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Free structures and limiting density

by

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Abstract. Gromov asked what a typical (finitely presented) group looks like, and he suggested a way to make the question precise in terms of *limiting density*. The typical finitely generated group is known to share some important properties with the non-Abelian free groups. The third author conjectured that for $n \geq 2$ and group presentations with a single relator, sentences true in the non-Abelian free groups have density 1. We state Gromov's question more generally, for structures in an arbitrary *algebraic variety* (in the sense of universal algebra), with presentations of a specific form, and focusing on elementary (i.e., finitary) first order sentences.

We give examples illustrating different behaviors. In the first, the sentences true in the typical structure are just those true in the free structure. In the second, each sentence has limiting density 0 or 1, but the theory of the typical structure is not that of the free structure. In the third, there is a sentence with limiting density strictly between 0 and 1. In the fourth, the limiting density lies strictly between 0 and 1 for some sentences, and does not exist for some other sentences.

Generalizing the first example, we consider *commutative generalized bijective varieties* in a language consisting of finitely many unary function symbols. We show that for presentations with a single generator and a single identity, if the free structure F is infinite, then the theory of the typical structure matches that of F. This also holds for presentations with m generators and one identity.

1. Introduction. In the paper where he introduced the notion of a hyperbolic group, Gromov [9] asked what a typical group looks like. He was thinking of finitely presented groups. He described, in terms of limiting density, what it might mean for a typical group to have some property Q, and he stated that the typical group is hyperbolic. Gromov's notion has been made precise in different ways; see, for instance, the survey [22]. Ol'shanskiĭ [23] cleaned up the statement and the proof that the typical group is hyperbolic.

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The third author conjectured that for the typical group obtained from a presentation consisting of $m \geq 2$ generators and a single relator, the elementary (i.e., finitary) first-order theory matches that of the non-Abelian free groups.

In this paper, we generalize Gromov's question to arbitrary *equational classes*, or *algebraic varieties* (in the sense of universal algebra). Here, as for groups, the notions of *finite presentation* and *free structure* make sense. Instead of *relators*, we use *identities.* For a variety V, a set P of *allowable* presentations (all finite), and a sentence φ in the language of V, we take the *limiting density* to be

$$\lim_{s \to \infty} \frac{P_s(\varphi)}{P_s},$$

where P_s is the number of presentations in P for which the identities have length at most s and $P_s(\varphi)$ is the number of these presentations for which the resulting structure satisfies φ. The limit may or may not exist.

We find examples, each a variety V together with a set P of allowable presentations, exhibiting different behavior.

Example 1.1 (Theorem 3.1). For the language with just two unary function symbols, let V_1 be the variety with axioms saying that the functions are inverses. Let P_1 be the set of presentations with generator a and a single identity. The free structure F is isomorphic to the integers with the successor function and its inverse. The typical structure satisfies the theory of F; that is, for an elementary first order sentence φ, the limiting density is 1 if $F \models \varphi$, and 0 otherwise.

Example 1.2 (Proposition 3.12). For the language with a single unary function symbol, let V_2 be the variety with no axioms, and let P_2 be the set of presentations with generator a and a single identity. The free structure F is isomorphic to the natural numbers with the successor function. All sentences have density 0 or 1. We describe a "limit structure" M whose theory is that of the typical structure. There is a sentence with density 1 that is false in F.

Example 1.3 (Theorem 3.26). Take the variety V_1, but let P_3 be the set of presentations with generator a and two identities. There is a sentence for which the limiting density does not exist.

Example 1.4 (Lemmas 3.32 and 3.33). Let V_4 be the variety of Abelian groups, and let P_4 be the set of presentations with generator a and a single relator, corresponding to an identity of the form $t(a) = 0$. There are sentences with density strictly between 0 and 1. There are also sentences for which the limiting density does not exist.

We generalize the first example, aiming at conditions guaranteeing that the theory of the typical structure matches that of the free structure. Our languages consist of finitely many unary function symbols. A variety is *commutative* if the axioms imply that the functions commute. A variety is *generalized bijective* if the axioms imply that each function is $1-1$ and onto. Here is our main result.

THEOREM 1.5 (Theorem 4.32). *Fix a natural number $m \geq 1$. For a commutative generalized bijective variety, and presentations with m generators and 1 identity, if the free structure F is infinite, then the theory of F matches that of the typical structure.*

The same conclusion of the previous theorem also holds if we add constants to name the generators, see Theorem 5.7.

1.1. Organization. We begin with Gromov's original question, which concerned finitely presented groups. We then generalize this question to finitely presented members of a variety V. Section 2 has some background on algebraic varieties. Section 3 has the four examples described above. This section also introduces key ideas that will appear in later proofs. In Section 4, we consider more general bijective varieties, and we prove the theorem stated above. For a language with unary function symbols $f_1, \ldots, f_n$, injective and commuting, we say how to find *elementary invariants*, i.e., sentences that distinguish the theory of one model from that of another. We use a version of Gaifman's Locality Theorem, which we prove using saturation.

In Section 5, we consider sentences with constants. Here we put no restrictions on the arity of the function symbols. For an arbitrary variety V and set P of presentations with a fixed tuple $\bar{a}$ of generators, we give conditions guaranteeing that, for sentences φ in the language with added constants naming the generators, φ has limiting density 1 if and only if it is true in the free structure on $\bar{a}$. In Section 6, we give further examples illustrating the results from Sections 4 and 5.

1.2. Gromov's question about groups. Here, we recall Gromov's original question and mention some prior work on typical or random groups. The usual language for groups has a binary operation symbol (for the group operation), a unary operation symbol (for the inverse), and a constant (for the identity). Let T be the theory of groups. Recall that a *group presentation* consists of a tuple $\bar{a}$ of *generators* and a set R of words $w_i(\bar{a})$ on these generators, called *relators*. In the group G with presentation $\langle \bar{a} | R \rangle$, $G \models t(\bar{a}) = e$ if and only if $T \cup \{w(\bar{a}) = e : w(\bar{a}) \in R\} \vdash t(\bar{a}) = e$. Suppose F is the free group on $\bar{a}$ and N is the subgroup of F consisting of the elements $t(\bar{a})$ such that $T \cup \{w(\bar{a}) = e\} \models t(\bar{a}) = e$. Then $G \cong F/N$.

The notion of limiting density depends not just on the variety, but also on the allowable group presentations. Ol'shanskiĭ [23] considered presentations with m generators and k relators, all reduced. Kapovich and Schupp [12] considered the special case where $k = 1$. In the Gromov "density" model, the number of relators may vary but is bounded in terms of the length of the relators and a parameter d. It is important to bound the number of relators in some way; otherwise, the typical group will almost surely be trivial [23].

Definition 1.6. Let Q be a property of interest. Let P_s be the number of presentations in which the relators have length at most s, and let $P_s(Q)$ be the number of these presentations for which the resulting group has property Q. The *limiting density* for Q is

$$\lim_{s\to\infty} \frac{P_s(Q)}{P_s},$$

if this limit exists.

We consider the *typical group* to have property Q if Q has density 1. We are particularly interested in elementary properties. For a sentence φ, the density of the property of satisfying φ will be called simply the density of φ. The typical group, in the sense of limiting density, is also called the *random group* (1). The typical group has some properties of free groups. Gromov introduced the property of hyperbolicity and stated that the typical group is hyperbolic. Ol'shanskiĭ [23] showed that for presentations with m generators and k reduced relators, the property of being hyperbolic has limiting density 1. Kapovich and Schupp [12] showed that for presentations with m generators and one reduced relator, the property that all minimal generating tuples are Nielsen equivalent has limiting density 1. *Nielsen equivalence* means that one tuple can be transformed into the other by a finite sequence of simple, obviously reversible, kinds of steps.

Benjamin Fine, in conversation with the third author at the JMM in January of 2013, made an off-hand comment to the effect that in the limiting density sense, all groups look free. Fine's comment gave rise to the conjecture below, saying that for presentations with $m \geq 2$ generators and one relator, the typical group has the same elementary first order theory as the the free group. By a result of Sela [25] (see also Kharlampovich–Miasnikov [13]), the elementary first-order theory of all non-Abelian free groups is the same. The conjecture is given in [11, Conjecture 2.2].

Conjecture 1.7 (Knight). For groups, let P be the set of group presentations with a fixed m-tuple of generators, for $m \geq 2$, and a single relator (the Kapovich–Schupp model). For all elementary first order sentences φ,

(1) Harrison-Trainor, Khoussainov, and Turetsky [10] took a different approach and considered *random structures* more along the lines of the Rado graph.

(1) the limiting density exists,
(2) the density has value 1 if φ is true in the non-Abelian free groups, and 0 otherwise.

There is some evidence for the conjecture. By a result of Arzhantseva and Ol'shanskiĭ [2, 1], a random group obtained from a presentation with m generators and k relators has many free subgroups. Thus, an existential sentence true in the free group is also true in a random group. Kharlampovich, Miasnikov, and Sklinos [14] (see also Kharlampovich–Sklinos [15, 16], and Massalha [20, 21]) used Gromov's density model with parameter d. In this setting, they showed the following.

Theorem 1.8 (Kharlampovich, Miasnikov, Sklinos). *A random group in Gromov's density model with $d \leq 1/2$ satisfies an elementary first-order sentence if and only if the sentence is true in the non-Abelian free groups.*

The Kharlampovich–Miasnikov–Sklinos result implies the conjecture. We will not give a proof here.

2. Generalizing Gromov's question. The question that Gromov asked about groups makes sense for other algebraic varieties as well. We begin by presenting our definition of an algebraic variety; then we discuss the types of presentations we will allow and give some basic lemmas.

2.1. Algebraic varieties

Definition 2.1. A language is *algebraic* if it consists only of function symbols and constants.

The term "algebraic variety" is used to mean different things in algebraic geometry and in universal algebra. The definition that we give below is the one from universal algebra.

Definition 2.2 (Algebraic variety). For a fixed algebraic language L, a class V of L-structures is an *algebraic variety*, or simply a *variety*, if it is closed under substructures, homomorphic images, and direct products.

For our purposes, it is convenient to use the following equivalent definition of "equational class."

Definition 2.3 (Equational class). For a fixed algebraic language L, a class V of L-structures is an *equational class* if it is axiomatized by sentences of the form $(\forall \bar{x})(t(\bar{x}) = t'(\bar{x}))$—universal quantifiers in front of an equation.

Birkhoff showed that these two definitions are equivalent. Mal'tsev defined a broader class of theories whose models have well-defined presentations. See [3] for a general overview of universal algebra, where the result below appears as Theorem 11.9.

THEOREM 2.4. *For a fixed algebraic language L, a class of L-structures is an equational class if and only if it is a variety.*

In the usual language for groups, namely $\{\cdot, ^{-1}, e\}$, the group axioms have the required form. Thus, groups form a variety.

Now we consider an arbitrary algebraic variety V defined by a set A of axioms. For a fixed generating tuple $\bar{a}$, there is a well-defined *free structure* F generated by $\bar{a}$. Recall that the *term algebra* on $\bar{a}$ is the set of all terms constructible from $\bar{a}$ in the language L. The free structure F is the quotient of the term algebra by the equivalence relation that two terms are equivalent exactly when A proves the two terms are equal.

For a structure $\mathcal{A}$ in V, we say $\mathcal{A}$ is *generated* by $\bar{a} \in \mathcal{A}$ if there is no proper substructure containing $\bar{a}$. In this case, $\mathcal{A}$ is a quotient of F under an appropriate equivalence relation on terms $t(\bar{a})$. This equivalence relation becomes equality in $\mathcal{A}$.

DEFINITION 2.5. For a variety V, a *presentation* has the form $\bar{a}|R$, where $\bar{a}$ is a generating tuple and R is a set of identities on $\bar{a}$. We write $\langle \bar{a}|R \rangle$ for the structure $\mathcal{A}$ such that the identities $t(\bar{a}) = t'(\bar{a})$ true in $\mathcal{A}$ are just the ones that follow logically from R and the axioms for V.

We ask what the typical behavior is for members of a variety given by presentations of a specific form.

