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ON A NON-MARKOVIAN MODEL WITH LINEAR TRANSITION RULE

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1. *INTRODUCTION*

The model we are concerned with is called a *linear OM-chain* (cf. [3], p. 133). It extends the concept of a simple Markov chain and is specified by a sequence of real numbers and by a stochastic matrix Λ. Our aim* is to investigate the classification of its states and its asymptotic behaviour, using the information provided by Λ. In other words, we examine to what extent properties of simple Markov chains can be transferred to a certain type of non-Markovian discrete parameter stochastic processes.

Linear OM-chains were introduced by Onicescu and Mihoc [8] and were rediscovered by Bush and Mosteller [1] when discussing certain aspects of learning theory.

2. *STOCHASTIC MODEL*

2.1. Linear OM-chains. Let I be a countable set and let $M(A)$ with $A \subset I$ be the set of all probability measures on A. Set $W = M(I)$. We say that a 4-tuple $\{W, I, u, P\}$ is a *linear OM-chain* (abbreviated to LOM) if

(i) u is a mapping from $W \times I$ into W given by

$$u(w, i) = a_i w + (1 - a_i)\Lambda_i,$$

where the a_i's are real numbers and $\Lambda_i \in W$;

(ii) P is a stochastic kernel from W to I given by

$$P(w, A) = \sum_{i \in A} w_i \quad \text{for } A \subset I,$$

where w_i is the i-th component of w.

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Note that an LOM is completely specified by the sequence of real numbers $\{a_i: i \in I\}$ and by the stochastic matrix $\Lambda = (\Lambda_{ij})$, where Λ_{ij} is the j-th component of Λ_i. For the sake of simplicity we assume that $0 \leqslant a_i \leqslant 1$ for all $i \in I$.

If an LOM is given, then (cf. [4], p. 64) for each $w \in W$ there exist a probability space $(\Omega, \mathscr{K}, \boldsymbol{P}_w)$ and a sequence of random variables $\{\xi_n: n \geqslant 1\}$, defined on Ω with values in I, such that

$$\boldsymbol{P}_w(\xi_1 \in A) = P(w, A) \quad \text{and} \quad \boldsymbol{P}_w(\xi_{n+1} \in A | \xi_\mu, 1 \leqslant \mu \leqslant n) = P(\zeta_n, A) \quad \boldsymbol{P}_w\text{-a.s.}$$

for all $A \subset I$, where $\zeta_0 = w$ and $\zeta_n = u(\zeta_{n-1}, \xi_n)$. Note that $\{\zeta_n: n \geqslant 0\}$ is a general Markov chain on W with transition probability function

$$Q(w, B) = P\big(w, \{i: u(w, i) \in B\}\big).$$

Note also that if $a_i = 0$ for all $i \in I$, then $\{\xi_n: n \geqslant 1\}$ reduces to a simple Markov chain on I with transition matrix Λ.

For $l, n \geqslant 1$ and $A \subset I^l$ let us define two stochastic kernels P_l and P_l^n by

$$P_l(w, A) = \boldsymbol{P}_w\big((\xi_1, \ldots, \xi_l) \in A\big)$$

and

$$P_l^n(w, A) = \boldsymbol{P}_w\big((\xi_n, \ldots, \xi_{n+l-1}) \in A\big),$$

respectively. Clearly, $P_1 = P$ and $P_l^1 = P_l$. Further, for $A' \subset I^m$ and $A \subset I^l$ we have

$$P_{m+l}(w, A' \times A) = \sum_{j^m \in A'} P_m(w, j^m) P_l(wj^m, A), \tag{1}$$

where $j^m = (j_1, \ldots, j_m)$ and wj^m is an abbreviated notation for

$$u_m(w, j_1, \ldots, j_m) = u\big(u_{m-1}(w, j_1, \ldots, j_{m-1}), j_m\big)$$

(in particular, $wj = u_1(w, j) = u(w, j)$). From (1) we obtain

$$P_l^{n+m}(w, A) = \sum_{j^m \in I^m} P_m(w, j^m) P_l^n(wj^m, A). \tag{2}$$

Note that $P(wj^m, k)$ can explicitly be calculated, namely

$$P(wj^m, k) = a_{j_m} \ldots a_{j_1} w_k + \sum_{r=1}^{m-1} a_{j_m} \ldots a_{j_{r+1}} A_{j_r k} + A_{j_m k}, \tag{3}$$

where $A = (A_{ij})$ with $A_{ij} = (1 - a_i)\Lambda_{ij}$.

