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NEW RESULTS ON PROJECTION ALGORITHMS FOR SOLVING SYSTEMS OF GENERAL VARIATIONAL INEQUALITIES

Abstract. In [Croatian Oper. Res. Rev. 13 (2022), 131–135], it was shown that the Lipschitz continuity condition with respect to the first and/or second variable has been misapplied in prior literature on systems of variational inequalities. This paper corrects errors in previous work by M. A. Noor and K. I. Noor by introducing a new iterative method.

1. Introduction. Variational inequalities play a fundamental role in a wide range of mathematical and applied problems, such as physics, finance, social sciences, ecology, industry, and economics. They include, as special cases, complementarity problems, systems involving nonlinear equations, optimization, and fixed-point problems. Another advantage of variational inequalities is that a large class of fluid mechanics problems, boundary value problems, transport, and equilibrium problems can be studied via variational inequalities.

Many works have been devoted to systems of variational inequalities. Among these, M. A. Noor and K. I. Noor [3] employed an incorrect definition of Lipschitz continuity in the first and/or second variable (further details can be found in [1]). In this paper, I present and study a new iterative method (3.1) in order to correct the main results of [3].

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a closed convex set in H. The authors of [3] studied a system of general variational inequalities involving

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a relaxed (α, β)-cocoercive operator; they considered the problem of finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, & \forall v \in H : g(v) \in K, \\ \langle \eta T_2(x^*, y^*) + y^* - h(x^*), h(v) - y^* \rangle \geq 0, & \forall v \in H : h(v) \in K, \end{cases} \tag{1.1}$$

where $T_1, T_2, g, h : H \to H$ are nonlinear operators and $\rho, \eta > 0$ are constant parameters. A system of type (1.1) is called a *system of general variational inequalities* (SGHVID).

2. Preliminaries. For nonlinear operators T_1, T_2, g, $h : H \to H$, we recall the following well known concepts.

DEFINITION 2.1. A mapping $T : H \to H$ is called λ-*Lipschitzian* if there exists a constant $\lambda > 0$ such that

$$\forall x, y \in H : \quad \|T(x) - T(y)\| \leq \lambda \|x - y\|. \tag{2.1}$$

DEFINITION 2.2. A mapping $T : H \to H$ is called *relaxed* (α, β)-*cocoercive* if there exist constants $\alpha, \beta > 0$ such that

$$\forall x, y \in H : \ \langle T(x) - T(y), x - y \rangle \geq -\alpha \|T(x) - T(y)\|^2 + \beta \|x - y\|^2. \tag{2.2}$$

PROPOSITION 2.3. *Let K be a closed convex set in H, and let $z \in H$ and $x \in K$. Then the condition*

$$\langle x - z, y - x \rangle \geq 0, \qquad \forall y \in K, \tag{2.3}$$

is equivalent to

$$x = P_K(z), \tag{2.4}$$

where P_K is the projection of H into K.

It is known that P_K is a *nonexpansive* mapping, i.e.

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \qquad \forall x, y \in H. \tag{2.5}$$

Using Proposition 2.3, we can easily show that finding a solution $(x^*, y^*) \in K \times K$ of SGHVID is equivalent to finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} x^* = (1 - \alpha_n) x^* + \alpha_n P_K[g(y^*) - \rho T_1(y^*, x^*)], \\ y^* = (1 - \alpha_n) y^* + \alpha_n P_K[h(x^*) - \eta T_2(x^*, y^*)], \end{cases} \tag{2.6}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

3. Main result. Now we analyze the following iterative method for solving SGHVID.

ALGORITHM 3.1. For any initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n, x_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[h(x_n) - \eta T_2(x_n, y_n)], \end{cases} \tag{3.1}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Special cases

(1) For $T_1 = T_2 = T$ in Algorithm 3.1, we arrive at

ALGORITHM 3.2. For any initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T(y_n, x_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[h(x_n) - \eta T(x_n, y_n)], \end{cases} \tag{3.2}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Then (x_n, y_n) is an approximate solution of the following system:

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \\ \langle \eta T(x^*, y^*) + y^* - h(x^*), h(v) - y^* \rangle \geq 0, \quad \forall v \in H : h(v) \in K. \end{cases} \tag{3.3}$$

(2) For $g = h$ in Algorithm 3.1, we get

ALGORITHM 3.3. For any initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n, x_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[g(x_n) - \eta T_2(x_n, y_n)], \end{cases} \tag{3.4}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Then (x_n, y_n) is an approximate solution of the system

