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UNIFORM STABILITY OF DYNAMIC SICA HIV TRANSMISSION MODELS ON TIME SCALES

Abstract. We consider a SICA model for HIV transmission on time scales. We prove permanence of solutions and we derive sufficient conditions for the existence and uniform asymptotic stability of a unique positive almost periodic solution of the system in terms of a Lyapunov function.

1. Introduction. In 2015, the deterministic SICA model was first presented as a subsystem of a TB-HIV/AIDS co-infection model [22]. One of the primary objectives of SICA models is to demonstrate how some of the fundamental relationships between epidemiological variables and the general pattern of the AIDS epidemic can be clarified using a straightforward mathematical model [18]. The celebrated SICA mathematical model [5, 6, 16, 25, 27] divides the total human population into four compartments, namely

- $S(t)$: susceptible individuals at time t;
- $I(t)$: HIV-infected individuals with no clinical symptoms of AIDS but able to transmit HIV to other individuals at time t;
- $C(t)$: HIV-infected individuals under antiretroviral therapy (ART), the so called chronic stage with a viral load remaining low at time t;
- $A(t)$: HIV-infected individuals with AIDS clinical symptoms at time t.

Under some realistic assumptions, the dynamics of the disease proliferation in a community is then translated into a mathematical model given by the

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following system of four ordinary differential equations [22–24]:

$$(1)\qquad \begin{cases} \dot{S}(t) = \Lambda - \beta\lambda(t)S(t) - \nu S(t), \\ \dot{I}(t) = \beta\lambda(t)S(t) - (\rho + \phi + \nu)I(t) + \gamma A(t) + \omega C(t), \\ \dot{C}(t) = \phi I(t) - (\omega + \nu)C(t), \\ \dot{A}(t) = \rho I(t) - (\gamma + \nu + d)A(t), \end{cases}$$

where Λ, β, ν, ρ, ϕ, γ, ω and d are real positive rates:

- Λ is the rate of new susceptibles;
- β is the HIV transmission rate;
- ν is the natural death rate;
- ρ is the default treatment rate for I individuals;
- ϕ is the HIV treatment rate for I individuals;
- γ is the AIDS treatment rate;
- ω is the default treatment rate for C individuals;
- d is the AIDS induced death rate;

and where the effective contact rate with people infected with HIV is given by

$$\lambda(t) = \frac{\beta}{N(t)}(I(t) + \eta_C C(t) + \eta_A A(t))$$

with

- $N(t)$ the total population at time t, that is,
$$N(t) = S(t) + I(t) + C(t) + A(t);$$
- $0 \leq \eta_C \leq 1$ the modification parameter;
- $\eta_A \geq 1$ the partial restoration parameter of immune function of individuals with HIV infection that use correctly the ART treatment.

The study of dynamical systems on time scales is now a very active area of research. The books of Bohner and Peterson [2, 3] offer a good introduction with applications to the time scale calculus along with some advanced topics. Applications of the time scale calculus can be found in many areas, including economics [1, 7, 26], ecology [19, 20, 30] and epidemics [4, 11, 21]. Here we generalize the SICA model (1) by considering dynamic equations on time scales and study it using the time-scale theory. By doing so, we unify the continuous and discrete-time models [27], generalizing them also to other contexts like the quantum [12, 17] or mixed/hybrid settings [8, 28].

Recently, Prasad and Khuddush [20] proved the existence and uniform asymptotic stability of positive and almost periodic solutions for a 3-species Lotka–Volterra competitive system on time scales. Moreover, they also studied the permanence and positive almost periodic solutions of an n-species Lotka–Volterra system on time scales [19]. Motivated by these works, here we investigate the permanence and uniform asymptotic stability of the unique positive almost periodic solution of the following SICA model on time scales:

$$(2)\qquad \begin{cases} x_1^{\Delta}(t) = \Lambda - \beta\lambda(t)x_1^{\sigma}(t) - \nu x_1^{\sigma}(t), \\ x_2^{\Delta}(t) = \beta\lambda(t)x_1(t) - (\rho + \phi + \nu)x_2^{\sigma}(t) + \gamma x_4(t) + \omega x_3(t), \\ x_3^{\Delta}(t) = \phi x_2(t) - (\omega + \nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) = \rho x_2(t) - (\gamma + \nu + d)x_4^{\sigma}(t), \end{cases}$$

where $t \in \mathbb{T}^+$, with $\mathbb{T}^+$ a nonempty closed subset of $\mathbb{R}^+ =]0, +\infty[$.

Note that, in our notation, $(x_1(t), x_2(t), x_3(t), x_4(t))$ is interpreted as $(S(t), I(t), C(t), A(t))$.

The paper is organized as follows. Section 2 gives a brief introduction to time-scale theory and we also recall there the definition of an almost periodic function with respect to time, uniformly for the state variables, and the lemma of Lyapunov that gives a sufficient condition for the existence of a unique almost periodic solution that is uniformly asymptotically stable. In Section 3, we derive sufficient conditions for system (2) to be permanent. Finally, in Section 4, we establish sufficient conditions for the existence and uniform asymptotic stability of the unique positive almost periodic solution to the SICA system (2). Our results seem to be new even for standard time scales. We end with an example (Example 4.2) and a brief conclusion (Section 5).