Since I is countable, the n-step transition probability function corresponding to $Q(w,\cdot)$, i.e. $Q^n(w,\cdot)$, is a discrete probability measure. Let us denote by $T^n(w)$ the countable set of its atomic points and take

$$T(w) = \bigcup_{n\geqslant 1} T^n(w), \quad T(B) = \bigcup_{w\in B} T(w) \text{ for } B \subset W.$$

2.2. Auxiliary properties. We start by giving

PROPOSITION 1. *Let $w \in W$, $j \in I$, and $m \geqslant 0$. We have*

$$P^{m+1}(w,j) \geqslant \sum_{i\in I} w_i A_{ij}^m, \tag{4}$$

where A_{ij}^m is the (i,j)-entry of the m-th power of the matrix A.

Proof. Using the inequality $P(wi,j) \geqslant A_{ij}$ we get (4) by induction on m.

PROPOSITION 2. *Let $w \in W$, $j \in I$, and $m \geqslant 1$. If $P^{m+1}(w,j) > 0$, then at least one of the following two statements holds true:*

(i) *$w_i a_i w_j > 0$ for some $i \in I$;*

(ii) *$w_k A_{kj}^n > 0$ for some $k \in I$ and $1 \leqslant n \leqslant m$.*

Proof. We proceed by induction on m. We can easily check that for $m = 1$ we get (i) or (ii). Suppose that our assertion holds true for some $m \geqslant 1$. Then, by (2),

$$P^{m+2}(w,j) = \sum_{l\in I} w_l P^{m+1}(wl,j) > 0,$$

so that there exists a $q \in I$ such that $w_q P^{m+1}(wq,j) > 0$. By our assumption we get at least one of the following two statements:

(a) $w_q(wq)_j > 0$ ($(wq)_j$ stands for the j-th component of wq);

(b) $w_q(wq)_k A_{kj}^n > 0$ for some $k \in I$ and $n \leqslant m$.

Since $(wq)_j = a_q w_j + (1-a_q)A_{qj}$, we remark that (a) as well as (b) reduces to (i) or to (ii).

3. CLASSIFICATION

3.1. Classes. Set $V_i = \{w: w = w'i \text{ for some } w' \text{ with } w'_i > 0\}$. We say that *$i$ leads to j* and write $i \to j$ if for all $w \in V_i$ there exists an $m = m(w) \geqslant 1$ such that $P^m(w,j) > 0$ (see [9]). This definition is slightly different from that suggested by Mihoc and Ciucu [5]. By setting $a_l = 0$ for all $l \in I$ we see that it is consistent with that used for Markov chains. We can also prove that the relation $\to$ is transitive.

We say that i and j *communicate* if $i \to j$ and $j \to i$. This relation divides I into disjoint subsets called *classes*. Denote by $C(i)$ the class containing i ($C(i) = \{i\}$ if i does not lead to i). Further, set

$$Z(i) = \{l: i \to l\} \cup \{i\}.$$

Clearly, $C(i) \subset Z(i)$.

We now want to obtain information on the classification of I by exploring the well-known classification of states of the Markov chain specified by Λ. In order to make it explicit in the notation when we deal with the Markov chain specified by Λ, we shall insert a Λ. We shall write $i \underset{\Lambda}{\to} j$, $C_\Lambda(i)$, $Z_\Lambda(i)$, Λ-essential, Λ-recurrent, and Λ-closed.

LEMMA 1. *Let $i, j \in I$ with $i \underset{\Lambda}{\to} j$. If $a_l < 1$ for all $l \in Z_\Lambda(i)$, then $i \to j$.*

Proof. If $i \underset{\Lambda}{\to} j$, then there exists an $m \geqslant 1$ for which $\Lambda_{ij}^m > 0$. By (4) we conclude that $P^m(w, j) > 0$ for all $w \in V_i$.

