{,r}\,dr\,dz + \int_\Omega \psi^2_{1,rz}\,dx + \int_\Omega \psi_{1,rz}\psi_{1,z}\,dr\,dz, \tag{2.14}$$

where

$$\int_{-a}^{a} \psi_{1,rz}\psi_{1,z}r|_{r=0}^{r=R}\,dz = 0$$

because $\psi_{1,rz}|_{r=0} = 0$ and $\psi_{1,z}|_{r=R} = 0$.

Integrating by parts in the second term in (2.13) and using (2.14) we obtain

$$\int_\Omega (\psi^2_{1,rz} + \psi^2_{1,zz})\,dx - 2\int_\Omega \psi_{1,zr}\psi_{1,z}\,dr\,dz = \int_\Omega \omega_{1,z}\psi_{1,z}\,dx. \tag{2.15}$$

The last term on the l.h.s. of (2.15) equals

$$-\int_{\Omega} \partial_r \psi_{1,z}^2 \, dr \, dz = \int_{-a}^{a} \psi_{1,z}^2(0) \, dz$$

because $\psi_{1,z}|_{r=R} = 0$.

Integrating by parts with respect to z in the r.h.s. of (2.15), using $\omega_1|_{S_2} = 0$ and applying the Hölder and Young inequalities we derive (2.12). ∎

From [9, Appendix A] we have

LEMMA 2.7 (Hardy's inequalities).

$$\Big(\int_0^\infty \Big(\int_0^x g(y)\, dy\Big)^p x^{-r-1}\, dx\Big)^{1/p} \le \frac{p}{r}\Big(\int_0^\infty |yg(y)|^p y^{-r-1}\, dy\Big)^{1/p}$$

for $g \ge 0$, $p \ge 1$ and $r > 0$.

REMARK 2.8. If we set $r = 1-\alpha$ and $f(x) = \int_0^x g(y)\, dy$ in Lemma 2.7, we obtain

$$\int_0^\infty x^{\alpha-2}|f(x)|^2\, dx \le \frac{4}{(1-\alpha)^2}\int_0^\infty x^\alpha |f'(x)|^2\, dx, \qquad \alpha < 1.$$

3. L_2-weighted estimates with respect to r for solutions to (1.14). In this section we derive various estimates with respect to r for solutions to (1.14) in weighted Sobolev spaces using the technique of Kondrat'ev (see [2]). These estimates lay foundations for the proofs of Theorems 1.1–1.4. The key idea is to treat the variable z as a parameter.

First, we rewrite (1.14) in the form

$$\begin{cases} -u_{,rr} - \dfrac{3}{r}u_{,r} = f + u_{,zz} & \text{in } \Omega^{(1)}, \\ u = 0 & \text{on } \partial\Omega^{(1)}. \end{cases} \tag{3.1}$$

For a fixed $z \in (-a, a)$ we treat (3.1) as an ordinary differential equation

$$-u_{,rr} - \frac{3}{r}u_{,r} = f + u_{,zz} \quad \text{in } \mathbb{R}_+. \tag{3.2}$$

Multiplying $(3.1)_1$ by r^2 we obtain

$$-r^2 u_{,rr} - 3ru_{,r} = r^2(f + u_{,zz}) \equiv g(r,z)$$

or equivalently

$$-r\partial_r(r\partial_r u) - 2r\partial_r u = g(r,z). \tag{3.3}$$

Introduce the new variable

$$\tau = -\ln r, \qquad r = e^{-\tau}.$$

Since $r\partial_r = -\partial_\tau$ we see that (3.3) takes the form

$$-\partial_\tau^2 u + 2\partial_\tau u = g(e^{-\tau}, z) \equiv g'(\tau, z). \tag{3.4}$$

Applying the Fourier transform (see (2.1)) to (3.4) we get

$$\lambda^2 \hat{u} + 2i\lambda\hat{u} = \hat{g}'.$$

For $\lambda \notin \{0, -2i\}$ we have

$$\hat{u} = \frac{1}{\lambda(\lambda + 2i)}\hat{g}' \equiv R(\lambda)\hat{g}'. \tag{3.5}$$

LEMMA 3.1. *Assume that $f + u_{,zz} \in H_\mu^k(\mathbb{R}_+)$, $k \in \mathbb{N}_0$, $\mu \in \mathbb{R}$. Assume that $R(\lambda)$ does not have poles on the line $\Im\lambda = 1 + k - \mu$. Then there exists a unique solution to* (3.2) *in $H_\mu^{k+2}(\mathbb{R}_+)$ such that*

$$\|u\|_{H_\mu^{k+2}(\mathbb{R}_+)} \le c\|f + u_{,zz}\|_{H_\mu^k(\mathbb{R}_+)}. \tag{3.6}$$

Proof. Since $R(\lambda)$ does not have poles on the line $\Im\lambda = 1 + k - \mu = h$, we can integrate (3.5) along the line $\Im\lambda = h$. Then

$$\begin{aligned}\int_{-\infty+ih}^{+\infty+ih} \sum_{j=0}^{k+2} |\lambda|^{2(k+2-j)}|\hat{u}|^2\,d\lambda &\le \int_{-\infty+ih}^{+\infty+ih} \sum_{j=0}^{k+2} |\lambda|^{2(k+2-j)}|R(\lambda)\hat{g}'|^2\,d\lambda \\ &\le c\int_{-\infty+ih}^{+\infty+ih} \sum_{j=0}^{k} |\lambda|^{2(k-j)}|\hat{g}'|^2\,d\lambda.\end{aligned} \tag{3.7}$$

By the Parseval identity (see (2.4)) inequality (3.7) becomes

$$\int_{\mathbb{R}} \sum_{j=0}^{k+2} |\partial_\tau^j u|^2 e^{2h\tau}\,d\tau \le c\int_{\mathbb{R}} \sum_{j=0}^{k} |\partial_\tau^j g'|^2 e^{2h\tau}\,d\tau.$$