2.2. Allowable presentations. In this paper, almost all of the languages of our varieties will be either the group language or a language with just unary function symbols. We consider presentations with a fixed generating tuple $\bar{a}$, say of length m. For the analogue of the Ol'shanskiĭ setting, we consider presentations with k identities for some fixed k. For the analogue of the Kapovich–Schupp setting, we set $k = 1$. This is the primary case we will consider. Where we do consider $k > 1$, our presentations have the form $\bar{a}|R$, where R is an *unordered* set of identities.

We may restrict the identities in certain natural ways. For groups, we do what the group theorists do; that is, we suppose that the identities have the form $w(\bar{a}) = e$, where $w(\bar{a})$ is a word representing a product of various a_i and a_i^{-1}. For the variety in the language consisting of two unary function symbols S, S^{-1} with axioms saying that the two functions are inverses, we may restrict in a similar way, allowing only identities of the form $t(a_i) = a_j$; that is, with function symbols only on the left. For the language with finitely many unary function symbols $f_1, \ldots, f_n$ and varieties that do not have axioms explicitly saying that one f_j is the inverse of another f_i, our identities have the form $t(a_i) = t'(a_j)$, where $t(x)$ and $t'(x)$ are terms built up from the function symbols.

2.2.1. *Length.* We will need to measure lengths of identities in our presentations. We will use the following conventions, based on the restrictions described above.

Definition 2.6.

- In the setting of groups, the *length* of an identity of the form $w(\bar{a}) = e$ is the number of occurrences of the various a_i and a_i^{-1} in the (potentially non-reduced) word $w(\bar{a})$. This is the usual length of the relator.
- For varieties in the language L with just the unary function symbols $f_1, \ldots, f_n$, the *length* of an identity of the form $t(a_i) = t'(a_j)$ is the sum of the total number of occurrences of the function symbols in the terms t and t'.

2.2.2. *Limiting density.* As for groups, we consider limiting density. Here is the formal definition of limiting density.

Definition 2.7. Fix a language, a variety, and a set of allowable presentations with an m-tuple $\overline{a}$ of generators and k identities. We write P_s for the number of allowable presentations in which all of the identities have length at most s. For a property Q, let $P_s(Q)$ be the number of these presentations for which the resulting structure has property Q. Then the *limiting density* of Q is $\lim_{s\to\infty} P_s(Q)/P_s$, provided that this limit exists.

We are particularly interested in the case where Q is the property of satisfying an elementary first-order sentence φ in the language of the variety, possibly with added constants for the generators. We write $P_s(\varphi)$ for the number of presentations in which the identities have length at most s and the resulting structure satisfies φ. We say that φ has *limiting density* d if $\lim_{s\to\infty} P_s(\varphi)/P_s = d$.

Definition 2.8. We say the variety V, with a specified set of allowable presentations, satisfies the *zero-one law* if for every elementary first-order sentence φ in L, φ has limiting density 1 or 0.

2.2.3. *Sets versus tuples of identities.* We have said that our presentations consist of a tuple of generators and an *unordered* set of distinct identities. Other possibilities would be to consider ordered tuples of identities, with or without repetition. In practice, most of the time, we will consider a single identity. When we do consider more than one identity, we show that the results would be the same for ordered tuples of identities allowing repetition, ordered tuples not allowing repetition, and unordered sets of identities.

As above, we write P_s for the number of unordered sets of k identities of length at most s. In the result below, we write P_s^* for the number of ordered k-tuples allowing repetition, and P_s^{**} for the number of ordered k-tuples not allowing repetition.

Proposition 2.9. *Let $N(s)$ be the number of identities in L of length at most s and suppose that* $\lim_{s\to\infty} N(s) = \infty$. *Then for any sentence φ,*

$$\lim_{s\to\infty} \frac{P_s(\varphi)}{P_s} = \lim_{s\to\infty} \frac{P_s^*(\varphi)}{P_s^*} = \lim_{s\to\infty} \frac{P_s^{**}(\varphi)}{P_s^{**}}.$$

Proof. By definition, $P_s = \binom{N(s)}{k}$, $P_s^* = N(s)^k$, and $P_s^{**} = k! \cdot P_s$. Each unordered set of k identities yields $k!$ ordered k-tuples of identities. Thus, it is clear that

$$\lim_{s\to\infty} \frac{P_s(\varphi)}{P_s} = \lim_{s\to\infty} \frac{P_s^{**}(\varphi)}{P_s^{**}}.$$

To compare $P_s(\varphi)/P_s$ and $P_s^*(\varphi)/P_s^*$, we need the following:

Claim.

(1) $\dfrac{k!P_s}{P_s^*} \to 1,$

(2) $\dfrac{N(s)^k - k!P_s}{P_s^*} \to 0.$

Proof of Claim. For (1), the denominator is $N(s)^k$, and the numerator is a polynomial in $N(s)$ with leading term $N(s)^k$. For (2), the numerator is a polynomial in $N(s)$ of degree less than k, and the denominator is $N(s)^k$. ∎

Now, we note that

$$k!P_s(\varphi) \le P_s^*(\varphi) \le k!P_s(\varphi) + N(s)^k - k!\binom{N(s)}{k}.$$

Dividing by $P_s^* = N(s)^k$ and letting $s \to \infty$, we get

$$\lim_{s\to\infty} \frac{k!P_s(\varphi)}{P_s^*} \le \lim_{s\to\infty} \frac{P_s^*(\varphi)}{P_s^*} \le \lim_{s\to\infty} \frac{k!P_s(\varphi) + N(s)^k - k!\binom{N(s)}{k}}{P_s^*}.$$

Using the claim, we see that the right-hand side is

$$\lim_{s\to\infty} \frac{k!P_s^*(\varphi)}{k!P_s^*} = \lim_{s\to\infty} \frac{P_s^*(\varphi)}{P_s^*}. \blacksquare$$

We can now phrase the questions we are interested in more formally.

Question 2.10.

(1) Which varieties (with allowable presentations involving a fixed m-tuple $\bar{a}$ of generators) satisfy the zero-one law?
(2) Given that the zero-one law holds, when do the sentences with limiting density 1 match those true in the free structure?

2.3. Basic lemmas. Before we begin, we state three lemmas that hold very generally.

Lemma 2.11. *$\varphi \vee \psi$ has limiting density 0 if and only if φ and ψ both have limiting density 0; in fact, this holds for any finite disjunction.*

Proof. We have

$$\frac{P_s(\varphi)}{P_s}, \frac{P_s(\psi)}{P_s} \le \frac{P_s(\varphi \vee \psi)}{P_s} \le \frac{P_s(\varphi)}{P_s} + \frac{P_s(\psi)}{P_s}.$$

From this, the lemma is clear. ∎

LEMMA 2.12. *φ has limiting density 0 if and only if $\neg\varphi$ has limiting density 1.*

Proof. We have $1 = \frac{P_s(\varphi)}{P_s} + \frac{P_s(\neg\varphi)}{P_s}$. Again, the lemma is clear. ∎

LEMMA 2.13. *Let S be the set of L-sentences with limiting density 1. Then S is consistent and is closed under logical implication: if $\varphi_1, \dots, \varphi_n \in S$ and $\varphi_1, \dots, \varphi_n \vdash \psi$, then $\psi \in S$.*

Proof. Suppose S is not consistent. By the Compactness Theorem, some finite subset is inconsistent. As every sentence in this set has density 1, there is a model of T that realizes all these (finitely many) sentences, giving us a contradiction. By the previous two lemmas, we know that each $\neg\varphi_i$ has limiting density 0, so $\bigvee \neg\varphi_i$ also has limiting density 0, and so $\bigwedge \varphi_i$ has limiting density 1. But $\bigwedge \varphi_i \vdash \psi$, so $\frac{P_s(\psi)}{P_s} \ge \frac{P_s(\bigwedge \varphi_i)}{P_s} = 1$, and the lemma follows. ∎

3. Illustrative examples. In this section, we consider some varieties and classes of presentations that illustrate different possibilities. Section 3.1 concerns the variety of bijective structures and presentations with a single generator and a single identity, in which the function symbols occur only on the left. We show that the sentences true in the free structure are exactly those with limiting density 1. In Section 3.2, the variety in a language with a single unary function symbol has no axioms. The presentations have a single generator and a single identity. We show that all sentences have limiting density 0 or 1, but there is a sentence with density 0 that is true in the free structure. In Section 3.3, we return to the variety of bijective structures. The presentations have a single generator and *two* identities. We show that there are sentences for which the limiting density does not exist. In Section 3.4, the variety is Abelian groups, and the presentations have a single generator a and a single identity of the form $t(a) = a$. We show that there are sentences for which the limiting density does not exist, and sentences for which the limiting density exists but is neither 0 nor 1.

We shall freely use notions from elementary model theory, such as *elementary equivalence*, and *saturation*, as presented in the old text of Chang and Keisler [4]. For a fixed language theory T, not necessarily complete, and a set E of sentences in the language of T, we say that E is a set of *elementary invariants* for T if any models of T that satisfy the same sentences in E are elementarily equivalent. It is well known that if E is a set of elementary invariants for T, then each sentence in the language is equivalent over T to a finite Boolean combination of sentences in S.

We note that Eklof and Fischer [7], building on work of Szmielew [25], gave elementary invariants for Abelian groups. Eklof and Fischer used saturation, and we shall do the same.

3.1. Bijective structures. For the variety V_1 of bijective structures, the language consists of two unary function symbols S, S^{-1}. The axioms are

$$(\forall x)(SS^{-1}(x) = x) \text{ and } (\forall x)(S^{-1}S(x) = x).$$

These axioms guarantee that the function S is $1-1$ and onto and that S^{-1} is the inverse of S. Let T be the theory of bijective structures. The models consist of infinite $\mathbb{Z}$-chains and finite cycles $\mathbb{Z}_m$. While bijective structures lack the mathematical interest and importance of groups, it is instructive to consider them because there are relatively simple elementary invariants. Using these invariants, we show that for presentations with a single generator a and a single identity, the analogue of Conjecture 1.7 holds.

Theorem 3.1. *For bijective structures and presentations with a single generator and a single identity, each sentence φ has limiting density equal to 1 if $\mathbb{Z} \models \varphi$ and 0 otherwise.*

The result below gives a simple set of elementary invariants for the theory of bijective structures.

Proposition 3.2. *Let T be the theory of bijective structures, and let E be the set of sentences of the two types below:*

(1) *$\alpha(n, k)$, saying that there are at least k cycles of size n,*
(2) *$\beta(n)$, saying that there is a chain of length at least n.*

Then E is a set of elementary invariants for T.

Proof. For any model $\mathcal{A}$ of T, we have a natural equivalence relation $\sim$ on the universe, where $a \sim b$ if $S^m(a) = b$ for some integer m. Each $\sim$-class is a copy of $\mathbb{Z}$ or a finite cycle. We show that if $\mathcal{A}$, $\mathcal{B}$ are models of T satisfying the same sentences from E, then $\mathcal{A} \equiv \mathcal{B}$, i.e., the structures are elementarily equivalent. We may suppose that $\mathcal{A}$, $\mathcal{B}$ are countable. The sentences $\alpha(k, n)$ guarantee that they agree on the number of n-cycles for all finite n. If there is a bound on the lengths of chains, then all elements lie on finite cycles, and $\mathcal{A} \cong \mathcal{B}$, so the theories are the same. If there is no bound on the lengths of chains, then $\mathcal{A}, \mathcal{B}$ may have different numbers of $\mathbb{Z}$-chains. Let $\mathcal{A}^* \equiv \mathcal{A}$, $\mathcal{B}^* \equiv \mathcal{B}$, where $\mathcal{A}^*$, $\mathcal{B}^*$ are saturated and still countable. By saturation, $\mathcal{A}^*$, $\mathcal{B}^*$ each have infinitely many $\mathbb{Z}$-chains, so $\mathcal{A}^* \cong \mathcal{B}^*$. It follows that the theory of $\mathcal{A}^*$ and $\mathcal{A}$ matches that of $\mathcal{B}^*$ and $\mathcal{B}$. ∎

For bijective structures generated by a single element a, there is a single $\sim$-class, which has the form $\mathbb{Z}$, the free structure, or $\mathbb{Z}_m$, a cycle of size m. We note that in either $\mathbb{Z}$ or $\mathbb{Z}_m$, all elements are automorphic. The following lemma is clear from the meanings of the sentences $\alpha(n,k)$ and $\beta(n)$.

