

Nonlinear twists and moments of L -functions

by

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*Dedicated to Henryk Iwaniec
with admiration and friendship*

Abstract. Let F belong to the extended Selberg class \mathcal{S}^\sharp . We show how a suitable hypothesis on the analytic continuation of a certain nonlinear twist of F^2 , namely the self-reciprocal twist, implies a sharp bound for the mean-square of $F(1/2 + it)$.

1. Introduction. Let \mathcal{S} and \mathcal{S}^\sharp denote the Selberg and the extended Selberg class, respectively, and let $F \in \mathcal{S}^\sharp$ be of degree $d \geq 1$ and conductor q . We shall briefly recall the basic notation and results in Section 2. In our papers [1, 4–7] we studied the analytic properties of a class of nonlinear twists of F . Moreover, in these and other papers, notably in [8, 9], we refined and applied such properties to the study of the structure of the Selberg classes.

A significant role in the above research is played by the *standard twist* $F(s, \alpha)$, a special nonlinear twist of F defined for $\sigma > 1$ by

$$F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n^{1/d}),$$

where $a(n)$ are the Dirichlet coefficients of F , $\alpha > 0$ and $e(x) = e^{2\pi i x}$. The analytic properties of $F(s, \alpha)$ are now rather well known (see Section 2 for some of them), and are crucial in several problems.

In this paper we introduce a more mysterious but equally important nonlinear twist, namely the *self-reciprocal twist* defined for $\sigma > 1$ by

$$F_{\text{self}}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\kappa_F n^{2/d}), \quad \kappa_F = \frac{1}{2} dq^{-1/d},$$

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and link it with a central problem in analytic number theory. Indeed, although at present the analytic properties of the self-reciprocal twist are essentially unknown, here we present an unexpected consequence of a seemingly mild hypothesis about its analytic continuation to the left of the line $\sigma = 1$. The name of this twist comes from the fact that the general transformation formula for nonlinear twists in [5] links the twists

$$F_\lambda(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n^\lambda), \quad \lambda > 1/d \text{ and } \alpha \neq 0,$$

to a certain combination of translates of so-called reciprocal (or dual) twists $\overline{F_{\lambda^*}}(s^*, \alpha^*)$. Here $\overline{F_{\lambda^*}}(s, \alpha)$ denotes the conjugate of $F_{\lambda^*}(s, \alpha)$ and

$$\lambda^* = \frac{\lambda}{d\lambda - 1}, \quad \alpha^* = (d\lambda - 1)(q^\lambda \lambda^{d\lambda} \alpha)^{-1/(d\lambda - 1)}$$

and, if $\theta_F = 0$,

$$s^* = \frac{s + \frac{d\lambda}{2} - 1}{d\lambda - 1},$$

where θ_F is the internal shift; see Section 2 for notation. Hence, when $\theta_F = 0$, we see that

$$(\lambda^*, \alpha^*, s^*) = (\lambda, \alpha, s) \quad \text{if and only if} \quad \lambda = 2/d \text{ and } \alpha = \kappa_F.$$

Therefore, in this case $\overline{F_{\lambda^*}}(s^*, \alpha^*) = \overline{F_{\text{self}}}(s)$, hence the name.

For powers of the Riemann zeta function, i.e. $F(s) = \zeta(s)^k$ with integer $k \geq 1$, we simply write

$$\zeta_k(s) = \sum_{n=1}^{\infty} \frac{d_{2k}(n)}{n^s} e(-kn^{1/k})$$

and our main result, Theorem 2 below, gives at once the following bound.

THEOREM 1. *Suppose that $\zeta_k(s)$ has holomorphic continuation to the half-plane $\sigma > \frac{1}{2} + \frac{1}{2k}$, with polynomial growth on every vertical strip inside this half-plane. Then for every $\varepsilon > 0$ we have*

$$\int_{-T}^T |\zeta(1/2 + it)|^{2k} dt \ll T^{1+\varepsilon}.$$

We expect that $\zeta_k(s)$ has meromorphic continuation over \mathbb{C} , and has a pole of order $k^2 + 1$ at $s = \frac{1}{2} + \frac{1}{2k}$; cf. the Conjecture below.

Returning to the general case, we denote by $a_2(n) = a * a(n)$ the coefficients of F^2 . Clearly, the degree and conductor of F^2 are $2d$ and q^2 , respectively, hence

$$F_{\text{self}}^2(s) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s} e(-\kappa_0 n^{1/d}), \quad \kappa_0 := \kappa_{F^2} = dq^{-1/d},$$

is the self-reciprocal twist of F^2 . For simplicity, in this paper we only consider L -functions $F \in \mathcal{S}^\sharp$ with integer degree d and $\theta_F = 0$. This is however the most interesting case, since the classical L -functions satisfy both conditions. Recalling that the conjugate function \overline{F} has the same degree and conductor as F and conjugate coefficients, we assume the following hypothesis on the analytic continuation of the self-reciprocal twist.

δ -HYPOTHESIS. *Let $F \in \mathcal{S}^\sharp$ be of integer degree $d \geq 1$ with conductor q and $\theta_F = 0$. Moreover, let $\delta \geq 0$ be fixed. Then the self-reciprocal twists $F_{\text{self}}^2(s)$ and $\overline{F}_{\text{self}}^2(s)$ have holomorphic continuation to the half-plane $\sigma > \frac{1}{2} + \frac{1}{2d} + \delta$ with polynomial growth on every vertical strip contained in this half-plane.*

Interestingly, the point κ_0 is related to the spectrum of F ; see Section 2. Moreover, the growth condition can be somewhat relaxed, still leading to the same conclusions in Theorem 2 below.

Turning to the applications of the self-reciprocal twist, for $T > 0$ sufficiently large we write

$$I_F(T) = \int_{-T}^T |F(1/2 + it)|^2 dt$$

and prove the following result.

THEOREM 2. *Let $F \in \mathcal{S}^\sharp$ be of integer degree $d \geq 1$ with $\theta_F = 0$, and suppose that the δ -Hypothesis holds true. Then for every $\varepsilon > 0$ we have*

$$I_F(T) \ll T^{1+\delta d+\varepsilon}.$$

Finally, some remarks are in order. First note that if the δ -Hypothesis holds with $\delta = 0$, then we have the optimal bound

$$I_F(T) \ll T^{1+\varepsilon}.$$

Thus, if the 0-Hypothesis holds true with F replaced by F^k for arbitrarily large integers k , then the Lindelöf Hypothesis holds for F . We have the following

CONJECTURE. *Let $F \in \mathcal{S}^\sharp$ be of integer degree $d \geq 1$ with $\theta_F = 0$, and satisfy the Ramanujan Conjecture. Then the self-reciprocal twist $F_{\text{self}}^2(s)$ has meromorphic continuation to \mathbb{C} with poles at most at the points*

$$s_k = \frac{1}{2} + \frac{1}{2d} - \frac{k}{d}, \quad k \geq 0 \text{ integer,}$$

and polynomial growth on vertical strips. Moreover, the pole at s_0 has order $d^2 + 1$, while the other poles have order either $d^2 + 1$ or 0.