Passing to the variable r yields

$$\int_{\mathbb{R}_+} \sum_{j=0}^{k+2} |r^j\partial_r^j u|^2 r^{2(\mu-k-1)}\frac{1}{r}\,dr \le c\int_{\mathbb{R}_+} \sum_{j=0}^{k} |r^j\partial_r^j g|^2 r^{2(\mu-k-1)}\frac{1}{r}\,dr.$$

Continuing, we get

$$\int_{\mathbb{R}_+} \sum_{j=0}^{k+2} |r^{j-(k+2)}\partial_r^j u|^2 r^{2\mu} r\,dr \le c\int_{\mathbb{R}_+} \sum_{j=0}^{k} |r^{j-k}\partial_r^j (f + u_{,zz})|^2 r^{2\mu} r\,dr,$$

where the relation $g = r^2(f + u_{,zz})$ was used. ∎

REMARK 3.2. Consider a solution u to (3.2). In light of Lemma 3.1 such a solution has certain regularity. Moreover, when we fix $\mu \in \mathbb{R}$ we expect from u a certain behavior near $r = 0$. We are interested in two cases: $k = 0$ and $k = 1$.

When $k = 0$ we have $h = 1 - \mu$. Hence

$$\begin{aligned} h_1 &= 1 - \mu_1 < 0 \quad \text{for some } \mu_1 \in (1,2),\\ h_2 &= 1 - \mu_2 > 0 \quad \text{for some } \mu_2 \in (0,1). \end{aligned}$$

Similarly, for $k = 1$ we have $h = 2 - \mu$ and

$$\begin{aligned} \bar h_1 &= 2 - \bar\mu_1 < 0 \quad \text{for some } \bar\mu_1 \in (2,3),\\ \bar h_2 &= 2 - \bar\mu_2 > 0 \quad \text{for some } \bar\mu_2 \in (0,2). \end{aligned}$$

The function $R(\lambda)$ has a pole for $h = \Im\lambda = 0$, and thus

$$1 - \mu_1 < 0 < 1 - \mu_2, \qquad 2 - \bar\mu_1 < 0 < 2 - \bar\mu_2.$$

By Lemma 3.1 we have four solutions:

$$\begin{aligned} k = 0: &\quad u_1 \in H^2_{\mu_1}(\mathbb{R}_+), \quad u_2 \in H^2_{\mu_2}(\mathbb{R}_+),\\ k = 1: &\quad \bar u_1 \in H^3_{\bar\mu_1}(\mathbb{R}_+), \quad \bar u_2 \in H^3_{\bar\mu_2}(\mathbb{R}_+). \end{aligned}$$

Our aim is to investigate the relations between these solutions.

We will be using the notation from Remark 3.2.

LEMMA 3.3. *Let $k = 0$. Then there exists a constant c_0 such that*

$$u_1 - u_2 = c_0. \tag{3.8}$$

If $k = 1$, then also

$$\bar u_1 - \bar u_2 = c_0. \tag{3.9}$$

Proof. Consider the case $k = 0$. The function $\hat g'$ is analytic for any $h \in (h_1, h_2)$ and

$$\int_{-\infty+ih}^{+\infty+ih} |\hat g'|^2\, d\lambda < \infty$$

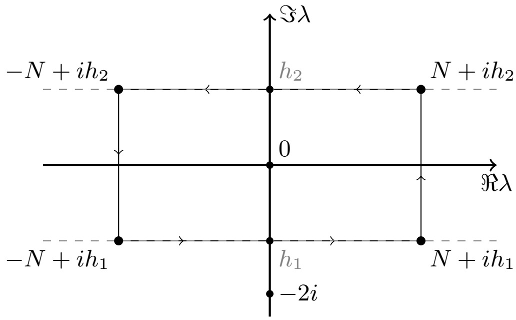

Fig. 1. Integration contour

for any $h \in [h_1, h_2]$. We also have (see Fig. 1)

$$u_1 = \lim_{N\to\infty} \int\limits_{-N+ih_1}^{N+ih_1} e^{i\lambda\tau} \hat{u}(\lambda)\, d\lambda = \lim_{N\to\infty} \int\limits_{-N+ih_1}^{N+ih_1} e^{i\lambda\tau} R(\lambda)\hat{g}'(\lambda)\, d\lambda$$
$$= \operatorname{Res}_0 e^{i\lambda\tau} R(\lambda)\hat{g}'(\lambda) - \lim_{N\to\infty} \Big(\int\limits_{N+ih_1}^{N+ih_2} e^{i\lambda\tau} R(\lambda)\hat{g}'(\lambda)\, d\lambda$$
$$- \int\limits_{-N+ih_1}^{-N+ih_2} e^{i\lambda\tau} R(\lambda)\hat{g}'(\lambda)\, d\lambda - \int\limits_{-N+ih_2}^{N+ih_2} e^{i\lambda\tau} R(\lambda)\hat{g}'(\lambda)\, d\lambda\Big).$$

Letting with $N \to \infty$ yields

$$u_1 = u_2 + \operatorname{Res}_0 e^{i\lambda\tau} R(\lambda)\hat{g}'(\lambda) = u_2 + c_0,$$

where

$$u_j = \int\limits_{-\infty+ih_j}^{+\infty+ih_j} e^{i\lambda\tau} R(\lambda)\hat{g}'(\lambda)\, d\lambda.$$

Hence (3.8) holds.

