

Genus theory of p -adic pseudo-measures, Hilbert kernels and abelian p -ramification

by

GEORGES GRAS

Abstract. We consider, for real abelian fields K , the Birch–Tate formula linking $\#\mathbf{K}_2(\mathbf{Z}_K)$ to $\zeta_K(-1)$; we compare, for quadratic and cyclic cubic fields with $p \in \{2, 3\}$, $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ with the order of the torsion group $\mathcal{T}_{K,p}$ of abelian p -ramification theory, given, for all p , by the residue of $\zeta_{K,p}(s)$ at $s = 1$. This is done, when $p \nmid [K : \mathbb{Q}]$, via the “genus theory” of p -adic pseudo-measures, inaugurated in the 1970/80’s (Theorem A). We apply this to prove a conjecture of Deng–Li giving the structure of $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$ for an interesting family of real quadratic fields (Theorem B). Then, for $p \geq 5$, we give a lower bound of $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K))$ in cyclic p -extensions K/\mathbb{Q} (Theorem C). Complements, numerical illustrations and PARI programs are given in the Appendices.

1. Introduction – Prerequisites. Let K be a real abelian number field. The aim of this paper is to utilize the link between p -adic \mathbf{L} -functions $\mathbf{L}_p(s, \chi')$ of even Dirichlet characters $\chi' = \varphi'\psi'$ of K (φ' of prime-to- p order, ψ' of p -power order) and classical arithmetic p -invariants of K , and then to show that there are some connections between the orders of these invariants, because of the characters ψ' (genus theory principle). Other links exist because of reflection theorems of class field theory, and this provides additional information about their p -ranks.

In what follows, $p \geq 2$ is a prime number, fixed once for all, μ_n denotes the group of n th roots of unity, and ζ_n is a generator of μ_n .

1.1. Notations. (i) Let $\chi =: \varphi\psi$ be an even character, where φ is of prime-to- p order and ψ of p -power order p^e , $e \geq 1$. For $a \in \mathbb{Z}_p^\times$, let $\theta(a)$ be the unique $\xi \in \text{tor}_{\mathbb{Z}_p}(\mathbb{Z}_p^\times)$ such that $a \equiv \xi \pmod{p}$ for $p \neq 2$ (resp. $a \equiv \xi \pmod{4}$ for $p = 2$). For $p \neq 2$, the character θ is the classical

2020 *Mathematics Subject Classification*: Primary 11R42; Secondary 11R70, 11R37, 11R29, 11R27.

Key words and phrases: abelian fields, \mathbf{K} -theory, Hilbert kernel, p -adic measures, p -adic ζ and \mathbf{L} -functions, class field theory, PARI programs.

Received 30 January 2024.

Published online 19 February 2025 in Open Access (under CC-BY license).

cyclotomic character ω of $\mathbb{Q}(\mu_p)$. For $p = 2$, we have $\omega = \chi_0$ (the unit character), and θ will be considered as the cyclotomic character of $\mathbb{Q}(\mu_4)$, since $\zeta_4^{\sigma_a} = \zeta_4^{\theta(a)}$ for the Artin symbols $\sigma_a = \left(\frac{\mathbb{Q}(\mu_4)/\mathbb{Q}}{a}\right)$.

Let $K_\chi =: K$ be the real cyclic field fixed by χ and let M be the subfield of \mathbb{Q}^{ab} fixed by the character $\theta^{-1}\chi$; since K is real, M is an imaginary field. Let M_0 be the maximal subfield of M of prime-to- p degree:

- if $p \neq 2$, then $M \subseteq K(\mu_p)$, and M_0 is the fixed field of $\theta^{-1}\varphi$ since θ is of order $p - 1$; if $\varphi = \chi_0$, then $M_0 = \mathbb{Q}(\mu_p)$;
- for $p = 2$, consider \widehat{K} with $[K : \widehat{K}] = 2$; then M is the quadratic extension of \widehat{K} in $K(\mu_4)$, distinct from K and $\widehat{K}(\mu_4)$, and M_0 is the fixed field of φ since θ is of order 2; if $\varphi = \chi_0$, then $M_0 = \mathbb{Q}$.

(ii) Let \mathfrak{m} be the maximal ideal of the field of values of χ over \mathbb{Q}_p (i.e., the ideal generated by $1 - \zeta_{p^e}$) and let $v_{\mathfrak{m}}$ be the corresponding valuation with image \mathbb{Z} (so $v_{\mathfrak{m}} = (p - 1)p^{e-1} \cdot v_p$).

(iii) For any prime number r , let p^{d_r} be the degree of the splitting field of r in M/M_0 ; let $\mathbf{D} = \sum_{\ell} p^{d_\ell}$, for $\ell \neq p$ ramified in M/M_0 and totally split in M_0/\mathbb{Q} , hence such that

- for $p \neq 2$: $\psi(\ell) = 0$ and $(\theta^{-1}\varphi)(\ell) = 1$;
- for $p = 2$: $\psi(\ell) = 0$ and $\varphi(\ell) = 1$.

1.2. Overview of the method. Genus theory does exist for \mathbf{L}_p -functions, as initiated in the 1970/80's with the genus theory of abelian p -adic pseudo-measures [Gra86^b, Théorème (0.3)], [Gra87, Théorème (0.1)] including the tricky case of p totally split in M_0 . A result of genus theory under consideration claims that either

$$(1.1) \quad \begin{cases} v_{\mathfrak{m}}\left(\frac{1}{2}\mathbf{L}_p(s, \chi)\right) > \mathbf{C}(s), \forall s \in \mathbb{Z}_p, \text{ or} \\ v_{\mathfrak{m}}\left(\frac{1}{2}\mathbf{L}_p(s, \chi)\right) = \mathbf{C}(s), \forall s \in \mathbb{Z}_p, \end{cases}$$

for an explicit $\mathbf{C}(s)$ essentially depending on \mathbf{D} (Theorem 4.4), which is a constant \mathbf{C} except when $\theta^{-1}\chi(p) = 1$ (total splitting of p in M).

This allows us to obtain, especially for $p \in \{2, 3\}$, similar properties for the following two arithmetic p -invariants depending on values of \mathbf{L}_p -functions:

(a) First (in Section 1.3), we will consider the torsion group $\mathcal{T}_{K,p}$ of the Galois group of the maximal abelian p -ramified (non-complexified) pro- p -extension $H_{K,p}^{\text{pr}}$ of K . In the real abelian case, $\mathcal{T}_{K,p} = \text{Gal}(H_{K,p}^{\text{pr}}/K_\infty)$, where K_∞ is the cyclotomic \mathbb{Z}_p -extension of K . Analytically, $\#\mathcal{T}_{K,p}$ is given, for any p , by the residue at $s = 1$ of the p -adic ζ -function:

$$(1.2) \quad \#\mathcal{T}_{K,p} \sim [K \cap \mathbb{Q}_\infty : \mathbb{Q}] \times \prod_{\chi' \neq 1} \frac{1}{2}\mathbf{L}_p(1, \chi'),$$

where \sim means equality “up to a p -adic unit factor” and χ' runs through the set of primitive Dirichlet characters of K , while class field theory elucidates it completely with formula (1.5) below, which explains that $\mathcal{T}_{K,p}$ plays a central role in this study.

(b) Then (in Section 1.4) we will focus on the tame Hilbert kernel $\mathbf{K}_2(\mathbf{Z}_K)$ of the ring of integers \mathbf{Z}_K of K , and, for $p = 2$, we will replace it by the regular kernel $\mathbf{R}_2(\mathbf{Z}_K)$ in the ordinary sense, linked to the Hilbert one via the exact sequence (see [Gar71])

$$(1.3) \quad 1 \rightarrow \mathbf{R}_2(\mathbf{Z}_K) \rightarrow \mathbf{K}_2(\mathbf{Z}_K) \rightarrow (\mathbb{Z}/2\mathbb{Z})^{[K:\mathbb{Q}]} \rightarrow 1.$$

Class field theory information about $\mathcal{T}_{K,p}$ and $\mathbf{R}_2(\mathbf{Z}_K)[p^\infty]$ comes, in general, from reflection theorems giving formulas (1.6), (1.7); we have refrained from citing all the articles re-proving well-known reflection theorems between $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$, $\mathcal{T}_{K,p}$ and $\mathcal{H}_{K,p}$, published in the 1980–90’s by several authors.

Finally, since $\#\mathbf{K}_2(\mathbf{Z}_K)$ is related to the complex $\mathbf{L}(-1, \chi')$ ’s, we utilize, whenever possible, the classical link between p -adic and complex \mathbf{L} -functions for some comparison of the two p -adic invariants $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\mathcal{T}_{K,p}$ ⁽¹⁾, with the use of property (1.1) when it applies, thus giving new information. We will consider the following analytic formula and the correspondence, for $p \in \{2, 3\}$, between p -adic and complex \mathbf{L} -functions of characters χ' of K :

$$(1.4) \quad \#\mathbf{K}_2(\mathbf{Z}_K) = \pm \left[\frac{2^{[K:\mathbb{Q}]-1}}{3} \times \prod_{[F:K]=2} \frac{\#\mu(F)}{2} \right] \times \prod_{\chi' \neq 1} \frac{1}{2} \mathbf{L}(-1, \chi'),$$

where $\mu(F)$ is the group of roots of unity of F (see, e.g., [W05]).

1.3. Torsion group of abelian p -ramification. Class field theory is especially convenient to state the arithmetic properties of $\mathcal{T}_{K,p}$ (e.g., [Gra05, §§III.2.c, III.4.b, IV.3], after some pioneering works [BP72, Gra86^a, Jau86, Ngu86, GJ89] and many others, like our old Crelle papers).

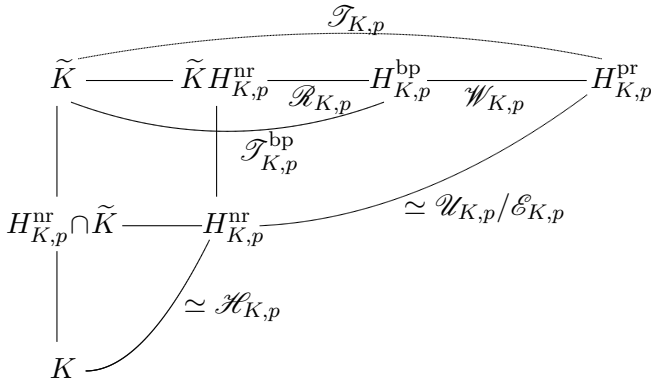
Numerical aspects are given in [Gra19^a] with PARI [PARI] programs, and the order of magnitude of $\#\mathcal{T}_{K,p}$ is studied in [Gra19^c].

We refer to Appendix A.1 for the program computing the structure of $\mathcal{T}_{K,p}$ in complete generality. The diagram below is valid for any number field K fulfilling Leopoldt’s conjecture, with K_∞ replaced by the compositum \tilde{K} of the \mathbb{Z}_p -extensions of K , and with the following notations:

- $\mathcal{H}_{K,p}$ is the p -class group and $H_{K,p}^{\text{nr}}$ the p -Hilbert class field;

⁽¹⁾ A finite p -invariant may be denoted $\mathbf{X}_K[p^\infty]$ when it is of the form $\mathbf{X}_K \otimes \mathbb{Z}_p$ for a finite global invariant \mathbf{X}_K , otherwise it is denoted $\mathcal{X}_{K,p}$. It is conjectured [Gra16^a, Section 8] that $\mathcal{T}_{K,p}$ can be written $\mathbf{T}_K \otimes \mathbb{Z}_p$; this conjecture may be coherent with the finiteness of \mathbf{H}_K (class group) and $\mathbf{K}_2(\mathbf{Z}_K)$, and is supported by many heuristics. For convenience, we will also use the notation $\mathcal{X}_{K,p}$ when \mathbf{X}_K does exist (e.g., $\mathcal{H}_{K,p} = \mathbf{H}_K[p^\infty]$).

- $\mathcal{R}_{K,p}$ is the *normalized* p -adic regulator, defined as a \mathbb{Z}_p -module in [Gra18, Section 5, Proposition 5.2]; its order is the classical p -adic regulator, up to an explicit factor;
- $\mathcal{U}_{K,p}$ is the group of principal local units at p , $\mathcal{E}_{K,p}$ the closure in $\mathcal{U}_{K,p}$ of the group of units E_K , and $\mathcal{W}_{K,p} := \text{tor}_{\mathbb{Z}_p}(\mathcal{U}_{K,p})/\mu_p(K)$;
- $H_{K,p}^{\text{bp}}$ (resp. $\mathcal{T}_{K,p}^{\text{bp}}$), named in [Ngu86] the *Bertrandias–Payan field* (resp. *module*), comes from [BP72], where it is shown that $H_{K,p}^{\text{bp}}$ is the compositum of the p -cyclic extensions of K embeddable in p -cyclic extensions of arbitrarily large degree; see [Gra16^b, Jau16, Ngu16] for complements.



Hence

$$(1.5) \quad \#\mathcal{T}_{K,p} = \frac{\#\mathcal{H}_{K,p}}{[H_{K,p}^{\text{nr}} \cap \tilde{K} : K]} \times \#\mathcal{R}_{K,p} \times \#\mathcal{W}_{K,p}.$$

In the real case, K_∞/K is in general totally ramified, so that the formula becomes $\#\mathcal{T}_{K,p} = \#\mathcal{H}_{K,p} \times \#\mathcal{R}_{K,p} \times \#\mathcal{W}_{K,p}$. From formula (1.2) and the arithmetic interpretation of the residue of $\zeta_{K,p}(s)$ ([Coa77, Appendix], under some assumptions, [Col88] for totally real fields, a more general proof in [CLL21, Appendix], and the previous algebraic interpretation), we can write, assuming $K \cap \mathbb{Q}_\infty = \mathbb{Q}$ to simplify,

$$\#\mathcal{T}_{K,p} \sim \#\mathcal{H}_{K,p} \frac{p}{\prod_{p|p} \mathbf{Np}} \frac{R_{K,p}}{\sqrt{D_K}},$$

where D_K is the discriminant and $R_{K,p}$ the usual p -adic regulator of K .

Hence we have the particular examples:

- $K = \mathbb{Q}(\sqrt{m})$ ($m > 0$ square-free, $m \neq 2$); then

$$\#\mathcal{T}_{K,2} \sim \#\mathcal{H}_{K,2} \times \frac{R_{K,2}}{2\sqrt{m}}.$$

- K cubic cyclic of conductor $f \neq 9$; then

$$\#\mathcal{T}_{K,3} \sim \#\mathcal{H}_{K,3} \times \frac{R_{K,3}}{9}.$$

1.4. Tame and regular Hilbert kernels. Let $S_p(K)$ be the set of p -places of K ; put $K' := K(\mu_p)$ and let $S_p(K')$ be the set of p -places of K' .

Recall that $\omega : \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \rightarrow \mu_{p-1}$ is the cyclotomic (or Teichmüller) character of $\mathbb{Q}(\mu_p)$, and is defined by the Galois action on μ_p (conductor p , order $p-1$ for $p \neq 2$); it is the unit character χ_0 for $p = 2$ since $\mu_2 \subset \mathbb{Q}$. For $p \neq 2$, we have $\omega = \theta$ (Notations 1.1(i)), but for $p = 2$, do not confuse ω with θ as cyclotomic character of $\mathbb{Q}(\mu_4)$ (conductor 4, order 2). Note that in [Gra87], ω always denotes the character θ .

1.4.1. First reflection theorem. The regular kernel $\mathbf{R}_2(\mathbf{Z}_K)$ (see (1.3)) fulfills, from Tate's results [Tat71, Tat76] and the reflection theorem [Gra98, Théorème 11.1] or [Gra05, Theorem II.7.7.3.1], the following p -rank formula, where $\mathbf{H}_{K'}^{S_p(K')\text{res}}$ is the $S_p(K')$ -class group $\mathbf{H}_{K'}^{\text{res}}/\langle c\ell_{K'}^{\text{res}}(S_p(K')) \rangle$ in the restricted sense, $\delta = 1$ or 0 , according as $\mu_p \subset K$ or not:

$$(1.6) \quad \begin{cases} p \neq 2: \text{rk}_p(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_p((\mathbf{H}_{K'}^{S_p(K')})_{\omega^{-1}}) \\ \quad \quad \quad + \#\{v \in S_p(K), v \text{ totally split in } K'/K\} - \delta, \\ p = 2: \text{rk}_2(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_2(\mathbf{H}_K^{S_2(K)\text{res}}) + \#S_2(K) - 1. \end{cases}$$

1.4.2. Second reflection theorem. Due to the reflection theorem between p -class groups and p -torsion groups of abelian p -ramification theory, we get similarly [Gra05, Proposition III.4.2.2]

$$(1.7) \quad \begin{cases} p \neq 2: \text{rk}_p(\mathcal{T}_{K,p}) = \text{rk}_p((\mathbf{H}_{K'}^{S_p(K')})_{\omega}) \\ \quad \quad \quad + \#\{v \in S_p(K), v \text{ totally split in } K'/K\} - \delta, \\ p = 2: \text{rk}_2(\mathcal{T}_{K,2}) = \text{rk}_2(\mathbf{H}_K^{S_2(K)\text{res}}) + \#S_2(K) - 1. \end{cases}$$

From the previous two relations, we get

$$(1.8) \quad \text{rk}_p(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_p(\mathcal{T}_{K,p})$$

as long as $\omega^2 = 1$, that is to say, if K contains the maximal real subfield of $\mathbb{Q}(\mu_p)$ (always fulfilled for $p \in \{2, 3\}$), a framework widely developed in [GJ89] where these fields are called “ p -regular”; these techniques yield some generalizations [Keu89, BG92, RØ00]. The relation (1.8) was in fact proven in [Gra86^a, Theorems 1, 2] (where $\mathbf{R}_2(\mathbf{Z}_K)$ is denoted $H_2^0 K$), to characterize, for $p \in \{2, 3\}$, the abelian p -extensions K/\mathbb{Q} such that $\mathbf{R}_2(\mathbf{Z}_K)[p^\infty] = 1$ (see the list in §1.6 and Remark 4.5 explaining that these are also the p -rational fields). For results with $p \geq 5$, see Section 6.

More generally, when K contains μ_p , [Gra98, Remarque 11.5] gives the relation (1.8) for the characters ρ of K , $\text{rk}_\rho(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_{\omega^2 \rho^{-1}}(\mathcal{T}_{K,p})$.

1.5. Main results of the article. We describe three results about the modules $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\mathcal{T}_{K,p}$.

(a) From the properties of p -adic \mathbf{L} -functions, regarding the constant \mathbf{C} , stated in §1.2 and described in Theorem 4.4, we obtain the following results for quadratic fields ($p = 2$) and cyclic cubic ones ($p = 3$):

THEOREM A (see Theorem 4.6).

- (i) Let $K = \mathbb{Q}(\sqrt{m})$, of character χ , be a real quadratic field with $m \neq 2$. Assume that $v_2(\#\mathcal{T}_{K,2}) = \mathbf{C}$ (equivalent to $v_2(\frac{1}{2}\mathbf{L}_2(1, \chi)) = \mathbf{C}$ from (1.2)). Then $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] = 2^{\mathbf{C}+2}$ (i.e., $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] = 2^{\mathbf{C}}$).
- (ii) Let K be a cyclic cubic field of conductor $f \neq 9$ with its two non-trivial characters χ and χ^2 . Assume that $v_3(\#\mathcal{T}_{K,3}) = \mathbf{C}$ (so $v_3(\mathbf{L}_3(1, \chi) \times \mathbf{L}_3(1, \chi^2)) = \mathbf{C}$ from (1.2)). Then $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] = 3^{\mathbf{C}}$.
- (iii) If $v_2(\mathcal{T}_{K,2}) > \mathbf{C}$ (resp. $v_3(\mathcal{T}_{K,3}) > \mathbf{C}$), then $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] > 2^{\mathbf{C}+2}$ (resp. $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] > 3^{\mathbf{C}}$).

In Appendices B.1 and C.1, we compute $\#\mathbf{R}_2(\mathbf{Z}_K)[p^\infty]$ and $\#\mathcal{T}_{K,p}$ for quadratic and cubic fields by means of explicit pseudo-measures yielding expressions of Theorem 4.1 and Corollary 4.2; this enables safe verifications.