Note the analogy between the conjectural polar structure of the above self-reciprocal twist and the known polar structure of the standard twist reported in Section 2; note also the main difference in the polar orders.

We already pointed out that at present the range of δ for which the δ -Hypothesis holds, and a fortiori the above conjecture, are open problems. However, some partial results can be obtained in this direction, and we shall return to this problem. Trivially, the hypothesis holds with $\delta = \frac{1}{2} - \frac{1}{2d}$, thus giving

$$I_F(T) \ll T^{(d+1)/2+\varepsilon}$$

for every $F \in \mathcal{S}^\sharp$. For $d \geq 2$ this is weaker than the classical bound

$$I_F(T) \ll T^{\max(1, d/2)+\varepsilon},$$

following from the approximate functional equation coupled with the mean-value theorem for Dirichlet polynomials. Nevertheless, we believe that our approach is interesting as it opens up a new attack to the moment problem for L -functions. Moreover, our method shows that a finer heuristic conjecture concerning the polar structure of the self-reciprocal twist could be used to derive a precise asymptotics for $I_F(T)$. This would allow a comparison with other conjectures on the behaviour of $I_F(T)$ already appearing in the literature.

2. Notation. Throughout the paper we write $s = \sigma + it$, and $\bar{f}(s)$ for $\overline{f(\bar{s})}$. The extended Selberg class \mathcal{S}^\sharp consists of non-identically-vanishing Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

absolutely convergent for $\sigma > 1$, such that $(s-1)^m F(s)$ is entire of finite order for some integer $m \geq 0$, and satisfying a functional equation of type

$$F(s)\gamma(s) = \omega\bar{\gamma}(1-s)\bar{F}(1-s),$$

where $|\omega| = 1$ and the γ -factor

$$\gamma(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j)$$

has $Q > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re(\mu_j) \geq 0$. The Selberg class \mathcal{S} is, roughly speaking, the subclass of \mathcal{S}^\sharp of the functions with a general Euler product and satisfying the Ramanujan conjecture $a(n) \ll n^\varepsilon$. We refer to our survey paper [2] for further definitions, examples and the basic theory of the Selberg class.

The degree d , conductor q , root number ω_F and ξ -invariant ξ_F of $F \in \mathcal{S}^\sharp$ are defined by

$$d = 2 \sum_{j=1}^r \lambda_j, \quad q = (2\pi)^d Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j},$$

$$\omega_F = \omega \prod_{j=1}^r \lambda_j^{-2i\Im(\mu_j)}, \quad \xi_F = 2 \sum_{j=1}^r (\mu_j - 1/2) := \eta_F + id\theta_F$$

with $\eta_F, \theta_F \in \mathbb{R}$; here θ_F is the internal shift of F .

We conclude this section by reporting some results on the meromorphic structure of the standard twist $F(s, \alpha)$; see [6]. The spectrum of $F \in \mathcal{S}^\sharp$ with $\theta_F = 0$ is defined as

$$\text{Spec}(F) = \{\alpha > 0 : a(n_\alpha) \neq 0\}, \quad n_\alpha = qd^{-d}\alpha^d, \quad a(n_\alpha) = 0 \text{ if } n_\alpha \notin \mathbb{N}.$$

Then $F(s, \alpha)$ is entire if $\alpha \notin \text{Spec}(F)$, while it is meromorphic over \mathbb{C} if $\alpha \in \text{Spec}(F)$. In the latter case, $F(s, \alpha)$ has at most simple poles at the points

$$s_k = \frac{1}{2} + \frac{1}{2d} - \frac{k}{d}, \quad k = 0, 1, \dots,$$

with $\text{res}_{s=s_0} F(s, \alpha) \neq 0$. Moreover, in all cases $F(s, \alpha)$ has polynomial growth on vertical strips.

3. Proof of Theorem 2. For $T > 0$ sufficiently large we also write

$$J_F(T) = \int_{-\infty}^{\infty} |F(1/2 + it)|^2 e^{-(t/T)^2} dt.$$

Clearly

$$I_F(T) \leq eJ_F(T).$$

Moreover, since by convexity $F(1/2 + it) \ll |t|^\xi$ for every $\xi > d/2$ as $|t| \rightarrow \infty$, in the opposite direction we have

$$J_F(T) \leq I_F(T\sqrt{c \log T}) + O(1)$$

for every $c > 1 + d$. By partial integration we also obtain

$$J_F(T) = \frac{1}{T} \int_0^\infty I_F(t) \varphi(t/T) dt,$$

where $\varphi(u) = 2ue^{-u^2}$. From the above relations it is easy to deduce that for any $A > 0$,

$$I_F(T) \ll T^A \iff J_F(T) \ll T^A,$$

$$I_F(T) \ll T \log^A T \iff J_F(T) \ll T \log^A T.$$

As a consequence, estimating $I_F(T)$ and $J_F(T)$ are essentially equivalent tasks; we shall deal with $J_F(T)$.

3.1. Set-up. In what follows, the implicit constants in the O - and \ll -symbols may always depend on F . The symbol c , with or without subscript, denotes complex constants depending on F , whose values will not necessarily be the same at each occurrence. Analogously, the symbol r , with or without suffix, denotes real constants with the same properties as c . Such constants c and r can be explicitly computed, but in this paper their value is not relevant.