For $k = 1$ the operator $R(\lambda)$ has the same pole in the interval $(\bar{h}_1, \bar{h}_2)$. Hence (3.9) holds. This ends the proof. ∎

REMARK 3.4. Let us compute c_0. Recall that $u_1 \in H^2_{\mu_1}(\mathbb{R}_+)$ with $\mu_1 \in (1,2)$. This means that $u_1|_{r=0} \neq 0$. But $u_2 = u_1 - c_0 \in H^2_{\mu_2}(\mathbb{R}_+)$ with $\mu_2 \in (0,1)$, so $u_2|_{r=0} = 0$. Hence

$$c_0 = u_1(0) = u_1|_{r=0}.$$

Similarly, $\bar{u}_2 = \bar{u}_1 - c_0 \in H^3_{\bar{\mu}_2}(\mathbb{R}_+)$ with $\bar{\mu}_2 \in (0,2)$, so

$$c_0 = \bar{u}_1(0) \equiv \bar{u}_1|_{r=0}.$$

Investigating $\bar{u}_2 \in H^3_{\bar{\mu}_2}(\mathbb{R}_+)$ with $\bar{\mu}_2 \in (0,1)$ we also need

$$\partial_r \bar{u}_2 = \partial_r \bar{u}_1 = 0 \quad \text{ for } r = 0.$$

The restriction follows from Remark 2.4.

The functions u_1 and $\bar{u}_1$ are good candidates for weak solutions to (3.2) because they do not vanish on $r = 0$.

Recall that $u = \psi_1\zeta^{(1)}$ and $f = \omega_1\zeta^{(1)} - 2\psi_{1,r}\dot{\zeta}^{(1)} - \psi_1\ddot{\zeta}^{(1)} - \frac{2}{r}\psi_1\dot{\zeta}^{(1)}$. Therefore Lemma 2.3 can be applied to solutions to (1.14). Hence we have

$$\|u\|^2_{H^1(\Omega)} + \int\limits_{-a}^{a} u^2(0)\, dz \le c\|f\|^2_{L_2(\Omega)} \tag{3.10}$$

and Lemmas 2.3 and 3.8 imply

$$\|u\|^2_{H^2(\Omega)} \le c\|f\|^2_{L_2(\Omega)}. \tag{3.11}$$

Weak solutions to problem (1.14) do not vanish on the axis of symmetry.

Hence, looking for increasing regularity of weak solutions to (1.14) in weighted Sobolev spaces we apply (3.6) for $u = u_1$ and $\mu = \mu_1 \in (1,2)$ (see notation in Remark 3.2).

Using (3.11), estimate (3.6) in this case has the form

$$\|u_1\|^2_{L_2(-a,a;H^2_{\mu_1}(\mathbb{R}_+))} \le c\|f\|^2_{L_2(-a,a;L_{2,\mu_1}(\mathbb{R}_+))},$$

where $\mu_1 \in (1,2)$. The above inequality reflects the increasing regularity of weak solutions to (1.14) in weighted Sobolev spaces because u_1 does not vanish on the axis of symmetry.

Recalling the properties of u_1, u_2 and assuming $f \in L_2(-a,a;L_{2,\mu}(\mathbb{R}_+))$, $\mu \in (0,1)$, we can conclude that

$$\|u - u(0)\|_{L_2(-a,a;H^2_\mu(\mathbb{R}_+))} \le c\|f\|_{L_2(-a,a;L_{2,\mu}(\mathbb{R}_+))},$$

where $u(0) = u|_{r=0}$.

Recalling the properties of $\bar{u}_1$ and $\bar{u}_2$ and assuming that

$$f + u_{,zz} \in L_2(-a,a;H^1_\mu(\mathbb{R}_+))$$

we conclude that

$$\|u - u(0)\|_{L_2(-a,a;H^3_\mu(\mathbb{R}_+))} \le c\|f + u_{,zz}\|_{L_2(-a,a;H^1_\mu(\mathbb{R}_+))}, \tag{3.12}$$

where $\mu \in (0,1)$.

Estimate (3.6) for $k = 1$ and $\mu = 0$ suggests for weak solutions to (1.14) the following inequality:

$$\|u - u(0)\|_{L_2(-a,a;H^3_0(\mathbb{R}_+))} \le c\|f + u_{,zz}\|_{L_2((-a,a)\times H^1_0(\mathbb{R}_+)},$$

The above estimate does not hold for the weak solutions to problem (1.14). The l.h.s. norm contains the term

$$I = \int_{-a}^{a}\int_0^R \frac{|u - u(0)|^2}{r^6} r\,dr\,dz.$$

In view of expansion (2.6) we have $I = \infty$ for $u - u(0) = a_2 r^2 + a_3 r^3 + \cdots$.

In view of Lemma 3.8 (see (3.26)) we need the estimate

$$\|u_{,zz}\|_{L_2(-a,a;H^1_0(\mathbb{R}_+))} \le c\|u_{,zzr}\|_{L_2(\Omega)}$$

but the Hardy inequality does not hold in this case.

Therefore, we introduce a new function $\eta(r,z)$ such that that

$$(u - u(0) - \eta(r,z))_{,rr}|_{r=0} = 0.$$

Moreover, we also need:

Lemma 3.5 (cf. [2, Lemma 4.12]). *Let $\bar{u} \in H^k(\mathbb{R}_+)$, $k \in \mathbb{N}$, $\frac{\partial^i}{\partial r^i} u\big|_{r=0} = 0$ for $i < k-1$ and $\partial_r^{k-1}\bar{u} \in H^1_0(\mathbb{R}_+)$. Then $\bar{u} \in H^k_0(\mathbb{R}_+)$ and*

$$\|\bar{u}\|_{H^k_0(\mathbb{R}_+)} \le c\|\partial_r^{k-1}\bar{u}\|_{H^1_0(\mathbb{R}_+)}. \tag{3.13}$$

Proof. Using the inequality from Remark 2.8 we infer that

$$\int_0^\infty r^{-2}|\partial_r^{k-1}\bar{u}(r)|^2 r\,dr \ge c\int_0^\infty r^{-4}|\partial_r^{k-2}\bar{u}(r)|^2 r\,dr \ge c\int_0^\infty r^{-2k}|\bar{u}|^2 r\,dr,$$

which holds for $\partial_r^i\bar{u}|_{r=0} = 0$, $i < k-1$. This implies (3.13) and concludes the proof. ∎