LEMMA 2. *Let $i, j \in I$ with $i \to j$. If $j \neq i$, then $i \underset{\Lambda}{\to} j$.*

Proof. For $w = a_i \delta_i + A_i$, where δ_i is the probability measure concentrated at i and $A_i = (1-a_i)\Lambda_i$, there exists an $m \geqslant 1$ such that $P^m(w, j) > 0$. If $m = 1$, we immediately obtain $\Lambda_{ij} > 0$. If $m > 1$, then, by Proposition 2, we have $w_k a_k w_j > 0$ for some $k \in I$ or $w_l \Lambda_{lj}^n > 0$ for some $l \in I$ and some $n \geqslant 1$. In both cases we conclude that $\Lambda_{ij}^r > 0$ for some $r \geqslant 1$.

3.2. Essential classes. We say that i is *essential* if $i \to j$ implies $j \to i$ for every $j \in I$; otherwise, we call i *inessential*. By transitivity, the property of being essential is a class property.

THEOREM 1. *Let $i \in I$ and let $a_l < 1$ for all $l \in Z_\Lambda(i)$. Then i is inessential if and only if i is Λ-inessential.*

Proof. If i is inessential, then there is a $j \neq i$ such that $i \to j$ but not $j \to i$. By Lemma 2 we get $i \underset{\Lambda}{\to} j$. Suppose, on the contrary, that $j \underset{\Lambda}{\to} i$. By transitivity we have $a_l < 1$ for all $l \in Z_\Lambda(j)$ and, therefore, by Lemma 1, we obtain $j \to i$. Hence we do not have $j \underset{\Lambda}{\to} i$.

Let now i be Λ-inessential. Then there is a $j \neq i$ such that $i \underset{\Lambda}{\to} j$ but not $j \underset{\Lambda}{\to} i$. By Lemma 1 we get $i \to j$. Suppose, on the contrary, hat $j \to i$. By Lemma 2 we have $j \underset{\Lambda}{\to} i$. Hence we do not have $j \to i$.

It follows from Lemma 2 that $C(i) \subset C_\Lambda(i)$ and $Z(i) \subset Z_\Lambda(i)$. If $a_l < 1$ for all $l \in Z_\Lambda(i)$, then by Lemma 1 we have $C(i) = C_\Lambda(i)$ and $Z(i) = Z_\Lambda(i)$. If we assume, in addition, that i is essential, then $C(i)$, $C_\Lambda(i)$, $Z(i)$, and $Z_\Lambda(i)$ coincide.

3.3. Period. We begin with

PROPOSITION 3. *Assume that $i, j \in I$, $w^* = a_i \delta_i + A_i$, and $m \geqslant 1$. If $P^m(w^*, j) > 0$, then $P^m(w, j) > 0$ for all $w \in V_i$.*

Proof. This follows from (1), (3), and from the fact that $w_k^* > 0$ implies $w_k > 0$ for all $w \in V_i$.

If $i \to j$, then by Proposition 3 there is an $m \geqslant 1$ such that $P^m(w, j) > 0$ for all $w \in V_i$, in which case we write $i \overset{m}{\to} j$. Moreover, for each $i \in I$ and $m \geqslant 1$ there exists some $j \in I$ with $i \overset{m}{\to} j$. It is also easily seen that $i \overset{m}{\to} j$ and $j \overset{n}{\to} k$ imply $i \overset{m+n}{\longrightarrow} k$.

For each $i \in I$ with $i \to i$ we define the *period* $d(i)$ of i as the greatest common divisor of those $n \geqslant 1$ for which $i \overset{n}{\to} i$. From transitivity it follows that the property of having a period equal to d is a class property and that the class $C(i)$ can be partitioned into $d(i)$ subclasses (for Markov chains, see [2], p. 13-15).

THEOREM 2. (i) *If $a_j > 0$ for some $j \in C(i)$, then $d(i) = 1$.*

(ii) *If $a_l < 1$ for all $l \in Z_A(i)$, then $d(i) \leqslant d_A(i)$.*

(iii) *Let $i \in I$ be essential and let $a_l < 1$ for all $l \in Z_A(i)$. If $a_j = 0$ for all $j \in C(i)$, then $d(i) = d_A(i)$.*

Proof. (i) If $a_i > 0$, then $w_i > 0$ for each $w \in V_i$, hence $d(i) = 1$.