(b) We apply the case of equality to a family of real quadratic fields introduced in [DL23, Theorem 1.2] and prove the Deng–Li conjecture on the structure of $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$:

THEOREM B (see Section 5). Let $K = \mathbb{Q}(\sqrt{m})$, $m = \ell_1 \cdots \ell_n$, n even, with

- $\ell_1 \equiv 3 \pmod{8}$, $\ell_i \equiv 5 \pmod{8}$ for $i \geq 2$,
- $\left(\frac{\ell_1}{\ell_2}\right) = -1$, $\left(\frac{\ell_1}{\ell_j}\right) = 1$ for $j \geq 3$, and $\left(\frac{\ell_i}{\ell_j}\right) = -1$ for $2 \leq i < j \leq n$.

Then $\mathcal{H}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$, $\mathcal{T}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-2} \times \mathbb{Z}/4\mathbb{Z}$, and $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^3\mathbb{Z}$ (Deng–Li conjecture).

(c) The cases $p \in \{2, 3\}$ being very specific because of the relation (2.1) for $m = 2$ between p -adic and complex \mathbf{L} -functions, we have obtained the following result when p is distinct from 2 and 3:

THEOREM C (see Theorem 6.1 for a more general statement). Let K/\mathbb{Q} be a cyclic p -extension for $p \geq 5$, and let S_{ta} be the set of primes $\ell \neq p$ ramified in K/\mathbb{Q} . Then $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \#S_{\text{ta}}$. In particular, $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] = 1$ if and only if K is contained in the cyclotomic \mathbb{Z}_p -extension \mathbb{Q}_∞ of \mathbb{Q} .

1.6. Some history and references. The class field theory approach to the structures of the p -groups $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\mathcal{T}_{K,p}$ has a huge literature, especially for rank computations, often restricted to $p \in \{2, 3\}$, using probably (1.8) when $\omega^2 = \chi_0$, to get practical results (e.g., [BS82, Gra86^a, CK89, Ber90, Bro92, Tho94, Qin95, YF00, Yue01, KM03, Bro05, Yue05, Qin05, GL09, DL23]). For example in [Gra86^a, Corollary to Theorem 2],

for $p \in \{2, 3\}$, we characterized the *abelian* p -extensions K/\mathbb{Q} with trivial $\mathbf{R}_2(\mathbf{Z}_K)[p^\infty]$, that is, the subfields of $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\sqrt{-\ell})$, $-\ell \equiv 5 \pmod{8}$, those of $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\sqrt{\sqrt{\ell} \frac{a-\sqrt{\ell}}{2}})$, $\ell = a^2 + 4b^2 \equiv 5 \pmod{8}$, then, for $p = 3$, those of $\mathbb{Q}_\infty k(\ell)$, $\ell \equiv 4, 7 \pmod{9}$, where $k(\ell)$ is the cyclic cubic field of conductor ℓ . This list reappeared in many later articles.

For more history and theoretical contributions, one may refer, among many other texts, to [Ba68, Tat71, Gar71, Tat76, Jau86, Ngu86, Hur89, GJ89, Keu89, Kol89, Ngu90, HK98, RW00, Yue02, Kol03, Gra05, Qin10]. For some generalizations to the wide étale kernels $\mathbf{WK}_{2i}(K)$, see for instance [Ngu92, JS01, JM06, AAM21, Lim23].

One may find (for instance in [Keu89, Sections 4, 5]) a thorough description of the properties of $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ in an extension E/F , by means of natural maps denoted, in Keune's paper (cf. [Keu89, (4.4), (4.5)]),

$$\mathbf{j}_{E/F} : \mathbf{K}_2(\mathbf{Z}_F)[p^\infty] \rightarrow \mathbf{K}_2(\mathbf{Z}_E)[p^\infty], \quad \mathbf{tr}_{E/F} : \mathbf{K}_2(\mathbf{Z}_E)[p^\infty] \rightarrow \mathbf{K}_2(\mathbf{Z}_F)[p^\infty],$$

having similar properties to those existing for $\mathcal{T}_{K,p}$ in E/F , except that the “transfer” \mathbf{tr} looks like, for us, the *arithmetic norm* \mathbf{N} of class field theory, while \mathbf{N} , in Keune's paper, denotes the *algebraic norm* \mathcal{V} with the usual relation $\mathcal{V} = \mathbf{J} \circ \mathbf{N}$ leading to the notion of \mathcal{G} -family defined in [Gra23^b]. Thus, the strong analogies between these two invariants may probably be largely improved in this direction. We note in particular that under the classical p -adic conjectures, these two modules never capitulate in p -extensions, in contrast to p -class groups and Jaulent's logarithmic class groups in the real case [Jau90, Jau94^b, Jau19].

But our goal is to return to simple effective p -adic methods allowing computations, by means of pseudo-measures. These pseudo-measures are the Mellin transforms (via a reflection principle) of Stickelberger elements, and give rise, after twisting, to p -adic measures of the form

$$(\mathcal{S}_{L_n}^*(c)(s))_n, \quad s \in \mathbb{Z}_p, \quad \mathcal{S}_{L_n}^*(c)(s) \in \mathbb{Z}_p/qp^n\mathbb{Z}_p[\text{Gal}(L_n/\mathbb{Q})]$$

for the layers $L_n := K(\mu_{qp^n})$ of the cyclotomic \mathbb{Z}_p -extension of $K(\mu_q)$ ($q = 4$ or p) [Gra78, Section II]. This was done after the main pioneering works of Kubota–Leopoldt [KL64], then Fresnel [Fre67], Amice–Fresnel [AF72], Iwasawa [Iwa72], Coates [Coa77], Serre [Ser78], giving many results, including the value of the residue of p -adic ζ -functions at $s = 1$, and some annihilation theorems for p -class groups, torsion groups $\mathcal{T}_{K,p}$ and logarithmic class groups [Gra79, BN05, Jau21, Jau24]; this was generalized to a totally real base field by Colmez [Col88] for residues of ζ -functions, Cassou–Noguès [CN78] and Deligne–Ribet [DR80] for \mathbf{L}_p -functions and p -adic pseudo-measures, giving the possibility to extend our results.

The Iwasawa framework of these constructions, defining the \mathbf{L}_p -functions, is very similar and is detailed in [Wa97^b, §§7.2, 7.3, Theorem 7.2]. The two

approaches lead to some relations in p -extensions, like Kida's formulas in K_∞ and K'_∞ , for $K = K_\chi$ and $K' = K_{\chi'}$ (χ even of conductor f , χ' of conductor $f' \mid f$), $\lambda_\chi = \lambda_{\chi'} + \sum_\ell p^{n(\ell)}$, where ℓ runs through the primes $\ell \neq p$, $\ell \mid f$, $\ell \nmid f'$, such that $\theta^{-1}\chi(\ell)$ is of p -power order, and where $p^{n(\ell)} \sim \frac{1}{q} \log_p(\ell)$ (see [Gra78, Propositions V.2, V.3] for more details).

2. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$, $p \in \{2, 3\}$

2.1. Birch–Tate formula and \mathbf{L}_p -functions. Let $\zeta_K(s)$ be the Dedekind zeta-function of a real abelian number field K . Then

$$\zeta_K(s) = \prod_{\chi'} \mathbf{L}(s, \chi'),$$

the product of the complex \mathbf{L} -functions where χ' runs through the set of primitive Dirichlet characters of K for which $\mathbf{L}(s, \chi_0) = \zeta_K(s)$. In the p -adic context, the definition of \mathbf{L}_p -functions is as follows ([Fre67, Section 5(a), Remarque], Amice–Fresnel [AF72, Section 1] or Washington's book [Wa97^b]):

$$(2.1) \quad \begin{cases} \mathbf{L}_p(1 - m, \chi) = (1 - p^{m-1}\chi(p))\mathbf{L}(1 - m, \chi), \\ m > 1 \ \& \ m \equiv 0 \pmod{p-1} \text{ if } p > 2, \\ m > 1 \ \& \ m \equiv 0 \pmod{2} \text{ if } p = 2. \end{cases}$$

This allows us to compute for instance $\mathbf{L}(-1, \chi)$ in terms of \mathbf{L}_p -functions for $p = 2$ and $p = 3$ since $m = 2$ fulfills the congruence conditions in (2.1).

The Birch–Tate formula is the following equality (proved as a consequence of the Mazur–Wiles “Main Theorem” in abelian theory and its complements in the case $p = 2$ [Kol89]):

$$\#\mathbf{K}_2(\mathbf{Z}_K) = w_2(K)\zeta_K(-1) = \frac{1}{3} \cdot \prod_{[F:K]=2} \frac{\#\mu(F)}{2} \prod_{\chi' \neq \chi_0} \mathbf{L}(-1, \chi'),$$

where

$$w_2(K) = 4 \cdot \prod_{[F:K]=2} \frac{\#\mu(F)}{2} \quad \text{and} \quad \zeta_K(-1) = \frac{-1}{12}.$$

Thus, the computation of the p -Sylow subgroup $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ of $\mathbf{K}_2(\mathbf{Z}_K)$ is possible for $p \in \{2, 3\}$ with the following formula (using (2.1) for $m = 2$):

$$\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] \sim \frac{1}{3} \cdot \prod_{[F:K]=2} \frac{\#\mu(F)}{2} \times \prod_{\chi' \neq \chi_0} \frac{1}{1 - p\chi'(p)} \mathbf{L}_p(-1, \chi'),$$

where \sim means equality up to a p -adic unit factor for the p considered.

Since the denominators $1 - p\chi'(p)$ are invertible in \mathbb{Z}_p , this yields

$$(2.2) \quad \begin{cases} p = 2: \#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \sim \prod_{[F:K]=2} \frac{\#\mu_2(F)}{2} \times \prod_{\chi' \neq \chi_0} \mathbf{L}_2(-1, \chi'), \\ p = 3: \#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \frac{1}{3} \cdot \prod_{[F:K]=2} \#\mu_3(F) \times \prod_{\chi' \neq \chi_0} \mathbf{L}_3(-1, \chi'). \end{cases}$$

2.2. Formulas for quadratic and cubic fields. We deduce the following particular cases with $p \in \{2, 3\}$, using $\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]$ for $p = 2$:

(a) Quadratic fields $K = K_\chi = \mathbb{Q}(\sqrt{m})$, $m \neq 2$, $p = 2$:

$$\prod_{[F:K]=2} \frac{\#\mu_2(F)}{2} = 2 \quad (F = K(\mu_4), F = K(\mu_6));$$

thus, $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \sim \frac{1}{2}\mathbf{L}_2(-1, \chi)$.

For $p = 3$,

$$\frac{1}{3} \prod_{[F:K]=2} \#\mu_3(F) = 1 \quad (F = K(\mu_6))$$

since $F = K(\mu_4)$ does not intervene); thus, $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \mathbf{L}_3(-1, \chi)$.

(b) Cyclic cubic fields $K = K_\chi$, conductor $f \neq 9$, $p = 3$:

$$\frac{1}{3} \prod_{[F:K]=2} \#\mu_3(F) = 1 \quad (F = K(\mu_6));$$

thus, $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \mathbf{L}_3(-1, \chi) \times \mathbf{L}_3(-1, \chi^2)$.

For $p = 2$,

$$\prod_{[F:K]=2} \frac{\#\mu_2(F)}{2} = 2 \quad (F = K(\mu_4)),$$

since $F = K(\mu_6)$ does not intervene; thus, $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \sim \frac{1}{2}\mathbf{L}_2(-1, \chi) \times \frac{1}{2}\mathbf{L}_2(-1, \chi^2)$.

3. Definition of a p -adic pseudo-measure $(\mathcal{S}_{L_n})_n$. Let K be a real abelian field, and set $L_n := K(\mu_{qp^n})$, $n \geq 0$, $q = p$ or 4 .

3.1. The Stickelberger elements. For n large enough, if p ramifies in K , the conductor of L_n is of the form $f_{L_n} = qp^n f'$ for a prime-to- p integer f' (otherwise, f' is the conductor f of K); in formulas we shall abbreviate f_{L_n} to f_n , $n \geq 0$. Then put, with all Artin symbols taken over \mathbb{Q} ,

$$(3.1) \quad \mathcal{S}_{L_n} := - \sum_{a=1}^{f_n} \left(\frac{a}{f_n} - \frac{1}{2} \right) \left(\frac{L_n}{a} \right)^{-1}$$

as the restriction to L_n of

$$\mathcal{S}_{\mathbb{Q}(\mu_{f_n})} := - \sum_{a=1}^{f_n} \left(\frac{a}{f_n} - \frac{1}{2} \right) \left(\frac{\mathbb{Q}(\mu_{f_n})}{a} \right)^{-1},$$

where a runs through the prime-to- f_n integers $a \in [1, f_n[$.

3.2. Norms of the Stickelberger elements. Let f and m be such that $m \mid f$; consider $\mathbb{Q}(\mu_f)$ and $\mathbb{Q}(\mu_m)$ and let $\mathbf{N}_{\mathbb{Q}(\mu_f)/\mathbb{Q}(\mu_m)}$ be the restriction map $\mathbb{Q}[\text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})] \rightarrow \mathbb{Q}[\text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})]$. We have

$$(3.2) \quad \mathbf{N}_{\mathbb{Q}(\mu_f)/\mathbb{Q}(\mu_m)}(\mathcal{S}_{\mathbb{Q}(\mu_f)}) = \prod_{\ell \mid f, \ell \nmid m} \left(1 - \left(\frac{\mathbb{Q}(\mu_m)}{\ell} \right)^{-1} \right) \cdot \mathcal{S}_{\mathbb{Q}(\mu_m)}.$$

Let L/K be an extension of abelian fields of conductors m and f ; we define $\mathcal{S}_L := \mathbf{N}_{\mathbb{Q}(\mu_f)/L}(\mathcal{S}_{\mathbb{Q}(\mu_f)})$ and $\mathcal{S}_K := \mathbf{N}_{\mathbb{Q}(\mu_m)/K}(\mathcal{S}_{\mathbb{Q}(\mu_m)})$, respectively; then

$$\mathbf{N}_{L/K}(\mathcal{S}_L) = \prod_{\ell \mid f, \ell \nmid m} \left(1 - \left(\frac{K}{\ell} \right)^{-1} \right) \cdot \mathcal{S}_K.$$

3.3. Twists of the Stickelberger elements. Let c be an integer prime to $2pf$; for $L_n = K(\mu_{qp^n})$, put

$$\mathcal{A}_{L_n}(c) := \left(1 - c \left(\frac{L_n}{c} \right)^{-1} \right) \cdot \mathcal{A}_{L_n}.$$

Then $\mathcal{A}_{L_n}(c) \in \mathbb{Z}[\text{Gal}(L_n/\mathbb{Q})]$. Indeed, we have

$$\mathcal{A}_{L_n}(c) = \frac{-1}{f_n} \sum_a \left[a \left(\frac{L_n}{a} \right)^{-1} - ac \left(\frac{L_n}{a} \right)^{-1} \left(\frac{L_n}{c} \right)^{-1} \right] + \frac{1-c}{2} \sum_a \left(\frac{L_n}{a} \right)^{-1}.$$

Let $a'_c \in [1, f_n[$ be the unique integer such that $a'_c \cdot c \equiv a \pmod{f_n}$. Put $a'_c \cdot c - a = \lambda_a^n(c) f_n$, $\lambda_a^n(c) \in \mathbb{Z}$. Then, using this relation, the bijection $a \mapsto a'_c$ in the summation, and the fact that $\left(\frac{L_n}{a'_c} \right) \left(\frac{L_n}{c} \right) = \left(\frac{L_n}{a} \right)$, we obtain

$$(3.3) \quad \mathcal{A}_{L_n}(c) = \sum_{a=1}^{f_n} \left(\lambda_a^n(c) + \frac{1-c}{2} \right) \left(\frac{L_n}{a} \right)^{-1} \in \mathbb{Z}[\text{Gal}(L_n/\mathbb{Q})].$$

4. The measure $(\mathcal{S}_{L_n}^*(c))_n$ defining $\mathbf{L}_p(s, \chi)$. Consider

$$A_n := \mathbb{Z}_p / qp^n \mathbb{Z}_p[\text{Gal}(L_n/\mathbb{Q})], \quad n \geq 0.$$

The Mellin transform is defined, on A_n , by the following image of any $\sigma \in G$, where a , defined modulo $f_n = qp^n f$, represents σ as an Artin symbol:

$$\sigma = \left(\frac{L_n}{a} \right) \mapsto \sigma^* := \theta(a) \langle a \rangle^s \left(\frac{L_n}{a} \right)^{-1} = a \langle a \rangle^{s-1} \left(\frac{L_n}{a} \right)^{-1} = a \langle a \rangle^{s-1} \sigma^{-1}$$

for all $s \in \mathbb{Z}_p$, and the image, in A_n , of the expression (3.3) of $\mathcal{S}_{L_n}(c)$ by this transform yields the p -adic measure $(\mathcal{S}_{L_n}^*(c)(s))_n \in \varprojlim_n A_n$ giving annihilation theorems and p -adic functions $\mathbf{L}_p(s, \chi)$; indeed, since $L_n = K(\mu_{qp^n})$, one gets, from the norm relations (3.2), $\mathbf{N}_{L_{n+1}/L_n}(\mathcal{S}_{L_{n+1}}^*(c)(s)) \equiv \mathcal{S}_{L_n}^*(c)(s) \pmod{qp^n}$, $n \geq 0$. So, we will use the following approximations:

$$\mathcal{S}_{L_n}^*(c)(s) \equiv \sum_{a=1}^{f_n} \left(\lambda_a^n(c) + \frac{1-c}{2} \right) a^{-1} \langle a \rangle^{1-s} \left(\frac{L_n}{a} \right) \pmod{qp^n}.$$

4.1. Approximations of $\mathbf{L}_p(s, \chi)$ modulo qp^n . The \mathbf{L}_p -functions of even Dirichlet characters χ are obtained by means of the Mellin transform of the Stickelberger pseudo-measures (3.1) (see, e.g., [Gra78, Section II] and [Fre67, p. 292] for the link with Bernoulli numbers). So, the Mellin transform of $1 - c \left(\frac{L_n}{c} \right)^{-1}$ is $1 - \langle c \rangle^{1-s} \left(\frac{L_n}{c} \right)$, and is a factor of the twisted p -adic measure $(\mathcal{S}_{L_n}^*(c)(s))_n$; thus the factor $1 - \langle c \rangle^{1-s} \chi(c)$ must be dropped when this makes sense (thus in any case, except $\chi = \chi_0$ and $s = 1$):

THEOREM 4.1 (Approximation of $\mathbf{L}_p(s, \chi)$ modulo p^n). *Let c be any odd integer, prime to p and to the conductor f of K . For all n large enough, let f_n be the conductor of $L_n = K(\mu_{qp^n})$, and for all $a \in [1, f_n[$ prime to f_n , let a'_c be the unique integer in $[1, f_n[$ such that $a'_c \cdot c - a = \lambda_a^n(c) f_n$, with $\lambda_a^n(c) \in \mathbb{Z}$. The \mathbf{L}_p -function of Dirichlet character χ of K is given by*

$$\mathbf{L}_p(s, \chi) = \frac{1}{1 - \chi(c) \langle c \rangle^{1-s}} \times \lim_{n \rightarrow \infty} \sum_{a=1}^{f_n} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a^{-1} \langle a \rangle^{1-s} \chi(a).$$

COROLLARY 4.2. *Assume that $p \in \{2, 3\}$; in this context, $\theta^2 = \chi_0$. Since for $s = -1$, $a^{-1} \langle a \rangle^{1-s} = a^{-1} \langle a \rangle^2 = a$, we get*

$$\mathbf{L}_p(-1, \chi) = \frac{1}{1 - \chi(c) \langle c \rangle^2} \times \lim_{n \rightarrow \infty} \sum_{a=1}^{f_n} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a \chi(a).$$

COROLLARY 4.3. *Assume $\chi \neq \chi_0$ and $\chi(c) \neq 1$. For $s = -1$, we have $\langle c \rangle \equiv 1 \pmod{q}$ and $\chi(c) \in \mu_{[K:\mathbb{Q}]} \setminus \{1\}$; thus, in the quadratic case with $p = 3$ and in the cyclic cubic case with $p = 2$, $1 - \chi(c) \langle c \rangle^2$ is invertible. Otherwise:*

- $1 - \chi(c) \langle c \rangle^2 \sim 2$ for the quadratic case and $p = 2$;
- $1 - \chi(c) \langle c \rangle^2 \sim 1 - \zeta_3$ for the cubic case and $p = 3$.