Recall that u is a solution to

$$u_{,rr} = -\left(\frac{3}{r}u_{,r} + u_{,zz} + f\right) \equiv g(r,z). \tag{3.14}$$

LEMMA 3.6. *Let u solve (3.14) and let $u|_{r=0} = u(0)$. Assume that $u \in L_2(-a,a;H^3(\mathbb{R}_+))$ and $f \in L_2(-a,a;H^1(\mathbb{R}_+))$. Then there exists a function*

$$\eta(r,z) = \int_0^r (r-\tau)g(\tau,z)(1+K(\tau))\,d\tau, \tag{3.15}$$

where $K(r)$ is a smooth function with compact support near $r=0$ such that

$$\lim_{r\to 0} K(r)r^{-2} = c_0 < \infty$$

and the function

$$u - \eta - u(0) \in L_2(-a,a;H_0^3(\mathbb{R}_+)) \tag{3.16}$$

satisfies

$$\begin{aligned} \|u-\eta-u(0)\|&_{L_2(-a,a;H_0^3(\mathbb{R}_+))} \\ &\le c(\|u\|_{L_2(-a,a;H^2(\mathbb{R}_+))} + \|f+u_{,zz}\|_{L_2(-a,a;H^1(\mathbb{R}_+))}). \end{aligned} \tag{3.17}$$

Proof. Since $u \in L_2(-a,a;H^3(\mathbb{R}_+))$, we can work with $\mathcal{C}(-a,a;\mathcal{C}_0^\infty(\mathbb{R}_+))$ and then use a density argument.

We construct a function η as a solution to the equation

$$\eta_{,rr} = g(r,z)(1+K(r)).$$

Integrating this equation we obtain (3.15).

To prove (3.16) and (3.17) we use Lemma 3.5 for $k=3$. To ensure its assumptions are met, we check that

$$\begin{aligned} (u-\eta-u(0))|_{r=0} &= -\eta|_{r=0} = 0, \\ \partial_r(u-\eta-u(0))|_{r=0} &= \partial_r(u-\eta)|_{r=0} = \partial_r u|_{r=0} - \partial_r\eta|_{r=0} = 0, \end{aligned}$$

where Remark 2.4 implies that $u_{,r}|_{r=0} = 0$ and

$$\partial_r\eta = \int_0^r g(\tau,z)(1+K(\tau))\,d\tau$$

gives $\partial_r\eta|_{r=0} = 0$.

Finally, we examine

$$
\begin{aligned}
(3.18)\qquad \|\partial_{rr}(u-\eta-u(0))\|_{H_0^1(\mathbb{R}_+)} &= \|gK\|_{H_0^1(\mathbb{R}_+)}\\
&= \left\|\left(\frac{3}{r}u_{,r}+u_{,zz}+f\right)K(r)\right\|_{H_0^1(\mathbb{R}_+)}\\
&\le c\|u\|_{H^2(\mathbb{R}_+)}+\|f+u_{,zz}\|_{H^1(\mathbb{R}_+))}.
\end{aligned}
$$

Applying Lemma 3.5 and integrating (3.18) with respect to z we derive (3.16) and (3.17). This ends the proof. ∎

Lemma 3.7. *Let u satisfy* (3.14), $u|_{r=0}=u(0)$, $u\in L_2(-a,a;H^2(\mathbb{R}_+))$ *and* $f\in L_2(-a,a;L_2(\mathbb{R}_+))$. *Then there exists a function*

$$
(3.19)\qquad \chi(r,z)=\int_0^r u_{,\tau}(1+K(\tau))\,d\tau,
$$

where K is defined in Lemma 3.6 *and the function*

$$
(3.20)\qquad u-\chi-u(0)\in L_2(-a,a;H_0^2(\mathbb{R}_+))
$$

satisfies

$$
(3.21)\qquad \|u-\chi-u(0)\|_{L_2(-a,a;H_0^2(\mathbb{R}_+))}\le c\|u\|_{L_2(-a,a;H^2(\mathbb{R}_+))}.
$$

Proof. Since $u\in L_2(-a,a;H^2(\mathbb{R}_+))$ we prove this lemma for functions from $\mathcal{C}(-a,a;\mathcal{C}_0^\infty(\mathbb{R}_+))$ and use a density argument.

We construct χ as a solution to

$$
(3.22)\qquad \chi_{,r}=u_{,r}(1+K(r)).
$$

Integrating (3.22) with respect to r yields (3.19).

To prove (3.20) and (3.21) we use Lemma 3.5 for $k=2$. We need to check its assumptions. We have

$$
(u-\chi-u(0))|_{r=0}=(u-u(0))|_{r=0}-\chi|_{r=0}=0
$$

and

$$
\begin{aligned}
(3.23)\qquad \|(u-\chi-u(0))_{,r}\|_{H_0^1(\mathbb{R}_+)} &= \left\|\partial_r\int_0^r u_{,\tau}(\tau,z)K(\tau)\,d\tau\right\|_{H_0^1(\mathbb{R}_+)}\\
&= \|u_{,r}K+uK_{,r}\|_{H_0^1(\mathbb{R}_+)}+\|uK_{,r}\|_{H_0^1(\mathbb{R}_+)}\le c\|u\|_{H^2(\mathbb{R}_+)}.
\end{aligned}
$$

Integrating (3.23) with respect to z and applying Lemma 3.5 for $k=2$ we conclude the proof. ∎

Recall that ψ_1 is a solution to

$$
(3.24)\qquad \begin{cases} -\psi_{1,rr}-\psi_{1,zz}-\dfrac{3}{r}\psi_{1,r}=\omega_1 & \text{in } \Omega,\\ \psi_1=0 & \text{on } S_1\cup S_2. \end{cases}
$$