LEMMA 3.3.

(1) *For* $k > 1$, $\alpha(n,k)$ *is false in both* $\mathbb{Z}$ *and* $\mathbb{Z}_m$,
(2) $\alpha(n,1)$ *is true only in* $\mathbb{Z}_n$,
(3) $\beta(n)$ *is true in* $\mathbb{Z}$; *it is true in* $\mathbb{Z}_m$ *if and only if* $m > n$.

For models with a single generator, $\alpha(n,k)$ is false for $k > 1$, and $\beta(n)$ is equivalent to $\bigwedge_{m\leq n} \neg\alpha(n,1)$. Thus, it is enough to consider the elementary invariants of the form $\alpha(n,1)$. Our presentations have a single identity of the form $t(a) = a$. For a single identity $t(a) = a$, we get $\mathbb{Z}$ if for some k, $t(a)$ has k occurrences of S and k occurrences of S^{-1}. We get $\mathbb{Z}_m$ if for some k, $t(a)$ has either $m+k$ occurrences of S and k of S^{-1}, or $m+k$ occurrences of S^{-1} and k of S.

We will show that for all $n \geq 1$, $\alpha(n,1)$ has limiting density 0. This implies that $\neg\alpha(n,1)$, which is true in the free structure, has limiting density 1. We will use several lemmas. The first is an approximation for $\binom{2k}{k}$, good for large k. The proof requires the use of Stirling's formula on all three factorials (see the website of Das [6]).

LEMMA 3.4. $\displaystyle\binom{2k}{k} = (1+o(1))\frac{2^{2k}}{\sqrt{\pi k}}$.

The second lemma is an inequality.

LEMMA 3.5. *For all* $n \geq 1$ *and all* k,

$$2\binom{n+2k}{k} < \binom{n+2(k+1)}{k+1}.$$

Proof. Recall Pascal's Identity

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

We prove the inequality by induction on k. First, for $k = 0$, the inequality just says that $2 < n+2$. Now, suppose $k > 0$. Applying Pascal's Identity to the right side of the inequality, we get

$$\binom{n+2(k+1)}{k+1} = \binom{n+2k+1}{k} + \binom{n+2k+1}{k+1}$$

and then

$$\binom{n+2k}{k-1} + \binom{n+2k}{k} + \binom{n+2k}{k} + \binom{n+2k}{k+1}.$$

This is clearly greater than $2\binom{n+2k}{k}$. ■

To show that $\alpha(n,1)$ has limiting density 0, we express P_s in closed form and give an upper bound for $P_s(\alpha(n,1))$, also in closed form.

LEMMA 3.6. $P_s = 2^{s+1} - 1$.

Proof. The number of terms of length m is 2^m, so the number of terms of length at most s is $1 + 2 + \cdots + 2^s = 2^{s+1} - 1$. ∎

LEMMA 3.7. $P_s(\alpha(n,1)) = \sum_{n+2k \le s} 2\binom{n+2k}{k}$.

Proof. For each $n \ge 1$ and each k, we have $\binom{n+2k}{k}$ terms with $n+k$ occurrences of S and k of S^{-1}, and the same number with the symbols switched. ∎

The next lemma bounds the sum $P_{n+2k}(\alpha(n,1))$ by a single term.

LEMMA 3.8. *For all $n \ge 1$ and all $k \ge 0$,*

$$P_{n+2k}(\alpha(n,1)) < \binom{n+2(k+1)}{k+1}.$$

Proof. We fix n and proceed by induction on k. For $k = 0$, the left side is $P_n(\alpha(n,1)) = \binom{n}{0} = 1$, and the right side is $\binom{n+2}{1} = n+2 > 1$. Supposing that the statement holds for k, we prove it for $k+1$. By Lemma 3.5,

$$\binom{n+2(k+2)}{k+2} > 2\binom{n+2(k+1)}{k+1} = \binom{n+2(k+1)}{k+1} + \binom{n+2(k+1)}{k+1}.$$

By the induction hypothesis, this is greater than

$$\binom{n+2(k+1)}{k+1} + P_{n+2k}(\alpha(n,1)) = P_{n+2(k+1)}(\alpha(n,1)). \blacksquare$$

We are ready to show that $\alpha(n,1)$ has limiting density 0.

PROPOSITION 3.9. *For $n \ge 1$,*

$$\lim_{s\to\infty} \frac{P_s(\alpha(n,1))}{P_s} = 0.$$

Proof. We must make an odd/even case distinction, because the only way to get presentations of different lengths of the same structure is for these lengths to differ by a multiple of 2, so $P_{n+2k+1}(\alpha(n,1))$ will equal $P_{n+2k}(\alpha(n,1))$. However, if $s = n+2k$ for some k, then $P_s(\alpha(n,1))$ has a new last term, namely $\binom{n+2k}{k}$, and $P_{s+1}(\alpha(n,1)) = P_s(\alpha(n,1))$. Therefore, it is enough to show that

$$\frac{P_{n+2k}(\alpha(n,1))}{P_{n+2k}} \to 0.$$

By Lemma 3.5, the last term of $P_{n+2k}(\alpha(n,1))$, given above, is greater than the sum of the earlier terms. Thus, $P_{n+2k}(\alpha(n,1)) < 2\binom{n+2k}{k}$. Recall that $P_{n+2k} = 2^{n+2k+1} - 1$, which is strictly greater than 2^{n+2k}, so $\frac{P_{n+2k}(\alpha(n,1))}{P_{n+2k}} < \frac{2}{2^{n+2k}}\binom{n+2k}{k}$. To prove that the limiting density of $\alpha(n,1)$ is 0, it is enough to prove that

$$\frac{2}{2^{n+2k}}\binom{n+2k}{k} \to 0.$$

We can express $\frac{2}{2^{n+2k}}\binom{n+2k}{k}$ as a product of two factors, one involving the fixed n and the other not. The first factor is

$$\frac{2}{2^n}\left(\frac{2k+n}{k+n}\right)\left(\frac{2k+(n-1)}{k+(n-1)}\right)\cdots\left(\frac{2k+1}{k+1}\right).$$

This is an $(n+1)$-fold product with limit 2 as $k \to \infty$. The second factor is $\frac{1}{2^{2k}}\binom{2k}{k}$. By Lemma 3.4 above, this is $(1+o(1))\frac{1}{\sqrt{\pi k}}$, which has limit 0. ∎

We are ready to complete the proof of Theorem 3.1. Recall the statement.

Theorem 3.1. *For the variety of bijective structures and presentations with generator* 1 *and a single identity, for all sentences* φ*, the limiting density is* 1 *if* φ *is true in the free structure, and* 0 *otherwise.*

Proof. We have shown that for the sentences in the set E of invariants, the limiting density is 1 if the sentence is true in the free structure, and 0 otherwise. An arbitrary sentence φ is equivalent over T to a Boolean combination of sentences in E, say $\bigvee_i \bigwedge_j \psi_{i,j}$, where each $\psi_{i,j}$ has limiting density 0 or 1. For each i, the disjunct $\bigwedge_j \psi_{i,j}$ has limiting density 1 if all conjuncts have density 1, and 0 if some conjunct has density 0. Then φ has limiting density 1 if some disjunct has density 1, and 0 if all disjuncts have density 0. We calculate the truth value of φ in the free structure in the same way. ∎

For later use, we state below another immediate consequence of Proposition 3.9. For a term $t(a)$, let X be the difference between the number of occurrences of S and the number of occurrences of S^{-1} in $t(a)$. The identities that make $\alpha(n,1)$ true are exactly those for which $|X| = n$.

Corollary 3.10. *For each* $k \in \mathbb{Z}$*, the set of presentations* $a, t(a) = a$ *for which* $X = k$ *has density* 0.

3.2. Structures with a single unary function. Next, we consider the variety V_2. The language has a single unary function symbol f, and the variety has no axioms. Let $\mathbb{P}_2$ be the set of presentations with a single generator a and a single identity. We show that every sentence has limiting density 0 or 1, and the sentence saying that f is not 1-1 has limiting density 1, although it is false in the free structure.

We begin by describing the structures in V_2 that result from presentations in $\mathbb{P}_2$. For an identity $f^{(r)}(a) = f^{(r')}(a)$, where $r, r' \geq 0$, the *length* is $r + r'$. If $r = r'$, then we get an ω-chain. If $0 < r < r'$, then we get a chain of length r leading to a cycle of length $r' - r$. If $0 = r < r'$, then we get a cycle of length r'. Similarly, if $0 < r' < r$, we get a chain of length r' leading to a cycle of length $r - r'$, and if $0 = r' < r$, we get a cycle of length r. We write $m + \mathbb{Z}_n$ for a chain of length m leading to a cycle of length n—we allow $m = 0$. Any structure in V_2 obtained from a presentation in $\mathbb{P}_2$ has one of the following forms:

- an ω-chain—this is the free structure,
- a finite chain leading to a finite cycle, or
- a finite cycle.

We focus first on the sentence saying that f is not 1-1. We show that this sentence has density 1 although it is false in the free structure. We then describe a special "limit structure" $\mathcal{A}$, and show that for all sentences φ, the limiting density is 1 if $\mathcal{A} \models \varphi$, and 0 otherwise.

PROPOSITION 3.11. *For the variety V_2 and presentations in $\mathbb{P}_2$, the sentence φ saying that f is not 1-1 has limiting density 1, while it is false in the free structure.*

Proof. We can see that φ is false in ω (the free structure). We must prove that φ has limiting density 1. Actually, we show that $\neg\varphi$ has limiting density 0. The sentence φ is false in the finite cycles $\mathbb{Z}_n$, as well as in ω, and it is true in the structures $m + \mathbb{Z}_n$ for $m \geq 1$. Thus, for the structure given by an identity $f^n(a) = f^{n'}(a)$, we see that φ is false if $n = n'$ or if one of n, n' is 0, and it is true otherwise.

We can calculate P_s and $P_s(\neg\varphi)$. The total number of identities of length m is $m + 1$. Then

$$P_s = 1 + 2 + \cdots + (s+1) = \tfrac{1}{2}(s+2)(s+1) = \tfrac{1}{2}(s^2 + 3s + 2).$$

We count the identities of length m that make φ false as follows: For $m = 0$, there is just one identity, and it makes φ false. For $m = 1$, there are two identities, and both make φ false. For $m \geq 2$, if m is odd, there are just two identities that make φ false, and if m is even, there are three identities that make φ false. Thus,

$$P_s(\neg\varphi) = 1 + 2 + 3 + 2 + \cdots + \left(\tfrac{5}{2} + \tfrac{1}{2}(-1)^s\right).$$

For $s \geq 1$, $P_s(\neg\varphi) \leq 3s$, and $\lim_{s\to\infty} \frac{3s}{P_s} = 0$. Therefore, $\neg\varphi$ has limiting density 0, and φ has limiting density 1. ■

We will show that every sentence has limiting density 1 or 0. In fact, we will describe a "limit" structure $\mathcal{A}$ such that for all sentences φ, φ has density 1 if it is true in $\mathcal{A}$, and 0 otherwise. We picture $\mathcal{A}$ as a limit of the

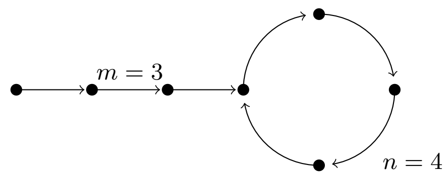

Fig. 1. $m + \mathbb{Z}_n$ where $m = 3$ and $n = 4$

structures $m + \mathbb{Z}_n$ (see Figure 1); $\mathcal{A}$ consists of an ω-chain together with two ω^*-chains and a single ω-chain that come together at a special point—this point is the end of the two ω^*-chains and the beginning of the ω-chain (see Figure 2). The chain of length m is replaced, in the limit, by an ω-chain plus one of the ω^*-chains, and the n-cycle is replaced, in the limit, by the other ω-chain and ω^*-chain. Note that $\mathcal{A}$ is not finitely generated.