From formulas of §2.2 one gets, dividing by 2 (resp. $1 - \zeta_3$):

(i) For $K = \mathbb{Q}(\sqrt{m})$ and the half summation giving $\frac{1}{2} \mathbf{L}_2(s, \chi)$,

$$\# \mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \sim \frac{1}{2} \mathbf{L}_2(-1, \chi) \sim \frac{1}{2} \sum_{a=1}^{f_n/2} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a \chi(a).$$

(ii) For the cubic case with $\mathbf{N} = \mathbf{N}_{\mathbb{Q}(\mu_3)/\mathbb{Q}}$,

$$\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \mathbf{L}_3(-1, \chi) \cdot \mathbf{L}_3(-1, \chi^2) \sim \frac{1}{3} \mathbf{N} \left(\sum_{a=1}^{f_n/2} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a \chi(a) \right).$$

4.2. Genus theory of p -adic pseudo-measures. We have obtained in [Gra87, Théorème (0.2)] the following result which does not seem much known but states the property of “stability” mentioned in the Introduction and may be linked to the existence (or not) of zeroes of \mathbf{L}_p -functions (for examples relating to zeroes of \mathbf{L}_p -functions, one may refer to [W81, W82, W95, Wa97^a, Wa97^b] among many others, and see Appendix C.4 for some illustrations):

THEOREM 4.4 (Fundamental theorem on “stability” of \mathfrak{m} -valuations). *Under Notations 1.1 for $K = K_\chi$, we have, with $\mathbf{D} = \sum_\ell p^{d_\ell}$ and $\epsilon = 0$ (resp. $\epsilon = 1$) when $\varphi \neq \chi_0$ (resp. $\varphi = \chi_0$):*

(a) $v_{\mathfrak{m}}(\frac{1}{2} \mathbf{L}_p(s, \chi)) \geq \mathbf{C}(s)$, $\forall s \in \mathbb{Z}_p$, where $\mathbf{C}(s)$ is as follows:

(i_p) If $p \neq 2$ and $\theta^{-1}\varphi(p) \neq 1$ (i.e., p is not totally split in M_0), then

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} - \epsilon.$$

(i₂) If $p = 2$ and $\varphi(2) \neq 1$ (i.e., 2 is not totally split in M_0), then

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D}.$$

(ii_p) If $p \neq 2$, $\theta^{-1}\varphi(p) = 1$ and $\theta^{-1}\chi(p) \neq 1$ (i.e., p is totally split in M_0 but not in M), then

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} + p^{d_p}.$$

(ii₂) If $p = 2$ and $\varphi(2) = 1$ and $\theta^{-1}\chi(2) \neq 1$ (i.e., 2 is totally split in M_0 but not in M), then

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} + 2^{d_2} - 2\epsilon.$$

(iii_p) If $p \neq 2$ and $\theta^{-1}\chi(p) = 1$ (i.e., p is totally split in M), then

$$\mathbf{C}(s) := \mathbf{D} + v_{\mathfrak{m}}(ps).$$

(iii₂) If $p = 2$ and $\theta^{-1}\chi(2) = 1$ (i.e., 2 is totally split in M), then

$$\mathbf{C}(s) := \mathbf{D} + v_{\mathfrak{m}}(qs) - 2\epsilon.$$

(b) We have either the equality $v_{\mathfrak{m}}(\frac{1}{2} \mathbf{L}_p(s, \chi)) = \mathbf{C}(s)$, $\forall s \in \mathbb{Z}_p$, or the strict inequality $v_{\mathfrak{m}}(\frac{1}{2} \mathbf{L}_p(s, \chi)) > \mathbf{C}(s)$, $\forall s \in \mathbb{Z}_p$ ($s \neq 0$ in cases (iii)).

REMARK 4.5. Let K be of prime-to- p degree. If $\psi = \chi_0$, $\chi = \varphi \neq \chi_0$, then $\epsilon = 0$; assuming $\theta^{-1}\chi(p) \neq 1$, we have $\mathbf{C} = \mathbf{D} = 0$; thus $\mathfrak{m} = (p)$, and $v_p(\frac{1}{2} \mathbf{L}_p(s, \chi)) = 0$ (resp. > 0) for all $s \in \mathbb{Z}_p$. If the valuation is trivial for all the characters χ' of K (e.g., if $K \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$), this is equivalent to $\mathcal{F}_{K,p} = 1$, from (1.2) (the case of the simplest structure giving the notion of

p -rational fields coming from [Gra86^a, Mov88] and largely developed in a lot of papers; see [Gra19^b, Appendix] for a survey about this notion and abelian p -ramification theory); of course, $v_p(\frac{1}{2}\mathbf{L}_p(s, \chi)) > 0, \forall s$, is also frequent.

The split case $\theta^{-1}\chi(p) = 1$ yields $\mathbf{C}(s) = v_p(qs) > 0$, whence never p -rationality; e.g., $p = 3, K = \mathbb{Q}(\sqrt{3m})$ with $-m \equiv 1 \pmod{9}$, and the examples $(m, \mathcal{T}_{K,3}) = (17, [3]), (107, [9]), (35, [27]), (305, [81]), (629, [3, 3]), (4967, [243]), (8378, [2187]), (188657, [3, 3, 3]), (216530, [243, 3, 3]), \dots$

So, in the case of stability $v_m(\frac{1}{2}\mathbf{L}_p(s, \chi)) = \mathbf{C}(s), \forall s \in \mathbb{Z}_p$, a computation may be done at a suitable value s for $\mathbf{L}_p(s, \chi)$, e.g., $s = 1$ giving a link with $\#\mathcal{T}_{K,p}$: for instance $\#\mathcal{T}_{K,2} \sim \frac{1}{2}\mathbf{L}_2(1, \chi)$ for K real quadratic, except for K of conductor 8, or $\#\mathcal{T}_{K,3} \sim \mathbf{L}_3(1, \chi) \times \mathbf{L}_3(1, \chi^2)$ for K cyclic cubic, except for K of conductor 9.

This gives the main process and Theorem A, where $\mathbf{C}(s) = \mathbf{C}$ as long as $s \in \mathbb{Z}_p^\times$, since $v_m(s) = 0$ in cases (iii) of Theorem 4.4 (e.g., $s = \pm 1$):

THEOREM 4.6 (Quadratic and cyclic cubic fields).

- (i) *Let K be a real quadratic field of character χ and conductor $\neq 8$, and set $p = 2$. Assume that the knowledge of $\#\mathcal{T}_{K,2}$ (e.g., using formula (1.5)) implies $v_2(\frac{1}{2}\mathbf{L}_2(1, \chi)) = \mathbf{C}$; then $v_2(\frac{1}{2}\mathbf{L}_2(s, \chi)) = \mathbf{C}, \forall s \in \mathbb{Z}_2$, whence in particular, $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] = 2^{\mathbf{C}+2}$. We have $\mathbf{C} = \mathbf{D}$ (resp. $\mathbf{C} = \mathbf{D} - 1$) if 2 splits (resp. does not split) in $M = \mathbb{Q}(\sqrt{-m})$.*
- (ii) *Let K be a cyclic cubic field of character χ and conductor $\neq 9$, and set $p = 3$. Assume that the knowledge of $\#\mathcal{T}_{K,3}$ implies $v_m(\mathbf{L}_3(1, \chi)) = \mathbf{C}$; then $v_m(\mathbf{L}_3(s, \chi)) = \mathbf{C}, \forall s \in \mathbb{Z}_3$, whence in particular $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] = 3^{\mathbf{C}}$. We have $\mathbf{C} = \mathbf{D} - 1$ since 3 cannot totally split in $M = K(\mu_3)$.*
- (iii) *If $v_2(\frac{1}{2}\mathbf{L}_2(1, \chi)) > \mathbf{C}$ (resp. $v_m(\mathbf{L}_3(1, \chi)) > \mathbf{C}$), then one obtains $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] > 2^{\mathbf{C}+2}$ (resp. $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] > 3^{\mathbf{C}}$).*

Since [DL23, Theorem 1.2] gives, for this specific family of real quadratic fields $K = \mathbb{Q}(\sqrt{m}), \mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^\delta\mathbb{Z}, \delta \geq 3$, the Deng–Li conjecture is $\delta = 3$; we intend to prove it (in §5) by means of the previous process, showing that $\#\mathcal{T}_{K,2} = 2^n$, hence $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] = 2^{n+2}$:

COROLLARY 4.7. *If $\#\mathcal{H}_{K,2} = 2^{n-1}, \mathcal{R}_{K,2} = 1, \#\mathcal{W}_{K,2} = 2$, then $\#\mathcal{T}_{K,2} = 2^n$, giving the valuation n for $\frac{1}{2}\mathbf{L}_2(1, \chi)$; thus, Theorem 4.6(i) yields $\mathbf{C} = \mathbf{D} = n$, since 2 splits in these $\mathbb{Q}(\sqrt{-m})$'s; hence, $\delta = 3$ follows.*

4.3. Modulus of continuity of $\mathbf{L}_p(s, \chi)$. Whatever p is, if χ is of prime-to- p order, genus theory is vacuous, and this raises the question of the “independence” (or not) of $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\#\mathcal{T}_{K,p}$. This is related to the rank formula (1.8) when K contains the maximal real subfield of $\mathbb{Q}(\mu_p)$, but is also a consequence of the existence of a non-trivial modulus of continuity for $\mathbf{L}_p(s, \chi)$, whatever $\chi = \varphi\psi$ and p are, as follows [Gra87, Théorème (0.3)]:

THEOREM 4.8. *With Notations 1.1, we have, for all $s, t \in \mathbb{Z}_p$,*

$$\frac{1}{2}\mathbf{L}_p(t, \chi) - \frac{1}{2}\mathbf{L}_p(s, \chi) \equiv a(t - s) \pmod{q\mathbf{m}^{\mathbf{V}}(t - s)},$$

with the computable constant a ($a = 0$ when $\varphi \neq \chi_0$) and \mathbf{V} of the form $\mathbf{V} = \mathbf{D} - \max_{\ell}(p^{d_{\ell}} - p^{d_p} + \epsilon, \epsilon) \geq 0$ ($\epsilon = 1$ if $\varphi = \chi_0$, $\epsilon = 0$ otherwise), where ℓ runs through the set of primes ramified in M/M_0 , totally split in M_0/\mathbb{Q} , and such that $\frac{1}{q}\log_p(\ell) \not\equiv 0 \pmod{p}$. In the case $\psi = \chi_0$, $\chi = \varphi \neq \chi_0$, the above formula becomes $\frac{1}{2}\mathbf{L}_p(t, \chi) \equiv \frac{1}{2}\mathbf{L}_p(s, \chi) \pmod{q(t - s)}$.

Thus, even if the genus principle is empty, there is a non-trivial congruence between the orders of the two corresponding invariants. For instance, we get $\frac{1}{2}\mathbf{L}_p(1, \chi) \equiv \frac{1}{2}\mathbf{L}_p(-1, \chi) \pmod{2q}$, whence (formulas (2.2))

$$\begin{cases} \#\mathbf{K}_2(\mathbf{Z}_K)[3^{\infty}] \equiv \#\mathcal{T}_{K,3} \pmod{3} & \text{for quadratic fields,} \\ \#\mathbf{R}_2(\mathbf{Z}_K)[2^{\infty}] \equiv \#\mathcal{T}_{K,2} \pmod{8} & \text{for cyclic cubic fields.} \end{cases}$$

For real quadratic fields and $p = 3$ (resp. for cyclic cubic fields and $p = 2$), see numerical results in Appendix B.3 (resp. C.3).

5. Proof of the Deng–Li conjecture $\delta = 3$. We consider the family defined in [DL23, Theorem 1.2]. We will prove the conjecture $\delta = 3$ in the form $\mathbf{K}_2(\mathbf{Z}_K)[2^{\infty}] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^{\delta}\mathbb{Z}$ and some other properties of this family of real quadratic fields.

Recall that $K = \mathbb{Q}(\sqrt{m})$, $m = \ell_1 \cdots \ell_n$, n even, with

- (i) $\ell_1 \equiv 3 \pmod{8}$, $\ell_i \equiv 5 \pmod{8}$ for $i \geq 2$,
- (ii) $\left(\frac{\ell_1}{\ell_2}\right) = -1$, $\left(\frac{\ell_1}{\ell_j}\right) = 1$ for $j \geq 3$,
- (iii) $\left(\frac{\ell_i}{\ell_j}\right) = -1$ for $2 \leq i < j \leq n$.

Condition (i) implies $m \equiv -1 \pmod{8}$ and K of discriminant $4m$; then $\left(\frac{\ell_i}{\ell_j}\right) = \left(\frac{\ell_j}{\ell_i}\right)$ for all i, j since for $i \neq j$, one of the two primes is congruent to 1 modulo 4. It also implies that the fundamental unit ε is of norm 1 since -1 is not norm in K/\mathbb{Q} ; then, if $\varepsilon = a + b\sqrt{m}$, one has $a^2 + b^2 \equiv 1 \pmod{8}$, whence either $a = 4a'$ with b odd, or $b = 4b'$ with a odd.

Let \mathfrak{p} be the prime ideal above (2) and let $K_{\mathfrak{p}}$ be the completion of K at \mathfrak{p} ; thus $K_{\mathfrak{p}} = \mathbb{Q}_2(\mu_4)$, proving that 2 is a local norm at 2 in K/\mathbb{Q} and that the norm group of local units is equal to $1 + 4\mathbb{Z}_2$.

The Hasse norm residue symbols, of the form $\left(\frac{\ell_i, K/\mathbb{Q}}{\ell_j}\right)$, characterizing the property “ ℓ_i is a local norm at ℓ_j in K/\mathbb{Q} ”, are given by the quadratic residue symbols $\left(\frac{\ell_i}{\ell_j}\right)$; indeed, ℓ_i being a norm in $\mathbb{Q}_{\ell_j}(\sqrt{m})/\mathbb{Q}_{\ell_j}$ is equivalent (for $i \neq j$) to ℓ_i being a square in $\mathbb{Q}_{\ell_j}^{\times}$ since the norm group of local units of $\mathbb{Q}_{\ell_j}(\sqrt{m})$ is of index 2 in $\mathbb{Z}_{\ell_j}^{\times} = \mu_{\ell_j-1} \oplus (1 + \ell_j\mathbb{Z}_{\ell_j})$, where $1 + \ell_j\mathbb{Z}_{\ell_j} = (1 + \ell_j\mathbb{Z}_{\ell_j})^2$.

We can add these properties:

- (iv) $\left(\frac{2}{\ell_i}\right) = (-1)^{(\ell_i^2-1)/8} = -1$ for all i , $\left(\frac{\ell_1}{2}\right) = -1$, $\left(\frac{\ell_i}{2}\right) = 1$ for $i \geq 2$ (in the sense that ℓ_i is a norm in $\mathbb{Q}_2(\mu_4)/\mathbb{Q}_2$).
- (v) $\left(\frac{\ell_i}{\ell_j}\right) = 1$ for all $i \neq 2$ and $\left(\frac{\ell_2}{\ell_i}\right) = -1$, obtained from the product formula of Hasse's norm residue symbols of fixed ℓ_i with n even. Put $G := \text{Gal}(K/\mathbb{Q}) =: \langle \sigma \rangle$ and $\Omega_K := \{(s_1, \dots, s_{n+1}) \in G^{n+1}, s_1 \cdots s_{n+1} = 1\} \simeq G^n$; let h_K be the map $\mathbb{Q}^\times \rightarrow \Omega_K$ defined, in terms of the Hasse norm residue symbols, by $h_K(x) = \left(\left(\frac{x, K/\mathbb{Q}}{v}\right)\right)_v$, where v runs through the $n+1$ places ramified in K/\mathbb{Q} , and for which $\prod_{v \text{ ramified}} \left(\frac{x, K/\mathbb{Q}}{v}\right) = 1$ where (x) is the norm of an ideal in K/\mathbb{Q} , hence local norm at every non-ramified place (e.g., $x \in \{-1, 1\}$). This gives the following matrix of symbols, with the product formula taken on each line:

$$(5.1) \quad \begin{array}{c|cccccccc} & 2 & \ell_1 & \ell_2 & \ell_3 & \dots & \ell_i & \dots & \ell_n \\ \hline 2 & 1 & -1 & -1 & -1 & \dots & -1 & \dots & -1 \\ \ell_1 & -1 & 1 & -1 & 1 & \dots & 1 & \dots & 1 \\ \ell_2 & 1 & -1 & -1 & -1 & \dots & -1 & \dots & -1 \\ \ell_3 & 1 & 1 & -1 & 1 & \dots & -1 & \dots & -1 \\ \vdots & & & & & & & & \\ \ell_i & 1 & 1 & -1 & -1 & \dots & 1 & \dots & -1 \\ \vdots & & & & & & & & \\ \ell_n & 1 & 1 & -1 & -1 & \dots & -1 & \dots & 1 \\ \hline -1 & -1 & -1 & 1 & 1 & \dots & 1 & \dots & 1 \end{array}$$

5.1. Proof of $\mathcal{H}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$. This is proven in [DL23], but we can bring more information and remarks.

The first one is given by the Chevalley–Herbrand formula [Che34, pp. 402–406] in K/\mathbb{Q} ,

$$\#\mathcal{H}_{K,2}^G = \frac{2^{n+1}}{[K:\mathbb{Q}](E_{\mathbb{Q}}:E_{\mathbb{Q}} \cap \mathbf{N}_{K/\mathbb{Q}}(K^\times))},$$

since $n+1$ primes ramify, with $E_{\mathbb{Q}} = \{\pm 1\}$; we shall write instead, since $-1 \notin \mathbf{N}_{K/\mathbb{Q}}(K^\times)$,

$$(5.2) \quad \#\mathcal{H}_{K,2}^G = \frac{\#\Omega_K}{\#h_K(E_{\mathbb{Q}})} = \frac{2^n}{\#h_K(\{\pm 1\})} = 2^{n-1}.$$

LEMMA 5.1. *The subgroup $\mathcal{H}_{K,2}^G$ is elementary of 2-rank $n-1$; it is generated by the classes of the prime ideals $\mathfrak{l}_i \mid \ell_i$, $i = 1, \dots, n$, and by the class of $\mathfrak{p} \mid (2)$. There are two independent relations of principality of the form $\mathfrak{p}^{a_0} \prod_{i=1}^n \mathfrak{l}_i^{a_i} = (\alpha)$ between the ramified primes (where the exponents are 0 or 1), the obvious one being $\prod_{i=1}^n \mathfrak{l}_i = (\sqrt{m})$.*

Proof. We have the following classical exact sequence, where $E_K = \langle -1, \varepsilon \rangle$ is the group of units of K and where $\mathcal{H}_{K,2}^{\text{ram}}$ is the subgroup of $\mathcal{H}_{K,2}^G$ generated by the classes of the ramified primes for which $(\mathcal{H}_{K,2}^{\text{ram}})^2 = 1$:

$$1 \rightarrow \mathcal{H}_{K,2}^{\text{ram}} \rightarrow \mathcal{H}_{K,2}^G \rightarrow \{\pm 1\} \cap \mathbf{N}_{K/\mathbb{Q}}(K^\times) / \mathbf{N}_{K/\mathbb{Q}}(E_K) \rightarrow 1,$$

giving here $\mathcal{H}_{K,2}^G = \mathcal{H}_{K,2}^{\text{ram}} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$, the right term being trivial; thus, there are exactly two independent relations of principality between the ramified primes of K .

Now, from [Gra17, §4.4] generalizing our earlier papers in Annales de l'Institut Fourier, for the second element of the filtration we have

$$\#((\mathcal{H}_{K,2}/\mathcal{H}_{K,2}^G)^G) = \frac{\#\Omega_K}{\#h_K(\Lambda)}, \quad \Lambda = \langle -1, 2, \ell_1, \dots, \ell_n \rangle \mathbb{Z};$$

then $\#h_K(\Lambda) = (\Lambda : \Lambda \cap \mathbf{N}_{K/\mathbb{Q}}(K^\times)) = 2^n$, by computing norm residue symbols with the Rédei matrix (5.1) which is of rank n ; hence $\mathcal{H}_{K,2} = \mathcal{H}_{K,2}^G = \mathcal{H}_{K,2}^{\text{ram}} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$. ■

A program computing these relations is given in Appendix A.2.