Recalling the definition of $J_F(T)$ and \bar{F} we have

$$J_F(T) = \frac{1}{i} \int_{(1/2)} F(s)\bar{F}(1-s)e^{(s-1/2)/T} ds.$$

Then we shift the line of integration to $\sigma = 3/2$, taking into account the residue of the integrand at the possible pole of F at $s = 1$. This is allowed by the polynomial growth of F and the exponential decay of $e^{((s-1/2)/T)^2}$. Estimating the residue we obtain

$$(3.1) \quad J_F(T) = \frac{1}{i} \int_{(3/2)} F(s)\bar{F}(1-s)e^{(s-1/2)/T} ds + O(1).$$

Recalling the functional equation of F , from (3.1) we deduce that

$$J_F(T) = \frac{1}{i\omega} \int_{(3/2)} F(s)^2 \frac{\gamma(s)}{\bar{\gamma}(1-s)} e^{(s-1/2)/T} ds + O(1).$$

Moreover, on the line $\sigma = 3/2$ we replace $F(s)^2$ by its Dirichlet series and switch summation and integration, thus getting

$$(3.2) \quad J_F(T) = \frac{1}{\omega} \sum_{n=1}^{\infty} \frac{a_2(n)}{n^{3/2}} \int_{-\infty}^{\infty} \frac{\gamma(3/2+it)}{\bar{\gamma}(-1/2-it)} n^{-it} e^{(1+it)/T} dt + O(1).$$

Recalling our assumption that $\theta_F = 0$, from the Stirling expansion in [3, equation (2.8)] we have, for fixed σ and large $|t|$,

$$(3.3) \quad \log \gamma(s) = As \log s + Bs + C \log s + \sum_{\nu=0}^N \frac{c_\nu}{s^\nu} + O(|s|^{-N-1}),$$

where $N \geq 0$ is a given integer and

$$(3.4) \quad A = d/2, \quad B = (\log q - d \log(2\pi e))/2, \quad C = \xi_F/2 \in \mathbb{R}, \quad c_\nu \in \mathbb{C}.$$

Observing that

$$\bar{\gamma}(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \bar{\mu}_j),$$

we see that the values of A, B, C computed from the data of $\gamma(s)$ and $\bar{\gamma}(s)$ coincide. Hence from (3.3) we deduce that

$$(3.5) \quad \begin{aligned} & \log \frac{\gamma(3/2 + it)}{\bar{\gamma}(-1/2 - it)} \\ &= A(3/2 + it) \log(3/2 + it) + B(3/2 + it) + C \log(3/2 + it) \\ & \quad + A(1/2 + it) \log(-1/2 - it) + B(1/2 + it) - C \log(-1/2 - it) + \Sigma_0(t), \end{aligned}$$

where $\Sigma_0(t)$ arises from the expansion of the terms in the sum over ν in (3.3) with $s = 3/2 + it$ and $s = -1/2 - it$. Recalling that the value of c_ν is not necessarily the same at each occurrence, $\Sigma_0(t)$ has the form

$$(3.6) \quad \Sigma_0(t) = \sum_{\nu=0}^N \frac{c_\nu}{t^\nu} + O(|t|^{-N-1}).$$

By further expansions we get

$$(3.7) \quad \begin{aligned} \log(3/2 + it) &= \log |t| + i \frac{\pi}{2} \operatorname{sgn}(t) + \Sigma_1(t), \\ \log(-1/2 - it) &= \log |t| - i \frac{\pi}{2} \operatorname{sgn}(t) + \Sigma_1(t). \end{aligned}$$

Here the form of $\Sigma_1(t)$ is similar to (3.6), but the summation starts with $\nu = 1$. Thus (3.4)–(3.7) give

$$\log \frac{\gamma(3/2 + it)}{\bar{\gamma}(-1/2 - it)} = d \log |t| + i(dt \log |t| + 2Bt + D \operatorname{sgn}(t)) + \Sigma_0(t),$$

where

$$(3.8) \quad D = \pi d/4 + \pi C \in \mathbb{R}.$$

Hence for large $|t|$, say $|t| > t_0$, we have

$$(3.9) \quad \frac{\gamma(3/2 + it)}{\bar{\gamma}(-1/2 - it)} = |t|^d e^{i\theta(t)} \Sigma_0(t),$$

where $\Sigma_0(t)$ is of the form (3.6) and

$$\theta(t) = dt \log |t| + 2Bt + D \operatorname{sgn}(t).$$

Recalling that $d \in \mathbb{N}$, choosing $N = d$ in (3.3), estimating trivially the part with $|t| \leq t_0$ of the integral over $(-\infty, \infty)$ and writing

$$f_n(t) = \theta(t) - t \log n + 2t/T^2,$$

from (3.2), (3.6) and (3.9) we obtain

$$\begin{aligned}
 (3.10) \quad J_F(T) &= \frac{e^{1/T^2}}{\omega} \sum_{n=1}^{\infty} \frac{a_2(n)}{n^{3/2}} \int_{-\infty}^{\infty} e^{if_n(t)} |t|^d e^{-(t/T)^2} \Sigma_0(t) dt + O(1) \\
 &= e^{1/T^2} \sum_{\nu=0}^d c_{\nu} \sum_{n=1}^{\infty} \frac{a_2(n)}{n^{3/2}} \int_{-\infty}^{\infty} e^{if_n(t)} |t|^d t^{-\nu} e^{-(t/T)^2} dt + O(\log T).
 \end{aligned}$$

Denoting the last integrand by $g_{\nu}(t)$, and observing that $f_n(-t) = -f_n(t)$ for $t \neq 0$, we have

$$\nu \text{ even} \implies g_{\nu}(-t) = \overline{g_{\nu}(t)} \quad \text{and} \quad \nu \text{ odd} \implies g_{\nu}(-t) = -\overline{g_{\nu}(t)}.$$

Hence, writing

$$(3.11) \quad J_{\nu}(n, T) = \int_0^{\infty} e^{iF_n(t)} t^{d-\nu} e^{-(t/T)^2} dt$$

with

$$(3.12) \quad F_n(t) = dt \log t + (2B - \log n + 2/T^2)t$$

and B as in (3.4), from (3.10) we finally deduce that

$$\begin{aligned}
 (3.13) \quad J_F(T) &= e^{1/T^2} \sum_{\substack{\nu=0 \\ \nu \text{ even}}}^d c_{\nu} \sum_{n=1}^{\infty} \frac{a_2(n)}{n^{3/2}} \Re(e^{iD} J_{\nu}(n, T)) \\
 &\quad + e^{1/T^2} \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^d c_{\nu} \sum_{n=1}^{\infty} \frac{a_2(n)}{n^{3/2}} \Im(e^{iD} J_{\nu}(n, T)) + O(\log T)
 \end{aligned}$$

with D as in (3.8).

3.2. Saddle point: preliminary reductions. Next we compute the saddle point of the exponential integrals $J_{\nu}(n, T)$ in (3.13) and then we use a suitable saddle point technique to extract their main contribution.