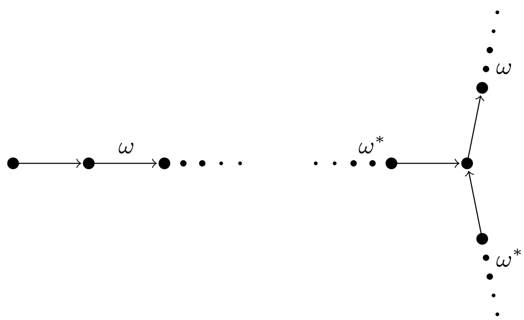

Fig. 2. The limit structure

We will prove the following.

PROPOSITION 3.12. *For all sentences φ, φ has limiting density* 1 *if $\mathcal{A} \models \varphi$, and* 0 *otherwise.*

The lemma below gives axioms for the theory of $\mathcal{A}$.

LEMMA 3.13. *The theory of $\mathcal{A}$ is generated* (*under logical implication*) *by the sentences below. In these sentences, the elements a and c are defined and not named by constants.*

(1) *ψ_a, saying that there is a unique element a with no f-pre-image,*
(2) *ψ_c, saying that there is a unique element c with two f-pre-images,*
(3) *ψ, saying that there is no element with more than two f-pre-images,*
(4) *for every $n \in \omega$, α_n saying that there is no cycle of length n,*
(5) *for every $n \in \omega$, β_n saying that a and c are not connected by a chain of length n.*

Proof. In any structure in the variety V_2, there is an equivalence relation $\sim$, where $x \sim y$ if there is a finite sequence $x_0, \ldots, x_n$ such that $x = x_0$,

$y = x_n$, and for each $i < n$, either $f(x_i) = x_{i+1}$ or $f(x_{i+1}) = x_i$. Let T be the theory generated (under logical implication) by the sentences above. We show that if $\mathcal{B}$ is another countable model of the theory, then it is elementarily equivalent to $\mathcal{A}$. For this, we again use saturation. Let $\mathcal{A}^*$, $\mathcal{B}^*$ be countable saturated models of the sentences above, such that $\mathcal{A}^* \equiv \mathcal{A}$ and $\mathcal{B}^* \equiv \mathcal{B}$. We show that $\mathcal{A}^* \cong \mathcal{B}^*$, so $\mathcal{A} \equiv \mathcal{B}$. In all models of T, the $\sim$-class of a is an ω-chain that does not include c. The $\sim$-class of c has two ω^*-chains ending in c and one ω-chain starting with c. We map the $\sim$-classes of the special elements a and c in $\mathcal{A}^*$ to the corresponding $\sim$-classes in $\mathcal{B}^*$. The other $\sim$-classes, not containing a or c, are $\mathbb{Z}$-chains. By saturation, $\mathcal{A}^*$ and $\mathcal{C}^*$ each have infinitely many of these $\mathbb{Z}$-chains, so we get the desired isomorphism. ∎

LEMMA 3.14. *The sentences from Lemma* 3.13 *that generate the theory of* $\mathcal{A}$ *all have limiting density* 1.

Proof. The sentence ψ, saying that there is no element with more than two f-pre-images, is true in all of the models that we get from a single generator a and a single identity, so this has limiting density 1. We have seen that the set of presentations that give models in which f is not 1-1 has limiting density 1. These models have the form $m + \mathbb{Z}_n$ for $m, n > 0$. The sentences ψ_a, saying that there is a unique element a with no f-pre-image, ψ_c, saying that there is a unique element c with two f-pre-images, and ψ are true in all of these models, so these sentences have limiting density 1.

Consider α_n, saying that there is no cycle of length n. Let B be the set of presentations that make α_n false—the resulting model *has* a cycle of length n. These identities have the form $f^{n+k}(a) = f^k(a)$ or $f^k(a) = f^{n+k}(a)$. The number of such identities of length m is 0 if $m-n$ is odd and 2 if $m-n$ is even. Then $P_s(\alpha_n) \leq 2s$. Since $P_s = O(s^2)$, the limiting density of B is 0, so the density of α_n is 1. Finally, consider β_n, saying that a and c are not connected by a chain of length n. Let C be the set of presentations that make this false. These identities have the form $f^n(a) = f^{n+k}(a)$ or $f^{n+k}(a) = f^n(a)$. The number of such identities of length m is 2 for all $m \geq 2n$. Then $P_s(C) \leq 2s$. Again the limiting density of C is 0, so the density of β_n is 1. ∎

3.3. Bijective structures with two identities. In this subsection, we return to the variety V_1 of bijective structures that we considered in Section 3.1. Recall that the language consists of unary function symbols S, S^{-1} and the axioms say that S and S^{-1} are inverses. We consider presentations in the set $\mathbb{P}_3$, with a single generator a and two identities of the form $t(a) = a$, with function symbols only on the left. Note that each identity is equivalent over the theory of V_1 to one of the form $S^m(a) = a$, for $m \in \mathbb{Z}$. We will show that the limiting density need not exist. Recall that a 1-cycle consists of a single element, x, with $S(x) = S^{-1}(x) = x$.

THEOREM 3.15. *For bijective structures, and presentations with a single generator a and two identities of form $t(a) = a$, the sentence φ saying that the structure is a 1-cycle does not have limiting density.*

The proof is somewhat involved. We begin with some elementary lemmas, but eventually we will consider a random walk on a group and appeal to results from random group theory that depend on the Central Limit Theorem. The lemma below tells us when the sentence φ is true.

LEMMA 3.16. *Let $\mathcal{A}$ be the structure given by an unordered set consisting of two identities, equivalent to $S^m(a) = a$ and $S^{m'}(a) = a$. Then $\mathcal{A}$ is a 1-cycle if and only if $\mathrm{GCD}(m, m') = 1$.*

Proof. Note that $S^k(a) = a$ if and only if $a = S^{-k}(a)$. Thus, we may suppose that both $m, m' \geq 0$. First, suppose that $\mathrm{GCD}(m, m') = 1$. In this case, there are $r, s \in \mathbb{Z}$ such that $mr + m's = 1$. Then we have

$$S(a) = S^{mr+m's}(a) = S^{mr}(S^{m's}(a)) = S^m \circ \cdots \circ S^m \circ S^{m'} \circ \cdots \circ S^{m'}(a) = a,$$

so $\mathcal{A}$ is a 1-cycle. Now, suppose that $\mathcal{A}$ is a 1-cycle, and let $\mathrm{GCD}(m, m') = d$. The axioms of T and the identities $S^m(a) = a$ and $S^{m'}(a) = a$ are both satisfied in a d-cycle, and $\mathcal{A}$ can only be a 1-cycle if $d = 1$. ∎

Our presentations have two identities, but we also need some facts about single identities. We indicate with $'$ that we are considering single identities, writing $P'_{=m}$ for the number of identities of length m and P'_s for the number of length at most s, and writing $P'_{=m}(B)$, $P'_s(B)$ for the number of these identities in a set B. We reserve P_s for the number of unordered pairs of identities of length at most s, and we write $P_s(B^2)$ for the number such that both identities are in B.

For a single identity of the form $t(a) = a$, let X be the difference between the number of occurrences of S and the number of occurrences of S^{-1} in t. Intuition may suggest that the statement $n \mid X$ should have limiting density $1/n$. This turns out to be true for odd n. However, for $n = 2$, we find that the limiting density for the statement $2 \mid X$ does not exist. Essentially, the reason is that the last term of $P'_s(2 \mid X) = \sum_{m \leq s} P'_{=m}(2 \mid X)$ may be greater than the sum of all earlier terms. Whether this happens depends on the parity of s. The lemma below says what happens to $P_{=s}(2 \mid X)$ as the parity of s changes.

LEMMA 3.17.

- *For even m, all identities of length m satisfy $2 \mid X$; none satisfy $2 \nmid X$.*
- *For odd m, all identities of length m satisfy $2 \nmid X$ and none satisfy $2 \mid X$.*

Proof. For $m = 0$, there is just one identity, and for this identity, $X = 0$. Supposing that the statements hold for m, if t has length m, then t has two extensions of length $m + 1$, and the parity of X changes. ∎

The next lemma gives the proportion of single identities of length at most s for which $2 \mid X$ holds. The value depends on the parity of s.

Lemma 3.18.

(1) $\lim_{s\to\infty} \frac{P'_{2s}(2 \mid X)}{P'_{2s}} = \frac{2}{3}$ *and* $\lim_{s\to\infty} \frac{P'_{2s}(2 \nmid X)}{P'_{2s}} = \frac{1}{3}$.

(2) $\lim_{s\to\infty} \frac{P'_{2s+1}(2 \mid X)}{P'_{2s+1}} = \frac{1}{3}$ *and* $\lim_{s\to\infty} \frac{P'_{2s+1}(2 \nmid X)}{P'_{2s+1}} = \frac{2}{3}$.

Proof. The calculation is based on Lemma 3.17. For (1), the even case, we have $P'_{2s} = 2^{2s+1} - 1 = 4^s \cdot 2 - 1$, and

$$P'_{2s}(2 \mid X) = \sum_{m\le s} 2^{2m} = \sum_{m\le s} 4^m = \frac{4^{s+1}-1}{3}.$$

Then

$$\frac{P'_{2s}(2 \mid X)}{P'_{2s}} = \frac{4^{s+1}-1}{3(4^s \cdot 2 - 1)} \to \frac{2}{3}.$$

For (2), the odd case, we have $P'_{2s+1} = 2^{2s+2} - 1 = 4^{s+1} - 1$ and

$$P'_{2s+1}(2 \mid X) = P'_{2s}(2 \mid X) = \frac{4^{s+1}-1}{3}.$$

Then

$$\frac{P'_{2s+1}(2 \mid X)}{P'_{2s+1}} = \frac{4^{s+1}-1}{3 \cdot (4^{s+1}-1)} = \frac{1}{3}. \quad \blacksquare$$

So far, the lemmas have involved only elementary calculations, but we shall use more. Below, we state a result from random group theory [5, 24] concerning a random walk on a group G of form $\mathbb{Z}_n$. In general, for a random walk, there are finitely many states, and given just the current state s, independent of prior history, there are fixed probabilities of passing next to state s'. We allow $s' = s$. Our states are group elements. We write $\mu(g)$ for the probability of passing in one step from the identity to g, and we write $\mu^{(k)}(g)$ for the probability of passing in k steps from the identity to g.

For the result below, the probability measure μ defined on G is "supported" on a special set Σ, which generates G in the usual sense. For μ to be supported on Σ means that μ assigns non-zero probability to each element of Σ. We think of Σ as the set of group elements reachable from the identity in one step, but it is also the set of differences $g' - g$ such that g' is reachable from g in one step. We have a tree consisting of finite sequences of elements of Σ, and we label each sequence σ by taking the sum of the sequence. In the tree, the base node is labeled 0. For a node σ labeled g, the label on each successor is obtained by adding to g the appropriate element of Σ. The values of $\mu^k(g)$, for $k > 0$, are obtained by considering the sequences in Σ^k with the last term g, multiplying the probabilities of the terms in each sequence, and then summing over the different sequences.

The result below, given in [24, Theorem 7.3], tells us that the probability of each remainder g in $\mathbb{Z}_n$ is approximately $1/n$, and that the convergence (as $k \to \infty$) has a great deal of uniformity.