2. Some preliminaries. For more background on the theory of time scales we recommend the books [2, 3]. A *time scale* $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. The time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. Let f be a function defined on $\mathbb{T}^+$. We set

$$f^L = \inf\{f(t) : t \in \mathbb{T}^+\}, \qquad f^U = \sup\{f(t) : t \in \mathbb{T}^+\}.$$

Definition 2.1 (see [2]). The *jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}, \quad \mu(t) = \sigma(t) - t$$

supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is *left-dense*, *left-scattered*, *right-dense*, or *right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then we define $\mathbb{T}_{\kappa} = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}_{\kappa} = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$, then $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is given by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. We define $f^{\Delta}(t)$ to be the number, provided it exists, with the property that given any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \le \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^{\Delta}(t)$ the *delta derivative* of f at t. Moreover, we say that f is delta (or *Hilger*) *differentiable* on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

Definition 2.3 (see [2]). A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ (a function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$) will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define

$$\mathcal{R}^{+} = \mathcal{R}^{+}(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Let now p, $q \in \mathcal{R}$. We define the circle plus addition $\oplus$ on $\mathcal{R}$ by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{for all } t \in \mathbb{T}$$

and the circle minus subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) := \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \text{for all } t \in \mathbb{T}.$$

The time scales exponential function $e_p(\cdot, t_0)$ is defined for $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$ as the unique solution of the initial value problem

$$y^{\Delta} = p(t)y \quad \text{and} \quad y(t_0) = 1 \quad \text{on } \mathbb{T}.$$

One has

$$e_p(\cdot, t_0)e_q(\cdot, t_0) = e_{p \oplus q}(\cdot, t_0), \qquad \frac{e_p(\cdot, t_0)}{e_q(\cdot, t_0)} = e_{p \ominus q}(\cdot, t_0)$$

and

$$e_{\ominus q}(\cdot, t_0) = \frac{1}{e_q(\cdot, t_0)}.$$

Lemma 2.4 (see [10]). *Assume that $a, b > 0$ and $-a \in \mathcal{R}^{+}$. Then*

$$y^{\Delta}(t) \ge (\le)\ b - a y^{\sigma}(t), \qquad y(t) > 0, \qquad t \in [t_0, \infty)_{\mathbb{T}},$$

implies that

$$y(t) \ge (\le)\ \frac{b}{a}\left[1 + \left(\frac{a y(t_0)}{b} - 1\right) e_{(\ominus a)}(t, t_0)\right], \qquad t \in [t_0, \infty)_{\mathbb{T}}.$$

Definition 2.5 (see [14]). A time scale $\mathbb{T}$ is said to be *almost periodic* if

$$\Pi = \{\tau \in \mathbb{R} : t + \tau \in \mathbb{T} \text{ for all } t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.6 (see [14]). Let $\mathbb{T}$ be an almost periodic time scale. A function $x \in C(\mathbb{T}, \mathbb{R}^n)$ is said to be *almost periodic* if the *ε-translation set* of x,

$$E\{\varepsilon, x\} = \{\tau \in \Pi : |x(t + \tau) - x(t)| < \varepsilon \text{ for all } t \in \mathbb{T}\},$$

is a relatively dense set in $\mathbb{T}$, that is, for all $\varepsilon > 0$ there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an $\eta(\varepsilon) \in E\{\varepsilon, x\}$ such that $|x(t+\tau) - x(t)| < \varepsilon$ for all $t \in \mathbb{T}$. Moreover, τ is called the *ε-translation number* of $x(t)$ and $l(\varepsilon)$ is called the *inclusion length* of $E\{\varepsilon, x\}$.

Definition 2.7 (see [14]). Let $\mathbb{D}$ be an open set in $\mathbb{R}^n$ and let $\mathbb{T}$ be a positive almost periodic time scale. A function $f \in C(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is said to be *almost periodic in $t \in \mathbb{T}$ uniformly for $x \in \mathbb{D}$*, if the *ε-translation set of f*,

$$E\{\varepsilon, f, \mathbb{S}\} = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } (t,x) \in \mathbb{T} \times \mathbb{S}\},$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset $\mathbb{S}$ of $\mathbb{D}$ there exists a constant $l(\varepsilon, \mathbb{S}) > 0$ such that each interval of length $l(\varepsilon, \mathbb{S})$ contains $\tau(\varepsilon, \mathbb{S}) \in E\{\varepsilon, f, \mathbb{S}\}$ such that

$$|f(t+\tau, x) - f(t,x)| < \varepsilon \quad \text{for all } (t,x) \in \mathbb{T} \times \mathbb{S}.$$

Consider the system

$$x^{\Delta}(t) = h(t,x) \tag{3}$$

and the associate product system

$$x^{\Delta}(t) = h(t,x), \qquad z^{\Delta}(t) = h(t,z),$$

where $h : \mathbb{T}^+ \times \mathbb{S}_B \to \mathbb{R}^n$, $\mathbb{S}_B = \{x \in \mathbb{R}^n : \|x\| < B\}$ and $h(t,x)$ is almost periodic in t uniformly for $x \in \mathbb{S}_B$ and continuous in x.