5.2. The non-trivial relation for n even. Consider the relations

$$(5.3) \quad \begin{cases} \mathfrak{p}^{a_0} \prod_{i=1}^n \ell_i^{a_i} =: \mathfrak{p}^{a_0} \prod_{i \in I} \ell_i = (\alpha_0), & \alpha_0 \in K^\times, \\ \mathbf{N}_{K/\mathbb{Q}}(\alpha_0) = s 2^{a_0} \prod_{i \in I} \ell_i, & s \in \{\pm 1\}, \end{cases}$$

where I is a subset of $[1, n]$; from the trivial relation $(\sqrt{m}) = \prod_{i=1}^n \ell_i$ and from $\mathbf{N}_{K/\mathbb{Q}}(\sqrt{m}) = -m = -\prod_{i=1}^n \ell_i$ we deduce $(\alpha_0 \sqrt{m}) = (c) \mathfrak{p}^{a_0} \prod_{i=1}^n \ell_i^{\bar{a}_i}$, where $\bar{a}_i = 1 - a_i$ and $c = \prod_{i \in I} \ell_i$; taking $\bar{\alpha}_0 := \alpha_0 \sqrt{m}/c$, one gets the following equivalent complementary relations, where $\bar{I} = [1, n] \setminus I$:

$$(5.4) \quad \mathfrak{p}^{a_0} \prod_{i \in \bar{I}} \ell_i = (\bar{\alpha}_0) \quad \& \quad \mathbf{N}_{K/\mathbb{Q}}(\bar{\alpha}_0) = -s 2^{a_0} \prod_{i \in \bar{I}} \ell_i.$$

LEMMA 5.2.

(i) In (5.3) and (5.4), one has $a_0 = 1$ and the two non-trivial equivalent relations may be written

- $\mathfrak{p} \cdot \prod_{j \in J} \ell_j = (\alpha)$, $J \subseteq \{2, \dots, n\}$, $\#J \equiv 1 \pmod{2}$,
- $\mathfrak{p} \cdot \ell_1 \prod_{j \in \bar{J}} \ell_j = (\bar{\alpha})$, $\bar{J} \subseteq \{2, \dots, n\}$, $\#\bar{J} \equiv 0 \pmod{2}$.

The first relation without ℓ_1 is $\mathfrak{p} \cdot \ell_2 = (\alpha)$.

- (ii) These relations are given by the ideals $(\varepsilon + 1)$ and $(\varepsilon - 1)$ for which $(\varepsilon + 1) = C \cdot \sqrt{m}(\varepsilon - 1)$, $C \in \mathbb{Q}^\times$.
- (iii) There exists a sign s_0 such that $(\varepsilon + s_0) = (c_{s_0}) \mathfrak{p} \cdot \ell_2$, $c_{s_0} \in \mathbb{Z}$.

Proof. (i) Assume $a_0 = 0$; let $\prod_{j \in J} \mathfrak{l}_j = (\alpha)$ be the first non-trivial relation with $J \subseteq [2, n]$ (so $J \neq \emptyset$). Then $\prod_{j \in J} \ell_j = \pm \mathbf{N}_{K/\mathbb{Q}}(\alpha)$; since $\mathbf{N}_{K/\mathbb{Q}}(\alpha) \equiv 1 \pmod{4}$ and the ℓ_j 's satisfy the same congruence, we get $\prod_{j \in J} \ell_j = \mathbf{N}_{K/\mathbb{Q}}(\alpha)$. Consider the symbols $(\frac{\prod_{j \in J} \ell_j}{\ell_i})$, $i \in [1, n]$, using (5.1):

- for $i = 2$, one gets $(\frac{\prod_{j \in J} \ell_j}{\ell_2}) = (-1)^{\#J}$, equal to 1 if and only if $\#J$ is even (which solves the case $n = 2$ since in that case $J = \{2\}$);
- for $i = 1$, $(\frac{\prod_{j \in J} \ell_j}{\ell_1}) = 1$ if and only if $j = 2 \notin J$;
- for any $i \geq 3$, $(\frac{\prod_{j \in J} \ell_j}{\ell_i}) = 1$ if and only if $j = i \notin J$, since $\#J$ is even.

So we get $J = \emptyset$ (absurd), whence $\mathfrak{p} \cdot \prod_{j \in J} \mathfrak{l}_j = (\alpha)$ for $J \subseteq [2, n]$.

Then $(\frac{2 \prod_{j \in J} \ell_j}{\ell_i}) = 1$ for all $i \in [1, n]$ since $\mathbf{N}_{K/\mathbb{Q}}(\alpha) = -2 \prod_{j \in J} \ell_j$ does not hold (-1 is not a local norm at 2); so this is equivalent to $(\frac{\prod_{j \in J} \ell_j}{\ell_i}) = -1$ for all $i \in [1, n]$.

The case $i = 2$ gives $\#J$ odd; the case $i = 1$ implies that $2 \in J$. Assume there exists $i \geq 3$ such that $i \in J$; then, since $(\frac{\ell_i}{\ell_i}) = 1$, $(\frac{\ell_2}{\ell_i}) = -1$, and $(\frac{\ell_j}{\ell_i}) = -1$ for all $j \in J \setminus \{2, i\}$, yielding $(-1)^{\#J-2} = -1$, one obtains a contradiction. So, the relation is $\mathfrak{p} \cdot \mathfrak{l}_2 = (\alpha)$; note that the $(n+1) \times (n+1)$ matrix obtained from (5.1) by omitting the last row is of rank n and the sum of the two rows corresponding to 2 and ℓ_2 confirms that $2\ell_2 \in \mathbf{N}_{K/\mathbb{Q}}(K^\times)$.

(ii) We have $(\varepsilon + s)^{1-\sigma} = s\varepsilon$; so $(\varepsilon + s)$ is a principal invariant ideal, of the form $(c_s)\mathfrak{p}\mathfrak{m}_s$, $c_s \in \mathbb{Z}$, $\mathfrak{p} \mid (2)$, $\mathfrak{m}_s \mid (\sqrt{m})$. Then $(\varepsilon + 1) = C \cdot \sqrt{m}(\varepsilon - 1)$ since $(\frac{\varepsilon+1}{\varepsilon-1} \cdot \frac{1}{\sqrt{m}})^{1-\sigma} = 1$.

(iii) This follows from the preceding points. ■

LEMMA 5.3. *The case b even does not occur. Then in Lemma 5.2(ii, iii), the integers c_s , C are odd.*

Proof. Assume that b is even. Then $b = 4b'$. There exists a sign $s = \pm 1$ such that $a + s \equiv 2 \pmod{4}$, so that $\varepsilon + s = a + s + 4b'\sqrt{m}$ and the ideal $(\varepsilon + s) = (a + s + 4b'\sqrt{m})$ is of the form $(2)(a'' + 2b'\sqrt{m})$ with a'' odd, which cannot give the non-trivial relations of Lemma 5.2(i) since $a'' + 2b'\sqrt{m}$ is “odd” (absurd). Thus, $a = 4a'$, and b is odd, and computing $(\varepsilon + s)^2$ gives immediately c_s and C odd. ■

LEMMA 5.4. *If we write $\varepsilon = a + b\sqrt{m}$, then $a = 4a'$ with a' odd.*

Proof. Assume that $a' = 2a''$, and put $\theta_{s'} := \varepsilon + s' = 8a'' + s' + b\sqrt{m}$, $s' \in \{\pm 1\}$. Using $mb^2 = a^2 - 1$, we have $\mathbf{N}_{K/\mathbb{Q}}(\theta_{s'}) = 2(8a''s' + 1)$. From Lemma 5.2(iii) the non-trivial relation may be written (where c is an odd integer as in Lemma 5.3) $(\theta_{s'}) = (c)\mathfrak{p}\mathfrak{l}_2$, giving, by taking the norm, the congruence $8a''s' + 1 = sc^2\mathfrak{l}_2 \equiv 5s \pmod{8}$, $s \in \{\pm 1\}$, which is absurd. ■

5.3. Structure of $\mathcal{T}_{K,2}$ and triviality of $\mathcal{R}_{K,2}$. Let us give the Kummer generators of the maximal subextension of $H_{K,2}^{\text{pr}}/K$ of exponent 2.

LEMMA 5.5. *The module $\mathcal{T}_{K,2}$ is of \mathbb{Z}_2 -rank $n - 1$ and the maximal subextension of $H_{K,2}^{\text{pr}}/K_\infty$, of exponent 2, is $K_\infty(\sqrt{\varepsilon}, \sqrt{\ell_1}, \dots, \sqrt{\ell_n})$, with the two independent relations $\sqrt{\ell_1} \cdots \sqrt{\ell_n} = \sqrt{m} \in K^\times$ and $\sqrt{2}\sqrt{\ell_2} = \alpha\sqrt{\varepsilon}$, $\alpha \in K^\times$.*

Proof. The maximal subextension of $H_{K,2}^{\text{pr}}/K$ of exponent 2 is the Kummer extension $K(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{\ell_1}, \dots, \sqrt{\ell_n})$ (all the radicals a in the \sqrt{a} 's are squares of ideals of K and are totally positive; there are no other radicals since $\mathcal{H}_{K,2} = \mathcal{H}_{K,2}^{\text{ram}}$). We utilize the fact that $\sqrt{2} \in K_\infty$, and the relations $\ell_1 \cdots \ell_n = m \in K^{\times 2}$ and $\mathfrak{pl}_2 = (\alpha)$ (Lemma 5.2), $\alpha \in K^\times$, giving, by squaring, $2\ell_2 = \alpha^2\eta > 0$, $\eta \in E_K$, $\eta \neq 1$; then we may assume that $\eta = \varepsilon$, whence $2\ell_2 = \alpha^2\varepsilon$.

The result follows since $\mathcal{T}_{K,2}$ is a direct factor in $\text{Gal}(H_{K,2}^{\text{pr}}/K)$. ■

THEOREM 5.6. *The normalized 2-adic regulator $\mathcal{R}_{K,2}$ is trivial, and $\mathcal{T}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-2} \times \mathbb{Z}/4\mathbb{Z}$.*

Proof. We have $\varepsilon = a + b\sqrt{m} = 4a' + b\sqrt{m}$ with $a^2 - mb^2 = 1$. Then

$$\varepsilon^2 = a^2 + 2ab\sqrt{m} + mb^2 = 1 + 2mb^2 + 8a'b\sqrt{m}.$$

Recall that a' is odd (Lemma 5.4), b is odd and $m \equiv -1 \pmod{8}$; thus $\varepsilon^2 \equiv 1 - 2 + 8\sqrt{m} \pmod{16}$, whence $\log_2(\varepsilon) \sim 4$.

From [Gra18, Proposition 5.2], we have, in our context,

$$\#\mathcal{R}_{K,2} \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_2 : \log(\mathbf{N}_{K/\mathbb{Q}}(\mathcal{U}_{K,2})))}{\#\mathcal{W}_{K,2} \cdot \prod_{\mathfrak{p}|2} \mathbf{N}\mathfrak{p}} \cdot \frac{R_K}{\sqrt{D_K}},$$

where $R_K = \log_2(\varepsilon)$ is the usual 2-adic regulator and D_K the discriminant of K . Since $(\mathbb{Z}_2 : \log(\mathbf{N}_{K/\mathbb{Q}}(\mathcal{U}_{K,2}))) = (\mathbb{Z}_2 : \log(1 + 4\mathbb{Z}_2)) = 4$, $R_K \sim 4$, $\#\mathcal{W}_{K,2} = 2$, $\prod_{\mathfrak{p}|2} \mathbf{N}\mathfrak{p} = 2$, $\sqrt{D_K} \sim 2$, this yields $\#\mathcal{R}_{K,2} \sim 1$.

Formula (1.5) gives $\#\mathcal{T}_{K,2} = 2^{n-1} \times 1 \times 2 = 2^n$, and Corollary 4.7 ends the proof of the Deng–Li conjecture $\delta = 3$ and gives the structure of $\mathcal{T}_{K,2}$. ■

6. On the p -rank of $\mathbf{K}_2(\mathbf{Z}_K)$ when $p \geq 5$. We consider a prime number $p \geq 5$ and a cyclic p -extension K/k fulfilling some conditions:

THEOREM 6.1 (Lower bound of $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K))$, $p \geq 5$). *Let $p \geq 5$ be a prime. Let k be a real abelian number field of prime-to- p degree, and let K be any cyclic p -extension of k , abelian over \mathbb{Q} . Assume that $k \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$ and p does not totally split in $k(\mu_p)/k$. Let S_{ta} be the set of primes $\ell \neq p$ ramified in K/k , and let $t_\ell \geq 1$ be the number of prime ideals above ℓ in k/\mathbb{Q} . Then $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \sum_{\ell \in S_{\text{ta}}} t_\ell$.*

In particular, for all primes $\ell \equiv 1 \pmod{p}$ and for any non-trivial p -extension K/\mathbb{Q} contained in $\mathbb{Q}(\mu_\ell)/\mathbb{Q}$, we have $\#\mathbf{K}_2(\mathbf{Z}_K) \equiv 0 \pmod{p}$.

Proof. Let ζ_p be a generator of μ_p , and set $\mathbb{Q}' := \mathbb{Q}(\zeta_p)$, $k' := k(\zeta_p)$ and $K' := K(\zeta_p)$; put $G := \text{Gal}(K'/k')$ and $g := \text{Gal}(K'/K)$. Recall that $\omega : g \rightarrow \mu_{p-1}(\mathbb{Z}_p)$ is the Teichmüller character (such that $\zeta_p^s =: \zeta_p^{\omega(s)}$ for all $s \in g$) and that any $\mathbb{Z}_p[g]$ -module X , of finite type, is the sum $\bigoplus_{j=1}^{p-1} X_{\omega^j}$ in the obvious sense. Since $p \geq 5$, we have $\omega \neq \omega^{-1}$, and the comparison of $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ with $\mathcal{T}_{K,p}$ made in [GJ89] and (1.8) does not hold.

The primes of $S_p(K)$ not being totally split in K'/K , the ω^{-1} -component of the $\mathbb{Z}_p[G]$ -module $\langle S_p(K') \rangle \otimes \mathbb{Z}_p$ generated by $S_p(K')$ above $S_p(K)$ is trivial; thus, formula (1.6) becomes $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) = \text{rk}_p((\mathcal{H}_{K',p})_{\omega^{-1}})$.

We note, from the local reciprocity law, that a prime $\ell \neq p$ ramified in K/\mathbb{Q} with ramification index divisible by p fulfills the condition $\ell \equiv 1 \pmod{p}$ (for generalizations to the non-Galois case, see [Gra05, §II.2.d]); hence ℓ totally splits in \mathbb{Q}'/\mathbb{Q} and, for $\mathfrak{l}|\ell$ in k , the $p-1$ prime ideals \mathfrak{L} above \mathfrak{l} in k' ramify in K'/k' with the same ramification index; this provides $(p-1)t_\ell$ ramified ideals in K'/k' , whence

$$\#\text{Sta}(k') = (p-1) \sum_{\ell \in \text{Sta}} t_\ell.$$

Let $G_v \subseteq G$ be the inertia group, in K'/k' , of a ramified place v of k' (so $v|\ell$ or $v|p$, but the ω^{-1} -component of $\langle S_p(k') \rangle \otimes \mathbb{Z}_p$ is trivial since p does not totally split in k'/k); put $S(k') = \text{Sta}(k') \cup S_p(k')$. Let $E_{k'}$ be the group of units of k' , $K'^{\text{nr}} := K' \cap H_{k'}^{\text{nr}}$, and let $\mathcal{H}_{k',p}^{\text{nr}} \subseteq \mathcal{H}_{k',p}$ correspond to $\text{Gal}(H_{k'}^{\text{nr}}/K' \cap H_{k'}^{\text{nr}})$; the Chevalley–Herbrand formula in K'/k' reads

$$\#\mathcal{H}_{K',p}^G = \frac{\#\mathcal{H}_{k',p} \times \prod_v \#G_v}{[K' : k'](E_{k'} : E_{k'} \cap \mathbf{N}_{K'/k'}(K'^{\times}))} = \#\mathcal{H}_{k',p}^{\text{nr}} \times \frac{\#\Omega_{k'}}{\#h_{k'}(E_{k'})},$$

where $\Omega_{k'} = \{(s_v)_v \in \bigoplus_{v \in S(k')} G_v, \prod_v s_v = 1\}$, $h_{k'} : E_{k'} \rightarrow \Omega_{k'}$ being defined by the family of norm residue symbols $((\frac{x, K'/k'}{v}))_v$ which fulfill the “product formula” on $E_{k'}$. This formula is associated to the exact sequences

$$(6.1) \quad \begin{cases} 1 \rightarrow \mathbf{J}_{K'/k'}(\mathcal{H}_{k',p}) \cdot \mathcal{H}_{K',p}^{\text{ram}} \rightarrow \mathcal{H}_{K',p}^G \\ \quad \quad \quad \rightarrow E_{k'} \cap \mathbf{N}_{K'/k'}(K'^{\times}) / \mathbf{N}_{K'/k'}(E_{K'}) \rightarrow 1, \\ 1 \rightarrow E_{k'}/E_{k'} \cap \mathbf{N}_{K'/k'}(K'^{\times}) \rightarrow \Omega_{k'} \rightarrow \text{Gal}(H_{K'/k'}^{\text{gen}}/K' H_{k'}^{\text{nr}}) \rightarrow 1, \end{cases}$$

where $\mathbf{J}_{K'/k'}$ is the extension of classes, $\mathcal{H}_{K',p}^{\text{ram}} \subseteq \mathcal{H}_{K',p}$ is generated by the ramified primes, and $H_{K'/k'}^{\text{gen}} \subseteq H_{K'}^{\text{nr}}$ is the genus field, fixed by $\mathcal{H}_{K',p}^{1-\sigma}$, where σ generates G [Gra05, Proposition IV.4.5]. The link between the two aspects (fixed points and genus exact sequences) is given by $\text{Gal}(H_{K'/k'}^{\text{gen}}/K') \simeq \mathcal{H}_{K',p}/\mathcal{H}_{K',p}^{1-\sigma}$ and by the exact sequence $1 \rightarrow \mathcal{H}_{K',p}^G \rightarrow \mathcal{H}_{K',p} \rightarrow \mathcal{H}_{K',p}^{1-\sigma} \rightarrow 1$.

More precisely, we need the ω^{-1} -components of $\mathcal{H}_{K',p}^G$ and of the terms of the exact sequences. For $\mathcal{H}_{K',p}^G$, the computation is done in Jaulent's thesis [Jau86, Chapitre III] and reproduced in [Gra23^a, §2.2, Theorem 2.1(ii)]; the difficulty comes from the fact that $\mathcal{H}_{k',p}$ is not necessarily isomorphic to a submodule of $\mathcal{H}_{K',p}$ (indeed, only $\mathbf{J}_{K'/k'}(\mathcal{H}_{k',p})$ makes sense as a submodule of $\mathcal{H}_{K',p}$, and $\mathbf{J}_{K'/k'}$ may be non-injective). In our context,

$$(6.2) \quad \#\mathcal{H}_{K',p,\omega^{-1}}^G = \#\mathcal{H}_{k',p,\omega^{-1}}^{\text{nr}} \times \frac{\#\Omega_{k',\omega^{-1}}}{\#h_{k'}(E_{k'})_{\omega^{-1}}}.$$

Since $E_{k'} = E_{k'+} \oplus \mu_p$ (up to a prime-to- p index), where the subgroup $E_{k'+}$ of real units of k' is of even character, and since μ_p is of character $\omega \neq \omega^{-1}$, the ω^{-1} -component of $h_{k'}(E_{k'})$ is trivial, and the exact sequences (6.1) and formula (6.2) reduce to

$$(6.3) \quad \begin{cases} \mathcal{H}_{K',p,\omega^{-1}}^G \simeq (\mathbf{J}_{K'/k'}(\mathcal{H}_{k',p}) \cdot \mathcal{H}_{K',p}^{\text{ram}})_{\omega^{-1}}, \\ \Omega_{k',\omega^{-1}} \simeq \text{Gal}(H_{K'/k'}^{\text{gen}}/K'H_{k'}^{\text{nr}})_{\omega^{-1}}, \\ \#\mathcal{H}_{K',p,\omega^{-1}}^G = \#\mathcal{H}_{k',p,\omega^{-1}}^{\text{nr}} \times \#\Omega_{k',\omega^{-1}}. \end{cases}$$

Hence

$$(6.4) \quad \text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) = \text{rk}_p(\mathcal{H}_{K',p,\omega^{-1}}) \geq \text{rk}_p((\mathcal{H}_{K',p}/\mathcal{H}_{K',p}^{1-\sigma})_{\omega^{-1}}) \\ \geq \text{rk}_p(\Omega_{k',\omega^{-1}}).$$

Let $G' := \langle G_v \rangle_v \simeq \text{Gal}(K'/K'^{\text{nr}})$; the product formula of class field theory is interpreted by means of the exact sequence $1 \rightarrow \Omega_{k'} \rightarrow \bigoplus_{v \in S(k')} G_v \rightarrow G' \subseteq G \rightarrow 1$, in which the character of G is the unit character and the ω^{-1} -component of $\bigoplus_{v \in S_p(k')} G_v$ is trivial since p does not totally split in k'/k . Hence the isomorphism $(\Omega_{k'})_{\omega^{-1}} \simeq \bigoplus_{v \in S_{\text{ta}}(k')} G_v$.