THEOREM 3.19. *There exist $\alpha, \beta > 0$ such that for any group G of the form $\mathbb{Z}_n$, any generating set Σ containing the group identity* 0*, and any probability measure μ supported on Σ, we have, for all $g \in G$ and all $k \in \omega$,*

$$\left| \mu^{(k)}(g) - \frac{1}{n} \right| < \alpha e^{-\beta k/n^2}.$$

To adapt this theorem to our setting, we will consider an odd n and, at least at first, identities of even length m. We break the identity into pieces of length 2, so each piece has 1/4 chance of being each of SS, $S^{-1}S^{-1}$, SS^{-1}, or $S^{-1}S$. These correspond to $2, -2$, and 0 in the random walk on $\mathbb{Z}_n$, and when n is odd, $\Sigma = \{-2, 0, 2\}$ generates $\mathbb{Z}_n$. For identities of fixed even length m and $k = \frac{m}{2}$, $\mu^k(g)$ approaches the uniform distribution on $g \in \mathbb{Z}$. The same is true for identities of odd length, and consequently so do the identities of length at most s. We have the following:

COROLLARY 3.20. *For any odd number n and any s,*

$$\lim_{s\to\infty} \frac{P'_s(n \,|\, X)}{P'_s} = \frac{1}{n} \quad \textit{and} \quad \lim_{s\to\infty} \frac{P'_s(n \nmid X)}{P'_s} = \frac{n-1}{n}.$$

Our presentations have an *unordered set* of two identities. However, it is easier to count ordered pairs, allowing repetition. By Proposition 2.9, we get the same limiting densities, so we will count ordered pairs, and allow repetition of elements. We are interested in the presentations for which the difference functions X_1, X_2 are both divisible by some prime p. It follows from Lemma 3.10 that $X_1 = 0$ and $X_2 = 0$ both have limiting density 0. We let C be the set of presentations such that $X_1 \neq 0$, $X_2 \neq 0$, and there is some prime p that divides both X_1 and X_2.

DEFINITION 3.21. For each s, let $p(s)$ be the greatest prime p such that $2 \cdot 3 \cdots p \leq \ln s$, i.e., the product of the primes up to p is bounded by the log of s.

For every s, we consider the presentations in C of length at most s and split them into two parts, which we call $C_{s,1}$ and $C_{s,2}$. For a presentation in C (of length at most s), let d be the least prime that divides both of the difference functions X_1, X_2. We put the presentation into $C_{s,1}$ if $d \leq p(s)$ and into $C_{s,2}$ if $d > p(s)$. We write $P_s(C_{s,i})$ for the number of presentations of length at most s that are in $C_{s,i}$, and we may refer to the limit as $s \to \infty$

of $P_s(C_{s,i})/P_s$ as the *limiting density* of C_i. Of course, whether a given presentation belongs to $C_{s,i}$ depends on s. Keeping this in mind, we will show that C_2 has limiting density 0, and the limiting density of C_1 does not exist—it toggles between two values, one for even s and the other for odd s. Among the primes, 2 behaves differently from the odd primes. We have shown that, for single identities with difference X, the limiting density of $2 \mid X$ does not exist. We will see later that this explains why the limiting density of C_1 does not exist.

Our first goal is to show that C_2 has limiting density 0. Toward this, we consider a single identity with difference X.

LEMMA 3.22. *For each odd prime p and all s,*

$$\frac{P'_s(p \mid X \ \&\ X \neq 0)}{P'_s} \leq \frac{2}{p+1}.$$

Proof. We will first prove that

$$(*) \qquad \frac{P'_{=m}(p \mid X \ \&\ X \neq 0)}{P'_{=m}} \leq \frac{2}{p+1} \quad \text{for all } m.$$

Note that the numbers $P'_{=m}(X = n)$, for $-m \leq n \leq m$, form a Pascal triangle. At the top, for $m = 0$, we have 1, corresponding to $X = 0$. For $m = 1$, we have 1's corresponding to $X = \pm 1$. In general, for even m, X takes the even values n in the interval $[-m, m]$, and for odd m, X takes the odd values in the interval $[-m, m]$. In both cases, $P'_{=m+1}(X = n) = P'_{=m}(X = n-1) + P'_{=m}(X = n+1)$. We can see that $P'_{=m}(X = n)$ decreases as $|n|$ increases, and that $P'_{=m}(X = n) = P'_{=m}(X = -n)$.

For odd m (so that X is odd), we have

$$\begin{gathered} P'_{=m}(X = \pm 1) \geq P'_{=m}(X = \pm 3) \geq \cdots \geq P'_{=m}(X = \pm p), \\ P'_{=m}(X = \pm(p+2)) \geq P'_{=m}(X = \pm(p+4)) \geq \cdots \geq P'_{=m}(X = \pm 3p), \\ \cdots \end{gathered}$$

Note that there are p terms in each of the lines except the first line, which has only $\frac{p+1}{2}$ terms. Therefore,

$$\begin{aligned} 1 = \sum_{n \,\mathrm{odd}} \frac{P'_{=m}(X = n)}{P'_{=m}} &= \frac{P'_{=m}(X = \pm 1)}{P'_{=m}} + \frac{P'_{=m}(X = \pm 3)}{P'_{=m}} + \cdots \\ &\geq \left(\frac{p+1}{2}\right) \frac{P'_{=m}(X = \pm p)}{P'_{=m}} + p \cdot \frac{P'_{=m}(X = \pm 3p)}{P'_{=m}} + \cdots \\ &\geq \left(\frac{p+1}{2}\right) \left(\frac{P'_{=m}(X = \pm p)}{P'_{=m}} + \frac{P'_{=m}(X = \pm 3p)}{P'_{=m}} + \cdots\right) \\ &= \left(\frac{p+1}{2}\right) \frac{P'_{=m}(p \mid X \ \&\ X \neq 0)}{P'_{=m}}. \end{aligned}$$

If m is even (so that X is even), then

$$P'_{=m}(X=\pm 2) \geq P'_{=m}(X=\pm 4) \geq \cdots \geq P'_{=m}(X=\pm 2p),$$
$$P'_{=m}(X=\pm(2p+2)) \geq P'_{=m}(X=\pm(2p+4)) \geq \cdots \geq P'_{=m}(X=\pm 4p),$$
$$\cdots$$

In this case, each line has p terms, and we get the following, slightly stronger inequality:

$$\begin{aligned}
1 &= \sum_{n\,\text{even}} \frac{P'_{=m}(X=n)}{P'_{=m}} \\
&= \frac{P'_{=m}(X=0)}{P'_{=m}} + \frac{P'_{=m}(X=\pm 2)}{P'_{=m}} + \frac{P'_{=m}(X=\pm 4)}{P'_{=m}} + \cdots \\
&\geq \frac{P'_{=m}(X=\pm 2)}{P'_{=m}} + \frac{P'_{=m}(X=\pm 4)}{P'_{=m}} + \cdots \\
&\geq p \cdot \frac{P'_{=m}(X=\pm 2p)}{P'_{=m}} + p \cdot \frac{P'_{=m}(X=\pm 4p)}{P'_{=m}} + \cdots \\
&= p \cdot \frac{P'_{=m}(p \mid X \ \&\ X \neq 0)}{P'_{=m}}.
\end{aligned}$$

Combining the even and odd cases, we get $(*)$.

Now, we turn our attention back to the inequality in the lemma, which concerns identities up to a certain length. The quotient $P'_s(p \mid X \,\&\, X \neq 0)/P'_s$ is a weighted average (weighted by the proportion of identities of each length) of the probabilities $P'_{=m}(p \mid X \ \&\ X \neq 0)/P'_{=m}$, where $m \leq s$. Thus, the lemma follows from the inequality $(*)$ on identities of a fixed length. ∎

We are now ready to consider both identities.

Lemma 3.23. $\displaystyle\lim_{s\to\infty} \frac{P_s(C_2)}{P_s} = 0.$

Proof. Below, we will appeal to Proposition 2.9 and consider, for each s, the probability space consisting of the ordered pairs of identities, each of length at most s. Then the random variables X_1, X_2 are independent. Counting ordered pairs of identities and allowing repetition, we see that for each s,

$$\begin{aligned}
P_s(C_2) &\leq \sum_{p>p(s)} P_s(X_1, X_2 \neq 0 \ \&\ p \mid X_1 \ \&\ p \mid X_2) \\
&= \sum_{p>p(s)} P'_s(X_1 \neq 0 \ \&\ p \mid X_1) P'_s(X_2 \neq 0 \ \&\ p \mid X_2).
\end{aligned}$$

So, it follows from the previous lemma that

$$\frac{P_s(C_2)}{P_s} \leq \sum_{p>p(s)} \frac{P'_s(X_1 \neq 0 \ \&\ p \mid X_1)}{P'_s} \frac{P'_s(X_2 \neq 0 \ \&\ p \mid X_2)}{P'_s} \leq \sum_{p>p(s)} \left(\frac{2}{p+1}\right)^2.$$

By a well-known fact from number theory, the sum of the squares of the reciprocals of primes (or of all natural numbers) converges. Since $\lim_{s\to\infty} p(s) = \infty$, we have

$$\lim_{s\to\infty} \sum_{p>p(s)} \left(\frac{2}{p+1}\right)^2 = 0.$$

Thus, C_2 has limiting density 0. ∎

We turn to C_1. Again, we consider first a single identity.

LEMMA 3.24. *Write D_s for the set of identities of length at most s but greater than $\sqrt{s}$. Then $P'_s(D_s)/P'_s \to 1$.*

Proof. We have $P'_s(D_s) = P'_s - P'_{\sqrt{s}}$, and $\frac{P'_s(D_s)}{P'_s} = 1 - \frac{2^{\sqrt{s}+1}-1}{2^{s+1}-1} \to 1$. ∎

Lemma 3.24 may be interpreted as saying that most identities of length at most s have length at least $\sqrt{s}$. We write $P_s(D_s^2)$ for the number of pairs of identities of length at most s such that both have length at least $\sqrt{s}$. The next lemma says that for most pairs of identities of length at most s, the length of both is at least $\sqrt{s}$.

COROLLARY 3.25. $P_s(D_s^2)/P_s \to 1$.

Now, $P_s(C_1)/P_s$ is the probability that, among pairs of identities of length at most s with difference functions X_1 and X_2, there is some prime $p \le p(s)$ such that $p \mid X_1$ & $p \mid X_2$. We may suppose that both identities have length greater than $\sqrt{s}$. We have seen that the limiting probability that 2 divides both X_1, X_2 does not exist—for even s, it approaches $\frac{4}{9}$, while for odd s, it approaches $\frac{1}{9}$ by Corollary 3.20. Here, we consider odd primes. We have justified thinking of the random variables X_1, X_2 (for identities of length at most s) as independent.

We would like to assume that for $i = 1, 2$, the events $p \mid X_i$ for *different* primes p are independent—this would simplify our calculations. The desired assumption, while not strictly true, turns out to be "approximately" true. The probability that X_1, X_2 are *not* both divisible by 3 is approximately $1 - \frac{1}{3^2}$. The probability that X_1, X_2 are *not* both divisible by 3 and not both divisible by 5 is approximately $\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)$. The probability that X_1, X_2 are *not* both divisible by any odd prime $p \le p(s)$ is approximately $\prod_{3\le p\le p(s)}(1 - 1/p^2)$. This formula matches what we would get by laborious inclusion-exclusion counting.

In fact, the divisibilities of X_i by different primes may *not* be independent. However, we can apply the Chinese Remainder Theorem and consider the residue of X_i modulo $N_s = \prod_{p\le p(s)} p$, which is $2\prod_{3\le p\le p(s)} p$. It follows from the definition of $p(s)$ (Definition 3.21) that $N_s \le \ln s$. This is where the random walk on the group comes in. We will use Theorem 3.19.

THEOREM 3.26. *Below, we let p range over all primes:*

(1) $\dfrac{P_{2s}(C^c_{s,1})}{P_{2s}} \to \dfrac{5}{9} \cdot \prod_{3\le p}\left(1-\dfrac{1}{p^2}\right)$.

(2) $\dfrac{P_{2s+1}(C^c_{s,1})}{P_{2s+1}} \to \dfrac{8}{9} \cdot \prod_{3\le p}\left(1-\dfrac{1}{p^2}\right)$.