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \\ \langle \eta T_2(x^*, y^*) + y^* - h(x^*), g(v) - y^* \rangle \geq 0, \quad \forall v \in H : g(v) \in K. \end{cases} \tag{3.5}$$

(3) For $T_1 = T_2 = T$ and $g = h$ in Algorithm 3.1, we get

ALGORITHM 3.4. For any initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T(y_n, x_n)], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K[g(x_n) - \eta T(x_n, y_n)], \end{cases} \tag{3.6}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Then (x_n, y_n) is an approximate solution of the system

$$(3.7)\qquad \begin{cases} \langle \rho T(y^*, x^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, & \forall v \in H : g(v) \in K, \\ \langle \eta T(x^*, y^*) + y^* - h(x^*), g(v) - y^* \rangle \geq 0, & \forall v \in H : g(v) \in K. \end{cases}$$

Now we present convergence criteria for Algorithm 3.1 under suitable conditions; this is the main result of this paper.

THEOREM 3.5. *Let (x^*, y^*) be a solution of SGHVID. Suppose that $T_1 : H \times H \to H$ is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian in the first variable and λ_1-Lipschitzian in the second variable. Let $T_2 : H \times H \to H$ be relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitzian in the first variable and λ_2-Lipschitzian in the second variable. Let g be relaxed (γ_3, r_3)-cocoercive and μ_3 Lipschitzian and let h be relaxed (γ_4, r_4)-cocoercive and μ_4-Lipschitzian. If*

$$(3.8)\qquad \begin{cases} k_1 < 1/2, \quad r_1 > \gamma_1 \mu_1^2 + \mu_1 \sqrt{3/4 - k_1^2 + k_1}, \\ \left| \rho - \dfrac{r_1 - \gamma_1 \mu_1^2}{\mu_1^2} \right| < \dfrac{\sqrt{(r_1 - \gamma_1 \mu_1^2)^2 - \mu_1^2 [3/4 - k_1^2 + k_1]}}{\mu_1^2}, \quad \rho < \dfrac{1}{2\lambda_1}, \end{cases}$$

$$(3.9)\qquad \begin{cases} k_2 < 1/2, \quad r_2 > \gamma_2 \mu_2^2 + \mu_2 \sqrt{3/4 - k_2^2 + k_2}, \\ \left| \eta - \dfrac{r_2 - \gamma_2 \mu_2^2}{\mu_2^2} \right| < \dfrac{\sqrt{(r_2 - \gamma_2 \mu_2^2)^2 - \mu_2^2 [3/4 - k_2^2 + k_2]}}{\mu_2^2}, \quad \eta < \dfrac{1}{2\lambda_2}, \end{cases}$$

where

$$k_1 = \sqrt{1 - 2(r_3 - \gamma_3 \mu_3^2) + \mu_3^2}, \quad k_2 = \sqrt{1 - 2(r_4 - \gamma_4 \mu_4^2) + \mu_4^2},$$

and $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, then for any initial points $x_0, y_0 \in K$, the x_n and y_n obtained from Algorithm 3.1 converge strongly to x^* and y^* respectively.

Proof. We first evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$:

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n)(x_n - x^*) \\ &\qquad + \alpha_n [P_K[g(y_n) - \rho T_1(y_n, x_n)] - P_K[g(y^*) - \rho T_1(y^*, x^*)]]\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n \|P_K[g(y_n) - \rho T_1(y_n, x_n)] - P_K[g(y^*) - \rho T_1(y^*, x^*)]\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| \\ &\quad + \alpha_n \|[g(y_n) - \rho T_1(y_n, x_n)] - [g(y^*) - \rho T_1(y^*, x^*)]\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x^*)]\| \\ &\quad + \alpha_n \|y_n - y^* - [g(y_n) - g(y^*)]\| \\ &\leq \alpha_n \|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x_n) + T_1(y^*, x_n) - T_1(y^*, x^*)]\| \\ &\quad + \alpha_n \|y_n - y^* - [g(y_n) - g(y^*)]\| + (1 - \alpha_n)\|x_n - x^*\| \end{aligned}$$

$$\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x_n)]\|$$
$$+ \rho\alpha_n\|T_1(y^*, x_n) - T_1(y^*, x^*)\| + \alpha_n\|y_n - y^* - [g(y_n) - g(y^*)]\|.$$