For $\ell \in S_{\text{ta}}$, let p^{e_ℓ} , $e_\ell \geq 1$, be the ramification index of ℓ in K'/k' . The inertia groups G_v do not depend on v , for $v|\ell$ in k' , and may all be denoted G_ℓ . Since each of the t_ℓ primes $\mathfrak{l}|\ell$ in k totally splits in k'/k into $p-1$ primes $\mathfrak{L}|\mathfrak{l}$ of k' , one may write $\bigoplus_{\mathfrak{L}} G_{\mathfrak{L}} = \bigoplus_{\ell \in S_{\text{ta}}} \bigoplus_{\mathfrak{l}|\ell} \bigoplus_{\mathfrak{L}|\mathfrak{l}} G_\ell \simeq \bigoplus_{\ell \in S_{\text{ta}}} (\bigoplus_{\mathfrak{L}|\mathfrak{l}} G_\ell)^{t_\ell}$, where each ℓ -component is isomorphic, as a g -module, to t_ℓ copies of the regular representation whose ω^{-1} -component is of order p^{e_ℓ} . Hence we get the group isomorphism $(\Omega_{k'})_{\omega^{-1}} \simeq \bigoplus_{\ell \in S_{\text{ta}}} (\mathbb{Z}/p^{e_\ell}\mathbb{Z})^{t_\ell}$, and, from (6.4), the inequality $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \sum_{\ell \in S_{\text{ta}}} t_\ell$. ■

COROLLARY 6.2. *We have the following consequences, still for $p \geq 5$:*

- (i) $\#\mathcal{H}_{k',p,\omega^{-1}}^{\text{nr}} \times \#\Omega_{k',\omega^{-1}} = \#(\mathbf{J}_{K'/k'}(\mathcal{H}_{k',p}) \cdot \mathcal{H}_{K',p}^{\text{ram}})_{\omega^{-1}}$.
- (ii) If $\mathbf{K}_2(\mathbf{Z}_k)[p^\infty] = 1$, $\mathcal{H}_{K',p,\omega^{-1}}^{\text{ram}} \simeq \bigoplus_{\ell \in S_{\text{ta}}} (\mathbb{Z}/p^{e_\ell}\mathbb{Z})^{t_\ell}$.
- (iii) If $k = \mathbb{Q}$, $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \#S_{\text{ta}}$ & $\mathcal{H}_{K',p,\omega^{-1}}^{\text{ram}} \simeq (\mathbb{Z}/p^{e_\ell}\mathbb{Z})^{\#S_{\text{ta}}(k)}$.
- (iv) $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \leq \text{rk}_p(\mathcal{H}_{K',p,\omega^{-1}}^G) \leq \text{rk}_p(\mathcal{H}_{k',p,\omega^{-1}}) + \text{rk}_p(\mathcal{H}_{K',p,\omega^{-1}}^{\text{ram}})$.

Proof. Clause (ii) holds since $\mathbf{K}_2(\mathbf{Z}_k)[p^\infty] = 1$ is equivalent to $\mathcal{H}_{k',p,\omega^{-1}} = 1$; then we apply (6.3). If $k = \mathbb{Q}$, $\sum_{\ell \in S_{\text{ta}}} t_\ell = \#S_{\text{ta}}$; then $\mathcal{H}_{\mathbb{Q}',p,\omega^{-1}} = 1$ from Ribet's reciprocal of Herbrand's criterion, since the Bernoulli number $B_2 = \frac{1}{6}$ is prime to p [Rib08, Theorem 5, p. 45], which gives (iii); and (iv) is clear. ■

REMARKS 6.3. (i) For $p = 3$, we have $\omega^{-1} = \omega$, so that $\mathcal{H}_{K',p,\omega^{-1}}^G$ depends on $\frac{\#\Omega_{\mathbb{Q}',\omega^{-1}}}{\#h_{\mathbb{Q}'}(\mu_3)}$, which may be trivial, when $S_{\text{ta}} = \{\ell\}$ and $[K : \mathbb{Q}] = 3$.

(ii) If we replace $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ by $\mathcal{T}_{K,p}$ (for whatever $p \geq 3$) and ω^{-1} by ω , $h_{\mathbb{Q}'}(\mu_p)$ may be non-trivial, being of character ω . So, if K/\mathbb{Q} is of prime conductor ℓ , then $\mathcal{H}_{K,p} = 1$ and we may have $\mathcal{T}_{K,p} = 1$ but $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] \neq 1$.

(iii) For a table of numerical examples, see Appendix D.1 for the computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$, $p \geq 3$, and Appendix D.2 for $\text{rk}_5(\mathbf{K}_2(\mathbf{Z}_K)[5^\infty])$. Numerical tables are also given in [Bro05] for cyclic cubic fields; other computations of p -ranks are given in [Qin95, Qin05, Qin10].

7. Conclusion. These explicit calculations lead to confirmation of the properties of the \mathbf{L}_p -functions when s varies in \mathbb{Z}_p and takes the values $s \in \{-1, 1\}$, giving annihilation theorems and the orders of the isotypic components of classical p -invariants. Invariants other than class groups, torsion groups and regular kernels exist, for instance the Jaulent logarithmic class group $\widetilde{\mathcal{H}}_{K,p}$ ⁽²⁾ [Jau94^b, Jau21, Jau24] linked to the wild Hilbert kernel [Jau94^a, Théorème 9] and in relation to Greenberg's conjecture [Jau19, Jau20] (see Appendices B.2, C.2 below for some computations), and invariants of the higher \mathbf{K} -theory (giving conjectural expressions suggested in [Gra98, Section 12, Conjecture 12.2] and some results of [JM06]).

An important problem is to find the structure of these invariants, it being understood that their orders, once the genus part has been taken into account, have standard densities in the spirit of the Cohen–Lenstra–Martinet–Malle distributions. A specific technique, of genus theory type, does exist in cyclic p -extensions, especially for p -class groups by determining the canonical filtration by means of a natural algorithm generalizing the Chevalley–Herbrand fixed points formula (large bibliography, synthesized in [Gra17]).

A main question is to find such algorithms for the other p -invariants, which unfortunately do not exist to our knowledge; the case of the most important groups $\mathcal{T}_{K,p}$ is very interesting; indeed, class field theory (by means

⁽²⁾ Also denoted $\widetilde{\mathcal{C}}_{K,p}$, or $\widetilde{\mathcal{T}}_{K,p}$ as a quotient of $\mathcal{T}_{K,p}$; but it looks more like $\mathcal{H}_{K,p}$ and gives rise to $\#\mathcal{T}_{K,p} = \widetilde{\mathcal{H}}_{K,p} \widetilde{\mathcal{R}}_{K,p} \widetilde{\mathcal{W}}_{K,p}$, for a suitable “logarithmic regulator” [Jau19, Schema, §2.3]. Moreover it may capitulate in real p -extensions of K (the key for Greenberg's conjecture [Gre76]), while $\mathcal{T}_{K,p}$ never capitulates under Leopoldt's conjecture.

of computation of suitable ray class groups) gives the group structure, but this computational aspect does not give a method to determine the filtration in cyclic p -extensions; only the fixed points formula is known [Gra05, Theorem IV.3.3].

From a cohomological viewpoint, we know that $\mathcal{T}_{K,p}$ is closely related to the Tate–Shafarevich group $\text{III}_{K,p}^2 := \text{Ker}[\text{H}^2(\mathcal{G}_{K,S}, \mathbb{F}_p) \rightarrow \bigoplus_{\mathfrak{p}|p} \text{H}^2(\mathcal{G}_{K_{\mathfrak{p}}}, \mathbb{F}_p)]$, and that $\mathcal{T}_{K,p} \simeq \text{H}^2(\mathcal{G}_{K,S}, \mathbb{Z}_p)^*$ [Ngu86, Théorème 1.1], where $\mathcal{G}_{K,S}$ is the Galois group of the maximal p -ramified pro- p -extension of K and $\mathcal{G}_{K_{\mathfrak{p}}}$ is the local analog over $K_{\mathfrak{p}}$; generalizations of the same kind exist in the literature, as in [Kol03, Theorems 0.4, 0.7, 0.11, . . .], [AAM21, Theorem 3.5, Example 4.3], [Lim23, Theorems 4.2, 4.5], with an analogous cohomological framework, without any workable algorithm.

A major difficulty is to interpret such cohomology groups in terms of ideals, global units and so on; the case of $\mathcal{T}_{K,p} \simeq \text{H}^2(\mathcal{G}_{K,S}, \mathbb{Z}_p)^*$ is edifying since $\mathcal{G}_{K,S}$ is inaccessible, while $\mathcal{T}_{K,p}$ depends on the p -class group and on the normalized p -adic regulator, for which heuristics are given in [Gra16^a], in complete generality for Galois fields. Of course, cohomology provides important and non-obvious relations (see, e.g., [Ngu86] for links between $\mathcal{T}_{K,p}$ and $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ in the framework of Iwasawa’s theory) but, in the previous example, it is the knowledge of $\mathcal{T}_{K,p}$ which yields important information on $\mathcal{G}_{K,S}$ (for instance the case of p -rationality, $\mathcal{T}_{K,p} = 1$, which we talked about).

Maybe the link we made in [Gra05, §III.2, Theorem III.2.6, Proposition III.3.3] and [GJ89, Appendice] to our Crellé’s papers, using logarithms of ideals for effectiveness of the arithmetic, may be generalized.

We point out that these analytic genus theory aspects of p -adic pseudo-measures are stated in [Gra86^b] for any totally real base field k replacing \mathbb{Q} (under Leopoldt’s conjecture for p , to get the important result of deployment [Gra05, Theorem III.4.1.5]). This allows the use of similar techniques with genus fields deduced from Deligne–Ribet pseudo-measures; these results are completed with (for instance) the article by Maire [Mai18] on the class field theory aspects and governing fields about the *existence* of prescribed p -cyclic ramified extensions K/k , for which an analog of Theorem 4.4 should apply, giving a natural generalization of some results to the non-abelian case.

Appendices: Numerical computations – PARI programs. All the PARI programs given in these Appendices can be copied and pasted from the pdf file of the article.

Appendix A. Data relating to the Deng–Li family. This section is concerned with the family of quadratic fields described and studied in Section 5; we illustrate and confirm various numerical aspects.

A.1. Computation of $\mathcal{H}_{K,p}$, $\mathcal{T}_{K,p}$ and $\widetilde{\mathcal{H}}_{K,p}$. We compute the three fundamental invariants for these families, it being understood that the structure of $\mathcal{H}_{K,p}$ and complex-analytic properties are given in [DL23] in order to study $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$.

A.1.1. General program for $\mathcal{H}_{K,p}$ and $\mathcal{T}_{K,p}$. It may be convenient to recall the program computing the structures of $\mathcal{H}_{K,p}$ and $\mathcal{T}_{K,p}$ for a number field K , for any $\mathfrak{p} \in [\mathfrak{bp}, \mathfrak{Bp}]$, as soon as a defining polynomial $P \in \mathbb{Z}[X]$ (monic, irreducible) is given (in \mathfrak{P}); for the computation of $\mathcal{T}_{K,p}$, the parameter nu must be chosen such that \mathfrak{p}^{nu} be larger than the exponent of the result, \mathfrak{T} [Gra19^a, §2.1]; the number \mathfrak{r} is the number $r_2 + 1$ of independent \mathbb{Z}_p -extensions of K :

```
{P=x^3-7*x+1;bp=2;Bp=5*10^5;K=bnfinit(P,1);r=K.sign[2]+1;
print("P=",P," H=",K.cyc);forprime(p=bp,Bp,nu=6;
Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
R=0;for(k=1,e-r,c=HKn[e-k+1];v=valuation(c,p);if(v>0,R=R+1;
listinsert(T,p^v,1));if(R>0,print("p=",p," rk(T)=",R," T=",T)))}
P=x^3-7*x+1    H=[]    P=x^8-8*x+1    H=[4]    P=x^8-5*x^6+6*x^2-1 H=[]
p=7    rk(T)=1 T=[7]    p=2    rk(T)=1 T=[4]    p=2    rk(T)=2    T=[8,2]
p=701  rk(T)=1 T=[701]  p=3    rk(T)=1 T=[3]    p=5    rk(T)=1    T=[5]
```

A.1.2. Case of the Deng–Li family. The following program determines the structure of $\mathcal{H}_{K,2}$ and $\mathcal{T}_{K,2}$ for the Deng–Li family of quadratic fields, for $n = 4$:

```
{L=List
([7215,26455,77415,119535,142935,153735,166335,171015,196359,226655,241215,
243295,257335,283855,311415,315055,420135,430495,447135,473415,475215,
490295,504295,545415,550615,552695,553335,563695,568815,592215,603655,
606615,633399,657735,665223,673215,685815,687895,727935,751335,755495,
757055,790495,798135,803751,807455,818935,833199,849615,878415,884455,
886015,886335,896415,905255,911495,934935,961935,973655,981695,990015]);
p=2;nu=9;for(j=1,61,P=x^2-L[j];K=bnfinit(P,1);HK=K.cyc;Kpn=bnrinit(K,p^nu);
HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];for(k=1,e-1,c=HKn[e-k+1];
v=valuation(c,p);if(v>0,listput(T,p^v));print("P=",P," T=",T," H=",HK))}
P=x^2-7215    T=[2,2,4] H=[2,2,2]    P=x^2-981695 T=[2,2,4] H=[10,2,2]
P=x^2-26455  T=[2,2,4] H=[2,2,2]    P=x^2-990015 T=[2,2,4] H=[18,2,2]
```

From (1.8), the 2-ranks of $\mathbf{R}_2(\mathbf{Z}_K)$ and $\mathcal{T}_{K,2}$ are equal to 3; this implies, since $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/2^3\mathbb{Z}$, that

$$\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/2^2\mathbb{Z} \simeq \mathcal{T}_{K,2}.$$

More generally, for the Deng–Li family, we have

$$\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \simeq \mathcal{T}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-2} \times \mathbb{Z}/2^2\mathbb{Z}, \quad \mathcal{H}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}.$$

From this, we verify that $\mathcal{W}_{K,2} \simeq \mathbb{Z}/2\mathbb{Z}$, which is in agreement with the formula $\#\mathcal{T}_{K,2} = \#\mathcal{H}_{K,2} \times \#\mathcal{R}_{K,2} \times \#\mathcal{W}_{K,2}$ since the normalized 2-adic regulator is trivial from Theorem 5.6.

A.1.3. Logarithmic class group. The computation of $\widetilde{\mathcal{H}}_{K,2}$ (first component of the command $\text{CLog} = \text{bnflog}(K, 2)$) for the Deng–Li family always gives the structure $(\mathbb{Z}/2\mathbb{Z})^2$ for $n = 4$; but we do not know if this invariant is given by the $\mathbf{L}_2(s, \chi)$'s at some $s \in \mathbb{Z}_2$ (see §B.2 for the computation of $\widetilde{\mathcal{H}}_{K,2}$ for quadratic fields and §C.2 for the cubic case):

```
P=x^2-7215      CLog=[[2,2], [], [2,2]]  T=[2,2,4] H=[2,2,2]
P=x^2-26455     CLog=[[2,2], [], [2,2]]  T=[2,2,4] H=[2,2,2]
(...)
P=x^2-981695    CLog=[[2,2], [], [2,2]]  T=[2,2,4] H=[10,2,2]
P=x^2-990015    CLog=[[2,2], [], [2,2]]  T=[2,2,4] H=[18,2,2]
```

Moreover, the third component of $\text{bnflog}(K, 2)$ gives the S_2 -class group $\mathcal{H}_{K,2}/\text{cl}(\mathfrak{p}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where $\mathfrak{p} \mid 2$ is ramified; this result confirms that necessarily \mathfrak{p} is non-principal of order 2 (Lemma 5.2(i)).

A.2. The non-trivial relation. The program provides the canonical non-trivial relation whose expression does not contain ℓ_1 (with $\ell_1 \equiv 3 \pmod{8}$), and then that obtained multiplying by (\sqrt{m}) . We have proven that this relation is always $\mathfrak{p}_2 = (\alpha)$ (Lemma 5.2). The list L gives the exponents a_0, a_1, a_2, \dots of the ideals $\mathfrak{p}, \ell_1, \ell_2, \dots$ in this order especially for the particular primes ℓ_1, ℓ_2 (which is not immediate since the PARI factorization of m gives primes in ascending order). The generator α of the relation of principality is $\alpha = u + v\sqrt{m}$; its norm N and the sign S of N are computed as a check.

```
{m=List
([7215,26455,77415,119535,142935,153735,166335,171015,196359,226655,241215,
243295,257335,283855,311415,315055,420135,430495,447135,473415,475215,
490295,504295,545415,550615,552695,553335,563695,568815,592215,603655,
606615,633399,657735,665223,673215,685815,687895,727935,751335,755495,
757055,790495,798135,803751,807455,818935,833199,849615,878415,884455,
886015,886335,896415,905255,911495,934935,961935,973655,981695,990015]);
p=2;for(j=1,61,M=m[j];D=quaddisc(M);r=omega(D);L0=List;
for(i=1,r,listput(L0,0));P=x^2-M;K=bnfinit(P,1);print();Lel=List;
Div=component(factor(D),1);listput(Lel,2,1);h=3;for(i=2,r,c=Div[i];
if(Mod(c,4)==-1,e1=c;listput(Lel,c,2)));for(i=2,r,c=Div[i];
if(c!=e1 & kronecker(c,e1)==-1,e2=c;listput(Lel,c,3)));
for(i=2,r,c=Div[i];if(c!=e2 & Mod(c,4)==1,h=h+1;listput(Lel,c,h)));
print("M=",M," D=",Lel);for(k=1,2^r-1,B=binary(k);t=#B;
LB=L0;for(i=1,t,listput(LB,B[i],r-t+i));F=idealfactor(K,D);
Fel=List;CFel=component(F,1);listput(Fel,CFel[1],1);h=3;
for(i=2,r,c=CFel[i][1];if(Mod(c,4)==-1,e1=c;listput(Fel,CFel[i],2)));
for(i=2,r,c=CFel[i][1];if(c!=e1 & kronecker(c,e1)==-1,e2=c;
listput(Fel,CFel[i],3)));
for(i=2,r,c=CFel[i][1];if(c!=e2 & Mod(c,4)==1,h=h+1;
listput(Fel,CFel[i],h)));
A=1;for(j=1,r,e=LB[j];if(e==0,next);A=idealmul(K,A,Fel[j]);
Q=bnfisprincipal(K,A);u=Q[2][1];v=Q[2][2];
```

```

if(Q[1]==0 & u!=0,print("L=",LB);N=u^2-M*v^2;S=sign(N);
print(component(factor(abs(N)),1));
print("u=",u," v=",v," S=",sign(S))))))}

```

Let us give some excerpts showing all possible cases:

M=7215 D=[2,3,5,13,37]	M=26455 D=[2,11,13,5,37]
L=[1,0,1,0,0]	L=[1,0,1,0,0]
[2,5]	[2,13]
u=-85 v=1 S=1	u=86654841 v=-532769 S=1
L=[1,1,0,1,1]	L=[1,1,0,1,1]
[2,3,13,37]	[2,5,11,37]
u=1443 v=17 S=-1	u=1084184915 v=-6665757 S=-1
M=504295 D=[2,11,173,5,53]	M=665223 D=[2,3,461,13,37]
L=[1,0,1,0,0]	L=[1,0,1,0,0]
[2,173]	[2,461]
u=7970980451749 v=11224562253 S=1	u=10603 v=13 S=1
L=[1,1,0,1,1]	L=[1,1,0,1,1]
[-1,2,5,11,53]	[2,3,13,37]
u=-32719598967495 v=46075031513 S=-1	u=18759 v=23 S=-1

Appendix B. Quadratic fields

B.1. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$, $\mathcal{T}_{K,2}$ and \mathbf{C} . The following program computes, instead, $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]$ for any real quadratic field (so its conductor f is m or $4m$) and $\mathcal{T}_{K,2}$ to illustrate Theorem 4.4 (cases (ii)₂) and (iii)₂) and property (b) about equality vs. inequality, for all $s \in \mathbf{Z}_p$.