Lemma 2.8 (see [29]). *Suppose that there exists a Lyapunov function $V(t,x,z)$ defined on $\mathbb{T}^+ \times \mathbb{S}_B \times \mathbb{S}_B$, that is, there exists a function $V(t,x,z)$ satisfying the following conditions:*

(1) $a(\|x-z\|) \le V(t,x,z) \le b(\|x-z\|)$, *where $a, b \in \mathbb{K}$ with*

$$\mathbb{K} = \{\alpha \in C(\mathbb{R}^+, \mathbb{R}^+) : \alpha(0) = 0, \text{ and } \alpha \text{ increasing}\};$$

(2) $|V(t,x,z) - V(t,x_1,z_1)| \le L(\|x-x_1\| + \|z-z_1\|)$, *where $L > 0$ is a constant;*

(3) $D^+V^{\Delta}(t,x,z) \le -cV(t,x,z)$, *where $c > 0$ and $-c \in \mathcal{R}^+$.*

Furthermore, if there exists a solution $x(t) \in \mathbb{S}$ of system (3) *for $t \in \mathbb{T}^+$, where $\mathbb{S} \cup \mathbb{S}_B$ is a compact set, then there exist a unique almost periodic solution $f(t) \in \mathbb{S}$ of system* (3), *which is uniformly asymptotically stable.*

Definition 2.9 (see [19]). System (2) is said to be *permanent* if there exist positive constants m and M such that

$$m \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le M, \quad i = 1,2,3,4,$$

for any solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of (2).

3. Permanence of positive solutions. The principal objective of this section is to establish sufficient conditions for system (2) to be permanent. Let $t_0 \in \mathbb{T}$ be a fixed positive initial time. We introduce the following assumption for (2):

(H_1) $\lambda(t)$ is a bounded and almost periodic function satisfying

$$0 < \lambda^L \le \lambda(t) \le \lambda^U.$$

LEMMA 3.1. *Suppose hypothesis* (H_1) *holds. Then, for any positive solution* $(x_1(t), x_2(t), x_3(t), x_4(t))$ *of system* (2), *there exist positive constants* M *and* T *such that* $x_i(t) \le M$, $i = 1,2,3,4$, *for* $t \ge T$.

Proof. Let $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be any positive solution of system (2). From the ith equation of (2), we have

$$\begin{cases} x_1^{\Delta}(t) \le \Lambda - (\beta\lambda^L + \nu)x_1^{\sigma}(t), \\ x_2^{\Delta}(t) \le \beta\lambda^U M_1 + (\gamma+\omega)\dfrac{\Lambda}{\nu} - (\rho+\phi+\nu)x_2^{\sigma}(t), \\ x_3^{\Delta}(t) \le \phi M_2 - (\omega+\nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) \le \rho M_2 - (\gamma+\nu+d)x_4^{\sigma}(t). \end{cases} \tag{4}$$

Hence, by Lemma 2.4, there exist positive constants M_i and T_i such that for any positive solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of (2), we have

$$x_1(t) \le \frac{\Lambda}{\beta\lambda^L + \nu}\left[1 + \left(\frac{(\beta\lambda^L+\nu)x_1(t_0)}{\Lambda} - 1\right)e_{\ominus(\beta\lambda^L+\nu)}(t,t_0)\right].$$

If $-(\beta\lambda^L + \nu) < 0$, then $e_{\ominus(\beta\lambda^L+\nu)}(t,t_0) \to 0$ as $t \to \infty$ and

$$\begin{cases} x_1(t) \le M_1 := \Lambda/(\beta\lambda^L+\nu) & \text{for } t \ge T_1, \\ x_2(t) \le M_2 := \beta\left(\lambda^U M_1 + (\gamma+\omega)\dfrac{\Lambda}{\nu}\right)/(\rho+\phi+\nu) & \text{for } t \ge T_2, \\ x_3(t) \le M_3 := \phi M_2/(\omega+\nu) & \text{for } t \ge T_3, \\ x_4(t) \le M_4 := \rho M_2/(\gamma+\nu+d) & \text{for } t \ge T_4. \end{cases} \tag{5}$$

Let $M = \max_{1\le i\le 4} M_i$ and $T = \max_{1\le i\le 4} T_i$. Then $x_i(t) \le M$, $i = 1,2,3,4$, for all $t \ge T$. ∎

LEMMA 3.2. *Suppose that* (H_1) *holds. Then system* (2) *is permanent.*

Proof. Let $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be any positive solution of system (2). From the ith equation of (2), we have

$$\begin{cases} x_1^{\Delta}(t) \ge \Lambda - (\beta\lambda^U + \nu)x_1^{\sigma}(t), \\ x_2^{\Delta}(t) \ge \beta\lambda^L m_1 - (\rho+\phi+\nu)x_2^{\sigma}(t), \\ x_3^{\Delta}(t) \ge \phi m_2 - (\omega+\nu)x_3^{\sigma}(t), \\ x_4^{\Delta}(t) \ge \rho m_2 - (\gamma+\nu+d)x_4^{\sigma}(t). \end{cases} \tag{6}$$