Since $F'_n(t) = d \log t - d \log(2\pi(n/q)^{1/d}) + 2/T^2$, see (3.12), we have

$$(3.14) \quad F'_n(t) = 0 \quad \text{if and only if} \quad t = t_n := 2\pi(n/q)^{1/d} e^{-2/(dT^2)}.$$

Accordingly, we shift the integration over $(0, \infty)$ in $J_{\nu}(n, T)$ in the following way. For a fixed $0 < \theta < \pi/4$ to be chosen later on, we consider the three lines in the complex z -plane

$$\ell_A : z = \rho e^{-i\theta}, \quad \ell_B : z = \rho e^{i\theta}, \quad \ell_n : z = t_n + \rho e^{i\pi/4} \quad \text{with } \rho \in \mathbb{R},$$

and define the two points

$$z_A = \ell_A \cap \ell_n \quad \text{and} \quad z_B = \ell_B \cap \ell_n$$

lying in the half-plane $\Re(z) > 0$. The functions $F_n(z)$ in (3.12) and $z^{d-\nu}$, $e^{-(z/T)^2}$ in (3.11) are holomorphic for $\Re(z) > 0$. Moreover, writing $z = \rho e^{i\phi}$, we have

$$e^{iF_n(z)} z^{d-\nu} e^{-(z/T)^2} \ll e^{-\rho(d \log \rho \sin \phi + d\phi \cos \phi + (2B - \log n + 2/T^2) \sin \phi)} \rho^{d-\nu} e^{-(\rho/T)^2 \cos(2\phi)},$$

thus the integrand in (3.11) has exponential decay as $\rho \rightarrow \infty$ uniformly for $0 \leq \phi \leq \theta$, i.e. in the sector between the half-lines $z = \rho$ and ℓ_B with $\rho > 0$, thanks to $0 < \theta < \pi/4$. Hence by Cauchy’s theorem we can shift the path of integration, thus getting

$$(3.15) \quad J_\nu(n, T) = \left(\int_0^{z_A} + \int_{z_A}^{z_B} + \int_{z_B}^{\infty e^{i\theta}} \right) e^{iF_n(z)} z^{d-\nu} e^{-(z/T)^2} dz = J_\nu^{(1)}(n, T) + J_\nu^{(2)}(n, T) + J_\nu^{(3)}(n, T),$$

say, where the paths of integration are along ℓ_A , ℓ_n and ℓ_B , respectively.

Before treating the integrals in (3.15) we compute z_A and z_B . Let h, h' be the distances of z_A, z_B from the real axis, respectively. Then

$$\begin{aligned} \tan \theta &= \frac{h}{t_n - h}, & h &= \frac{\tan \theta}{1 + \tan \theta} t_n, \\ |z_A| &= \frac{h}{\sin \theta} = \frac{\tan \theta}{1 + \tan \theta} \frac{1}{\sin \theta} t_n = \frac{t_n}{\sin \theta + \cos \theta} \end{aligned}$$

and hence

$$(3.16) \quad z_A = \frac{e^{-i\theta}}{\sin \theta + \cos \theta} t_n.$$

Arguing in a similar way we obtain

$$(3.17) \quad z_B = \frac{e^{i\theta}}{\cos \theta - \sin \theta} t_n.$$

In view of the definition of ℓ_A and recalling that $0 \leq \nu \leq d$, we have

$$J_\nu^{(1)}(n, T) \ll \int_0^{|z_A|} e^{-\Im F_n(\rho e^{-i\theta})} (1 + \rho^d) d\rho.$$

Moreover, from (3.4), (3.12) and (3.14) we get

$$(3.18) \quad \begin{aligned} \Im F_n(\rho e^{-i\theta}) &= -d\rho \log \rho \sin \theta - d\rho \cos \theta - \rho(2B - \log n + 2/T^2) \sin \theta \\ &= -d\rho \sin \theta \log \left(\frac{\rho(q/n)^{1/d} e^{2/(dT^2)} e^{\theta \cot \theta}}{2\pi e} \right) \\ &= d\rho \sin \theta \log \left(\frac{et_n}{\rho e^{\theta \cot \theta}} \right). \end{aligned}$$

But, thanks to (3.16), for $0 \leq \rho \leq |z_A|$ we obtain

$$\frac{et_n}{\rho e^{\theta \cot \theta}} \geq e^{\frac{\sin \theta + \cos \theta}{e^{\theta \cot \theta}}} = 1 + \theta + O(\theta^2)$$

as $\theta \rightarrow 0^+$, therefore

$$\Im F_n(\rho e^{-i\theta}) \geq d\rho\theta^2/2$$

for $0 \leq \rho \leq |z_A|$ and $0 < \theta < \theta_0$, where $\theta_0 > 0$ is sufficiently small. As a consequence we have

$$(3.19) \quad J_\nu^{(1)}(n, T) \ll \int_0^{|z_A|} e^{-d\rho\theta^2/4}(1 + \rho^d) d\rho \ll 1$$

uniformly in n, ν and T , with any fixed $0 < \theta < \theta_0$ and a sufficiently small $\theta_0 > 0$.

The treatment of $J_\nu^{(3)}(n, T)$ is similar. Thanks to (3.14), (3.17) and (3.18), for $\rho \geq |z_B|$ we have

$$\Im F_n(\rho e^{i\theta}) = d\rho \sin \theta \log \left(\frac{\rho e^{\theta \cot \theta}}{et_n} \right) \geq d\rho \sin \theta \log \left(\frac{e^{\theta \cot \theta}}{e(\cos \theta - \sin \theta)} \right),$$

and as $\theta \rightarrow 0^+$,

$$\frac{e^{\theta \cot \theta}}{e(\cos \theta - \sin \theta)} = 1 + \theta + O(\theta^2).$$

Hence, as before, we deduce that

$$J_\nu^{(3)}(n, T) \ll \int_{|z_B|}^\infty e^{-d\rho\theta^2/4} \rho^d d\rho \ll 1,$$

thus from (3.15) and (3.19) also that

$$(3.20) \quad J_\nu(n, T) = J_\nu^{(2)}(n, T) + O(1),$$

with the same uniformity and conditions stated after (3.19).