In those cases, $\epsilon = 1$, and \mathbf{D} is the number of odd ramified primes, and $\mathbf{C} = \mathbf{D}$ (resp. $\mathbf{C} = \mathbf{D} - 1$) if $\theta^{-1}\chi(2) = 1$ (resp. $\theta^{-1}\chi(2) \neq 1$).

Recall that nu and n must be chosen large enough (below we take $n = 10$ for $m > 100$, $n = 13$ for $m > 10^3$).

We do not write out the trivial cases $\mathcal{T}_{K,2} = \mathbf{R}_2(\mathbf{Z}_K)[2^\infty] = 1$.

```

{p=2;q=4;nu=10;n=6;for(m=5,2000,if(core(m)!=m,next);f=quaddisc(m);
D=omega(m);if(Mod(m,2)==0,D=D-1);C=D;if(Mod(m,8)!=-1,C=D-1);
P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^nu);H=Kpn.cyc;T=List;
e=matsize(H)[2];valT=0;for(k=1,e-1,h=H[e-k+1];v=valuation(h,p);
if(v>0,valT=valT+v;listput(T,p^v));f=quaddisc(m);fn=q*p^n*f;c=1;
while(gcd(c,2*m)!=1 || kronecker(f,c)!=-1,c=c+1);S=0;
forstep(a=1,fn/2,2,if(gcd(a,m)!=1,next);aa=lift(Mod(a/c,fn));
la=(aa*c-a)/fn;eps=kronecker(m,a);S=S+a*eps*(la+(1-c)/2));
valR=valuation(S,2)-1;
if(valT==C,print("m=",m," c=",c," T=",T," v_2(T)=",valT,
" val(R_2Z)=",valR," C=",C," Equality"));
if(valT>C,print("m=",m," c=",c," T=",T," v_2(T)=",valT,
" v_2(R_2Z)=",valR," C=",C," Inequality")))}
m=7   c=5   T=[4]   v_2(T)=2   v_2(R_2Z)=2   C=1   Inequality
m=14  c=3   T=[2]   v_2(T)=1   v_2(R_2Z)=1   C=0   Inequality
m=15  c=13  T=[4]   v_2(T)=2   v_2(R_2Z)=2   C=2   Equality

```

m=17	c=3	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=0	Inequality
m=21	c=11	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=1	Equality
m=23	c=3	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=1	Inequality
m=30	c=11	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=1	Equality
m=31	c=7	T=[8]	v_2(T)=3	v_2(R_2Z)=3	C=1	Inequality
m=33	c=5	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=1	Equality
m=34	c=7	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=0	Inequality
m=35	c=3	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=1	Equality
m=39	c=11	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=2	Equality
m=41	c=3	T=[16]	v_2(T)=4	v_2(R_2Z)=3	C=0	Inequality
(...)						
m=1001	c=3	T=[2,2]	v_2(T)=2	v_2(R_2Z)=2	C=2	Equality
m=1002	c=5	T=[16]	v_2(T)=4	v_2(R_2Z)=5	C=1	Inequality
m=1003	c=5	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=1	Inequality
m=1005	c=29	T=[2,8]	v_2(T)=4	v_2(R_2Z)=5	C=2	Inequality
m=1006	c=7	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=0	Inequality
m=1007	c=3	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=2	Equality
m=1009	c=11	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=0	Inequality
m=1010	c=3	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=1	Equality
m=1011	c=7	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=1	Inequality
m=1015	c=17	T=[2,4]	v_2(T)=3	v_2(R_2Z)=3	C=3	Equality
m=1022	c=3	T=[2,16]	v_2(T)=5	v_2(R_2Z)=9	C=1	Inequality
m=1023	c=5	T=[2,64]	v_2(T)=7	v_2(R_2Z)=7	C=3	Inequality

Equality $v_2(\#\mathcal{I}_{K,2}) = \mathbf{C}$ often occurs, and when $v_2(\#\mathcal{I}_{K,2}) > \mathbf{C}$, then $v_2(\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]) > \mathbf{C}$, with cases of distinct valuations ($m = 41, 65, 66, 114$). Numerical examples given in [DL23, table, p. 15] yield equalities with $\mathbf{C} = \mathbf{D} + v_2(4s) - 2\epsilon = 4$ for $s = -1$, since 2 splits in $M = \mathbb{Q}(\sqrt{-m})$ (Theorem 4.4(iii₂) and Corollary 4.7).

B.2. Computation of $\widetilde{\mathcal{H}}_{K,2}$. We give a program for the computation of the logarithmic class group to compare with the invariants $\mathcal{H}_{K,2}$, $\mathcal{I}_{K,2}$. There are many structures but, the relation between $\mathcal{H}_{K,2}$ and $\mathcal{I}_{K,2}$ being well-known, we only give an excerpt of interesting examples concerning $\widetilde{\mathcal{H}}_{K,2}$. For $p = 2$, the first component of $\text{bnflog}(K, 2)$ gives $\widetilde{\mathcal{H}}_{K,2}^0$ of index 1 or 2 in $\widetilde{\mathcal{H}}_{K,2}$ [BJ16, §3, Remark 3.1].

```
{bf=10^4;Bf=10^4+10^3;for(m=bf,Bf,if(core(m)!=m,next);PK=x^2-m;
F=component(factor(m),1);nu=12;K=bnfinit(PK,1);HK=K.clgp;
Kpn=bnrinit(K,2^nu);HKn=Kpn.cyc;T=List;w=0;e=matsize(HKn)[2];
for(k=1,e-1,CK=HKn[e-k+1];v=valuation(CK,2);if(v>0,w=w+v;
listput(T,2^v));K=bnfinit(PK,1);HK=K.clgp;Kpn=bnrinit(K,2^nu);
HKn=Kpn.cyc;T=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,CK=HKn[k+1];
v=valuation(CK,2);if(v>0,w=w+v;listput(T,2^v));CLK=bnflog(K,2);
print("m=",F," P=",PK," H=",HK[2]," T=",T," CLog=",CLK)}
m=[3,5,23,29] x^2-10005 H=[2,2] T=[4,2,2] CLog=[[2,2],[],[2,2]]
m=[3,47,71] x^2-10011 H=[4,2] T=[4,2] CLog=[[4],[],[4]]
m=[3,5,11,61] x^2-10065 H=[2,2] T=[16,2,2] CLog=[[4],[2],[2]]
```

$m=[2,3,23,73]$	$x^2-10074$	$H=[8,2]$	$T=[32,4]$	$CLog=[[8],[],[8]]$
$m=[2,3,19,89]$	$x^2-10146$	$H=[4,2]$	$T=[8,2]$	$CLog=[[4],[],[4,2]]$
$m=[7,31,47]$	$x^2-10199$	$H=[4,2]$	$T=[4,4,2]$	$CLog=[[4,2],[],[4,2]]$
$m=[10337]$	$x^2-10337$	$H=[]$	$T=[8]$	$CLog=[[4],[4],[]]$
$m=[2,5297]$	$x^2-10594$	$H=[8]$	$T=[32]$	$CLog=[[4],[],[8]]$
$m=[3,11,17,19]$	$x^2-10659$	$H=[2,2,2]$	$T=[2,2,2]$	$CLog=[[2,2,2],[],[2,2,2]]$
$m=[17,641]$	$x^2-10897$	$H=[2]$	$T=[64,2]$	$CLog=[[2,2],[2],[2]]$
$m=[3,5,17,47]$	$x^2-11985$	$H=[2,2]$	$T=[1024,2,2]$	$CLog=[[128],[64],[2]]$
$m=[12161]$	$x^2-12161$	$H=[11]$	$T=[512]$	$CLog=[[16],[16],[]]$
$m=[2,3,13,167]$	$x^2-13026$	$H=[2,2]$	$T=[128,2]$	$CLog=[[],[],[2]]$

B.3. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$ and $\mathcal{T}_{K,3}$. For $p = 3$, genus theory does not apply; we only give the valuation of $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$ and the structure of $\mathcal{T}_{K,3}$; trivial cases are not listed:

```
{p=3;nu=5;n=6;for(m=2,10^2,if(core(m)!=m,next));P=x^2-m;
K=bnfinit(P,1);Kpn=bnrinit(K,p^nu);H=Kpn.cyc;T=List;e=matsize(H)[2];
for(k=1,e-1,C=H[e-k+1];v=valuation(C,p);if(v>0,listput(T,p^v));
f=quaddisc(m);fn=p^(n+1)*f;c=1;while(Mod(c,p)==0 ||
kronecker(f,c)!=-1,c=c+1);S=0;forstep(a=1,fn/2,[1,2],
if(gcd(a,f)!=1,next);aa=lift(Mod(a/c,fn));la=(aa*c-a)/fn;
eps=kronecker(f,a);S=S+a*eps*(la+(1-c)/2));print
("m=",m," v_3(K_2Z)=",valuation(S,3)," T=",T)}
m=6 v_3(K_2Z)=1 T=[3] m=69 v_3(K_2Z)=1 T=[3]
m=15 v_3(K_2Z)=1 T=[3] m=74 v_3(K_2Z)=1 T=[9]
m=29 v_3(K_2Z)=1 T=[9] m=77 v_3(K_2Z)=1 T=[3]
m=33 v_3(K_2Z)=1 T=[3] m=78 v_3(K_2Z)=1 T=[3]
m=42 v_3(K_2Z)=3 T=[9] m=79 v_3(K_2Z)=1 T=[9]
m=43 v_3(K_2Z)=2 T=[3] m=82 v_3(K_2Z)=4 T=[3]
m=51 v_3(K_2Z)=1 T=[3] m=83 v_3(K_2Z)=1 T=[3]
(...)
m=10187 v_3(K_2Z)=2 T=[81] m=10673 v_3(K_2Z)=4 T=[27]
m=10239 v_3(K_2Z)=2 T=[3,9] m=10771 v_3(K_2Z)=6 T=[9]
m=10281 v_3(K_2Z)=1 T=[27] m=10842 v_3(K_2Z)=4 T=[3,3]
m=10297 v_3(K_2Z)=4 T=[3] m=10942 v_3(K_2Z)=2 T=[3,243]
m=10351 v_3(K_2Z)=1 T=[243] m=11062 v_3(K_2Z)=3 T=[27]
```

Appendix C. Cyclic cubic fields

C.1. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$, $\mathcal{T}_{K,3}$ and \mathbf{C} . We consider all cyclic cubic fields, which requires calculating the Artin group giving the Galois group $\text{Gal}(\mathbb{Q}(\mu_f)/K)$; indeed, a composite conductor f gives rise to several fields, so that we must work with the defining polynomial depending on suitable integers a, b such that $f = (a^2 + 27b^2)/4$ (another method would be to work in the group $(\mathbb{Z}/f\mathbb{Z})^\times$, leading to other difficulties).

We compare $v_m(\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty])$ to the constant $\mathbf{C} = \mathbf{D} - 1$ to see if we are, or not, in the case of equality $v_m(\mathbf{L}_3(s, \chi)) = \mathbf{C}$, $\forall s \in \mathbb{Z}_3$.

C.1.1. General program. The program provides at first the list of all cyclic cubic fields, of conductor f (in f , $f \in [\text{bf}, \text{Bf}]$), with a defining polynomial P_K (in PK). We compute the structures of the 3-class group (in H), of the 3-torsion group (in T), and $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$ (in $\#\mathbf{K_2Z}$).

Except when the first layer of the cyclotomic \mathbb{Z}_3 -extension of K is non-ramified (e.g., $f = 657$, $\text{P} = x^3 - 219 * x + 730$, $\text{H}_K = [3, 3]$, $\text{T} = [3]$), the relation $\mathcal{T}_{K,3} = 1$ implies $\mathcal{H}_{K,3} = \mathcal{R}_{K,3} = 1$ since $\mathcal{W}_{K,3} = 1$ (formula (1.5)).

```
{bf=7;Bf=10^3;Q=y^2+y+1;Y=Mod(y,Q);for(f=bf,Bf,
if(isprime(f)==1,next);h=valuation(f,3);if(h!=0 & h!=2,next);
F=f/3^h;if(core(F)!=F,next);FD=factor(F);Div=component(FD,1);
d=matsize(FD)[1];for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));
C=d-1;Df=component(factor(f),1);for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
nu=8;K=bnfinit(PK,1);HK=K.clgp;Kpn=bnrinit(K,3^nu);HKn=Kpn.cyc;
T=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,CK=HKn[e-k+1];
v=valuation(CK,3);if(v>0,w=w+v;listput(T,3^v));forprime(q=2,10^2*f,
if(Mod(3*f,q)!=0 & polisirreducible(PK+O(q))==1,q0=q;break));
A0=List;A1=List;A2=List;c=q0;n=3;fn=3^n*f;N=eulerphi(fn)/(2*3);
for(s=1,fn/2,if(gcd(s,f*3)!=1,next);for(k=0,fn,r=s+k*fn;
if(isprime(r)==0,next);if(polisirreducible(PK+O(r))==0,
listput(A0,s);break));S0=0;S1=0;S2=0;g=(1-c)/2;
for(j=1,N,s=A0[j];listput(A1,lift(Mod(s*q0,fn)))));
for(j=1,N,s=A1[j];listput(A2,lift(Mod(s*q0,fn)))));
for(j=1,N,s=A0[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S0=S0+s*(ls+g));
for(j=1,N,s=A1[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S1=S1+s*(ls+g));
for(j=1,N,s=A2[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S2=S2+s*(ls+g));
S=S0+Y*S1+Y^2*S2;K2=norm(S)/3;v=valuation(K2,3);
if(v==C,print("f=",Df," ",PK," H=",HK[2],
" T=",T," #K_2Z=",3^v," C=",C," Equality"));
if(v>C,print("f=",Df," ",PK," H=",HK[2],
" T=",T," #K_2Z=",3^v," C=",C," Inequality")))}
}
```

C.1.2. Numerical results. We give an excerpt of the results ($\text{Ram} = [3, 7]$ means that 3 and 7 are ramified, hence $f = 63 = [9, 7]$):

Ram=[3,7]	$x^3-21*x-35$	H=[3]	T=[]	#K_2Z=1	C=0	Equality
Ram=[3,7]	$x^3-21*x+28$	H=[3]	T=[]	#K_2Z=1	C=0	Equality
Ram=[7,13]	$x^3+x^2-30*x-64$	H=[3]	T=[3]	#K_2Z=3	C=1	Equality
Ram=[7,13]	$x^3+x^2-30*x+27$	H=[3]	T=[3]	#K_2Z=3	C=1	Equality
Ram=[3,19]	$x^3-57*x-152$	H=[3]	T=[3]	#K_2Z=3	C=0	Inequality
Ram=[3,19]	$x^3-57*x+19$	H=[3]	T=[3]	#K_2Z=3	C=0	Inequality
Ram=[7,31]	$x^3+x^2-72*x+209$	H=[3]	T=[3,3]	#K_2Z=9	C=1	Inequality
Ram=[7,31]	$x^3+x^2-72*x-225$	H=[3]	T=[3]	#K_2Z=3	C=1	Equality
Ram=[3,37]	$x^3-111*x+370$	H=[3]	T=[3]	#K_2Z=3	C=0	Inequality
Ram=[3,37]	$x^3-111*x+37$	H=[3]	T=[3]	#K_2Z=3	C=0	Inequality

```

Ram=[13,31] x^3+x^2-134*x-597 H=[3] T=[3] #K_2Z=3 C=1 Equality
Ram=[13,31] x^3+x^2-134*x+209 H=[3] T=[3] #K_2Z=3 C=1 Equality
Ram=[7,61] x^3+x^2-142*x+601 H=[3] T=[3] #K_2Z=3 C=1 Equality
Ram=[7,61] x^3+x^2-142*x-680 H=[3] T=[9,9] #K_2Z=9 C=1 Inequality
Ram=[7,67] x^3+x^2-156*x-799 H=[3] T=[3,3] #K_2Z=27 C=1 Inequality
Ram=[7,67] x^3+x^2-156*x+608 H=[3] T=[3] #K_2Z=3 C=1 Equality
(...)
Ram=[3,73] x^3-219*x-1241 H=[3,3] T=[3] #K_2Z=3 C=0 Inequality

```

C.2. Computation of $\widetilde{\mathcal{H}}_{K,3}$. We use a part of the previous program, adding the computation of the logarithmic class group, and we give some data obtained up to conductors $\leq 10^3$, and then for larger conductors. The first component of $\text{bnflog}(K, 3)$ gives $\widetilde{\mathcal{H}}_{K,3}$, the second one (often trivial) is the subgroup $\widetilde{\text{cl}}(\langle S_3 \rangle)$ generated by S_3 , and the third component is $\mathcal{H}_{K,3}^{S_3} = \widetilde{\mathcal{H}}_{K,3}/\widetilde{\text{cl}}(\langle S_3 \rangle)$; we have $\widetilde{\mathcal{H}}_{K,3}/\widetilde{\text{cl}}(\langle S_3 \rangle) \simeq \mathcal{H}_{K,3}^{S_3}$ [BJ16].

```

Ram=[19] x^3+x^2-6*x-7 H=[] T=[3] CLog=[[ ], [ ], [ ]]
Ram=[7,13] x^3+x^2-30*x-64 H=[3] T=[3] CLog=[[3], [ ], [3]]
Ram=[3,61] x^3-183*x-854 H=[3] T=[] CLog=[[ ], [ ], [3]]
Ram=[19,37] x^3+x^2-234*x-729 H=[2,6] T=[3,3] CLog=[[3], [3], [ ]]
Ram=[7,181] x^3+x^2-422*x+3191 H=[3,3] T=[3,3] CLog=[[3,3], [ ], [3,3]]
(...)
Ram=[7,19,73,103] x^3+x^2-333342*x+73964960
H=[3,3,3,3] T=[3,3,3,3] CLog=[[3,3,3,3], [ ], [3,3,3,3]]
Ram=[7,19,73,103] x^3+x^2-333342*x-69038901
H=[6,6,3] T=[3,3,3,3] CLog=[[3,3,3], [3], [3,3]]
Ram=[7,19,73,103] x^3+x^2-333342*x-49038361
H=[3,3,3] T=[3,3,3] CLog=[[3,3,3], [ ], [3,3,3]]
Ram=[397,251893] x^3+x^2-33333840*x+41296924413
H=[3] T=[9,3,3] CLog=[[3,3], [3,3], [ ]]
Ram=[7,14286007] x^3+x^2-33334016*x-56482638787
H=[54,18] T=[27,27] CLog=[[27,9], [ ], [27,9]]
Ram=[7,163,87643] x^3+x^2-33333554*x-214816239
H=[3,3] T=List([3,3,3]) CLog=[[3,3], [3], [3]]
Ram=[100001053] x^3+x^2-33333684*x-17437220649
H=[] T=List([9,3]) CLog=[[9,3], [9,3], [ ]]
Ram=[31,103,31321] x^3+x^2-33335984*x+68190607953
H=[6,6,3] T=List([9,9,3]) CLog=[[9,3], [3], [3,3]]