From hypothesis (H_1) and Lemma 2.4, there exist positive constants $m_i > 0$ such that for any positive solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of (2) there exists $\hat{T}_i$ such that

$$(7)\qquad \begin{cases} x_1(t) \geq m_1 := \Lambda/(\beta\lambda^U + \nu) & \text{for } t \geq \hat{T}_1, \\ x_2(t) \geq m_2 := (\beta\lambda^L m_1)/(\rho + \phi + \nu) & \text{for } t \geq \hat{T}_2, \\ x_3(t) \geq m_3 := \phi m_2/(\omega + \nu) & \text{for } t \geq \hat{T}_3, \\ x_4(t) \geq m_4 := \rho m_2/(\gamma + \nu + d) & \text{for } t \geq \hat{T}_4. \end{cases}$$

Let $m = \min_{1 \leq i \leq 4} m_i$ and $\hat{T} = \max_{1 \leq i \leq 4} \hat{T}_i$. We conclude that $x_i(t) \geq m$, $i = 1, 2, 3, 4$, for all $t \geq \hat{T}$. ■

4. Uniform asymptotic stability. In this section, we prove sufficient conditions for the existence and uniform asymptotic stability of the unique positive almost periodic solution to system (2). Let us define

$$\begin{aligned} \Omega := \{ & Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \in (\mathbb{R}^+)^4 : (x_1(t), x_2(t), x_3(t), x_4(t)) \\ & \text{is a solution of (2) with } 0 < m \leq x_i \leq M,\ i = 1, \ldots, 4, \text{ and } N(t) \leq \Lambda/\nu \}. \end{aligned}$$

It is clear that Ω is an invariant set of system (2) and, by Lemma 3.2, we have $\Omega \neq \emptyset$. We introduce some more notation. Let

$$\begin{aligned} a_1 &:= \beta\lambda^L + \nu, & b_1 &:= \beta\lambda^U + \frac{\beta^2 M}{2m}, \\ a_2 &:= \rho + \phi + \nu, & b_2 &:= \rho + \phi, \\ a_3 &:= \omega + \nu, & b_3 &:= \omega + \frac{\beta^2 M}{2m}\eta_C, \\ a_4 &:= \gamma + \nu + d, & b_4 &:= \gamma + \frac{\beta^2 M}{2m}\eta_A. \end{aligned}$$

Moreover, let

$$\Gamma_1 := \min_{1 \leq i \leq 4} a_i \quad \text{and} \quad \Gamma_2 := \max_{1 \leq i \leq 4} b_i.$$

In our next result (Theorem 4.1) we assume the following additional hypothesis:

(H_2) $\Gamma_2 < \Gamma_1$ with $\Gamma_1, \Gamma_2 \in \mathcal{R}^+$.

THEOREM 4.1. *Suppose that* (H_1) *and* (H_2) *hold. Then the dynamic system* (2) *has a unique almost periodic solution* $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \in \Omega$ *that is uniformly asymptotically stable.*

Proof. According to Lemma 2.4, every solution

$$Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$$

of (2) satisfies $x_i^L \leq x_i(t) \leq x_i^U$, $i = 1, \ldots, 4$, and $|x_i| \leq K_i$, $i = 1, \ldots, 4$.

Denote

$$\begin{aligned}\|Z\| &= \|(x_1(t), x_2(t), x_3(t), x_4(t))\| \\ &= \sup_{t\in\mathbb{T}^+} (|x_1(t)| + |x_2(t)| + |x_3(t)| + |x_4(t)|).\end{aligned}$$

Let $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ and $\hat{Z}(t) = (\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t), \hat{x}_4(t))$ be two positive solutions of (2). Then $\|Z\| \leq K$ and $\|\hat{Z}\| \leq K$, where

$$K = \sum_{i=1}^{4} K_i.$$

In view of (2), we have

$$\begin{cases} x_1^\Delta(t) = \Lambda - \beta\lambda(t)x_1^\sigma(t) - \nu x_1^\sigma(t), \\ x_2^\Delta(t) = \beta\lambda(t)x_1(t) - (\rho+\phi+\nu)x_2^\sigma(t) + \gamma x_4(t) + \omega x_3(t), \\ x_3^\Delta(t) = \phi x_2(t) - (\omega+\nu)x_3^\sigma(t), \\ x_4^\Delta(t) = \rho x_2(t) - (\gamma+\nu+d)x_4^\sigma(t), \end{cases}$$

and

$$\begin{cases} \hat{x}_1^\Delta(t) = \Lambda - \beta\hat{\lambda}(t)\hat{x}_1^\sigma(t) - \nu \hat{x}_1^\sigma(t), \\ \hat{x}_2^\Delta(t) = \beta\hat{\lambda}(t)\hat{x}_1(t) - (\rho+\phi+\nu)\hat{x}_2^\sigma(t) + \gamma \hat{x}_4(t) + \omega \hat{x}_3(t), \\ \hat{x}_3^\Delta(t) = \phi \hat{x}_2(t) - (\omega+\nu)\hat{x}_3^\sigma(t), \\ \hat{x}_4^\Delta(t) = \rho \hat{x}_2(t) - (\gamma+\nu+d)\hat{x}_4^\sigma(t). \end{cases}$$