We conclude this subsection by rewriting $J_\nu^{(2)}(n, T)$ in a more convenient form. To this end we write $z \in [z_A, z_B]$ as

$$(3.21) \quad z = t_n(1 + ue^{i\pi/4}) \quad \text{with } -A(\theta) \leq u \leq B(\theta)$$

and compute $A(\theta)$ and $B(\theta)$. From (3.16) we see that $A(\theta)$ satisfies

$$\frac{\cos \theta - i \sin \theta}{\sin \theta + \cos \theta} = 1 - A(\theta)e^{i\pi/4},$$

hence

$$(3.22) \quad A(\theta) = (1 + i) \frac{\sin \theta}{\sin \theta + \cos \theta} e^{-i\pi/4} = \sqrt{2} \frac{\sin \theta}{\sin \theta + \cos \theta} = \frac{\sqrt{2}}{1 + \cot \theta}.$$

Similarly, from (3.17) we get

$$(3.23) \quad B(\theta) = (1 + i) \frac{\sin \theta}{\cos \theta - \sin \theta} e^{-i\pi/4} = \frac{\sqrt{2}}{\cot \theta - 1}.$$

Therefore, in view of (3.15) and (3.21), $J_\nu^{(2)}(n, T)$ can be rewritten as

$$\begin{aligned}
 (3.24) \quad J_\nu^{(2)}(n, T) &= t_n e^{i\pi/4} \int_{-A(\theta)}^{B(\theta)} e^{iF_n(t_n(1+ue^{i\pi/4}))} t_n^{d-\nu} (1+ue^{i\pi/4})^{d-\nu} e^{-(t_n/T)^2(1+ue^{i\pi/4})^2} du \\
 &= t_n^{d+1-\nu} e^{i\pi/4} e^{iF_n(t_n)} \int_{-A(\theta)}^{B(\theta)} e^{i(F_n(t_n(1+ue^{i\pi/4})) - F_n(t_n))} \\
 &\quad \times (1+ue^{i\pi/4})^{d-\nu} e^{-(t_n/T)^2(1+ue^{i\pi/4})^2} du,
 \end{aligned}$$

where $A(\theta)$ and $B(\theta)$ are given by (3.22) and (3.23), respectively. Moreover, θ is fixed with $0 < \theta < \theta_0$ and $\theta_0 > 0$ sufficiently small. Therefore, in what follows we assume that $|u|$ is sufficiently small as well, since $A(\theta), B(\theta) \rightarrow 0$ as $\theta \rightarrow 0^+$.

3.3. Saddle point: further reductions. From (3.4), (3.12) and (3.14) we see that

$$(3.25) \quad F_n(z) = dz \log\left(\frac{z}{et_n}\right),$$

hence

$$F_n(t_n(1+ue^{i\pi/4})) - F_n(t_n) = dt_n((1+ue^{i\pi/4}) \log(1+ue^{i\pi/4}) - ue^{i\pi/4}).$$

But for $|w| \leq 1/2$ we have

$$(1+w) \log(1+w) - w = (1+w) \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} w^m - w = \sum_{m=2}^{\infty} \frac{(-1)^m}{m(m-1)} w^m,$$

therefore for any integer $M \geq 2$ we deduce, with the obvious meaning of the real constants r_m , that

$$(3.26) \quad F_n(t_n(1+ue^{i\pi/4})) - F_n(t_n) = dt_n \sum_{m=2}^M r_m e^{im\pi/4} u^m + O\left(\frac{t_n|u|^{M+1}}{M}\right).$$

As a consequence, from (3.24) we obtain

$$\begin{aligned}
 (3.27) \quad J_\nu^{(2)}(n, T) &= t_n^{d+1-\nu} e^{iF_n(t_n)} e^{-(t_n/T)^2} \sum_{\mu=0}^{d-\nu} \binom{d-\nu}{\mu} e^{i(\mu+1)\pi/4} \\
 &\times \int_{-A(\theta)}^{B(\theta)} e^{-\frac{1}{2}dt_n u^2} e^{idt_n \sum_{m=3}^M r_m e^{im\pi/4} u^m} e^{O\left(\frac{t_n|u|^{M+1}}{M}\right)} u^\mu e^{-(t_n/T)^2(2e^{i\pi/4}u+iu^2)} du \\
 &= t_n^{d+1-\nu} e^{iF_n(t_n)} e^{-(t_n/T)^2} \sum_{\mu=0}^{d-\nu} \binom{d-\nu}{\mu} e^{i(\mu+1)\pi/4} \tilde{\mathfrak{J}}_\mu(n, T),
 \end{aligned}$$

say.

By the substitution

$$(3.28) \quad \sqrt{dt_n/2} u = \xi$$

we have

$$(3.29) \quad \begin{aligned} \tilde{\mathfrak{J}}_\mu(n, T) &= \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} \int_{-\sqrt{dt_n/2} A(\theta)}^{\sqrt{dt_n/2} B(\theta)} e^{-\xi^2} \xi^\mu e^{idt_n \sum_{m=3}^M r_m \left(\frac{2}{dt_n}\right)^{m/2} e^{im\pi/4} \xi^m} \\ &\quad \times e^{O\left(\frac{t_n}{M} (2\xi^2/(dt_n))^{(M+1)/2}\right)} e^{-(t_n/T)^2 (2e^{i\pi/4} \sqrt{\frac{2}{dt_n}} \xi + i \frac{2}{dt_n} \xi^2)} d\xi \\ &= \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} \left(\int_{-\sqrt{dt_n/2} A(\theta)}^{\sqrt{dt_n/2} B(\theta)} e^{-\xi^2} \xi^\mu e^{\Sigma(\xi, n)} e^{-g(\xi, n, T)} d\xi + R_\mu(n, T) \right) \\ &= \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} (\mathfrak{J}_\mu(n, T) + R_\mu(n, T)). \end{aligned}$$

Here $R_\mu(n, T)$ is the error term arising from the replacement of $e^{O\left(\frac{t_n}{M} (2\xi^2/(dt_n))^{(M+1)/2}\right)}$ by 1 inside the first integral, and

$$(3.30) \quad \Sigma(\xi, n) = idt_n \sum_{m=3}^M r_m \left(\frac{2}{dt_n}\right)^{m/2} e^{im\pi/4} \xi^m,$$

$$(3.31) \quad g(\xi, n, T) = (t_n/T)^2 \left(2e^{i\pi/4} \sqrt{\frac{2}{dt_n}} \xi + i \frac{2}{dt_n} \xi^2 \right),$$

$$(3.32) \quad \mathfrak{J}_\mu(n, T) = \int_{-\sqrt{dt_n/2} A(\theta)}^{\sqrt{dt_n/2} B(\theta)} e^{-\xi^2 + \Sigma(\xi, n) - g(\xi, n, T)} \xi^\mu d\xi.$$