Proof. We prove (1). Take $\alpha, \beta > 0$ as in Theorem 3.19. Recall that X_1 and X_2 are the difference functions associated with the first and second identities where, in each identity, the function symbols are all on the left. Fixing s, we consider the residue of X_1 and X_2 modulo $N_s = 2\prod_{3\le p\le p(s)} p$, where N_s is at most $\ln s$. For a single identity, we consider a string t of length k. For the fixed s, let G_s be the group of possible remainders after division of X_1 by N_s. Theorem 3.19 tells us that for every $0 \le a < N_s$ and every k,

$$\left|\frac{P'_{=k}(X_1 = a \pmod{N_s})}{P'_{=k}} - \frac{2}{N_s}\right| < \alpha e^{-\beta k/N_s^2} \quad \text{if } a \text{ and } k \text{ have the same parity,}$$

$$\frac{P'_{=k}(X_1 = a \pmod{N_s})}{P'_{=k}} = 0 \qquad \text{if } a \text{ and } k \text{ have different parities.}$$

When we sum up the identities of length at most s, the previous lemma says that most of them will have length some $k \ge \sqrt{s}$. Thus, we may assume that $k \ge \sqrt{s}$. The previous inequality yields

$$\left|\frac{P'_{=k}(X_1 = a \pmod{N_s})}{P'_{=k}} - \frac{2}{N_s}\right| < \alpha e^{-\beta k/N_s^2} < \alpha e^{-\beta\sqrt{s}/(\ln s)^2}.$$

By Lemma 3.18, among the identities of length up to $2s$, $2/3$ of them are even, and $1/3$ of them are odd. The probability $P'_{2s}(X_1 = a \pmod{N_s})/P'_{2s}$ is a weighted sum of $P'_{=k}(X_1 = a \pmod{N_s}))/P'_{=k}$. Noticing that the rest of the previous inequality does not depend on k, doing a weighted sum gives

$$\left|\frac{P'_{2s}(X_1 = a \pmod{N_s})}{P'_{2s}} - \frac{2}{3}\cdot\frac{2}{N_s}\right| < \alpha e^{-\beta\sqrt{s}/(\ln s)^2} \qquad \text{if } a \text{ is even,}$$

$$\left|\frac{P'_{2s}(X_1 = a \pmod{N_s})}{P'_{2s}} - \frac{1}{3}\cdot\frac{2}{N_s}\right| < \alpha e^{-\beta\sqrt{s}/(\ln s)^2} \qquad \text{if } a \text{ is odd.}$$

By the independence of X_1, X_2, we see that for all sufficiently large s,

$$\left|\frac{P_{2s}(X_1 = a \pmod{N_s}\ \&\ X_2 = b \pmod{N_s})}{P_{2s}} - c\frac{4}{N_s^2}\right| < \alpha e^{-\beta\sqrt{s}/(\ln s)^2}$$

where

$$c = \begin{cases} \frac{1}{9} & \text{if } a, b \text{ are both odd,} \\ \frac{2}{9} & \text{if one of } a, b \text{ is odd and the other is even,} \\ \frac{4}{9} & \text{if } a, b \text{ are both even.} \end{cases}$$

Now, we consider pairs (a, b) modulo N_s such that no $p \leq p(s)$ divides both a and b. Note that (a, b) cannot both be even. As we have seen, up to the parities of a and b, for large s, the distribution of X_i is approximately uniform, and the distribution of ordered pairs (X_1, X_2) is also approximately uniform. For each odd prime p, the fraction of the pairs (a, b) such that a, b are both divisible by p is approximately $1/p^2$. Thus, considering all primes, approximately $\prod_{3 \leq p \leq p(s)}(1 - 1/p^2)$ of the possible pairs do not have a common odd prime factor $\leq p(s)$. More precisely,

$$\left|\frac{P_{2s}(\text{no odd prime } p \leq p(s) \text{ divides both } X_1, X_2)}{P_{2s}} - \prod_{3 \leq p \leq p(s)} \left(1 - \frac{1}{p^2}\right)\right| < N_s^2 \alpha e^{-\beta\sqrt{s}/(\ln s)^2}.$$

Finally, considering $p = 2$, the probability that both a, b are even is approximately $\frac{4}{9}$. Thus, we see $P_{2s}(C^c_{s,1})/P_{2s}$, the probability that no prime $p \leq p(s)$ divides both X_1 and X_2, satisfies

$$\left|\frac{P_{2s}(C^c_{s,1})}{P_{2s}} - \frac{5}{9} \cdot \prod_{3 \leq p \leq p(s)} \left(1 - \frac{1}{p^2}\right)\right| < N_s^2 \alpha e^{-\beta\sqrt{s}/(\ln s)^2} \leq (\ln s)^2 \alpha e^{-\beta\sqrt{s}/(\ln s)^2}.$$

Note that the right-hand side of the inequality goes to 0 as $s \to \infty$. Thus,

$$\lim_{s \to \infty} \frac{P_{2s}(C^c_{s,1})}{P_{2s}} = \frac{5}{9} \cdot \prod_{3 \leq p} \left(1 - \frac{1}{p^2}\right).$$

This completes the proof of (1). The proof of (2) is similar. ∎

Now, for presentations with a single generator a and two identities of the form $t(a) = a$, we consider the sentence φ saying that the structure is a 1-cycle. The presentations not satisfying this sentence are exactly the presentations in C. We have seen that C splits into two parts, C_1 and C_2, and C_1 does not have limiting density (Theorem 3.26) while C_2 has limiting density 0 (Lemma 3.23). Thus, φ does not have limiting density, proving Theorem 3.15.

3.4. Abelian groups. Let V_4 be the variety of Abelian groups. The language consists of a binary function symbol $+$, a unary function symbol $-$, and a constant 0. We write $x + y$ instead of $+(x, y)$. To axiomatize V_4, we add to the usual group axioms the sentence $(\forall x)(\forall y)(x + y = y + x)$. Let $\mathbb{P}_4$ be the set of presentations with a single generator a and a single identity of the form $t(a) = 0$, where $t(a)$ is the sum of a's and $-a$'s. We prove the following.

THEOREM 3.27. *For the variety V_4 of Abelian groups and presentations in $\mathbb{P}_4$,*

(1) *there are sentences for which the limiting density does not exist,*
(2) *there are sentences for which the limiting density is strictly between* 0 *and* 1.

To prove the theorem, we begin with the Szmielew invariants, which have been known for many years.

3.4.1. *Elementary invariants.* Szmielew [26] carried out an elimination of quantifiers for Abelian groups, and she gave elementary invariants. Later, Eklof and Fischer [7] used saturation to give elementary invariants for modules. Their methods also yield the Szmielew invariants for Abelian groups. We give the invariants below. For a prime p, we write $G[p]$ for the set $\{x \in G : px = 0\}$, which consists of the identity and the elements of order p. We write $p^n G$ for the set $\{p^n g : g \in G\}$. Then $p^n G[p]$ is the set of $x \in G$ such that x is divisible by p^n and $px = 0$.

THEOREM 3.28 (Szmielew, Eklof–Fischer). *The following sentences form a set of elementary invariants for Abelian groups:*

(1) $\alpha(p, n, k)$, *saying* $|p^n G| \geq k$,
(2) $\beta(p, n, k)$, *saying* $\dim(p^n G/p^{n+1} G) \geq k$,
(3) $\gamma(p, n, k)$, *saying* $\dim(p^n G[p]) \geq k$,
(4) $\delta(p, n, k)$, *saying* $\dim(p^n G[p]/p^{n+1} G[p]) \geq k$.

Our presentations have one generator a and a single identity of the form $t(a) = 0$. The free Abelian group on one generator is $\mathbb{Z}$, and the other Abelian groups on one generator are the finite cyclic groups C_m. We focus on the sentences $\beta(p, n, 1)$, which say that there is an element divisible by p^n and not by p^{n+1}. We will see that these sentences are true in $\mathbb{Z}$. Moreover, for $p = 2$, the limiting density does not exist, while for odd primes p, the limiting density exists and has a value strictly between 0 and 1.

LEMMA 3.29.

(1) $\beta(p, n, 1)$ *is true in* $\mathbb{Z}$.
(2) $\beta(p, n, 1)$ *is true in* C_m *if and only if* $p^{n+1} \mid m$.

Proof. For (1), we note that in $\mathbb{Z}$, the element p^n witnesses the truth of $\beta(p, n, 1)$. For (2), consider C_m. For some r (possibly 0) and some m' relatively prime to p, we have $m = p^r \cdot m'$, and then $C_m \cong C_{p^r} \oplus C_{m'}$. If $r > n$, then C_{p^r} has an element divisible by p^n and not by p^{n+1}, and otherwise, there is no such element. Furthermore, all elements of $C_{m'}$ are divisible by all powers of p. So, C_m has elements divisible by p^n if and only if $r > n$. ∎

For an identity $t(a) = 0$, the *length* is the number of occurrences of a, $-a$ in $t(a)$. We consider the empty sum to be 0.

Lemma 3.30. $P_s = 1 + 2 + \cdots + 2^s = 2^{s+1} - 1$.

Proof. There is just one identity of length 0. For $m \geq 1$, there are 2^m identities of length m. Then $P_s = 1+2+2^2+\cdots+2^s = \frac{1-2^{s+1}}{1-2} = 2^{s+1}-1$. ∎

We consider the limiting density for $\beta(p, n, 1)$ for various combinations of p and n. Recall that the sentence $\beta(p, n, 1)$ is true in $\mathbb{Z}$, and it is true in C_m provided that $p^{n+1} \mid m$. Let A be the set of identities that give $\mathbb{Z}$, and for $r < p^{n+1}$, let B_r be the set of identities that give $\mathbb{Z}_m$ for the various $m \equiv_{p^{n+1}} r$. We write $P_s(A)$ and $P_s(B_r)$ for the number of identities of length at most s in the sets A and B_r.

Lemma 3.31.

(1) $P_s(A) = 1 + \displaystyle\sum_{0<2m\leq s} \binom{2m}{m}$,

(2) *A has limiting density* 0.

Proof. For (1), note that the identity $t(a) = 0$ gives $\mathbb{Z}$ just in case $t(a)$ has even length $2m$ for some m and a and $-a$ each occur m times. For (2), look back at Section 3.1 where we encountered the same quantities as the current P_s and $P_s(A)$. There, we saw that $\lim_{s\to\infty} P_s(A)/P_s = 0$. ∎

Recall that for an identity $t(a) = 0$, X is the difference between the number of occurrences of a and the number of occurrences of a^{-1} in the term $t(a)$. For an identity of even length $2m$, X takes only even values $0, \pm 2, \ldots, \pm 2m$. For an identity of odd length $2m+1$, X takes only odd values $\pm 1, \pm 3, \ldots, \pm(2m+1)$. Using arguments similar to those in Section 3.3, we will show that the limiting density of $2^{n+1} \mid X$ does not exist, and for odd primes p, the limiting density of $p^{n+1} \mid X$ is $\frac{1}{p^{n+1}}$.

Lemma 3.32. *For $p = 2$ and $n \geq 0$, the limiting density of $2^{n+1} \mid X$ does not exist. In particular,*

$$\frac{P_{2s}(2^{n+1} \mid X)}{P_{2s}} \to \frac{2}{3} \cdot \frac{1}{2^n} \quad \textit{and} \quad \frac{P_{2s+1}(2^{n+1} \mid X)}{p_{2s+1}} \to \frac{1}{3} \cdot \frac{1}{2^n}.$$

Proof. We begin with the case where $n = 0$. Here the calculations are straightforward. We have $P_{=2m}(2 \mid X) = 2^{2m}$, and $P_{=2m+1}(2 \mid X) = 0$. Then

$$P_{2s}(2 \mid X) = 1 + 2^2 + 2^4 + \cdots + 2^{2s} = 1 + 4 + 4^2 + \cdots + 4^s = \frac{4^{s+1}-1}{3}.$$

Therefore,

$$\frac{P_{2s}(2 \mid X)}{P_{2s+1}} = \frac{\frac{4^{s+1}-1}{3}}{2^{2s+2}-1} \to \frac{2}{3}.$$

Since $P_{2s+1}(2 \mid X) = P_{2s}$, we have

$$\frac{P_{2s+1}(2 \mid X)}{P_{2s+1}} = \frac{\frac{4^{s+1}-1}{3}}{2^{2s+2}-1} \to \frac{1}{3}.$$

What we have shown is that if E is the set of identities of even length, then $P_{2s}(E)/P_{2s} \to \frac{2}{3}$ and $P_{2s+1}(E)/P_{2s+1} \to \frac{1}{3}$.