Define the Lyapunov function $V(t, Z, \hat{Z})$ on $\mathbb{T}^+ \times \Omega \times \Omega$ as

$$V(t, Z, \hat{Z}) = \sum_{i=1}^{4} |x_i(t) - \hat{x}_i(t)| = \sum_{i=1}^{4} V_i(t),$$

where $V_i(t) = |x_i(t) - \hat{x}_i(t)|$. The two norms

$$\|Z(t) - \hat{Z}(t)\| = \sup_{t\in\mathbb{T}^+} \sum_{i=1}^{4} |x_i(t) - \hat{x}_i(t)|$$

and

$$\|Z(t) - \hat{Z}(t)\|_* = \sup_{t\in\mathbb{T}^+} \Big(\sum_{i=1}^{4} (x_i(t) - \hat{x}_i(t))^2\Big)^{1/2}$$

are equivalent, that is, there exist constants $\eta_1, \eta_2 > 0$ such that

$$\eta_1\|Z(t) - \hat{Z}(t)\|_* \leq \|Z(t) - \hat{Z}(t)\| \leq \eta_2\|Z(t) - \hat{Z}(t)\|_*.$$

Hence,

$$\eta_1\|Z(t) - \hat{Z}(t)\|_* \leq V(t, Z, \hat{Z}) \leq \eta_2\|Z(t) - \hat{Z}(t)\|_*.$$

Let $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = \eta_1 x$ and $b(x) = \eta_2 x$. Then assumption (i) of Lemma 2.8 is satisfied. Moreover,

$$\begin{aligned}
|V(t, Z, \hat{Z}) - V(t, Z^*, \hat{Z}^*)| &= \Big|\sum_{i=1}^{4} |x_i(t) - \hat{x}_i(t)| - \sum_{i=1}^{4} |x_i^*(t) - \hat{x}_i^*(t)|\Big| \\
&\leq \sum_{i=1}^{4} |(x_i(t) - \hat{x}_i(t)) - (x_i^*(t) - \hat{x}_i^*(t))| \\
&\leq \sum_{i=1}^{4} |(x_i(t) - x_i^*(t)) + (\hat{x}_i^*(t) - \hat{x}_i(t))| \\
&\leq \sum_{i=1}^{4} |(x_i(t) - x_i^*(t))| + \sum_{i=1}^{4} |(\hat{x}_i^*(t) - \hat{x}_i(t))| \\
&\leq L\big(\|Z(t) - Z^*(t)\| + \|\hat{Z}(t) - \hat{Z}^*(t)\|\big),
\end{aligned}$$

where $L = 1$, so that condition (ii) of Lemma 2.8 is also satisfied. Now, let $v_i(t) = x_i(t) - \hat{x}_i(t)$, $i = 1, \dots, 4$. We compute and estimate the Dini derivative D^+V^Δ of V along the associated product system (2). Using [15, Lemma 4.1], it follows that $D^+V_1^\Delta(t) \leq \operatorname{sign}(v_1^\sigma(t))(v_1(t))^\Delta$. For more details on $D^+V_1^\Delta(t)$ see [9, 13]. Now, let us begin computing:

$$\begin{aligned}
D^+V_1^\Delta(t) &\leq \operatorname{sign}(v_1^\sigma(t))(v_1(t))^\Delta \\
&= \operatorname{sign}(v_1^\sigma(t))[-\beta\lambda(t)x_1^\sigma(t) - \nu x_1^\sigma(t) + \beta\hat{\lambda}(t)\hat{x}_1^\sigma(t) + \nu\hat{x}_1^\sigma(t)] \\
&= \operatorname{sign}(v_1^\sigma(t))[-\beta\lambda(t)(x_1^\sigma(t) - \hat{x}_1^\sigma(t)) - \nu(x_1^\sigma(t) - \hat{x}_1^\sigma(t))] \\
&\quad - \beta\hat{x}_1^\sigma(t)(\lambda(t) - \hat{\lambda}(t))] \\
&= \operatorname{sign}(v_1^\sigma(t))[-(\beta\lambda(t) + \nu)(x_1^\sigma(t) - \hat{x}_1^\sigma(t)) \\
&\quad - \beta\hat{x}_1^\sigma(t)(\lambda(t) - \hat{\lambda}(t))] \\
&\leq -(\beta\lambda(t) + \nu)|x_1^\sigma(t) - \hat{x}_1^\sigma(t))| + \beta\hat{x}_1^\sigma(t)|\lambda(t) - \hat{\lambda}(t)| \\
&\leq -(\beta\lambda^L(t) + \nu)|x_1^\sigma(t) - \hat{x}_1^\sigma(t))| + \beta\hat{x}_1^\sigma(t)|\lambda(t) - \hat{\lambda}(t)| \\
&\leq -(\beta\lambda^L(t) + \nu)|x_1^\sigma(t) - \hat{x}_1^\sigma(t))| + \beta M|\lambda(t) - \hat{\lambda}(t)|.
\end{aligned}$$