Next we estimate the contribution to $J_\nu^{(2)}(n, T)$ of the error terms $R_\mu(n, T)$ in (3.29). Recalling (3.28), the value of r_m in (3.26) and the fact that $|u|$ is assumed to be sufficiently small (see after (3.24)), from (3.30) and (3.31) we have that

$$(3.33) \quad \Sigma(\xi, n) \ll \frac{|\xi|^3}{\sqrt{t_n}} \leq \xi^2/2 \quad \text{and} \quad |g(\xi, n, T)| \leq \frac{(t_n/T)^2}{2}$$

for $-\sqrt{dt_n/2} A(\theta) \leq \xi \leq \sqrt{dt_n/2} B(\theta)$. Then we choose $M = M(n)$ so large that

$$(3.34) \quad \frac{t_n}{M2^M} < ct_n^{-d-1/2},$$

with a sufficiently small constant $c > 0$. Hence, recalling that $A(\theta), B(\theta)$ are

sufficiently small and $\mu \geq 0$, from (3.29), (3.33) and (3.34) we obtain

$$(3.35) \quad \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} R_\mu(n, T) \ll \frac{1}{t_n^{d+1}} e^{(t_n/T)^2/2} \int_{-\infty}^{\infty} e^{-\xi^2/2} d\xi \\ \ll \frac{1}{t_n^{d+1}} e^{(t_n/T)^2/2}.$$

Since $F_n(t_n) \in \mathbb{R}$, by (3.27), (3.29) and (3.35) the contribution to $J_\nu^{(2)}(n, T)$ of the error terms $R_\mu(n, T)$ is

$$(3.36) \quad O(1)$$

uniformly in ν, n and T , provided $\theta_0 > 0$ is sufficiently small and M satisfies (3.34). Therefore, from (3.20), (3.27), (3.29) and (3.36) we deduce that

$$(3.37) \quad J_\nu(n, T) \\ = e^{iF_n(t_n)} t_n^{d+1-\nu} e^{-(t_n/T)^2} \sum_{\mu=0}^{d-\nu} \binom{d-\nu}{\mu} e^{i(\mu+1)\pi/4} \left(\frac{2}{dt_n}\right)^{(\mu+1)/2} \mathfrak{J}_\mu(n, T) \\ + O(1)$$

with the same uniformity and conditions after (3.36), where $\mathfrak{J}_\mu(n, T)$ is defined by (3.32).

3.4. Saddle point: computing the main terms. Now we study the integrals $\mathfrak{J}_\mu(n, T)$. We may assume that n is sufficiently large, say $n \geq n_0$, since for $n < n_0$ we have $J_\nu(n, T) = O(1)$ and their contribution to $J_F(T)$ amounts to $O(1)$. We first show that the range of integration in $\mathfrak{J}_\mu(n, T)$ can be replaced, up to a negligible quantity, by $|\xi| \leq c(\theta) \log t_n$, where $c(\theta) > 0$ is such that

$$\min(\sqrt{kt_n/2} A(\theta), \sqrt{kt_n/2} B(\theta)) > c(\theta) \log t_n$$

for every $n \geq n_0$. To this end we write

$$(3.38) \quad I_\mu(n, T) = \int_{-c(\theta) \log t_n}^{c(\theta) \log t_n} e^{-\xi^2 + \Sigma(\xi, n) - g(\xi, n, T)} \xi^\mu d\xi,$$

and arguing as in (3.35), from (3.33) we find that

$$(3.39) \quad \mathfrak{J}_\mu(n, T) - I_\mu(n, T) \ll \left(\int_{-\infty}^{-c(\theta) \log t_n} + \int_{c(\theta) \log t_n}^{\infty} \right) e^{-\xi^2/2 + (t_n/T)^2/2} |\xi|^\mu d\xi \\ \ll t_n^{-A} e^{(t_n/T)^2/2}$$

for every $A > 0$. Therefore, arguing as for (3.36), from (3.37) and (3.39) we get

$$(3.40) \quad J_\nu(n, T) = e^{iF_n(t_n)} e^{-(t_n/T)^2} \sum_{\mu=0}^{d-\nu} \alpha_{\nu,\mu} t_n^{d+1-\nu-(\mu+1)/2} I_\mu(n, T) + O(1)$$

with the same uniformity and conditions after (3.36), where $I_\mu(n, T)$ is defined by (3.38) and

$$(3.41) \quad \alpha_{\nu,\mu} = \binom{d-\nu}{\mu} \left(\frac{2}{d}\right)^{(\mu+1)/2} e^{i(\mu+1)\pi/4}.$$

Next we estimate the contribution of the integrals $I_\mu(n, T)$ to $J_\nu(n, T)$ for the values $n \geq n_0$ such that $t_n > T \log t_n$. In this case, again thanks to (3.33), we have

$$I_\mu(n, T) \ll \int_{-c(\theta) \log t_n}^{c(\theta) \log t_n} e^{(t_n/T)^2/2} |\xi|^\mu d\xi \ll e^{(t_n/T)^2/2} \log^{\mu+1} t_n.$$

Hence such a contribution is

$$(3.42) \quad O(1),$$

once more with the same uniformity and conditions after (3.36).

Finally, we compute the contribution of $I_\mu(n, T)$ to $J_\nu(n, T)$ for $n \geq n_0$ with $t_n \leq T \log t_n$. From (3.30), (3.31) and the first estimate in (3.33) we see that for $|\xi| \leq c(\theta) \log t_n$,

$$|\Sigma(\xi, n)|, |g(\xi, n, T)| \leq \frac{\log^3 t_n}{\sqrt{t_n}},$$

provided $c(\theta)$ is sufficiently small. Hence, given an arbitrarily large constant $A > 0$, there exist integers $Q = Q(A) > 0$ and $0 \leq p, k, \ell \leq Q$, and coefficients $\beta_{p,k,\ell} \in \mathbb{C}$, such that for $|\xi| \leq c(\theta) \log t_n$ we have

$$e^{\Sigma(\xi,n)-g(\xi,n,T)} = \sum_{0 \leq p \leq Q} \sum_{0 \leq k \leq Q} \sum_{0 \leq \ell \leq Q} \beta_{p,k,\ell} \frac{\xi^p}{t_n^{k/2}} \left(\frac{t_n}{T}\right)^{2\ell} + O(t_n^{-A-1}).$$

Note that $\beta_{0,0,0} = 1$, while the other coefficients $\beta_{p,k,\ell}$ are computable from the expressions in (3.30) and (3.31). Note also that, although M in (3.30) now depends on n , the integer Q is independent of n . Indeed, the contribution of the terms in (3.30) with m sufficiently large in terms of A is directly absorbed

into the error term $O(t_n^{-A-1})$. As a consequence,

$$(3.43) \quad I_\mu(n, T) = \sum_{0 \leq p \leq Q} \sum_{0 \leq k \leq Q} \sum_{0 \leq \ell \leq Q} \beta_{p,k,\ell} \frac{1}{t_n^{k/2}} \left(\frac{t_n}{T}\right)^{2\ell} \int_{-c(\theta) \log t_n}^{c(\theta) \log t_n} e^{-\xi^2} \xi^{\mu+p} d\xi + O(t_n^{-A-1}).$$