For $n \geq 1$, we again use Theorem 3.19. For every presentation in E^c, we have $2^{n+1} \nmid X$, so we may condition on E when we consider even length $2m$. In the relation $w(a) = \sum_{i\leq 2m} d_i a$, we can consider the sums $d_1 + d_2, d_3 + d_4, \dots, d_{2m-1} + d_{2m}$. This gives us an m-step random walk on $\mathbb{Z}_{2^{n+1}}$ with each step being 2 with probability $\frac{1}{4}$, -2 with probability $\frac{1}{4}$, and 0 with probability $\frac{1}{2}$. Dividing everything by 2, we get a random walk with support $\{1, 0, -1\}$ on $\mathbb{Z}_{2^n}$. Thus, Theorem 3.19 applies. We have $2^{n+1} \mid X$ exactly when the random walk ends at $0 \in \mathbb{Z}_{2^n}$. The probability of this is $P_{=2m}(2^{n+1} \mid X)/P_{=2m} \to 1/2^n$.

Now, as in the proof of Theorem 3.26, most identities of length at most s will have length at least $\geq \sqrt{s}$. Since the rate of convergence in Theorem 3.19 is exponential and all identities in E have even length, we can pass from the probability for identities of a fixed length to the probability for identities with length $\leq s$, and we get

$$\frac{P_{2s}(2^{n+1} \mid X \mid E)}{P_{2s}(E)} \to \frac{1}{2^n}.$$

Since $2^{n+1} \mid X$ only when X is even, i.e., the identity is in E, and the above probability $1/2^n$ was conditioned to E, we have the desired

$$\frac{P_{2s}(2^{n+1} \mid X)}{P_{2s}} = \frac{P_{2s}(2^{n+1} \mid X \mid E)}{P_{2s}} = \frac{P_{2s}(2^{n+1} \mid X \mid E)}{P_{2s}(E)} \cdot \frac{P_{2s}(E)}{P_{2s}} \to \frac{1}{2^n} \cdot \frac{2}{3}.$$

The odd case can be proved similarly. ∎

Lemma 3.33. *For odd primes p, $p^{n+1} \mid X$ has limiting density $\frac{1}{p^{n+1}}$.*

Proof. For a fixed even length $2m$, we get a random walk on $\mathbb{Z}_{p^{n+1}}$ supported on $\{2, 0, -2\}$—a single step increases the length by 2. By Theorem 3.19, we have $P_{2m}(p^{n+1} \mid X)/P_{2n} \to 1/p^{n+1}$. This random walk converges to the uniform distribution for even lengths, and the same is true for the odd lengths.

As before, we split the set of identities of length at most s into two parts, those of length less than $\sqrt{s}$ and those of length at least $\sqrt{s}$. Let S_s be the number of relators of length at most s for which the length is less than $\sqrt{s}$, and let L_s be the number for which the length is at least $\sqrt{s}$.

Then $S_s/P_s \to 0$, so $L_s/P_s \to 1$. Then the exponential rate of convergence of Theorem 3.19 gives

$$\frac{P_s(p^{n+1} \mid X)}{P_s} \to \frac{1}{p^{n+1}}. \blacksquare$$

4. Generalizing. In this section, we give general conditions that imply some of the behaviors that we saw in Section 3. Our main general result, Theorem 4.32, says that for a commutative generalized bijective variety V and presentations with finitely many generators and a single identity, the zero-one law holds. Moreover, the sentences with density 1 are those true in the free structure.

We will consider languages with finitely many unary function symbols.

4.1. Generalized bijective varieties. In Section 3.1, we considered the variety with a language consisting of a pair of unary function symbols S, S^{-1} and axioms saying that they are inverses, and we showed that for presentations with a single generator a and a single identity of the form $t(a) = a$, the sentences true in the free structure are exactly those with limiting density 1. In this subsection and the next, we turn our attention to varieties of structures with multiple bijective unary functions, possibly with additional axioms. We might suppose that the language has unary function symbols $g_1, g_1^{-1}, \ldots, g_n, g_n^{-1}$, and that our varieties have axioms saying that for each i, g_i and g_i^{-1} are inverses. However, the assumption that the functions have inverses named by function symbols turns out to be unnecessary once we know that the functions are 1-1 and onto.

Definition 4.1. Let L be a language with unary function symbols $f_1, \ldots, f_n$, and let V be an algebraic variety with theory T. The variety is *generalized bijective* if for all i,

$$T \vdash (\forall x, y)(f_i(x) = f_i(y) \to x = y) \quad \text{and} \quad T \vdash (\forall y)(\exists x)(f_i(x) = y).$$

The result below says that for a generalized bijective variety, the basic functions have inverses named by terms.

Proposition 4.2. *Let T be the theory of a generalized bijective variety in the language $\{f_1, \ldots, f_n\}$. Then for each f_i, there is some word u_i such that $T \vdash (\forall x)(f_i \circ u_i(x) = u_i \circ f_i(x) = x)$.*

Proof. Fix i, and let F be the free structure on one generator a. There is some $b \in F$ with $f_i(b) = a$. We can express b as $u_i(a)$ for some word u_i. Then $F \models f_i \circ u_i(a) = a$. Recall that in a variety, if an atomic formula is true of the generating tuple in the free structure, then it holds on all tuples in all structures [3, Theorem 11.4]. Thus, $T \vdash (\forall x)(f_i \circ u_i(x) = x)$. In F, let $x = f_i(a)$. We have $f_i \circ u_i \circ f_i(a) = f_i(a)$. Since f_i is injective, this means that $F \models u_i \circ f_i(a) = a$. Hence, $T \models (\forall x)(u_i \circ f_i(x) = x)$. This completes the proof. ∎

Definition 4.3. Let V be a variety in the language $\{f_1, \dots, f_n\}$, where each f_i is unary. The variety is called *commutative* if the axioms imply that $(\forall x)(f_i(f_j(x)) = f_j(f_i(x)))$ for all i, j.

We now turn to the proof of our main general result. To prove Theorem 4.26, which addresses the case of a commutative generalized bijective variety with a single generator and a single identity, we will use a version of Gaifman's Locality Theorem, which we discuss below. We will then extend this theorem to the case with multiple generators in Theorem 4.32.

4.2. Gaifman's Locality Theorem. We state a special version of Gaifman's Locality Theorem for generalized bijective varieties, and we sketch a proof using saturation. Fix a language L consisting of unary function symbols $f_1, \dots, f_m$. Below, we define the *Gaifman graph* of an L-structure. Gaifman defined the graph for structures in a finite relational language. When convenient, we treat the unary functions as binary relations.

Definition 4.4. Let $\mathcal{A}$ be an L-structure. The *Gaifman graph* of $\mathcal{A}$ is the undirected graph with universe equal to that of $\mathcal{A}$, and with an edge between x and y if and only if $f_i(x) = y$ or $f_i(y) = x$ for some i.

We define an equivalence relation $\sim$ on $\mathcal{A}$ such that $x \sim y$ if x and y belong to the same connected component in the Gaifman graph, i.e., there is a finite path leading from x to y.

Definition 4.5 (distance, $d(x, y)$). For $x, y \in \mathcal{A}$, the *distance* between x and y is the least r such that there is a path of length r from x to y. We write $d(x, y) \geq r$, $d(x, y) > r$ to indicate that the distance is, respectively, at least r or greater than r.

Remark. Elements x, y lie in different connected components if and only if $d(x, y) > r$ for all r.

We consider substructures of $\mathcal{A}$. Note that two connected components, thought of as substructures, are isomorphic if there is a map from one onto the other that preserves the unary functions f_i, which we think of as binary relations. The structure $\mathcal{A}$ is determined, up to isomorphism, by the number of connected components of different isomorphism types.

Definition 4.6 (r-ball, $B_r(a)$, $B_r(\bar{a})$). Let $\mathcal{A}$ be a structure and let $r \in \omega$.

(1) For $a \in \mathcal{A}$, the *r-ball around a* is $B_r(a) = \{x \in \mathcal{A} : d(a, x) \leq r\}$.
(2) For $\bar{a} \in \mathcal{A}^n$, we write $B_r(\bar{a})$ for the set $\bigcup_{i<n} B_r(a_i)$.
(3) We write $B_\infty(a)$ for the connected component of a, or $\bigcup_r B_r(a)$.

(4) We write $B_\infty(\bar{a})$ for the union of the connected components of elements of $\bar{a}$, or $\bigcup_i B_\infty(a_i)$.

Let V be a generalized bijective variety for the language L. For $\mathcal{A} \in V$, each element has a unique image and a unique pre-image under each f_i. We show that for each r and n, there is a finite set of formulas $\alpha(\bar{x})$ that describe, for all $\mathcal{A} \in V$, the possible substructures $B_r(\bar{a})$ for n-tuples $\bar{a}$.

LEMMA 4.7. *Let V be a generalized bijective variety for the language L. For each r and n, there is a finite set $C_{r,n}$ of formulas $\alpha(\bar{x})$ such that*

(1) *for each $\mathcal{A} \in V$, each n-tuple $\bar{a}$ in $\mathcal{A}$ satisfies a unique formula $\alpha(\bar{x}) \in C_{r,n}$,*
(2) *for all $\mathcal{A}, \mathcal{A}' \in V$, if n-tuples $\bar{a}$ in $\mathcal{A}$ and $\bar{a}'$ in $\mathcal{A}'$ satisfy the same formula $\alpha(\bar{x}) \in C_{r,n}$, then there is an isomorphism from $B_r(\bar{a})$ onto $B_r(\bar{a}')$ that takes $\bar{a}$ to $\bar{a}'$.*

Moreover, we may take the formulas $\alpha(\bar{x})$ in $C_{r,n}$ to be existential. We may equally well take them to be universal.

Proof. We describe the *possible elements* of $B_r(\bar{x})$ inductively as follows. The set $B_0(\bar{x})$ has just the members of the n-tuple $\bar{x}$ as possible elements. Now, suppose we have the possible elements of $B_r(\bar{x})$ for some $r \geq 0$. We will set the possible elements of $B_{r+1}(\bar{x})$ to be the elements of $B_r(\bar{x})$ together with additional possible elements z obtained as follows: Take some $y \in B_r(\bar{x})$ corresponding to a node at a distance r from some $x \in \bar{x}$ and follow an arrow labeled f_i or f_i^{-1} from y to z; note that f_i^{-1} is shorthand for the term that acts as an inverse to f_i from Proposition 4.2.

We may think of the possible elements of $B_r(\bar{x})$ as terms $u(x)$, where u is a string of f_i, f_i^{-1} of length at most r. For an actual structure in our generalized bijective variety, with an actual tuple $\bar{a}$ corresponding to $\bar{x}$, we may have equalities—different paths may lead to the same point. For $\mathcal{A} \in V$ generated by $\bar{a}$, the elements of $B_r(\bar{a})$ are equivalence classes of terms $u(a_i)$, where u is a string of f_i, f_i^{-1} of length at most r. We have an existential formula saying that there exist y's corresponding to the possible elements of $B_r(\bar{x})$ such that the structure has a specific atomic diagram. We also have a universal formula saying that for all y's corresponding to the possible elements of $B_r(\bar{x})$, the structure has a specific atomic diagram. ∎

We fix sets of formulas $C_{r,n}$ as in the lemma. Gaifman's Locality Theorem says that any formula $\varphi(\bar{x})$ (in a relational language) can be expressed as a finite Boolean combination of "local" formulas and "local" sentences (see [18, 8, 17]). For our setting, we take the local formulas and local sentences to be as follows.

Definition 4.8.

(1) The *r-local formulas* $\bar{x}$ are those in $C_{r,n}$ for various n.
(2) The *r-local sentences* have one of the following forms:
   (a) $(\exists v_1, \ldots, v_s)\Big(\bigwedge_i \alpha_i(v_i) \; \& \bigwedge_{i<j} d(v_i, v_j) > 2r\Big)$
   for some s and $\alpha_i(x) \in C_{r,1}$,
   (b) $(\exists v)(\alpha(v))$ for some $\alpha \in C_{r,1}$.

Remark. This definition is similar to Gaifman's, except that we allow only special formulas in $C_{r,n}$. Note that the formulas in $C_{r,n}$ already give information on whether the distance between x_i and x_j is greater than $2r$. Indeed, if $d(x_i, x_j) \leq 2r$, the formula will contain a conjunct that says (in the relational language) $t(x_i) = t'(x_j)$ for some t, t' of length at most r. Thus, we may equivalently replace (2)(a) by $(\exists v_1, \ldots, v_s)(\alpha(v_1, \ldots, v_s))$ for some $\alpha \in C_{r,s}$. We chose the form above to stay closer to Gaifman's definition.