Let $v_2(t) = x_2(t) - \hat{x}_2(t)$. Similarly, we have

$$\begin{aligned}
&D^+V_2^\Delta(t) \leq \operatorname{sign}(v_2^\sigma(t))(v_2(t))^\Delta \\
&\quad= \operatorname{sign}(v_2^\sigma(t))[\beta\lambda(t)x_1(t) - (\rho + \phi + \nu)x_2^\sigma(t) + \gamma x_4(t) \\
&\qquad + \omega x_3(t) - \beta\hat{\lambda}(t)\hat{x}_1(t) + (\rho + \phi + \nu)\hat{x}_2^\sigma(t) - \gamma\hat{x}_4(t) - \omega\hat{x}_3(t)] \\
&\quad= \operatorname{sign}(v_2^\sigma(t))[(\beta\lambda(t)x_1(t) - \beta\hat{\lambda}(t)\hat{x}_1(t)) - (\rho + \phi + \nu)(x_2^\sigma(t) - \hat{x}_2^\sigma(t)) \\
&\qquad + \gamma(x_4(t) - \hat{x}_4(t)) + \omega(x_3(t) - \hat{x}_3(t))]
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{sign}(v_2^\sigma(t))[\beta(\lambda(t)(x_1(t)-\hat{x}_1(t))+\hat{x}_1(t)(\lambda(t)-\hat{\lambda}(t))) \\
&\quad -(\rho+\phi+\nu)(x_2^\sigma(t)-\hat{x}_2^\sigma(t))+\gamma(x_4(t)-\hat{x}_4(t))+\omega(x_3(t)-\hat{x}_3(t))] \\
&= \operatorname{sign}(v_2^\sigma(t))[\beta\lambda(t)(x_1(t)-\hat{x}_1(t))+\beta\hat{x}_1(t)(\lambda(t)-\hat{\lambda}(t)) \\
&\quad -(\rho+\phi+\nu)(x_2^\sigma(t)-\hat{x}_2^\sigma(t))+\gamma(x_4(t)-\hat{x}_4(t))+\omega(x_3(t)-\hat{x}_3(t))] \\
&\le \beta\lambda^U|x_1(t)-\hat{x}_1(t)|+\beta M|\lambda(t)-\hat{\lambda}(t)| \\
&\quad -(\rho+\phi+\nu)|x_2^\sigma(t)-\hat{x}_2^\sigma(t)|+\gamma|x_4(t)-\hat{x}_4(t)| \\
&\quad +\omega|x_3(t)-\hat{x}_3(t)|;
\end{aligned}
$$

for $v_3(t)=x_3(t)-\hat{x}_3(t)$ one has

$$
\begin{aligned}
D^+V_3^\Delta(t) &\le \operatorname{sign}(v_3^\sigma(t))(v_3(t))^\Delta \\
&= \operatorname{sign}(v_3^\sigma(t))[\phi x_2(t)-(\omega+\nu)x_3^\sigma(t)-\phi\hat{x}_2(t)+(\omega+\nu)\hat{x}_3^\sigma(t)] \\
&\le \phi|x_2(t)-\hat{x}_2(t)|-(\omega+\nu)|x_3^\sigma(t)-\hat{x}_3^\sigma(t)|;
\end{aligned}
$$

and for $v_4(t)=x_4(t)-\hat{x}_4(t)$ we have

$$D^+V_4^\Delta(t)\le \rho|x_2(t)-\hat{x}_2(t)|-(\gamma+\nu+d)|x_4^\sigma(t)-\hat{x}_4^\sigma(t)|.$$

Since

$$\lambda(t)=\frac{\beta}{N(t)}(x_1(t)+\eta_C x_3(t)+\eta_A)x_4(t),$$

it follows that

$$
\begin{aligned}
D^+V^\Delta(t) &\le -(\beta\lambda^L(t)+\nu)|x_1^\sigma(t)-\hat{x}_1^\sigma(t))|+(\beta M)|\lambda(t)-\hat{\lambda}(t)| \\
&\quad +\beta\lambda^U|x_1(t)-\hat{x}_1(t)|+\beta M|\lambda(t)-\hat{\lambda}(t)| \\
&\quad -(\rho+\phi+\nu)|x_2^\sigma(t)-\hat{x}_2^\sigma(t)|+\gamma|x_4(t)-\hat{x}_4(t)| \\
&\quad +\omega|x_3(t)-\hat{x}_3(t)|+\phi|x_2(t)-\hat{x}_2(t)| \\
&\quad -(\omega+\nu)|x_3^\sigma(t)-\hat{x}_3^\sigma(t)| \\
&\quad +\rho|x_2(t)-\hat{x}_2(t)|-(\gamma+\nu+d)|x_4^\sigma(t)-\hat{x}_4^\sigma(t)|.
\end{aligned} \tag{8}
$$