Since T_1 is relaxed (γ_1, r_1)-cocoercive in the first variable, we have

$$\begin{aligned}
&\|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x_n)]\|^2 \\
&\quad = \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n, x_n) - T_1(y^*, x_n), y_n - y^*\rangle \\
&\qquad + \rho^2\|T_1(y_n, x_n) - T_1(x_n, y^*)\|^2 \\
&\quad \leq -2\rho[-\gamma_1\|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 + r_1\|y_n - y^*\|^2] + \|y_n - y^*\|^2 \\
&\qquad + \rho^2\|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 \\
&\quad \leq 2\rho\gamma_1\|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2 - 2\rho r_1\|y_n - y^*\|^2 + \|y_n - y^*\|^2 \\
&\qquad + \rho^2\|T_1(y_n, x_n) - T_1(y^*, x_n)\|^2.
\end{aligned}$$

As T_1 is μ_1-Lipschitzian in the first variable, we get

$$\|y_n - y^* - \rho[T_1(y_n, x_n) - T_1(y^*, x_n)]\|^2 \leq [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]\|y_n - y^*\|^2.$$

In a similar way, since g is (γ_3, r_3)-cocoercive and μ_3-Lipschitzian, we have

$$\|y_n - y^* - [g(y_n) - g(y^*)]\| \leq k_1\|y_n - y^*\|.$$

As T_1 is λ_1-Lipschitzian in the second variable, we find that

$$\|T_1(y^*, x_n) - T_1(y^*, x^*)\| \leq \lambda_1\|x_n - x^*\|.$$

As a result, we obtain

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| + \alpha_n\rho\lambda_1\|x_n - x^*\|, \tag{3.10}$$

where

$$\theta_1 = k_1 + [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]^{1/2}.$$

In the same manner, we arrive at

$$\|y_{n+1} - y^*\| \leq (1-\alpha_n)\|y_n - y^*\| + \alpha_n\theta_2\|x_n - x^*\| + \alpha_n\eta\lambda_2\|y_n - y^*\|, \tag{3.11}$$

where

$$\theta_2 = k_2 + [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]^{1/2}.$$

Conditions (3.8) and (3.9) make it clear that

$$\theta_1 + \eta\lambda_2 < 1 \quad \text{and} \quad \theta_2 + \rho\lambda_1 < 1.$$

Then, from (3.10) and (3.11),

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\qquad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| + \alpha_n\rho\lambda_1\|x_n - x^*\| \\
&\qquad\quad + (1-\alpha_n)\|y_n - y^*\| + \alpha_n\theta_2\|x_n - x^*\| + \alpha_n\eta\lambda_2\|y_n - y^*\| \\
&\qquad \leq (1-\alpha_n)[\|x_n - x^*\| + \|y_n - y^*\|] \\
&\qquad\quad + \sigma\alpha_n[\|x_n - x^*\| + \|y_n - y^*\|],
\end{aligned}$$

where

$$\sigma = \max(\theta_1 + \eta\lambda_2, \theta_2 + \rho\lambda_1) < 1.$$

Set

$$z_n = \|x_n - x^*\| + \|y_n - y^*\|.$$

Then

$$z_{n+1} \leq (1 - (1-\sigma)\alpha_n)z_n,$$

which means that

$$z_{n+1} \leq \prod_{k=0}^{n} (1 - (1-\sigma)\alpha_k)z_0.$$

Since $0 < \sigma < 1$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$, this implies in light of [4] that

$$\lim_{n\to\infty} \prod_{k=0}^{n} (1 - (1-\sigma)\alpha_k) = 0,$$

and therefore $x_n \to x^*$ and $y_n \to y^*$. This completes the proof. ∎

COROLLARY 3.6. *We can replace conditions* (3.8) *and* (3.9) *by* (3.12) *and* (3.13) *below, where* $0 < p_1, p_2 < 1$, *and*

$$\begin{cases} k_1 < p_1, \quad r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{-k_1^2 + 2p_1k_1 + 1 - p_1^2}, \\ \left|\rho - \dfrac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \dfrac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2[-k_1^2 + 2p_1k_1 + 1 - p_1^2]}}{\mu_1^2}, \\ \rho < \dfrac{1 - p_2}{\lambda_1}, \end{cases} \tag{3.12}$$

$$\begin{cases} k_2 < p_2, \quad r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{-k_2^2 + 2p_2k_2 + 1 - p_2^2}, \\ \left|\eta - \dfrac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \dfrac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2[-k_2^2 + 2p_2k_2 + 1 - p_2^2]}}{\mu_2^2}, \\ \eta < \dfrac{1 - p_1}{\lambda_2}. \end{cases} \tag{3.13}$$

REMARK 3.7. If $T_1, T_2 : H \to H$ are univariate operators, then Algorithm 3.1 can be replaced by the following algorithm:

ALGORITHM 3.8. For any initial points $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n)], \\ y_n = P_K[h(x_n) - \eta T_2(x_n)], \end{cases}$$

where $\alpha_n \in [0,1]$ for all $n \geq 0$.