```

C.3. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$ and $\mathcal{T}_{K,2}$. We still compute

$$\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \sim \frac{1}{2}\mathbf{L}_2(-1, \chi) \times \frac{1}{2}\mathbf{L}_2(-1, \chi^2)$$

for prime conductors, noting that genus theory does not apply for cyclic cubic fields with $p = 2$. We compute $\mathcal{T}_{K,2}$, first with the classical program using ray class groups, and then we use the formula of Theorem 4.1 in the following complete program (we only print an example of the six cases of structures that occur up to $f = 3500$):

```

{p=2;Q=y^2+y+1;Y=Mod(y,Q);forprime(f=7,10^3,
if(Mod(f,3)!=1,next);P=polsubcyclo(f,3);K=bnfinit(P,1);nu=8;
Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
for(k=1,e-1,CK=HKn[e-k+1];v=valuation(CK,p);if(v>0,listput(T,p^v));
g=znprimroot(f);c=lift(g);if(Mod(c,p)==0,c=c+f);n=12;ex=(f-1)/3;
cex=Mod(c,f)^ex;R0=0;R1=0;R2=0;T0=0;T1=0;T2=0;fn=p^(n+2)*f;
forstep(a=1,fn/2,2,if(Mod(a,f)==0,next);aa=lift(Mod(a/c,fn));
am=lift(Mod(a,fn)^-1);la=(aa*c-a)/fn;XR=a*(la+(1-c)/2);XT=am*la;
aex=Mod(a,f)^ex;if(aex==1,R0=R0+XR;T0=T0+XT);if(aex==cex,
R1=R1+XR;T1=T1+XT);if(aex==cex^2,R2=R2+XR;T2=T2+XT));
AR=lift(Mod(R0-R2,fn));BR=lift(Mod(R1-R2,fn));
AT=lift(Mod(T0-T2,fn));BT=lift(Mod(T1-T2,fn));
NR=norm(AR+Y*BR);wR=valuation(NR,p);
NT=norm(AT+Y*BT);wT=valuation(NT,p);print();print("f=",f," P=",P);
print("AR=",AR," BR=",BR," #R_2Z=",p^wR);print("T=",T);
print("AT=",AT," BT=",BT," #T=",p^wT)}}
f=7 P=x^3+x^2-2*x-1 f=739 P=x^3+x^2-246*x-520
AR=114683 BR=6 #R_2Z=1 AR=7232 BR=4072 #R_2Z=64
T=[] T=[8,8]
AT=45707 BT=57070 #T=1 AT=2395776 BT=8911992 #T=64
f=31 P=x^3+x^2-10*x-8 f=2689 P=x^3+x^2-896*x+5876
AR=24 BR=507870 #R_2Z=4 AR=1733440 BR=461096 #R_2Z=64
T=[2,2] T=[16,16]
AT=82976 BT=73118 #T=4 AT=29791312 BT=37866256 #T=256
f=277 P=x^3+x^2-92*x+236 f=3163 P=x^3+x^2-1054*x-13472
AR=3256 BR=4537692 #R_2Z=16 AR=432 BR=51769200 #R_2Z=256
T=[4,4] T=[16,16]
AT=718480 BT=1119412 #T=16 AT=11946416 BT=40520656 #T=256
f=3457 P=x^3+x^2-1152*x+13700 f=6163 P=x^3+x^2-2054*x+17576
AR=286536 BR=3460600 #R_2Z=64 AR=6140160 BR=6195120 #R_2Z=256
T=List([16,16]) T=List([8,8])
AT=2216784 BT=1012224 #T=256 AT=2754032 BT=3116408 #T=64

```

We know from (1.8) that $\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]$ and $\mathcal{T}_{K,2}$ have the same 2-rank, but we see that most often the whole structures coincide (the first exceptions are $f = 2689, 3457$, and $f = 6163$ gives an interesting case).

C.4. Zeroes of p -adic \mathbf{L} -functions and $\mathbf{C}(s)$. A method to study the influence of the zeroes, close to $s = 1$, of p -adic \mathbf{L} -functions is to construct families of degree- p cyclic fields K_n such that the p -adic regulator tends to 0 as $n \rightarrow \infty$, so $\mathbf{L}_p(1, \chi_n) \rightarrow 0$ and $\#\mathcal{T}_{K_n, p} \rightarrow 0$ from (1.5); thus, invariants given by some $\mathbf{L}_p(s_0, \chi_n)$, $s_0 \in \mathbb{Z}_p^\times$, have the same behavior because of the role of $\mathbf{C}(s) = \mathbf{C}$ when $s \in \mathbb{Z}_p^\times$ (Theorem 4.4).

C.4.1. The Lecacheux–Washington cyclic cubic fields. The following family of cyclic cubic fields K illustrates, for $p = 3$, the fact that, in a genus theory context, the bound \mathbf{C} in $v_m(\mathbf{L}_3(s, \chi)) > \mathbf{C}$ for all $s \in \mathbb{Z}_3$ (Theorem 4.4(b)) can be much increased because $\mathbf{L}_3(1, \chi)$ is very close to 0. Using the

family of cubic polynomials

$$P_n = x^3 - (n^3 - 2n^2 + 3n - 3)x^2 - n^2x - 1$$

(see [Lec93, Wa97^a]), for some values of n tending to 1 in \mathbb{Z}_3 , one obtains 3-adic \mathbf{L} -functions having a zero near $s = 1$, which of course provides large modules $\mathcal{T}_{K_n,3}$. The required conditions are given in [Wa97^a, Theorem 3].

C.4.2. Numerical examples. The constant \mathbf{C} associated to these fields is 2, so that we must obtain $v_3(\mathcal{T}_{K,3}) > 2$ as long as n is close to 1 in \mathbb{Z}_3 . We give the program and some examples suggesting that $\#\mathcal{T}_{K_n,3}$ is unbounded in such families as $n \rightarrow \infty$; but these fields have a huge discriminant D_n , and the function $\mathcal{C}_p(K_n) := \log(\#\mathcal{T}_{K_n,p})/\log(\sqrt{D_n})$, introduced in [Gra19^c], is rather small and fulfills the various conjectures of this article. The modulus M gives the integers $n \equiv 1 \pmod{M}$, $n \rightarrow 1$ in \mathbb{Z}_3 .

```
{M=3^5;forstep(n=1,10^6,M,if(isprime(n^2+3))==0,next);A=n^2-3*n+3;
if(numdiv(A)!=4,next);F=factor(A);p=component(F,1)[1];q=component(F,1)[2];
if(Mod(p^2,9)==1 || Mod(q^2,9)==1,next);P=x^3-(n^3-2*n^2+3*n-3)*x^2-n^2*x-1;
K=bnfinit(P,1);nu=18;HK=K.clgp;Kpn=bnrinit(K,3^nu);HKn=Kpn.cyc;T=List;
w=0;e=matsize(HKn)[2];for(k=1,e-1,CK=HKn[e-k+1];
v=valuation(CK,3);if(v>0,w=w+v;listput(T,3^v)));
print("n=",n," p=",p," q=",q," T=",T," v_3(T)=",w)}
n=2674 p=43 q=166099 T=[3,81,243] v_3(T)=10
n=40096 p=3271 q=491461 T=[3,729,729] v_3(T)=13
n=43498 p=6547 q=288979 T=[3,81,243] v_3(T)=10
n=50788 p=28111 q=91753 T=[3,243,243] v_3(T)=11
n=56134 p=3613 q=872089 T=[3,729,2187] v_3(T)=14
n=76546 p=31 q=189001951 T=[3,2187,2187] v_3(T)=15
n=78490 p=7 q=880063519 T=[3,81,243] v_3(T)=10
n=96958 p=79 q=118994467 T=[3,9,243,729] v_3(T)=14
n=124660 p=7 q=2219963089 T=[3,6561,6561] v_3(T)=17
n=158194 p=135301 q=184957 T=[3,243,729] v_3(T)=12
(...)
n=570808 p=643 q=506718601 T=[3,6561,19683] v_3(T)=18
n=649540 p=229 q=1842359227 T=[3,3,19683,59049]v_3(T)=21
```

Note that $n = 570808 \equiv 1 \pmod{3^9}$ and $n = 649540 \equiv 1 \pmod{3^{10}}$. For small values of M we obtain many minimal valuations $2v_3(M)$:

```
M=3
n=340 p=7 q=16369 T=[3,3] v_3(T)=2
n=736 p=79 q=6829 T=[3,3] v_3(T)=2
n=970 p=7 q=133999 T=[3,3] v_3(T)=2 (...)
M=9
n=874 p=7 q=108751 T=[3,3,9] v_3(T)=4
n=2926 p=1777 q=4813 T=[3,3,9] v_3(T)=4
n=4042 p=3067 q=5323 T=[3,3,9] v_3(T)=4 (...)
```

Appendix D. Numerical results for $p \geq 5$. When $p \geq 5$, and for a degree- p cyclic extension K/\mathbb{Q} , the trick used for $p \in \{2, 3\}$ does not exist, and we only have available the complex-analytic formulas for $\#\mathbf{K}_2(\mathbf{Z}_K)$ and the rank formula (1.6) giving Theorem 6.1.

D.1. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$. The computation of $\#\mathbf{K}_2(\mathbf{Z}_K)$ uses Bernoulli polynomials [Wa97^b, Proposition 4.1, Theorem 4.2]. The following program computes the p -valuation of the product of the $p-1$ conjugates of $\mathbf{L}(-1, \chi)$, for χ of order p and prime conductor ℓ with the formula $\mathbf{L}(-1, \chi) \sim \sum_{a=1}^{\ell-1} \chi(a)(a^2 - \ell a)$, where $\chi(a)$ is computed from $\chi(g) := \zeta_p$ where g is a primitive root modulo ℓ , and from $a \equiv g^k \pmod{\ell}$, $k \in [1, \ell-1]$; by comparison we compute the structure of $\mathcal{F}_{K,p}$. The case $p=3$ is included to recall that it may give trivial modules. As expected, for $p \geq 5$, $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] \neq 1$ while $\mathcal{F}_{K,p} \neq 1$ if and only if $\ell \equiv 1 \pmod{p^2}$ (fixed point formula for $\#\mathcal{F}_{K,p}^G$ [Gra05, Theorem IV.3.3, Exercise IV.3.3.1]).

```
{forprime(p=3,100,Q=polcyclo(p);X=Mod(x,Q);print();
forprime(ell=1,10^4,if(Mod(ell,p)!=1,next);g=znprimroot(ell);
L=0;for(k=1,ell-1,a=lift(g^k);E=a^2-a*ell;L=L+E*X^k);
w=valuation(norm(L),p);P=polsubcyclo(ell,p);K=bnfinit(P,1);
nu=8;Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
for(k=1,e-1,c=HKn[e-k+1];v=valuation(c,p);if(v>0,listput(T,p^v)));
print("p=",p," ell=",ell," T=",T," v(K_2Z)=",w))}
p=3 ell=7 T=[] v(K_2Z)=0 p=11 ell=23 T=[] v(K_2Z)=1
p=3 ell=19 T=[3] v(K_2Z)=1 p=11 ell=727 T=[11] v(K_2Z)=1
p=3 ell=199 T=[3,3] v(K_2Z)=2 p=11 ell=1321 T=[] v(K_2Z)=3
p=3 ell=4177 T=[9,9] v(K_2Z)=3 p=11 ell=1453 T=[11] v(K_2Z)=1
p=3 ell=2593 T=[3,9] v(K_2Z)=6 p=11 ell=3631 T=[11] v(K_2Z)=2
p=3 ell=21997 T=[27,81] v(K_2Z)=7 p=11 ell=4357 T=[11,11] v(K_2Z)=1
p=5 ell=11 T=[] v(K_2Z)=1 p=13 ell=53 T=[] v(K_2Z)=1
p=5 ell=101 T=[5,5] v(K_2Z)=1 p=13 ell=677 T=[13] v(K_2Z)=1
p=5 ell=181 T=[] v(K_2Z)=3 p=13 ell=1483 T=[] v(K_2Z)=2
p=5 ell=401 T=[5,5] v(K_2Z)=2 p=13 ell=2029 T=[13] v(K_2Z)=1
p=5 ell=3001 T=[5] v(K_2Z)=7 p=13 ell=6761 T=[13] v(K_2Z)=4
p=5 ell=5351 T=[5,5,5] v(K_2Z)=1 p=13 ell=11831 T=[13] v(K_2Z)=2
p=7 ell=29 T=[] v(K_2Z)=1 p=17 ell=103 T=[] v(K_2Z)=2
p=7 ell=127 T=[] v(K_2Z)=2 p=17 ell=137 T=[] v(K_2Z)=1
p=7 ell=197 T=[7,7] v(K_2Z)=1 p=17 ell=3469 T=[17] v(K_2Z)=1
p=7 ell=491 T=[7] v(K_2Z)=1 p=17 ell=3571 T=[] v(K_2Z)=1
p=7 ell=4159 T=[] v(K_2Z)=3 p=17 ell=3673 T=[] v(K_2Z)=1
p=7 ell=4229 T=[] v(K_2Z)=4 p=17 ell=3911 T=[] v(K_2Z)=1
```

D.2. Computations of $\text{rk}_5(\mathbf{K}_2(\mathbf{Z}_K))$. The next programs compute the ω^{-1} -component of the 5-class group \mathcal{H}_{K_2} of $K_2 := K(\mu_5)$, for degree-5 cyclic fields K , of prime conductor $f = \ell$ in the first section, and with $f = 11 \cdot 31$ in the second section, thus giving the 5-rank of $\mathbf{K}_2\mathbf{Z}_K[5^\infty]$ (formula (1.6), where $\langle S_5(K') \rangle$ is of character χ_0). One computes the 5-class

group $\mathcal{H}_{K_1} \simeq (\mathcal{H}_{K_2})_{\omega^2}$ of $K_1 := K(\sqrt{5})$, so that

$$\mathcal{H}_{K_2} \simeq \mathcal{H}_{K_1} \oplus (\mathcal{H}_{K_2})_{\omega^{-1}} \oplus (\mathcal{H}_{K_2})_{\omega},$$

where $\text{rk}_5((\mathcal{H}_{K_2})_{\omega}) = \text{rk}_5(\mathcal{T}_{K,5})$ (formula (1.7)), the structure of $\mathcal{T}_{K,5}$ being computed as usual via the program of Section A.1. Hence

$$(D.1) \quad \text{rk}_5(\mathbf{K}_2(\mathbf{Z}_K)) = \text{rk}_5(\mathcal{H}_{K_2}) - \text{rk}_5(\mathcal{H}_{K_1}) - \text{rk}_5(\mathcal{T}_{K,5}).$$

D.2.1. Case of prime conductors. So, f is a prime number $\ell \equiv 1 \pmod{5}$; thus, $\mathcal{H}_K = 1$. We give an excerpt of each structure obtained:

```
{p=5;forprime(e11=11,10^5,if(Mod(e11,p)!=1,next);P=polsubcyclo(e11,p);
print("e11=",e11," ",P);nu=8;K=bnfinit(P,1);H=K.clgp[2];
Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;w=0;e=matsize(HKn)[2];
for(k=1,e-1,Ck=HKn[e-k+1];v=valuation(Ck,p);if(v>0,listput(T,p^v)));
R1=polcompositum(P,polsubcyclo(p,2))[1];LK1=bnfinit(R1,1);
R2=polcompositum(P,polcyclo(p))[1];LK2=bnfinit(R2,1);
print("T=",T," H1=",LK1.clgp[2]," H2=",LK2.clgp[2]))}
e11=11 x^5+x^4-4*x^3-3*x^2+3*x+1 T=[] H1=[] H2=[5]
e11=31 x^5+x^4-12*x^3-21*x^2+x+5 T=[] H1=[] H2=[10,10,2,2]
e11=101 x^5+x^4-40*x^3+93*x^2-21*x-17 T=[5,5] H1=[] H2=[2005,5,5]
e11=151 x^5+x^4-60*x^3-12*x^2+784*x+128 T=[5] H1=[] H2=[5305,5]
e11=251 x^5+x^4-100*x^3-20*x^2+1504*x+1024 T=[5] H1=[] H2=[11810,10,2,2]
e11=281 x^5+x^4-112*x^3-191*x^2+2257*x+967 T=[] H1=[5] H2=[261005,5,5]
e11=401 x^5+x^4-160*x^3+369*x^2+879*x-29 T=[5,5] H1=[] H2=[42505,5,5,5]
e11=421 x^5+x^4-168*x^3+219*x^2+3853*x-3517 T=[] H1=[5] H2=[225915,15,3,3]
```

The rank 2 is obtained for $\ell = 31, 281, 401$.

D.2.2. Examples of composite conductor. The program computes the ω^{-1} -component for the four degree-5 fields K of conductors $f = 11 \cdot 31$ and $f = 11 \cdot 101$, and then for some f 's that are products of primes $\ell_i \equiv 1 \pmod{5}$:

```
{p=5;f=11*31;LP=polsubcyclo(f,p);for(i=1,6,P=LP[i];D=nfdisc(P);if(D!=f^(p-1),next);
print();print(P);K=bnfinit(P,1);H=K.clgp[2];nu=8;Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;
T=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,Ck=HKn[e-k+1];v=valuation(Ck,p);
if(v>0,listput(T,p^v)));R1=polcompositum(P,polsubcyclo(p,2))[1];LK1=bnfinit(R1,1);
R2=polcompositum(P,polcyclo(p))[1];LK2=bnfinit(R2,1);
print("H=",H," T=",T," H1=",LK1.clgp[2]," H2=",LK2.clgp[2]))}
f=11*31
x^5-x^4-136*x^3+641*x^2-371*x-67 H=[5] T=[] H1=[5,5] H2=[218305,5,5,5,5]
x^5-x^4-136*x^3-723*x^2-1053*x-67 H=[5] T=[] H1=[5,5] H2=[145505,5,5,5,5]
x^5-x^4-136*x^3-41*x^2+3039*x-1431 H=[5] T=[] H1=[5,5] H2=[3805,5,5,5,5,5,5]
x^5-x^4-136*x^3+300*x^2+2016*x-3136 H=[5] T=[] H1=[5,5] H2=[12605,5,5,5,5,5]
f=11*101
x^5-x^4-444*x^3+3644*x^2+80*x-32000 H=[5] T=[5] H1=[5,5] H2=[1196410,10,10,10,5]
x^5-x^4-444*x^3+311*x^2+32299*x+26883 H=[5] T=[5] H1=[5,5] H2=[15020305,5,5,5,5]
x^5-x^4-444*x^3-1911*x^2+12301*x-8669 H=[5] T=[5] H1=[5,5] H2=[1405,1405,5,5,5,5,5]
x^5-x^4-444*x^3+311*x^2+43409*x+4663 H=[5] T=[5] H1=[5,5] H2=[5559605,5,5,5,5,5]
f=151*101 x^5-x^4-6100*x^3+77475*x^2+3766875*x-37515625 H=[5]
T=[5,5,5] H1=[3410,10,2,2] H2=[44818318820,220,220,20,5,5,5]
f=11*31*41 x^5+x^4-5592*x^3+46417*x^2+4301003*x-26664769 H=[5,5]
T=[5,5] H1=[55,5,5,5] H2=[476241615355,5,5,5,5,5,5,5]
```

The ranks may indeed be larger than the number of tame ramified primes, but all cases illustrate Theorem 6.1 with rank relation (D.1).

D.3. Computation of $\mathbf{L}(-1 - (p-3)p^n, \chi)$. Let K/\mathbb{Q} be a degree- p cyclic extension. For $p \geq 5$, the conditions in (2.1) ($m > 1$ and $m \equiv 0 \pmod{p-1}$) are not satisfied for $m = 2$, which does not allow expressing $\mathbf{L}(-1, \chi)$ by means of the \mathbf{L}_p -function at $s = -1$. But with $m := 2 + (p-3)p^n > 1$ and $(-1)_n := 1 - m = -1 - (p-3)p^n$, we may compute the expression, for n large enough,

$$\mathbf{L}_p(-1 - (p-3)p^n, \chi) = (1 - p^{1+(p-3)p^n})\mathbf{L}((-1)_n, \chi) \sim \mathbf{L}_p(-1, \chi).$$

This does not give $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ but is related to $\#\mathbf{K}_{2m-2}(\mathbf{Z}_K)[p^\infty]$ of higher \mathbf{K} -theory: from the Quillen–Lichtenbaum conjecture, via the $\mathbf{L}((-1)_n, \chi)$'s (see, e.g., [Kol03, Ngu92]), we get

$$\zeta_K((-1)_n) = \zeta_K(1 - m) = \pm \frac{\#\mathbf{K}_{2m-2}(\mathbf{Z}_K)}{w_m(K)},$$

where $w_m(K)$ is the largest integer N such that $[K(\zeta_N) : K] \mid m$. For $p \neq 2$, we have $w_m(K) \sim p$, giving $v_p(\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]) = v_p(\zeta_K(1 - m)) + 1$.