Therefore,

$$
\begin{aligned}
|\lambda(t)-\hat{\lambda}(t)| \le \frac{\beta}{4m}[&|x_1(t)-\hat{x}_1(t)|+\eta_C|x_3(t)-\hat{x}_3(t)| \\
&+\eta_A|x_4(t)-\hat{x}_4(t)|],
\end{aligned}
$$

where $0\le\eta_C\le 1$, $\eta_A\ge 1$, and $\beta>0$. Inequality (8) becomes

$$
\begin{aligned}
D^+V^\Delta(t) &\le -(\beta\lambda^L+\nu)|x_1^\sigma(t)-\hat{x}_1^\sigma(t)|-(\rho+\phi+\nu)|x_2^\sigma(t)-\hat{x}_2^\sigma(t)| \\
&\quad -(\omega+\nu)|x_3^\sigma(t)-\hat{x}_3^\sigma(t)|-(\gamma+\nu+d)|x_4^\sigma(t)-\hat{x}_4^\sigma(t)| \\
&\quad +2\beta M|\lambda(t)-\hat{\lambda}(t)|+(\beta\lambda^U)|x_1(t)-\hat{x}_1(t)|+\gamma|x_4(t)-\hat{x}_4(t)| \\
&\quad +\omega|x_3(t)-\hat{x}_3(t)|+(\rho+\phi)|x_2(t)-\hat{x}_2(t)|
\end{aligned}
$$

$$
\begin{aligned}
&= -(\beta\lambda^L + \nu)|x_1^\sigma(t) - \hat{x}_1^\sigma(t)| - (\rho + \phi + \nu)|x_2^\sigma(t) - \hat{x}_2^\sigma(t)| \\
&\quad - (\omega + \nu)|x_3^\sigma(t) - \hat{x}_3^\sigma(t)| - (\gamma + \nu + d)|x_4^\sigma(t) - \hat{x}_4^\sigma(t)| \\
&\quad + \beta\left(\lambda^U + \frac{\beta M}{2m}\right)|x_1(t) - \hat{x}_1(t)| + (\rho + \phi)|x_2(t) - \hat{x}_2(t)| \\
&\quad + \left(\omega + \frac{\beta^2 M}{2m}\eta_C\right)|x_3(t) - \hat{x}_3(t)| \\
&\quad + \left(\gamma + \frac{\beta^2 M}{2m}\eta_A\right)|x_4(t) - \hat{x}_4(t)|.
\end{aligned}
$$

From the previous inequality we can write

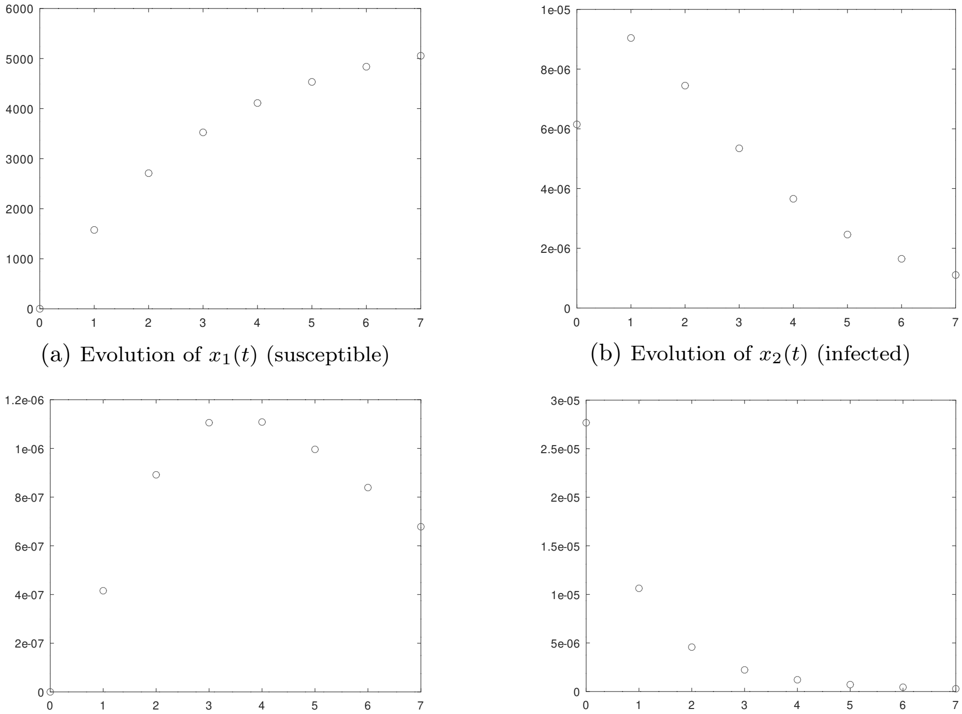
$$
\begin{aligned}
D^+V^\Delta(t) &\le -(\beta\lambda^L + \nu)|x_1^\sigma(t) - \hat{x}_1^\sigma(t)| - (\rho + \phi + \nu)|x_2^\sigma(t) - \hat{x}_2^\sigma(t)| \\
&\quad - (\omega + \nu)|x_3^\sigma(t) - \hat{x}_3^\sigma(t)| - (\gamma + \nu + d)|x_4^\sigma(t) - \hat{x}_4^\sigma(t)| \\
&\quad + \left(\beta\lambda^U + \frac{\beta^2 M}{2m}\right)|x_1(t) - \hat{x}_1(t)| \\
&\quad + (\phi + \rho)|x_2(t) - \hat{x}_2(t)| + \left(\omega + \frac{\beta^2 M}{2m}\eta_C\right)|x_3(t) - \hat{x}_3(t)| \\
&\quad + \left(\gamma + \frac{\beta^2 M}{2m}\eta_A\right)|x_4(t) - \hat{x}_4(t)| \\
&= -a_1|x_1^\sigma(t) - \hat{x}_1^\sigma(t)| - a_2|x_2^\sigma(t) - \hat{x}_2^\sigma(t)| \\
&\quad - a_3|x_3^\sigma(t) - \hat{x}_3^\sigma(t)| - a_4|x_4^\sigma(t) - \hat{x}_4^\sigma(t)| \\
&\quad + b_1|x_1(t) - \hat{x}_1(t)| + b_2|x_2(t) - \hat{x}_2(t)| \\
&\quad + b_3|x_3(t) - \hat{x}_3(t)| + b_4|x_4(t) - \hat{x}_4(t)| \\
&= -\Gamma_1 V(\sigma(t)) + \Gamma_2 V(t) \\
&= (\Gamma_2 - \Gamma_1)V(t) - \Gamma_1\mu(t)D^+V^\Delta(t)
\end{aligned}
$$