(ii) If $i \underset{A}{\overset{m}{\to}} i$, then by (4) we get $i \overset{m}{\to} i$, hence $d(i) \leqslant d_A(i)$.

(iii) If $i \overset{m}{\to} i$, then $P_m(A_i, i_1 \ldots i_m) > 0$ for some $i_1 \ldots i_m$, $i_m = i$ with $i_l \in C(i)$, $1 \leqslant l \leqslant m$. It follows that $i \underset{A}{\overset{m}{\to}} i$, hence $d(i) \geqslant d_A(i)$.

3.4. Recurrence and transience. For $n \geqslant 1$, $w \in W$, and $j \in I$ let us define the first entrance probabilities:

$$f_{w,j}^n = \begin{cases} \boldsymbol{P}_w(\xi_1 = j) & \text{for } n = 1, \\ \boldsymbol{P}_w(\xi_\nu \neq j, 1 \leqslant \nu < n, \xi_n = j) & \text{for } n > 1. \end{cases}$$

Set

$$f_{w,j} = \sum_{n \geqslant 1} f_{w,j}^n = \boldsymbol{P}_w(\xi_n = j \text{ for some } n \geqslant 1).$$

Further, let $V \subset W$; we say that i is *V-recurrent* (*V-transient*) if $f_{w,i} = 1$ for all $w \in V$ ($f_{w,i} < 1$ for all $w \in V$). If $a_l = 0$ for all $l \in I$, then the notions of V_i-transience and $M(Z(i))$-recurrence of i are identical with the standard notions of transience and recurrence of i occurring in the theory of Markov chains.

Let us examine now the relation between transience and the property of being inessential.

THEOREM 3. *Let $i \in I$ and let $a_l \leqslant a < 1$ for all $l \in I$. If i is A-inessential, then i is V_i-transient.*

Proof. There are $j \neq i$, $m \geqslant 1$, and a path $ij_1 \ldots j_{m-1}j_m$, $j_l \neq i$, $1 \leqslant l \leqslant m$, $j_m = j$, such that

$$A_{ij_1} A_{j_1 j_2} \ldots A_{j_{m-1} j_m} > 0 \quad \text{and} \quad i \notin Z_A(j).$$

Therefore, for $w \in V_i$ we get

$$D_{w,i} = P_m(w, j_1 \dots j_{m-1} j_m) \geqslant A_{ij_1} A_{j_1 j_2} \dots A_{j_{m-1} j_m} > 0.$$

Further, for $N \geqslant 2$ by (1) we can write

$$\boldsymbol{P}_w(\xi_\nu \neq i, 1 \leqslant \nu \leqslant m+N) \geqslant D_{w,i} \boldsymbol{P}_{w^*}(\xi_\nu \neq i, 1 \leqslant \nu \leqslant N)$$

$$\geqslant D_{w,i} \sum_{i_1, \dots, i_N \in Z_A(j)} P(w^*, i_1) P(w^* i_1, i_2) \dots P(w^* i_1 \dots i_{N-1}, i_N),$$

where $w^* = w j_1 \dots j_{m-1} j$. Since $Z_A(j)$ is A-closed, by (3) we have

$$\sum_{i_n \in Z_A(j)} P(w^* i_1 \dots i_{n-1}, i_n) \geqslant (1 - a_{i_1} \dots a_{i_{n-1}}) \geqslant 1 - a^{n-1}$$

for $2 \leqslant n \leqslant N$ and $P(w^*, Z_A(j)) \geqslant 1 - a_j$. Thus

$$1 - f_{w,i} \geqslant D_{w,i}(1 - a_j) \prod_{n \geqslant 1} (1 - a^n) > 0.$$

In order to exhibit a V which can be used when discussing recurrence we give

PROPOSITION 4. *If $i \in I$ and $V = M(Z(i))$, then $T(V) \subset V$.*

Proof. Let $w \in V$ and $Q(w, \{w'\}) > 0$ for some $w' \in W$. It suffices to show that $w' \in V$. Indeed, we have $w' = a_k w + A_k$ with $k \in Z(i)$, so that $w'_j > 0$ implies $j \in Z(i)$.