LEMMA 3.8. *For solutions to* (3.24) *the following estimates hold:*

$$\int_\Omega (\psi_{1,rr}^2+\psi_{1,rz}^2+\psi_{1,zz}^2)\,dx+\int_\Omega \frac{1}{r^2}\psi_{1,r}^2\,dx\le c\|\omega_1\|_{L_2(\Omega)}^2, \tag{3.25}$$

$$\int_\Omega (\psi_{1,rrz}^2+\psi_{1,zzr}^2+\psi_{1,zzz}^2)\,dx+\int_{-a}^{a}\psi_{1,zz}^2|_{r=0}\,dz+\int_{-a}^{a}\psi_{1,rz}^2|_{r=R}\,dz \le c\|\omega_{1,z}\|_{L_2(\Omega)}^2. \tag{3.26}$$

Proof. First we show (3.25). Multiplying (3.24) by $\psi_{1,zz}$ and integrating over Ω yields

$$-\int_\Omega \psi_{1,rr}\psi_{1,zz}\,dx-\int_\Omega \psi_{1,zz}^2\,dx-3\int_\Omega \frac{1}{r}\psi_{1,r}\psi_{1,zz}\,dx=\int_\Omega \omega_1\psi_{1,zz}\,dx. \tag{3.27}$$

The first term in (3.27) equals

$$\begin{aligned}
&-\int_\Omega (\psi_{1,rr}\psi_{1,z})_{,z}\,dx+\int_\Omega \psi_{1,rrz}\psi_{1,z}\,dx\\
&=-\int_\Omega (\psi_{1,rr}\psi_{1,z})_{,z}\,dx+\int_\Omega (\psi_{1,rz}\psi_{1,z}r)_{,r}\,dr\,dz-\int_\Omega \psi_{1,rz}^2\,dx-\int_\Omega \psi_{1,rz}\psi_{1,z}\,dr\,dz,
\end{aligned}$$

where the first term is equal to

$$-\int_0^R \psi_{1,rr}\psi_{1,z}|_{S_2}r\,dr=0,$$

because $\psi_{1,rr}|_{S_2}=0$, and the second

$$\int_{-a}^{a}\psi_{1,rz}\psi_{1,z}|_{S_1}\,dz=0,$$

which follows from $\psi_{1,z}|_{S_1}=0$.

Consider the last term on the l.h.s. of (3.27). We have

$$\begin{aligned}
-3\int_\Omega \psi_{1,r}\psi_{1,zz}\,dr\,dz&=-3\int_\Omega (\psi_{1,r}\psi_{1,z})_{,z}\,dr\,dz+3\int_\Omega \psi_{1,rz}\psi_{1,z}\,dx\\
&=-\frac{3}{2}\int_{-a}^{a}\psi_{1,z}^2|_{r=0}^{r=R}\,dz,
\end{aligned}$$

where we have used

$$\int_0^R \psi_{1,r}\psi_{1,z}|_{S_2}\,dr=0$$

because $\psi_{1,r}\big|_{S_2}=0$.

Using the above considerations in (3.27) implies

$$-\int_\Omega(\psi_{1,rz}^2+\psi_{1,zz}^2)\,dx+\int_{-a}^a\psi_{1,z}^2|_{r=0}^{r=R}\,dz=\int_\Omega\omega_1\psi_{1,zz}\,dx. \tag{3.28}$$

Since $\psi_{1,z}|_{r=R}=0$, equality (3.28) can be written in the form

$$\int_\Omega(\psi_{1,rz}^2+\psi_{1,zz}^2)\,dx+\int_{-a}^a\psi_{1,z}^2|_{r=0}\,dz=-\int_\Omega\omega_1\psi_{1,zz}\,dx. \tag{3.29}$$

Applying the Hölder and Young inequalities to the r.h.s. of (3.29) we obtain

$$\int_\Omega(\psi_{1,rz}^2+\psi_{1,zz}^2)\,dx+\int_{-a}^a\psi_{1,z}^2|_{r=0}\,dz\le c\int_\Omega\omega_1^2\,dx. \tag{3.30}$$

Multiplying (3.24) by $\frac{1}{r}\psi_{1,r}$ and integrating over Ω yields

$$3\int_\Omega\left|\frac{1}{r}\psi_{1,r}\right|^2dx=-\int_\Omega\psi_{1,rr}\frac{1}{r}\psi_{1,r}\,dx-\int_\Omega\psi_{1,zz}\frac{1}{r}\psi_{1,r}\,dx-\int_\Omega\omega_1\frac{1}{r}\psi_{1,r}\,dx. \tag{3.31}$$

The first term on the r.h.s. of (3.31) equals

$$-\int_\Omega\psi_{1,r}\psi_{1,rr}\,dr\,dz=-\frac{1}{2}\int_\Omega\partial_r(\psi_{1,r}^2)\,dr\,dz=-\frac{1}{2}\int_{-a}^a\psi_{1,r}^2|_{r=R}\,dz$$

because $\psi_{1,r}|_{r=0}=0$ (see Remark 2.4).

Applying the Hölder and Young inequalities to the last two terms on the r.h.s. of (3.31) we finally obtain

$$3\int_\Omega\left|\frac{1}{r}\psi_{1,r}\right|^2dx+\frac{1}{2}\int_{-a}^a\psi_{1,r}^2(R,z)\,dz\le c(\|\psi_{1,zz}\|_{L_2(\Omega)}^2+\|\omega_1\|_{L_2(\Omega)}^2). \tag{3.32}$$

From (3.24) we infer that

$$\|\psi_{1,rr}\|_{L_2(\Omega)}^2\le\|\psi_{1,zz}\|_{L_2(\Omega)}^2+3\left\|\frac{1}{r}\psi_{1,r}\right\|_{L_2(\Omega)}^2+\|\omega_1\|_{L_2(\Omega)}^2.$$

Combining the above inequality with (3.30) and (3.32) yields (3.25).