Indeed, the condition $m \equiv 0 \pmod{p-1}$ of definition (2.1) is fulfilled for $m = 2 + (p-3)p^n$, so that a relation between p -adic and complex \mathbf{L} -functions does exist and leads, for n large enough, by continuity of $\mathbf{L}_p(s, \chi)$, to $\prod_{\chi \neq \chi_0} \mathbf{L}((-1)_n, \chi) \sim \prod_{\chi \neq \chi_0} \mathbf{L}_p(-1, \chi)$, since the Euler factors $1 - p^{m-1}\chi(p)$ are p -adic units.

Thus, the formula of Theorem 4.1 for $s = (-1)_n$ makes sense for computing $\mathbf{L}((-1)_n, \chi)$, of constant valuation for $n \geq n_0$, only depending on the computation of $\mathbf{L}_p(-1, \chi)$ related to $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\#\mathcal{T}_{K,p}$.

The complex-analytic computation of $\mathbf{L}((-1)_n, \chi)$ is done by the following program, with $m = 2 + (p-3)p^n$, but with less immediate Bernoulli polynomials, so we use their series definition; we also give the computation of the structure of $\mathcal{T}_{K,p}$:

```
{n=2;forprime(p=5,50,m=2+(p-3)*p^n;print();
forprime(e11=1,60000,if(Mod(e11,p)!=1,next);Q=polcyclo(p);X=Mod(x,Q);
g=znprimroot(e11);P=polsubcyclo(e11,p);K=bnfinit(P,1);nu=8;
Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;w=0;e=matsize(HKn)[2];
for(k=1,e-1,Ck=HKn[e-k+1];v=valuation(Ck,p);if(v>0,listput(T,p^v));
Factm=1;for(i=1,m,Factm=i*Factm);
B=(x+0(x^(m+2)))*exp(x*y/e11+0(x^(m+2)))/(exp(x+0(x^(m+2)))-1);
Bm=e11^(m-1)/m*Factm*polcoeff(B,m);L=0;dm=poldegree(Bm);
for(k=1,e11-1,a=lift(g^k);E=0;for(j=0,dm,cm=polcoeff(Bm,j);E=E+cm*a^j);
L=L+E*X^k);vm=valuation(norm(L),p);
print("p=",p," e11=",e11," T=",T," v(K_(2m-2)Z)=",vm+1))}
```

For instance, for $p = 5$ and cyclic quintic fields of prime conductor ℓ , this provides the following examples where $m = 52$ (computing $\mathbf{L}_5((-1)_n, \chi)$ by

means of the formula of Theorem 4.1 gives the same results):

$$\begin{array}{llll} \text{e11=101 } T=[5,5] & v(K_{-}(2m-2)\mathbb{Z})=3 & \text{e11=401 } T=[5,5] & v(K_{-}(2m-2)\mathbb{Z})=3 \\ \text{e11=151 } T=[5] & v(K_{-}(2m-2)\mathbb{Z})=2 & \text{e11=5351 } T=[5,5,5] & v(K_{-}(2m-2)\mathbb{Z})=4 \\ \text{e11=251 } T=[5] & v(K_{-}(2m-2)\mathbb{Z})=2 & \text{e11=29251}T=[5,5,5,5] & v(K_{-}(2m-2)\mathbb{Z})=5 \end{array}$$

In the selected interval, $\#\mathcal{T}_{K,5}$ and $\prod_{\chi \neq \chi_0} \mathbf{L}_5(-1, \chi)$ have same valuation, giving $v_5(\#\mathbf{K}_2(\mathbf{Z}_K)) = v_5(\#\mathcal{T}_{K,5}) + 1$. The minimal ℓ giving distinct 5-valuations is $\ell = 56401$ with $\mathcal{T}_{K,5} \simeq (\mathbb{Z}/5\mathbb{Z})^3 \times \mathbb{Z}/25\mathbb{Z}$, and $v_5(\#\mathbf{K}_2(\mathbf{Z}_K)) = 7$.

Acknowledgements. I thank with pleasure Litong Deng and Yongxiong Li for interesting exchanges about their paper, making me revisit my old papers on genus theory of p -adic pseudo-measures in the hope that arithmetic invariants of new families of real abelian fields can be elucidated via this method. Some of their comments were indeed helpful.

References

- [AF72] Y. Amice et J. Fresnel, *Fonctions zêta p -adiques des corps de nombres abéliens réels*, Acta Arith. 20 (1972), 353–384.
- [AAM21] H. Asensouyis, J. Assim and Y. Mazigh, *A genus formula for the positive étale wild kernel*, J. Number Theory 218 (2021), 161–179.
- [Ba68] H. Bass, *Algebraic K -Theory*, Benjamin, New York, 1968.
- [BJ16] K. Belabas and J.-F. Jaulent, *The logarithmic class group package in PARI/GP*, Publ. Math. Besançon Algèbre Théor. Nombres 2016, 5–18.
- [BN05] J.-R. Belliard and T. Nguyen Quang Do, *On modified circular units and annihilation of real classes*, Nagoya Math. J. 177 (2005), 77–115.
- [Ber90] R. I. Berger, *Quadratic extensions of number fields with elementary abelian 2-prim $K_2(O_F)$ of smallest rank*, J. Number Theory 34 (1990), 284–292.
- [BG92] R. I. Berger and G. Gras, *Regular fields: normic criteria in p -extensions*, Publ. Math. Besançon Théor. Nombres 1992, art. 1, 5 pp.
- [BP72] F. Bertrandias et J.-J. Payan, *Γ -extensions et invariants cyclotomiques*, Ann. Sci. École Norm. Sup. (4) 5 (1972), 517–543.
- [Bro92] J. Browkin, *On the p -rank of the tame kernel of algebraic number fields*, J. Reine Angew. Math. 432 (1992), 135–149.
- [Bro05] J. Browkin, *Tame kernels of cubic cyclic fields*, Math. Comp. 74 (2005), 967–999.
- [BS82] J. Browkin and A. Schinzel, *On Sylow 2-subgroups of K_2O_F for quadratic number fields F* , J. Reine Angew. Math. 331 (1982), 104–113.
- [CK89] A. Candiotti and K. Kramer, *On the 2-Sylow subgroup of the Hilbert kernel of K_2 of number fields*, Acta Arith. 52 (1989), 49–65.
- [CN78] P. Cassou-Noguès, *Valeurs aux entiers négatifs des fonctions L et fonctions L p -adiques d’un corps de nombres totalement réel*, Sémin. Théor. Nombres Grenoble 6 (1978), exp. 5, 8 pp.
- [Che34] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux*, Thèses de l’entre-deux-guerres 155 (1934), 365–476.
- [Coa77] J. Coates, *p -adic L -functions and Iwasawa’s theory*, in: Algebraic Number Fields: L -functions and Galois Properties (Durham, 1975), Academic Press, London, 1977, 269–353.

- [CLL21] J. Coates, J. Li and Y. Li, *Classical Iwasawa theory and infinite descent on a family of abelian varieties*, *Selecta Math. (N.S.)* 27 (2021), no. 2, art. 28, 36 pp.
- [Col88] P. Colmez, *Résidu en $s = 1$ des fonctions zêta p -adiques*, *Invent. Math.* 91 (1988), 371–389.
- [DR80] P. Deligne and K. A. Ribet, *Values of abelian L -functions at negative integers over totally real fields*, *Invent. Math.* 59 (1980), 227–286.
- [DL23] L.-T. Deng and Y.-X. Li, *An application of Birch–Tate formula to tame kernels of real quadratic number fields*, *Proc. Math. Sci.* 134 (2024), art. 26, 21pp.
- [Fre66] J. Fresnel, *Nombres de Bernoulli et fonctions L p -adiques*, *Sém. Delange–Pisot–Poitou Théor. Nombres* 7 (1965/66), exp. 14, 15 pp.
- [Fre67] J. Fresnel, *Nombres de Bernoulli et fonctions L p -adiques*, *Ann. Inst. Fourier (Grenoble)* 17 (1967), no. 2, 281–333.
- [Gar71] H. Garland, *A finiteness theorem for K_2 of a number field*, *Ann. of Math. (2)* 94 (1971), 534–548.
- [Gra78] G. Gras, *Sur la construction des fonctions L p -adiques abéliennes*, *Sém. Delange–Pisot–Poitou Théor. Nombres* 20 (1978/79), exp. 22, 20 pp.
- [Gra79] G. Gras, *Annulation du groupe des ℓ -classes généralisées d’une extension abélienne réelle de degré premier à ℓ* , *Ann. Inst. Fourier (Grenoble)* 29 (1979), no. 1, 15–32.
- [Gra86^a] G. Gras, *Remarks on K_2 of number fields*, *J. Number Theory* 23 (1986), 322–335.
- [Gra86^b] G. Gras, *Théorie des genres analytique des fonctions L p -adiques des corps totalement réels*, *Invent. Math.* 86 (1986), 1–17.
- [Gra87] G. Gras, *Pseudo-mesures p -adiques associées aux fonctions L de \mathbb{Q}* , *Manuscripta Math.* 57 (1987), 373–415.
- [Gra98] G. Gras, *Théorèmes de réflexion*, *J. Théor. Nombres Bordeaux* 10 (1998), 399–499.
- [Gra05] G. Gras, *Class Field Theory: From Theory to Practice*, 2nd ed., Springer Monogr. Math., Springer, 2005.
- [Gra16^a] G. Gras, *Les θ -régulateurs locaux d’un nombre algébrique : conjectures p -adiques*, *Canad. J. Math.* 68 (2016), 571–624; English transl.: arXiv:1701.02618.
- [Gra16^b] G. Gras, *Sur le module de Bertandias–Payan dans une p -extension – noyau de capitulation*, *Publ. Math. Sci. Besançon Algèbre Théor. Nombres* 2016, 25–44.
- [Gra17] G. Gras, *Invariant generalized ideal classes – structure theorems for p -class groups in p -extensions*, *Proc. Indian Acad. Sci. Math. Sci.* 127 (2017), 1–34.
- [Gra18] G. Gras, *The p -adic Kummer–Leopoldt constant: normalized p -adic regulator*, *Int. J. Number Theory* 14 (2018), 329–337.
- [Gra19^a] G. Gras, *On p -rationality of number fields. Applications – PARI/GP programs*, *Publ. Math. Besançon Algèbre Théor. Nombres* 2019, 29–51.
- [Gra19^b] G. Gras, *Practice of the incomplete p -ramification over a number field – history of abelian p -ramification*, *Comm. Adv. Math. Sci.* 2 (2019), 251–280.
- [Gra19^c] G. Gras, *Heuristics and conjectures in the direction of a p -adic Brauer–Siegel theorem*, *Math. Comp.* 88 (2019), 1929–1965.
- [Gra23^a] G. Gras, *The Chevalley–Herbrand formula and the real abelian Main Conjecture (new criterion using capitulation of the class group)*, *J. Number Theory* 248 (2023), 78–119.
- [Gra23^b] G. Gras, *Notion of abelian arithmetic φ -objects for the study of p -class groups and p -ramified torsion groups*, *North-West. Eur. J. Math.* 9 (2023), 109–199.
- [GJ89] G. Gras et J.-F. Jaulent, *Sur les corps de nombres réguliers*, *Math. Z.* 202 (1989), 343–365.

- [Gre76] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math. 98 (1976), 263–284.
- [GL09] R. Griffiths and M. Lescop, *On the 2-rank of the Hilbert kernel of number fields*, Canad. J. Math. 61 (2009), 1073–1091.
- [Hur89] J. Hurrelbrink, *Class numbers, units and K_2* , in: Algebraic K -theory: Connections with Geometry and Topology (Lake Louise, AB, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 279, Kluwer, Dordrecht, 1989, 87–102.
- [HK98] J. Hurrelbrink and M. Kolster, *Tame kernels under relative quadratic extensions and Hilbert symbols*, J. Reine Angew. Math. 499 (1998), 145–188.
- [Iwa72] K. Iwasawa, *Lectures on p -Adic L -Functions*, Ann. of Math. Stud. 74, Princeton Univ. Press, Princeton, NJ, 1972.
- [Jau86] J.-F. Jaulent, *L'arithmétique des ℓ -extensions* (thèse d'état), Publ. Math. Fac. Sci. Besançon Théor. Nombres 1986, no. 1, 357 pp.
- [Jau90] J.-F. Jaulent, *La théorie de Kummer et le K_2 des corps de nombres*, Sém. Théor. Nombres Bordeaux 2 (1990), 377–411.
- [Jau94^a] J.-F. Jaulent, *Sur le noyau sauvage des corps de nombres*, Acta Arith. 67 (1994), 335–348.
- [Jau94^b] J.-F. Jaulent, *Classes logarithmiques des corps de nombres*, J. Théor. Nombres Bordeaux 6 (1994), 301–325.
- [Jau16] J.-F. Jaulent, *Sur la capitulation pour le module de Bertrandias–Payan*, Publ. Math. Besançon Algèbre Théor. Nombres 2016, 45–58.
- [Jau19] J.-F. Jaulent, *Note sur la conjecture de Greenberg*, J. Ramanujan Math. Soc. 34 (2019), 59–80.
- [Jau20] J.-F. Jaulent, *Normes universelles et conjecture de Greenberg*, Acta Arith. 194 (2020), 99–109.
- [Jau21] J.-F. Jaulent, *Annulateurs de Stickelberger des groupes de classes logarithmiques*, Acta Arith. 203 (2021), 241–253.
- [Jau24] J.-F. Jaulent, *Annulateurs circulaires des groupes de classes logarithmiques*, Ann. Inst. Fourier (Grenoble) (online, 2024).
- [JM06] J.-F. Jaulent et A. Michel, *Approche logarithmique des noyaux étales sauvages des corps de nombres*, J. Number Theory 120 (2006), 72–91.
- [JS01] J.-F. Jaulent et F. Soriano, *Sur le noyau sauvage des corps de nombres et le groupe des classes logarithmiques*, Math. Z. 238 (2001), 335–354.
- [Keu89] F. Keune, *On the structure of the K_2 of the ring of integers in a number field*, K -Theory 2 (1989), 625–645.
- [Kol89] M. Kolster, *A relation between the 2-primary parts of the Main Conjecture and the Birch–Tate-Conjecture*, Canad. Math. Bull. 32 (1989), 248–251.
- [Kol03] M. Kolster, *K -theory and arithmetic*, in: Contemporary Developments in Algebraic K -theory 2002 (Trieste, 2002), ICTP Lecture Notes 15, Abdus Salam Int. Centre Theoret. Phys., Trieste, 2003, 191–258.
- [KM03] M. Kolster and A. Movahhedi, *Bi-quadratic number fields with trivial 2-primary Hilbert kernels*, Proc. London Math. Soc. 87 (2003), 109–136.
- [KL64] T. Kubota und H. W. Leopoldt, *Eine p -adische Theorie der Zetawerte. Teil I: Einführung der p -adischen Dirichletschen L -Funktionen*, J. Reine Angew. Math. 214-215 (1964), 328–339.
- [Lec93] O. Lécacheux, *Units in number fields and elliptic curves*, in: Advances in Number Theory (Kingston, ON, 1991), Oxford Sci. Publ., Oxford Univ. Press, New York, 1993, 293–301.
- [Lim23] M. F. Lim, *On the structure of even K -groups of rings of algebraic integers*, Acta Arith. 211 (2023), 345–361.

- [Mai18] C. Maire, *Genus theory and governing fields*, New York J. Math. 24 (2018), 1056–1067.
- [Mov88] A. Movahhedi, *Sur les p -extensions des corps p -rationnels*, thèse de doctorat, Univ. Paris VII, 1988.
- [Tho94] H. Thomas, *Trivialité du 2-rang du noyau hilbertien*, J. Théor. Nombres Bordeaux 6 (1994), 459–483.
- [Ngu86] T. Nguyen Quang Do, *Sur la \mathbf{Z}_p -torsion de certains modules galoisiens*, Ann. Sci. Inst. Fourier (Grenoble) 36 (1986), no. 3, 27–46.
- [Ngu90] T. Nguyen Quang Do, *Une étude cohomologique de la partie 2-primaire de $K_2\mathcal{O}$* , K -Theory 3 (1990), 523–542.
- [Ngu92] T. Nguyen Quang Do, *Analogues supérieurs du noyau sauvage*, Sém. Théor. Nombres Bordeaux 4 (1992), 263–271.
- [Ngu16] T. Nguyen Quang Do, *Descente galoisienne et capitulation entre modules de Bertrandias–Payan*, Publ. Math. Besançon Théor. Nombres 2016, 59–79.
- [PARI] The PARI Group, *PARI/GP, version 2.5.3 (2013)*, Univ. Bordeaux, <http://pari.math.u-bordeaux.fr/download.html>.
- [Qin95] H. R. Qin, *The 4-rank of $K_2\mathcal{O}_F$ for real quadratic fields F* , Acta Arith. 72 (1995), 323–333.
- [Qin05] H. R. Qin, *The 2-Sylow subgroup of $K_2\mathcal{O}_F$ for number fields F* , J. Algebra 284 (2005), 494–519.
- [Qin10] H. R. Qin, *Reflection theorems and the p -Sylow subgroup of $K_2\mathcal{O}_F$ for a number field F* , J. Pure Appl. Algebra 214 (2010), 1181–1192.
- [Rib08] K. A. Ribet, *Bernoulli numbers and ideal classes*, Gazette Math. 118 (2008), 42–49,
- [RØ00] J. Rognes and P. A. Østvær, *Two-primary algebraic K -theory of two-regular number fields*, Math. Z. 233 (2000), 251–263.
- [RW00] J. Rognes and C. Weibel, *Two-primary algebraic K -theory of rings of integers in number fields*, J. Amer. Math. Soc. 13 (2000), 1–54.
- [Ser78] J.-P. Serre, *Sur le résidu de la fonction zêta p -adique d'un corps de nombres*, C. R. Acad. Sci. Paris Sér. A 287 (1978), 183–188; reprinted in: Oeuvres–Collected Papers III, pos. 116 (1972–1984), Springer, Berlin, 453–458.
- [Tat71] J. Tate, *Symbols in arithmetic*, in: Actes du Congrès International de Mathématiciens (Nice, 1970), Tome 1, Gauthier-Villars, 1971, 201–211.
- [Tat76] J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. 36 (1976), 257–274.
- [W82] L. C. Washington, *Zeros of p -adic L -functions*, in: Séminaire de théorie des nombres, Paris 1980-81, Sém. Delange–Pisot–Poitou, Birkhäuser, Boston, 1982, 337–357.
- [W81] L. C. Washington, *Zeros of p -adic L -functions*, Sém. Théor. Nombres Bordeaux 1980-1981, exp. 25, 4 pp.
- [W95] L. C. Washington, *Siegel zeros for 2-adic L -functions*, in: Number Theory (Halifax, NS, 1994), CMS Conf. Proc. 15, Amer. Math. Soc., Providence, RI, 1995, 393–396.
- [Wa97^a] L. C. Washington, *A family of cubic fields and zeros of 3-adic L -functions*, J. Number Theory 63 (1997), 408–417.
- [Wa97^b] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Grad. Texts in Math. 83, Springer, New York, 1997.
- [W05] C. Weibel, *Algebraic K -theory of rings of integers in local and global fields*, in: Handbook of K -Theory, Vol. 1, Springer, Berlin, 2005, 139–190.

- [Yue01] Q. Yue, *Elements of order 4 of the Hilbert kernel in quadratic number fields*, Acta Arith. 97 (2001), 295–311.
- [Yue02] Q. Yue, *On tame kernel and class group in terms of quadratic forms*, J. Number Theory 96 (2002), 373–387.
- [Yue05] Q. Yue, *The formula of 8-ranks of tame kernels*, J. Algebra 286 (2005), 1–25.
- [YF00] Q. Yue and K. Q. Feng, *The 4-rank of the tame kernel versus the 4-rank of the narrow class group in quadratic number fields*, Acta Arith. 96 (2000), 155–165.

Georges Gras
38520 Le Bourg d’Oisans, France
ORCID: 0000-0002-1318-4414
E-mail: g.mn.gras@wanadoo.fr