But for an arbitrarily large constant $B > 0$ we have

$$\begin{aligned} \int_{-c(\theta) \log t_n}^{c(\theta) \log t_n} e^{-\xi^2} \xi^{\mu+p} d\xi &= \int_{-\infty}^{\infty} e^{-\xi^2} \xi^{\mu+p} d\xi + O(t_n^{-B}) \\ &= \frac{1}{2} \Gamma\left(\frac{\mu+p+1}{2}\right) (1 + (-1)^{\mu+p}) + O(t_n^{-B}). \end{aligned}$$

Hence, by a suitable choice of $B = B(A)$ and after summation over p , (3.43) becomes

$$(3.44) \quad I_\mu(n, T) = \sum_{0 \leq k \leq Q} \sum_{0 \leq \ell \leq Q} \gamma_{k,\ell,\mu} \frac{1}{t_n^{k/2}} \left(\frac{t_n}{T}\right)^{2\ell} + O(t_n^{-A})$$

for $n \geq n_0$ with $t_n \leq T \log t_n$, where

$$(3.45) \quad \gamma_{k,\ell,\mu} = \frac{1}{2} \sum_{0 \leq p \leq Q} \beta_{p,k,\ell} \Gamma\left(\frac{\mu+p+1}{2}\right) (1 + (-1)^{\mu+p}).$$

Now we fix a sufficiently small $\theta > 0$ such that all the above estimates hold, and a sufficiently large A . Hence from the remark at the beginning of this subsection and from equations (3.40), (3.42), (3.44) we deduce, for the values of n such that $n < n_0$ or $t_n > T \log t_n$, that

$$(3.46) \quad J_\nu(n, T) = O(1).$$

Moreover, for the values of n with $n \geq n_0$ and $t_n \leq T \log t_n$ we have

$$(3.47) \quad J_\nu(n, T) = e^{iF_n(t_n)} e^{-(t_n/T)^2} \sum_{\mu=0}^{d-\nu} \sum_{k=0}^Q \sum_{\ell=0}^Q \delta_{\nu,\mu,k,\ell} t_n^{d+1/2-\nu-(\mu+k)/2} \left(\frac{t_n}{T}\right)^{2\ell} + O(1)$$

uniformly in ν, n and T . Here

$$(3.48) \quad \delta_{\nu,\mu,k,\ell} = \alpha_{\nu,\mu} \gamma_{k,\ell,\mu},$$

where $\alpha_{\nu,\mu}$ is given by (3.41) and $\gamma_{k,\ell,\mu}$ is as in (3.45). Note that $\delta_{\nu,\mu,k,\ell}$ are complex numbers due to the powers of $e^{i\pi/4}$ involved in $\alpha_{\nu,\mu}$ and $\beta_{p,k,\ell}$.

3.5. Entering the self-reciprocal twist and completion of the proof. From (3.13), (3.46), (3.47) and a simple estimate for the terms with

$n < n_0$ we obtain

$$(3.49) \quad J_F(T) = e^{1/T^2} \sum_{\substack{\nu=0 \\ \nu \text{ even}}}^d c_\nu \sum_{\mu=0}^{d-\nu} \sum_{k=0}^Q \sum_{\ell=0}^Q \tilde{S}_{\text{Re}}(T) \\ + e^{1/T^2} \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^d c_\nu \sum_{\mu=0}^{d-\nu} \sum_{k=0}^Q \sum_{\ell=0}^Q \tilde{S}_{\text{Im}}(T) + O(\log T),$$

where the coefficients c_ν are as in (3.13),

$$(3.50) \quad \tilde{S}_{\text{Re}}(T) = \sum_{t_n \leq T \log t_n} \frac{a_2(n)}{n^{3/2}} \Re(\delta_{\nu,\mu,k,\ell} e^{iD} e^{iF_n(t_n)}) t_n^{d-\nu+\frac{1-\mu-k}{2}} \phi_{2\ell}(t_n/T), \\ \phi_{2\ell}(\xi) = e^{-\xi^2} \xi^{2\ell},$$

the coefficients $\delta_{\nu,\mu,k,\ell} \in \mathbb{C}$ are given by (3.48), and D is as in (3.8). Moreover, $\tilde{S}_{\text{Im}}(T)$ is similar to $\tilde{S}_{\text{Re}}(T)$, but with the real part replaced by the imaginary part.

Recalling the value of t_n in (3.14) we have

$$t_n^{d-\nu+\frac{1-k-\mu}{2}} = (2\pi)^{d-\nu+\frac{1-k-\mu}{2}} \left(\frac{n}{q}\right)^{\frac{d-\nu}{d}+\frac{1-k-\mu}{2d}} \left(\sum_{h=0}^d \frac{r_h}{T^{2h}} + O\left(\frac{1}{T^{2d+2}}\right)\right),$$

where $r_h \in \mathbb{R}$ are easily computed and $r_0 = 1$. Moreover, by (3.14) and (3.25) we also get

$$F_n(t_n) = -dt_n = -2\pi d \left(\frac{n}{q}\right)^{1/d} \left(\sum_{h=0}^d \frac{r_h}{T^{2h}} + O\left(\frac{1}{T^{2d+2}}\right)\right)$$

with certain coefficients r_h . Therefore, since $t_n \leq T \log t_n$ implies that $n^{1/d} \ll T \log T$, by a further expansion of the exponential we obtain

$$(3.51) \quad e^{iF_n(t_n)} t_n^{d-\nu+\frac{1-k-\mu}{2}} = (2\pi)^{d-\nu+\frac{1-k-\mu}{2}} \left(\frac{n}{q}\right)^{\frac{d-\nu}{d}+\frac{1-k-\mu}{2d}} e^{(-\kappa_0 n^{1/d})} \\ \times \left(\sum_{h=0}^d \sum_{j=0}^h \eta_{h,j} \frac{n^{j/d}}{T^{2h}} + O(T^{-d})\right)$$

uniformly in ν, μ and k , where $\eta_{h,j} \in \mathbb{C}$ can be computed from the above expressions and

$$(3.52) \quad \kappa_0 = dq^{-1/d}.$$

Plugging (3.51) into (3.50) and then completing to ∞ the resulting sum over n , thanks to the decay of the function $\phi_{2\ell}(t_n/T)$ we obtain

$$(3.53) \quad \tilde{S}_{\text{Re}}(T) = \sum_{h=0}^d \sum_{j=0}^h \frac{1}{T^{2h}} S_{\text{Re}}(T) + O(1),$$

where

$$(3.54) \quad S_{\text{Re}}(T) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^{\frac{1}{2} - \frac{1}{2d} + \frac{\nu-j}{d} + \frac{k+\mu}{2d}}} \Re(\omega_{\nu,\mu,k,\ell,h,j} e^{-\kappa_0 n^{1/d}}) \phi_2 \ell(t_n/T),$$