Next we show (3.26). Differentiating (3.24) with respect to z, multiplying by $-\psi_{1,zzz}$ and integrating over Ω we obtain

$$\begin{aligned}\int_\Omega\psi_{1,rrz}\psi_{1,zzz}\,dx+\int_\Omega\psi_{1,zzz}^2\,dx+3\int_\Omega\frac{1}{r}\psi_{1,rz}\psi_{1,zzz}\,dx&\\=-\int_\Omega\omega_{1,z}\psi_{1,zzz}\,dx.&\end{aligned} \tag{3.33}$$

Integrating by parts in the first term yields

$$
\begin{aligned}
&\int_\Omega (\psi_{1,rrz}\psi_{1,zz})_{,z}\,dx - \int_\Omega \psi_{1,rrzz}\psi_{1,zz}\,dx \\
&\qquad = \int_0^R \psi_{1,rrz}\psi_{1,zz}|_{z=-a}^{z=a} r\,dr - \int_\Omega (\psi_{1,rzz}\psi_{1,zz}r)_{,r}\,dr\,dz + \int_\Omega \psi_{1,rzz}^2\,dx \\
&\qquad\quad + \int_\Omega \psi_{1,rzz}\psi_{1,zz}\,dr\,dz \\
&\qquad = \int_0^R \psi_{1,rrz}\psi_{1,zz}|_{z=-a}^{z=a} r\,dr - \int_{-a}^{a} \psi_{1,rzz}\psi_{1,zz}r|_{r=0}^{r=R}\,dz \\
&\qquad\quad + \int_\Omega \psi_{1,rzz}^2\,dx + \int_\Omega \psi_{1,rzz}\psi_{1,zz}\,dr\,dz \equiv I.
\end{aligned}
$$

Since $\psi_{1,zz}|_{r=R} = 0$ and $\psi_{1,rzz}|_{r=0} = 0$, the second term in I vanishes. To examine the first term in I we project (3.24) onto S_2. Then we have

$$\psi_{1,zz}|_{S_2} = -\psi_{1,rr}|_{S_2} - \frac{3}{r}\psi_{1,r}|_{S_2} - \omega_1|_{S_2}.$$

Since $\omega_1|_{S_2} = 0$ and $\psi_1|_{S_2} = 0$ it follows that $\psi_{1,zz}|_{S_2} = 0$. Therefore I becomes

$$I = \int_\Omega \psi_{1,rzz}^2\,dx + \int_\Omega \psi_{1,rzz}\psi_{1,zz}\,dr\,dz.$$

The second term in I is equal to

$$-\frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2|_{r=0}\,dz,$$

where it is used that $\psi_{1,zz}|_{r=R} = 0$.

The last term on the l.h.s. of (3.33) equals

$$-3\int_\Omega \psi_{1,rzz}\psi_{1,zz}\,dr\,dz = -\frac{3}{2}\int_{-a}^{a} \psi_{1,zz}^2|_{r=0}^{r=R}\,dz = \frac{3}{2}\int_{-a}^{a} \psi_{1,zz}^2|_{r=0}\,dz,$$

where we have used $\psi_{1,zz}|_{S_2} = 0$ and $\psi_{1,zz}|_{r=R} = 0$.

In view of the above calculations equality (3.33) takes the form

$$\int_\Omega (\psi_{1,rzz}^2 + \psi_{1,zzz}^2)\,dx + \int_{-a}^{a} \psi_{1,zz}^2|_{r=0}\,dz = -\int_\Omega \omega_{1,z}\psi_{1,zzz}\,dx. \tag{3.34}$$

Applying the Hölder and Young inequalities to the r.h.s. of (3.34) gives

$$\int_\Omega (\psi_{1,rzz}^2 + \psi_{1,zzz}^2)\,dx + \int_{-a}^{a} \psi_{1,zz}^2|_{r=0}\,dz \le \int_\Omega |\omega_{1,z}|^2\,dx. \tag{3.35}$$

Differentiate $(3.24)_1$ with respect to z, multiply by $\psi_{1,rrz}$ and integrate over Ω. Then we have

$$(3.36)\qquad -\int_\Omega \psi_{1,rrz}^2\,dx - \int_\Omega \psi_{1,zzz}\psi_{1,rrz}\,dx - 3\int_\Omega \frac{1}{r}\psi_{1,rz}\psi_{1,rrz}\,dx = \int_\Omega \omega_{1,z}\psi_{1,rrz}\,dx.$$

Integrating by parts with respect to z in the second term in (3.36) implies

$$(3.37)\qquad -\int_\Omega \psi_{1,zzz}\psi_{1,rrz}\,dx = -\int_\Omega (\psi_{1,zz}\psi_{1,rrz})_{,z}\,dx + \int_\Omega \psi_{1,zz}\psi_{1,rrzz}\,dx$$
$$= -\int_0^R \psi_{1,zz}\psi_{1,rrz}\big|_{z=-a}^{z=a} r\,dr + \int_\Omega (\psi_{1,zz}\psi_{1,rzz}r)_{,r}\,dr\,dz - \int_\Omega \psi_{1,rzz}^2\,dx - \int_\Omega \psi_{1,zz}\psi_{1,rzz}\,dr\,dz,$$

where the first term on the r.h.s. of (3.37) vanishes because $\psi_{1,zz}|_{S_2} = 0$ and the second vanishes also because $\psi_{1,rzz}|_{r=0} = 0$ and $\psi_{1,zz}|_{r=R} = 0$.