THEOREM 4. *Let $i \in I$ be A-essential. Suppose that for all $j \in C_A(i)$ there exists an $n = n(j) \geqslant 1$ such that $A^n_{ji} \geqslant c > 0$. Then i is $M(Z(i))$-recurrent.*

Proof. Let $w \in M(Z(i))$ and $w' \in T(w)$. Proposition 4 yields $w' = a_l w'' + A_l$ with $l \in Z(i)$. Then by (4) we have

$$f_{w',i} \geqslant \sup_{m \geqslant 1} P^m(w', i) \geqslant \sup_{m \geqslant 1} \sum_{k \in I} w'_k A^{m-1}_{ki} \geqslant \sup_{m \geqslant 1} A^m_{li} \geqslant c > 0,$$

since $Z(i) = C_A(i)$.

Take now $n_0 = n_0(w') \geqslant 1$ such that

$$\sum_{n=1}^{n_0} f^n_{w',i} \geqslant c/2.$$

Then by (1) we get

$$1 - f_{w',i} = \boldsymbol{P}_{w'}(\xi_\nu \neq i, \nu \geqslant 1)$$

$$\leqslant \boldsymbol{P}_{w'}(\xi_\nu \neq i, 1 \leqslant \nu \leqslant n_0) \sup \{\boldsymbol{P}_{w''}(\xi_\nu \neq i, \nu \geqslant 1) : w'' \in T(w')\}$$

$$\leqslant (1 - c/2) \sup \{1 - f_{w'',i} : w'' \in T(w)\}.$$

Hence $f_{w',i} = 1$ for all $w' \in T(w)$ and, therefore, $f_{w,i} = 1$.

Let us apply our previous results when I is finite. We have

THEOREM 5. *Let I be finite and $a_l < 1$ for all $l \in I$.*

(i) *i is A-recurrent if and only if i is $M(Z(i))$-recurrent.*

(ii) *i is A-transient if and only if i is V_i-transient.*

Proof. (i) If i is A-recurrent, then i is A-essential. Hence, by Theorem 4, i is $M(Z(i))$-recurrent. Conversely, if i is $M(Z(i))$-recurrent, then i is A-essential by Theorem 3, since $a_i\delta_i + A_i \in V_i \cap M(Z(i))$. Thus i is A-recurrent.

(ii) If i is A-transient, then i is A-inessential. Hence, by Theorem 3, i is V_i-transient. Conversely, if i is V_i-transient, then i is also $\{w^*\}$-transient with $w^* = a_i\delta_i + A_i$. Since $w^* \in M(Z(i))$, it follows from Theorem 4 that i is A-inessential. Hence i is A-transient.

4. ASYMPTOTIC BEHAVIOUR

4.1. Ergodicity and regularity. We say (cf. [7], p. 37 and 101) that an LOM is *uniformly ergodic* if there exist a sequence $\varepsilon_n \downarrow 0$ and, for each $l \geq 1$, a probability measure P_l^∞ on I^l such that

$$|\bar{P}_l^n(w, A^l) - P_l^\infty(A^l)| \leq \varepsilon_n \tag{5}$$

for all $w \in W$, $n, l \geq 1$, and $A^l \subset I^l$, where

$$\bar{P}_l^n = (1/n) \sum_{j=1}^{n} P_l^j.$$

If $\bar{P}_l^n$ is replaced in (5) by P_l^n, we say that the LOM is *uniformly regular*.

Consistent with the terminology above, we say that a finite stochastic matrix A is *ergodic* if it has a unique essential class. If this essential class is aperiodic, we say that A is *regular*. Note that A may have transient states.

4.2. Limit theorem. We show that the asymptotic behaviour of the n-step transition probabilities P_l^n is determined by that of A^n.

THEOREM 6. *Let I be finite and suppose that $a_l < 1$ for all $l \in I$. Then the LOM is*

(i) *uniformly ergodic if A is ergodic;*

(ii) *uniformly regular if A is regular.*

Proof. We start by remarking that a finite LOM is a distance diminishing model (cf. [7], p. 31) with compact W.

(i) Let Λ be ergodic. It can be shown by arguments similar to those used by Norman in his Theorem 6.1 ([7], p. 61) that it suffices to check

$$d\big(T(w), T(w')\big) = 0 \quad \text{for all } w, w' \in W, \tag{6}$$

where d is the Euclidean distance.