Then (x_n, y_n) is an approximate solution of the system

$$\begin{cases} \langle \rho T_1(y^*) + x^* - g(y^*), g(v) - x^* \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \\ \langle \eta T_2(x^*) + y^* - h(x^*), h(v) - y^* \rangle \geq 0, \quad \forall v \in H : h(v) \in K. \end{cases} \tag{3.14}$$

For the system (3.14), we use Algorithm 3.8 and present the following theorem which uses fewer conditions than the previous theorem.

Theorem 3.9. *Let* (x^*, y^*) *be a solution of* (3.14). *Suppose* $T_1, T_2, g, h : H \to H$ *are relaxed-cocoercive with constants* (γ_1, r_1), (γ_2, r_2), (γ_3, r_3), (γ_4, r_4) *and Lipschitzian with constants* μ_1, μ_2, μ_3, μ_4 *respectively. If*

$$\text{(3.15)} \qquad \begin{cases} k_1 < 1, \quad r_1 > \gamma_1\mu_1^2 + \mu_1\sqrt{-k_1^2 + 2k_1}, \\ \left|\rho - \dfrac{r_1 - \gamma_1\mu_1^2}{\mu_1^2}\right| < \dfrac{\sqrt{(r_1 - \gamma_1\mu_1^2)^2 - \mu_1^2[-k_1^2 + 2k_1]}}{\mu_1^2}, \end{cases}$$

and

$$\text{(3.16)} \qquad \begin{cases} k_2 < 1, \quad r_2 > \gamma_2\mu_2^2 + \mu_2\sqrt{-k_2^2 + 2k_2}, \\ \left|\eta - \dfrac{r_2 - \gamma_2\mu_2^2}{\mu_2^2}\right| < \dfrac{\sqrt{(r_2 - \gamma_2\mu_2^2)^2 - \mu_2^2[-k_2^2 + 2k_2]}}{\mu_2^2}, \end{cases}$$

where

$$k_1 = \sqrt{1 - 2(r_3 - \gamma_3\mu_3^2) + \mu_3^2}, \qquad k_2 = \sqrt{1 - 2(r_4 - \gamma_4\mu_4^2) + \mu_4^2},$$

and $\alpha_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_k = \infty$, *then for any initial points* $x_0 \in K$, *the* x_n *and* y_n *obtained from Algorithm* 3.8 *converge strongly to* x^* *and* y^* *respectively.*

Proof. We first evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$:

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \|(1-\alpha_n)x_n + \alpha_n P_K[g(y_n) - \rho T_1(y_n)] - (1-\alpha_n)x^* \\
&\qquad + \alpha_n P_K[g(y^*) - \rho T_1(y^*)]\| \\
&\quad \leq (1-\alpha_n)\|x_n - x^*\| \\
&\qquad + \alpha_n\|P_K[g(y_n) - \rho T_1(y_n)] - P_K[g(y^*) - \rho T_1(y^*)]\| \\
&\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|[g(y_n) - \rho T_1(y_n)] - [g(y^*) - \rho T_1(y^*)]\| \\
&\quad \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho[T_1(y_n) - T_1(y^*)]\| \\
&\qquad + \|y_n - y^* - [g(y_n) - g(y^*)]\|.
\end{aligned}$$

Since T_1 is relaxed (γ_1, r_1)-cocoercive, we have

$$\begin{aligned}
\|y_n - y^* - \rho[T_1(y_n) - T_1(y^*)]\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle T_1(y_n) - T_1(y^*), y_n - y^*\rangle \\
&\quad + \rho^2\|T_1(y_n) - T_1(y^*)\|^2 \\
&\leq -2\rho[-\gamma_1\|T_1(y_n) - T_1(y^*)\|^2 + r_1\|y_n - y^*\|^2] \\
&\quad + \|y_n - y^*\|^2 + \rho^2\|T_1(y_n) - T_1(y^*)\|^2 \\
&\leq 2\rho\gamma_1\|T_1(y_n) - T_1(y^*)\|^2 - 2\rho r_1\|y_n - y^*\|^2 \\
&\quad + \|y_n - y^*\|^2 + \rho^2\|T_1(y_n) - T_1(y^*)\|^2.
\end{aligned}$$

As T_1 is μ_1-Lipschitzian, we find that

$$\|y_n - y^* - \rho[T_1(y_n) - T_1(y^*)]\|^2 \leq [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]\|y_n - y^*\|^2.$$