Applying (3.37) in (3.36) yields

$$(3.38)\qquad \int_\Omega (\psi_{1,rrz}^2 + \psi_{1,rzz}^2)\,dx + \int_\Omega \psi_{1,zz}\psi_{1,rzz}\,dr\,dz + 3\int_\Omega \psi_{1,rz}\psi_{1,rrz}\,dr\,dz = -\int_\Omega \omega_{1,z}\psi_{1,rrz}\,dx.$$

The second term in (3.38) equals

$$\frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2\big|_{r=0}^{r=R}\,dx = -\frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2|_{r=0}\,dx$$

because $\psi_{1,zz}|_{r=R} = 0$, and the last term has the form

$$\frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2\big|_{r=0}^{r=R}\,dz = \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2|_{r=R}\,dz,$$

where expansion (2.6) is used.

Exploiting the above expressions in (3.38) and applying the Hölder and Young inequalitites to the r.h.s. of (3.38) we obtain

$$(3.39)\qquad \int_\Omega (\psi_{1,rrz}^2 + \psi_{1,rzz}^2)\,dx - \frac{1}{2}\int_{-a}^{a} \psi_{1,zz}^2|_{r=0}\,dx + \frac{3}{2}\int_{-a}^{a} \psi_{1,rz}^2|_{r=R}\,dz \le c\|\omega_{1,z}\|_{L_2(\Omega)}^2.$$

Inequalities (3.35) and (3.39) imply (3.26), which concludes the proof. ∎

REMARK 3.9. Up to now we have considered problem (3.1) treating z as a parameter. It describes solutions to (1.9) only in a neighborhood of the axis of symmetry. Solutions to (1.9) in a domain $r > r_0 > 0$ are described by problem (1.15). From (2.6), (3.25) and $(1.13)_3$ we obtain for solutions to (1.15) the estimate

$$\|\omega\|_{H^{2+k}(\Omega^{(2)})} \le c\|\omega_1\|_{H^k(\Omega^{(2)})} \tag{3.40}$$

for some $k \in \{0,1\}$. Since $\operatorname{supp}\omega \subset \Omega^{(2)}$ we see that (3.40) can also be deduced for weighted spaces:

$$\|w\|_{H^{2+k}_\mu(\Omega^{(2)})} \le c\|\omega_1\|_{H^k_\mu(\Omega^{(2)})}, \qquad \mu \ge 0.$$

4. Estimates with respect to z for solutions to (1.9). Consider problem (1.9) in the form

$$\begin{cases} -\psi_{1,rr} - \dfrac{3}{r}\psi_{1,r} - \psi_{1,zz} = \omega_1 & \text{in } \Omega, \\ \psi_1 = 0 & \text{for } z \in \{-a, a\}, \\ \psi_1 = 0 & \text{for } r = R. \end{cases} \tag{4.1}$$

LEMMA 4.1. *Fix $\mu \in [0,1)$. Assume that $\omega_1 \in L_{2,\mu}(\Omega)$. Then*

$$\int_\Omega (\psi_{1,zz}^2 + \psi_{1,zr}^2) r^{2\mu}\, dx + 2\mu(1-\mu)\int_\Omega \psi_{1,z}^2 r^{2\mu-2}\, dx \le c\int_\Omega \omega_1^2 r^{2\mu}\, dx. \tag{4.2}$$

Proof. Multiply $(4.1)_1$ by $-\psi_{1,zz}r^{2\mu}$ and integrate over Ω. Then we have

$$\int_\Omega \psi_{1,zz}^2 r^{2\mu}\, dx + \int_\Omega \psi_{1,rr}\psi_{1,zz} r^{2\mu}\, dx + 3\int_\Omega \frac{1}{r}\psi_{1,r}\psi_{1,zz} r^{2\mu}\, dx = -\int_\Omega \omega_1\psi_{1,zz} r^{2\mu}\, dx. \tag{4.3}$$

Integrating by parts in the second term on the l.h.s. we obtain

$$\begin{aligned} -\int_\Omega \psi_{1,rrz}\psi_{1,z} r^{2\mu}\, dx &= -\int_\Omega \psi_{1,rrz}\psi_{1,z} r^{2\mu+1}\, dr\, dz \\ &= -\int_\Omega (\psi_{1,rz}\psi_{1,z} r^{2\mu+1})_{,r}\, dr\, dz \\ &\quad + \int_\Omega \psi_{1,rz}^2 r^{2\mu}\, dx + (2\mu+1)\int_\Omega \psi_{1,rz}\psi_{1,z} r^{2\mu}\, dr\, dz \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

We easily see that

$$I_1 = -\int_{-a}^{a} \psi_{1,rz}\psi_{1,z} r^{2\mu}\big|_{r=0}^{r=R}\, dz = 0$$

because $\psi_{1,z}|_{r=R} = 0$ and Remark 2.4 implies that $\psi_{1,rz}|_{r=0} = 0$.

Using the above results in (4.3) and integrating by parts in the last term on the l.h.s. of (4.3) we derive

$$\int_\Omega(\psi_{1,zz}^2+\psi_{1,rz}^2)r^{2\mu}\,dx-(1-\mu)\int_\Omega\partial_r(\psi_{1,z}^2r^{2\mu})\,dr\,dz+2\mu(1-\mu)\int_\Omega\psi_{1,z}^2r^{2\mu-1}\,dr\,dz = -\int_\Omega\omega_1\psi_{1,zz}r^{2\mu}\,dx,$$

where the second integral vanishes by the same arguments as for I_1.