κ_0 is given by (3.52) and

$$\omega_{\nu,\mu,k,\ell,h,j} = e^{iD} \left(\frac{(2\pi)^d}{q} \right)^{\frac{d-\nu}{d} + \frac{1-k-\mu}{2d}} \delta_{\nu,\mu,k,\ell,h,j}.$$

Clearly, a completely analogous expression holds for $\tilde{S}_{\text{Im}}(T)$, with the imaginary part in place of the real part. Therefore, inserting (3.53) into (3.49) we finally obtain

$$(3.55) \quad J_F(T) = e^{1/T^2} \sum_{\substack{\nu=0 \\ \nu \text{ even}}}^d \sum_{\mu=0}^{d-\nu} \sum_{k=0}^Q \sum_{\ell=0}^Q \sum_{h=0}^d \sum_{j=0}^h \frac{c_\nu}{T^{2h}} S_{\text{Re}}(T) \\ + e^{1/T^2} \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^d \sum_{\mu=0}^{d-\nu} \sum_{k=0}^Q \sum_{\ell=0}^Q \sum_{h=0}^d \sum_{j=0}^h \frac{c_\nu}{T^{2h}} S_{\text{Im}}(T) + O(\log T).$$

Now we recall that $a_2(n)$ are the coefficients of $F(s)^2$, whose degree is $2d$. Thus the above quantities $S_{\text{Re}}(T)$ and $S_{\text{Im}}(T)$, and hence also $J_F(T)$ thanks to (3.55), are closely related to the self-reciprocal twists $F_{\text{self}}^2(s)$ and $\overline{F_{\text{self}}^2}(s)$. More precisely, for $\sigma > 1$ and $\alpha \neq 0$ we write

$$(3.56) \quad F_{\text{cos}}^2(s) := \frac{1}{2} (F_{\text{self}}^2(s) + \overline{F_{\text{self}}^2}(s)) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s} \cos(-2\pi\kappa_0 n^{1/d}),$$

$$(3.57) \quad F_{\text{sin}}^2(s) := \frac{1}{2i} (F_{\text{self}}^2(s, \alpha) - \overline{F_{\text{self}}^2}(s)) = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s} \sin(-2\pi\kappa_0 n^{1/d}).$$

Hence, writing for simplicity

$$a = a_{\nu,\mu,k,\ell,h,j} := \Re(\omega_{\nu,\mu,k,\ell,h,j}), \quad b = b_{\nu,\mu,k,\ell,h,j} := \Im(\omega_{\nu,\mu,k,\ell,h,j})$$

and

$$(3.58) \quad \sigma_0 := \frac{1}{2} - \frac{1}{2d} + \frac{\nu-j}{d} + \frac{k+\mu}{2d} = \frac{1}{2} - \frac{1}{2d} - \frac{j}{d} + f,$$

say, in view of (3.54), (3.56), (3.57) the quantities $S_{\text{Re}}(T)$ and $S_{\text{Im}}(T)$ are closely related to

$$aF_{\text{cos}}^2(\sigma_0) - bF_{\text{sin}}^2(\sigma_0) \quad \text{and} \quad bF_{\text{cos}}^2(\sigma_0) + aF_{\text{sin}}^2(\sigma_0),$$

respectively. But, thanks to the δ -Hypothesis, the functions $F_{\text{cos}}(s)$ and $F_{\text{sin}}(s)$ have holomorphic continuation to the half-plane $\sigma > 1/2 + 1/(2d) + \delta$, with polynomial growth on vertical strips. Thus we may apply a Mellin transform technique to get bounds for $S_{\text{Re}}(T)$ and $S_{\text{Im}}(T)$.

To this end we first compute the Mellin transform of $\phi_{2\ell}(\xi)$, namely

$$(3.59) \quad \tilde{\phi}_{2\ell}(s) = \int_0^\infty \phi_{2\ell}(\xi)\xi^{s-1} d\xi = \frac{1}{2}\Gamma\left(\frac{s+2\ell}{2}\right),$$

and let

$$(3.60) \quad Y = \frac{q^{1/d}}{2\pi} T e^{2/(dT^2)} = \frac{q^{1/d}}{2\pi} T \left(1 + O\left(\frac{1}{T^2}\right)\right),$$

so that by (3.14) we have

$$(3.61) \quad \phi_{2\ell}(t_n/T) = \phi_{2\ell}(n^{1/d}/Y).$$

Thus from (3.59), (3.61) and the inverse Mellin transform we obtain

$$(3.62) \quad S_{\text{Re}}(T) = \frac{1}{2\pi i} \int_{(c)} (aF_{\cos}^2(s/d + \sigma_0) - bF_{\sin}^2(s/d + \sigma_0)) \frac{1}{2}\Gamma\left(\frac{s+2\ell}{2}\right) Y^s ds$$

with a sufficiently large constant $c > 0$, and similarly for $S_{\text{Im}}(T)$. Let $\varepsilon > 0$ be arbitrarily small. Recalling the value of σ_0 in (3.58), thanks to our hypothesis and the decay of the Γ function we can shift the integration in (3.62) to the line $\sigma = 1 + \delta d + j - df + \varepsilon$. Indeed, on this line we have

$$\Re(s/d + \sigma_0) = \frac{1}{2} + \frac{1}{2d} + \delta + \frac{\varepsilon}{d},$$

and hence in view of (3.56), (3.57), (3.60) and (3.62) we get

$$(3.63) \quad S_{\text{Re}}(T), S_{\text{Im}}(T) \ll Y^{1+\delta d+j-df+\varepsilon} \ll T^{1+\delta d+j-df+\varepsilon}.$$

Finally, from (3.55) and (3.58) we see that the worst case in (3.63) happens when $\nu \leq 1$ and $\mu = k = h = j = 0$, so $f = 0$ as well, thus

$$J_F(T) \ll T^{1+\delta d+\varepsilon},$$

and Theorem 2 follows.

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