In a similar way, since g is (γ_3, r_3)-cocoercive and μ_3-Lipschitzian, we obtain

$$\|y_n - y^* - [g(y_n) - g(y^*)]\| \leq k_1\|y_n - y^*\|.$$

As a result, we obtain

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\|, \tag{3.17}$$

where

$$\theta_1 = k_1 + [1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \rho^2\mu_1^2]^{1/2}.$$

Now we evaluate $\|y_{n+1} - y^*\|$ for all $n \geq 0$:

$$\begin{aligned}\|y_n - y^*\| &= \|P_K[h(x_n) - \eta T_2(x_n)] - P_K[h(x^*) - \eta T_1(x^*)]\| \\ &\leq \|[h(x_n) - \eta T_2(x_n)] - [h(x^*) - \eta T_2(x^*)]\| \\ &\leq \|x_n - x^* - \eta[T_2(x_n) - T_2(x^*)]\| + \|x_n - x^* - [h(x_n) - h(x^*)]\|.\end{aligned}$$

Since T_2 is relaxed (γ_2, r_2)-cocoercive, we have

$$\begin{aligned}&\|x_n - x^* - \eta[T_2(x_n) - T_2(x^*)]\|^2 \\ &\quad= \|x_n - x^*\|^2 - 2\eta\langle T_2(x_n) - T_2(x^*), x_n - x^*\rangle + \eta^2\|T_2(x_n) - T_2(x^*)\|^2 \\ &\quad\leq -2\eta[-\gamma_2\|T_2(x_n) - T_2(x^*)\|^2 + r_2\|x_n - x^*\|^2] \\ &\qquad+ \|x_n - x^*\|^2 + \eta^2\|T_2(x_n) - T_2(x^*)\|^2 \\ &\quad\leq 2\eta\gamma_2\|T_2(x_n) - T_2(x^*)\|^2 - 2\eta r_2\|x_n - x^*\|^2 \\ &\qquad+ \|x_n - x^*\|^2 + \eta^2\|T_2(x_n) - T_2(x^*)\|^2.\end{aligned}$$

Since h is (γ_4, r_4)-cocoercive and μ_4-Lipschitzian, we have

$$\|x_n - x^* - [h(x_n) - h(x^*)]\| \leq k_2\|x_n - x^*\|.$$

Then

$$\|y_n - y^*\| \leq \theta_2\|x_n - x^*\|, \tag{3.18}$$

where

$$\theta_2 = k_2 + [1 + 2\eta\gamma_2\mu_2^2 - 2\eta r_2 + \eta^2\mu_2^2]^{1/2}.$$

Conditions (3.15) and (3.16) make it clear that

$$\theta_1 < 1 \quad \text{and} \quad \theta_2 < 1.$$

Then, from (3.17) and (3.18),

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\|,$$

which implies that

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_k)\|x_0 - x^*\|.$$

Since $0 < \theta_1\theta_2 < 1$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$, this implies in light of [4] that

$$\lim_{n\to\infty} \prod_{k=0}^{n} (1 - (1 - \theta_1\theta_2)\alpha_k) = 0,$$

and therefore $x_n \to x^*$ and $y_n \to y^*$. This completes the proof. ■

REMARK 3.10. (1) For $g = h$ and $T_1 = T_2 = T$ in Algorithm 3.8, we arrive at

ALGORITHM 3.11. For any initial points $x_0 \in K$, compute the sequences $\{x_n\}$ by using

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K[h(y_n) - \rho T(y_n)], \\ y_n = P_K[h(x_n) - \eta T(x_n)], \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Then (x_n, y_n) is an approximate solution of the following system:

$$\begin{cases} \langle \rho T(y^*) + x^* - h(y^*), h(v) - x^* \rangle \geq 0, & \forall v \in H : h(v) \in K, \\ \langle \eta T(x^*) + y^* - h(x^*), h(v) - y^* \rangle \geq 0, & \forall v \in H : h(v) \in K. \end{cases}$$

(2) For $g = h = I$ and $T_1 = T_2 = T$ in Algorithm 3.8, we get

ALGORITHM 3.12. For any initial points $x_0 \in K$, compute the sequences, $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K[y_n - \rho T(y_n)], \\ y_n = P_K[x_n - \eta T(x_n)], \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Then (x_n, y_n) is an approximate solution of the following system of variational inequalities (SNVI):

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, v - x^* \rangle \geq 0, & \forall v \in K, \\ \langle \eta T(x^*) + y^* - x^*, v - y^* \rangle \geq 0, & \forall v \in K, \end{cases}$$

studied by Benhadid and Brahimi [2].

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