Using the above results in (4.3) and applying the Hölder and Young inequalities to the r.h.s. yields

$$\int_\Omega(\psi_{1,zz}^2+\psi_{1,zr}^2)r^{2\mu}\,dx+2\mu(1-\mu)\int_\Omega\psi_{1,z}^2r^{2\mu-2}\,dx\le c\int_\Omega\omega_1^2r^{2\mu}\,dx.$$

This inequality implies (4.2) and concludes the proof. ∎

Lemma 4.2. *Fix $\mu\in[0,1)$. Assume that $\omega_{1,z}\in L_{2,\mu}(\Omega)$. Then*

$$\int_\Omega(\psi_{1,zzz}^2+\psi_{1,rzz}^2)r^{2\mu}\,dx+2\mu(1-\mu)\int_\Omega\psi_{1,zz}^2r^{2\mu-2}\,dx\le c\int_\Omega\omega_{1,z}^2r^{2\mu}\,dx. \tag{4.4}$$

Proof. Differentiate (4.1) with respect to z, multiply by $-\psi_{1,zzz}r^{2\mu}$ and integrate over Ω. Then we obtain

$$\int_\Omega\psi_{1,rrz}\psi_{1,zzz}r^{2\mu}\,dx+\int_\Omega\psi_{1,zzz}^2r^{2\mu}\,dx+3\int_\Omega\frac{1}{r}\psi_{1,rz}\psi_{1,zzz}r^{2\mu}\,dx = -\int_\Omega\omega_{1,z}\psi_{1,zzz}r^{2\mu}\,dx. \tag{4.5}$$

From $(4.1)_1$ it follows that

$$\psi_{1,zz}|_{z\in\{-a,a\}}=0 \tag{4.6}$$

because $\psi_1|_{z\in\{-a,a\}}=0$ and $\omega_1|_{z\in\{-a,a\}}=0$.

In view of (4.6) the first integral on the l.h.s. of (4.5) equals

$$-\int_\Omega\psi_{1,rrzz}\psi_{1,zz}r^{2\mu}\,dx=-\int_\Omega(\psi_{1,rzz}\psi_{1,zz}r^{2\mu+1})_{,r}\,dr\,dz + \int_\Omega\psi_{1,rzz}^2r^{2\mu}\,dx+(2\mu+1)\int_\Omega\psi_{1,rzz}\psi_{1,zz}r^{2\mu}\,drdz. \tag{4.7}$$

In virtue of the boundary condition $\psi_1|_{r=R}=0$ and Remark 2.4 the first integral on the r.h.s. of (4.7) vanishes.

Integrating by parts in the last term on the l.h.s. of (4.5) and using (4.7), we obtain

$$\int_\Omega(\psi_{1,zzz}^2+\psi_{1,rzz}^2)r^{2\mu}\,dx+(2\mu-2)\int_\Omega\psi_{1,rzz}\psi_{1,zz}r^{2\mu}\,dr\,dz = -\int_\Omega\omega_{1,z}\psi_{1,zzz}r^{2\mu}\,dx. \tag{4.8}$$

The second term on the l.h.s. equals

$$
\begin{aligned}
(4.9)\qquad (\mu-1)\int_\Omega \partial_r(\psi_{1,zz}^2)r^{2\mu}\,dr\,dz
&= (\mu-1)\int_\Omega \partial_r(\psi_{1,zz}^2 r^{2\mu})\,dr\,dz + 2\mu(1-\mu)\int_\Omega \psi_{1,zz}^2 r^{2\mu-1}\,dr\,dz\\
&= (\mu-1)\int_{-a}^{a} \psi_{1,zz}^2 r^{2\mu}\big|_{r=0}^{r=R}\,dz + 2\mu(1-\mu)\int_\Omega \psi_{1,zz}^2 r^{2\mu-2}\,dx,
\end{aligned}
$$

where the first term on the r.h.s. equals $(1-\mu)\int_{-a}^{a}\psi_{1,zz}^2 r^{2\mu}|_{r=0}dz$ because $\psi_{1,zz}|_{r=R}=0$. Using (4.9) in (4.8) implies (4.4). This ends the proof. ∎

5. Proofs of theorems. Let $\mu\in(0,1)$. Combining Lemma 3.1 with $k=0$ and $k=1$ with Lemmas 4.1 and 4.2 we obtain

$$
\begin{aligned}
\int_{-a}^{a} \|\psi_1-\psi_1^{(1)}(0)\|^2_{H^2_\mu(0,R)}\,dz + \|\psi_{1,zz}\|^2_{L_{2,\mu}(\Omega)} + \|\psi_{1,rz}\|^2_{L_{2,\mu}(\Omega)}&\\
+ 2\mu(1-\mu)\int_\Omega \psi_{1,z}^2 r^{2\mu-2}\,dx &\le c\|\omega_1\|^2_{L_{2,\mu}(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-a}^{a} \|\psi_1-\psi_1^{(1)}(0)\|^2_{H^3_\mu(0,R)}\,dz + \|\psi_{1,zzz}\|^2_{L_{2,\mu}(\Omega)} + \|\psi_{1,rzz}\|^2_{L_{2,\mu}(\Omega)}&\\
+ 2\mu(1-\mu)\int_\Omega \psi_{1,zz}^2 r^{2\mu-2}\,dx &\le c\|\omega_1\|^2_{H^1_\mu(\Omega)}.
\end{aligned}
$$

This proves Theorems 1.1 and 1.2.

Lemmas 3.7 and 3.8 used with (1.13) and (3.40) for $k=0$ yield

$$
\int_{-a}^{a} \|\psi_1-\psi_1^{(1)}(0)-\chi\|^2_{H^2_0(0,R)}\,dz + \int_\Omega (\psi_{1,zz}^2+\psi_{1,zr}^2)\,dx \le c\|\omega_1\|^2_{L_2(\Omega)}
$$

and Lemmas 3.6, 2.4 and 3.8 along with (1.13) and (3.40) for $k=1$ give

$$
\begin{aligned}
\int_{-a}^{a} \|\psi_1-\psi_1^{(1)}(0)-\eta\|^2_{H^3_0(0,R)}\,dz + \int_\Omega (\psi_{1,zzz}^2+\psi_{1,zzr}^2)\,dx + \|\psi_1\|^2_{H^2(\Omega)}&\\
\le c\|\omega_1\|^2_{H^1(\Omega)},&
\end{aligned}
$$

and thus Theorems 1.3 and 1.4 follow.

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