Let $\alpha = \max\{\alpha_l : l \in I\}$ and take $j^l \in I^l$. Then by (3) we first get $d(wj^l, w'j^l) \leqslant \sqrt{2}\alpha^l$ for all $w, w' \in W$. Secondly, if $P_l^{m+1}(w, j^l) > 0$ for $m, l \geqslant 1$, then there exists a $j^m \in I^m$ with

$$P_{m+l}(w, j^m j^l) = P_m(w, j^m) P_l(w^*, j^l) > 0,$$

where $w^* = wj^m$ and $j^m j^l \in I^{m+l}$, so that $w^* j^l \in T^{m+l}(w)$. Combining these two facts, we conclude that $P_l^m(w, j^l) P_l^{m'}(w', j^l) > 0$ implies

$$d\big(T^{m-1+l}(w), T^{m'-1+l}(w')\big) \leqslant \sqrt{2}\alpha^l.$$

Thus (6) is fulfilled if for all $w, w' \in W$ and $l \geqslant 1$ there exist $j^l \in I^l$ and $m, m' \geqslant 1$ such that

$$P_l^m(w, j^l) P_l^{m'}(w', j^l) > 0. \tag{7}$$

Since $P(w'i, j) \geqslant A_{ij}$, by (1) we have

$$P_l^m(w, j^l) \geqslant P^m(w, j_1) A_{j_1 j_2} \ldots A_{j_{l-1} j_l}.$$

Consequently, (7) and hence (6) are fulfilled if for all $w, w' \in W$ there exist $k \in I$ and $m, m' \geqslant 1$ such that

$$P^m(w, k) P^{m'}(w', k) > 0. \tag{8}$$

Λ being ergodic, there exists a $k \in I$ and for each $i \in I$ there exists an $m(i) \geqslant 1$ such that $A_{ik}^{m(i)} > 0$. For an arbitrary $w \in W$ we choose $i \in I$ with $w_i > 0$ and, by (4), we obtain $P^{m(i)+1}(w, k) > 0$. Thus (8) is established, and so is (6).

(ii) By Theorem 6.1 of Norman [7], p. 61 (see also Theorem 2.4 of Norman [6], p. 68) we have to check

$$\lim_{n\to\infty} d\big(T^n(w), T^n(w')\big) = 0 \quad \text{for all } w, w' \in W. \tag{9}$$

Since Λ is regular, we can choose, once again using (4), $m \geqslant 1$ and $k \in I$ such that $P^{m+1}(w, k) > 0$ for all $w \in W$. Thus (8) is fulfilled with $m = m'$, and so is (7), where we can take $m = m'$ independently of l. Hence (9) holds true.

We have the following converse of Theorem 6 (i):

THEOREM 7. *Let I be finite. If $\lim_{n\to\infty} \overline{P}^n(w, i)$ exists for all $w \in W$ and $i \in I$ and does not depend on w, then Λ is ergodic.*

Proof. It suffices to show that there exists a $k \in I$ such that for all $i \in I$ we have $\Lambda_{ik}^{m} > 0$ for some $m = m(i) \geqslant 1$. We can choose $k \in I$ with

$$\lim_{n\to\infty} \overline{P}^{n}(w, k) > 0 \quad \text{for all } w \in W.$$

For each $i \in I$ there exists an $n = n(i) \geqslant 2$ with $P^{n}(\delta_i, k) > 0$. By Proposition 2 we conclude that for $i \neq k$ there exists an $m = m(i) \geqslant 1$ with $\Lambda_{ik}^{m} > 0$.

It is easily seen that the converse of Theorem 6 (ii) is not true. Take, for instance, $I = \{1, 2\}$, $\Lambda_{12} = \Lambda_{21} = 1$, and $a_1 = a_2 = \frac{1}{2}$. The matrix Λ is ergodic but not regular, and, by (2), $P^{n}(w, i) = \frac{1}{2}$ for all $w \in W$, $i \in I$, and $n \geqslant 2$. Hence (8) is fulfilled with $m = m'$ and, therefore, we get uniform regularity.

Finally, if I is finite and $a_l < 1$ for all $l \in I$, then Theorems 1, 6 (i), and 7 show that a linear OM-chain is uniformly ergodic if and only if it has a unique essential class.

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