and it follows that

$$
D^+V^\Delta(t) \le \frac{\Gamma_2 - \Gamma_1}{1 + \Gamma_1\mu}V(t) = -\Psi V(t).
$$

By hypothesis (H_2), one has $-\Psi \in \mathcal{R}^+$ and $\Psi = \frac{\Gamma_1 - \Gamma_2}{1+\Gamma_1\mu} > 0$. Thus, assumption (iii) of Lemma 2.8 is satisfied. Furthermore, conditions (i) and (ii) of Lemma 2.8 also hold. For condition (i) we consider two functions $a, b \in C(\mathbb{R}^+)$ with $a(x) = 0.5x$ and $b(x) = 2x$. For condition (ii) we put $L = 1$. So there exists a unique uniformly asymptotically stable almost periodic solution $Z(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ of the dynamic system (2) with $Z(t) \in \Omega$. ■

Example 4.2. Motivated by [16] and the case of Morocco, we consider system (2) for $\mathbb{T} = \mathbb{Z}^+$:



(a) Evolution of $x_1(t)$ (susceptible)

(b) Evolution of $x_2(t)$ (infected)

(c) Evolution of $x_3(t)$ (chronic)

(d) Evolution of $x_4(t)$ (AIDS)

Fig. 1. Example 4.2: solution of (9) during seven years

$$(9) \quad \begin{cases} \Delta x_1(t) = \Lambda - \beta\lambda(t)x_1(t+1) - \nu x_1(t+1), \\ \Delta x_2(t) = \beta\lambda(t)x_1(t) - (\rho+\phi+\nu)x_2(t+1) + \gamma x_4(t) + \omega x_3(t), \\ \Delta x_3(t) = \phi x_2(t) - (\omega+\nu)x_3(t+1), \\ \Delta x_4(t) = \rho x_2(t) - (\gamma+\nu+d)x_4(t+1), \end{cases}$$

where

$$x_1(0) = 1 - \frac{11}{N_0}, \quad x_2(0) = \frac{2}{N_0}, \quad x_3(0) = 0, \quad x_4(0) = \frac{9}{N_0},$$

with $N_0 = 325235$, $\Lambda = 2190$, $\beta = 2.710^{-7}$, $\nu = 0.39$, $\rho = 0.2$, $\phi = 0.1$, $\gamma = 0.33$, $\omega = 0.09$, and $d = 1$. System (9) is permanent. Furthermore, $m_1 = 5615.381462$, $m_2 = 5.47870412010^{-9}$, $m_3 = 1.14139669210^{-9}$, $m_4 = 6.37058618610^{-10}$, $M_1 = 5615.384615$, $M_2 = 0.002773104793$, $M_3 = 0.0005777301652$, and $M_4 = 0.0003224540457$. The conditions of Theorem 4.1 are verified with $0.37 = \mathit{\Gamma}_2 < \mathit{\Gamma}_1 = 0.39$ and $\mathit{\Psi} = 0.01391941151$. Therefore, system (9) has a unique positive almost periodic solution, which is uniformly asymptotic stable. In Figure 1 we plot the solution for the first seven years with $\eta_C = 0.5$ and $\eta_A = 1.5$.

5. Conclusion. We have investigated the uniform stability of the unique positive solution in an HIV/AIDS epidemic model, specifically the SICA model on an arbitrary time scale. The purpose of incorporating time scales is to integrate both continuous and discrete time models. We established the permanence of each solution and, using a suitable Lyapunov function, we derived a sufficient condition for uniform asymptotic stability of the solution. Additionally, we presented an illustrative example to substantiate our analytical results.

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