Definition 4.9. A formula or sentence is *local* if it is r-local for some r.

Here is our special version of Gaifman's Locality Theorem, where the local formulas and sentences are as defined above.

Theorem 4.10. *Let V be a generalized bijective variety with theory T.*

(1) *Any elementary first order sentence φ is equivalent over T to a sentence φ^* that is a finite Boolean combination of local sentences.*
(2) *Any elementary first order formula $\varphi(\bar{x})$ with free variables $\bar{x}$ is equivalent over T to a formula $\varphi^*(\bar{x})$ that is a finite Boolean combination of local sentences and local formulas. In fact, we may take $\varphi^*(\bar{x})$ to be a finite disjunction of formulas $\alpha_i(\bar{x})$ & β, where for each i, $\alpha_i(\bar{x})$ is a single local formula and β is a finite conjunction of local sentences and negations of local sentences.*

We sketch a proof using saturation. We begin with some definitions and lemmas.

Definition 4.11.

- For $\mathcal{A} \in V$, the *local theory* of $\mathcal{A}$ is the set of all local sentences and negations of local sentences that are true in $\mathcal{A}$.
- For $\bar{a}$ in $\mathcal{A}$, the *local type* of $\bar{a}$ is the set of formulas generated (under logical implication) by the local theory and the set of local formulas true of $\bar{a}$ in $\mathcal{A}$.

Note that for $\bar{a}$ in $\mathcal{A}$ and $\bar{a}'$ in $\mathcal{A}'$ of the same length, if the local type of $\bar{a}$ in $\mathcal{A}$ is contained in the local type of $\bar{a}'$ in $\mathcal{A}'$, then the local types are the same.

Lemma 4.12. *Let $\mathcal{A}, \mathcal{A}' \in V$. If n-tuples $\bar{a}$ in $\mathcal{A}$ and $\bar{a}'$ in $\mathcal{A}'$ satisfy the same local type, then there is a partial isomorphism f from $B_\infty(\bar{a})$ onto $B_\infty(\bar{a}')$ such that $f(\bar{a}) = \bar{a}'$.*

Proof. The fact that the tuples $\bar{a}$ and $\bar{a}'$ satisfy the same local type means that the structures $\mathcal{A}$ and $\mathcal{A}'$ satisfy the same local theory, and for each r, the tuples $\bar{a}, \bar{a}'$ satisfy the same unique formula $\alpha(\bar{x}) \in C_{r,n}$. By Lemma 4.7, for each r, there is an isomorphism p from $B_r(\bar{a})$ onto $B_r(\bar{a}')$ taking $\bar{a}$ to $\bar{a}'$. We have a tree of these finite partial isomorphisms p between $B_\infty(\bar{a}) \subseteq \mathcal{A}$ and $B_\infty(\bar{a}') \subseteq \mathcal{A}'$, where at level r, we put the isomorphisms from $B_r(\bar{a})$ onto $B_r(\bar{a}')$ that take $\bar{a}$ to $\bar{a}'$, and at level $r+1$, the successors of a given partial isomorphism p from level n are the extensions of p taking $B_{r+1}(\bar{a})$ isomorphically onto $B_{r+1}(\bar{a}')$. If $B_\infty(\bar{a})$ is infinite, then the tree is infinite, and it is finitely branching, so by König's Lemma, there is a path $(p_r)_{r\in\omega}$. The desired isomorphism is $\bigcup_r p_r$. If the substructure $B_\infty(\bar{a})$ is finite, then it is contained in $B_r(\bar{a})$ for some r, and p_r is the desired isomorphism. ∎

For any $\mathcal{A} \in V$, the isomorphism type of $\mathcal{A}$ is determined by the number of connected components of each isomorphism type. Suppose $\mathcal{A}$ is saturated and of infinite cardinality κ. In $\mathcal{A}$, a local type $\Gamma(\bar{x})$ is satisfied if it is finitely satisfied. For a local type $\Gamma(x) = \{\alpha_r(x) : r \in \omega\}$, there are at least n realizations of $\Gamma(x)$ on different connected components if and only if for all r, $\mathcal{A}$ satisfies the r-local sentence saying that there are at least n elements satisfying $\alpha_r(x)$ and at a distance greater than $2r$. The number of connected components with an element satisfying $\Gamma(x)$ is either finite or κ. This yields the following.

Lemma 4.13. *Suppose $\mathcal{A}, \mathcal{A}' \in V$ are saturated and of the same cardinality κ. If $\mathcal{A}, \mathcal{A}'$ satisfy the same local sentences, then $\mathcal{A} \cong \mathcal{A}'$.*

Proof. Since $\mathcal{A}, \mathcal{A}'$ are saturated, of the same cardinality, and satisfy the same local sentences, they realize the same local types, and they have the same number of connected components of each isomorphism type. Hence, they are isomorphic. ∎

Knowing what the saturated structures in the variety V look like, we see that for any countable $\mathcal{A} \in V$, there exists a saturated structure $\mathcal{A}^*$ of cardinality $2^{\aleph_0}$ such that $\mathcal{A}, \mathcal{A}^*$ satisfy the same local sentences.

Lemma 4.14. *If $\mathcal{A}, \mathcal{A}' \in V$ have the same local theory, then they are elementarily equivalent.*

Proof. Let $\mathcal{A}^*$ and $(\mathcal{A}')^*$ be saturated models of the common local theory of $\mathcal{A}, \mathcal{A}'$ such that $\mathcal{A}^*, (\mathcal{A}')^*$ both have cardinality $2^{\aleph_0}$. Applying Lemma 4.13, we see that $\mathcal{A}^* \cong (\mathcal{A}')^*$. Hence, $\mathcal{A}, \mathcal{A}'$ are elementarily equivalent. ∎

LEMMA 4.15. *Take n-tuples $\bar{a}, \bar{a}'$ in $\mathcal{A}$. If $\bar{a}, \bar{a}'$ satisfy the same local type, then there is an automorphism of $\mathcal{A}$ that takes $\bar{a}$ to $\bar{a}'$.*

Proof. We have a partial isomorphism f from $B_\infty(\bar{a})$ onto $B_\infty(\bar{a}')$ such that $f(\bar{a}) = \bar{a}'$. This extends to an automorphism that agrees with f on $B_\infty(\bar{a})$, with f^{-1} on $B_\infty(\bar{a}')$, and with the identity on the rest of $\mathcal{A}$. ∎

LEMMA 4.16. *If $\mathcal{A} \models \varphi$, then there is a sentence ψ true in $\mathcal{A}$ such that ψ is a finite conjunction of local sentences and negations of local sentences and $T \vdash (\psi \to \varphi)$.*

Proof. If S is the local theory of $\mathcal{A}$, then $T \cup S \vdash \varphi$. Then there is some ψ, the conjunction of a finite subset of S, such that $T \vdash (\psi \to \varphi)$. ∎

For a formula $\varphi(\bar{x})$ with an n-tuple $\bar{x}$ of variables, we have the following.

LEMMA 4.17. *If $\mathcal{A} \models \varphi(\bar{a})$, then there is a formula $\psi(\bar{x}) = \alpha(\bar{x})\ \&\ \beta$ such that β is a finite conjunction of sentences in the local theory of $\mathcal{A}$, $\alpha(\bar{x})$ is a local formula satisfied by $\bar{a}$ in $\mathcal{A}$, and $T \vdash (\forall \bar{x})(\psi(\bar{x}) \to \varphi(\bar{x}))$.*

Proof. We have a saturated model $\mathcal{B}$ of cardinality $2^{\aleph_0}$ with a tuple $\bar{b}$ satisfying the type of $\bar{a}$. If $\mathcal{B}'$ is saturated and satisfies the local theory of $\mathcal{A}$ and $\mathcal{B}$, there is an isomorphism f from $\mathcal{B}$ onto $\mathcal{B}'$. If $\bar{b}'$ is an n-tuple in $\mathcal{B}'$ satisfying the local type of $\bar{a}$ and $\bar{b}$, we may suppose that $f(\bar{b}) = \bar{b}'$. Hence, $\bar{b}'$ realizes the complete type of $\bar{a}$. This shows that the local theory of $\mathcal{A}$ and the local type of $\bar{a}$ generate the full theory and type. If $\chi(\bar{x})$ is a finite conjunction of local formulas and negations of local formulas in the local type of $\bar{a}$, then there is a single formula $\alpha(\bar{x})$ in the local type of $\bar{a}$ that implies $\chi(\bar{x})$—take $\alpha(\bar{x}) \in C_{r,n}$ for sufficiently large r. ∎

A standard model-theoretic argument gives the following.

PROPOSITION 4.18. *Any elementary first-order sentence φ is equivalent over T to a finite disjunction of local sentences and negations of such sentences.*

Proof. For each $\mathcal{A} \in V$ satisfying φ, choose ψ as in Lemma 4.16, a finite conjunction of local sentences and negations, true in $\mathcal{A}$, such that $T \vdash (\psi \to \varphi)$. Let S be the set of chosen sentences. Now, $T \cup \{\neg\psi : \psi \in S\} \cup \{\varphi\}$ is inconsistent, so there is a finite set $S' \subseteq S$ such that $T \vdash (\varphi \to \bigvee_{\psi \in S'} \psi)$. Then φ is equivalent over T to the disjunction of the sentences in S'. ∎

Here is the companion result for formulas with free variables.

PROPOSITION 4.19. *For any formula $\varphi(\bar{x})$ with free variables among $\bar{x}$, there is a formula $\varphi^*(\bar{x})$ equivalent over T to $\varphi(\bar{x})$ such that $\varphi^*(\bar{x})$ is a finite disjunction of formulas $\alpha(\bar{x})\ \&\ \beta$, where β is a conjunction of local sentences and negations and $\alpha(\bar{x})$ is a local formula.*

Proof. We replace $\bar{x}$ with a tuple of constants $\bar{c}$. For each model $\mathcal{A}$ of T and each tuple $\bar{a}$ satisfying $\varphi(\bar{x})$, choose a formula $\psi(\bar{x})$ in the local type of $\bar{a}$ such that $T \vdash (\psi(\bar{c}) \to \varphi(\bar{c}))$. Let S be the set of chosen formulas. Now, $T \cup \{\neg\psi(\bar{c}) : \psi(\bar{c}) \in S\} \cup \{\varphi(\bar{c})\}$ is inconsistent, so for some finite $S' \subseteq S$, $T \vdash (\varphi(\bar{c}) \to \bigvee_{\psi(\bar{c}) \in S'} \psi(\bar{c}))$. We may take ψ of the form $\alpha(\bar{x})$ & β, where β is the conjunction of the local sentences in S' and $\alpha(\bar{x})$ is the local formula in $C_{r,n}$ that is true of $\bar{a}$, where r is greatest such that S' contains a formula in $C_{r,n}$. ∎

Remark. For our special version of Gaifman's Locality Theorem, the local formulas may be taken to be either existential or universal. Thus, over a completion of T (or over the set of local sentences in the complete theory), each formula is equivalent to an existential formula and to a universal formula.

4.3. The group associated to a generalized bijective variety. Let V be a generalized bijective variety with theory T. There is an equivalence relation on strings of function symbols such that strings t, t' are equivalent if $T \vdash (\forall x)(t(x) = t'(x))$. We will associate to the variety V the Gaifman group $G(V)$, whose elements are the equivalence classes of strings under this relation.

Definition 4.20 (Gaifman group, $G(V)$). For a generalized bijective variety V with theory T, the *Gaifman group* $G(V)$ is defined by the following: the domain of $G(V)$ consists of equivalence classes of strings of symbols, where t and t' are equivalent if $T \vdash (\forall x)(t(x) = t'(x))$. The group operation is induced by concatenation of strings (2)(3).

The identity in $G(V)$ is the equivalence class of the empty string. For each function symbol f_i, we fix a term u_i that names the inverse, as in Proposition 4.2. We may write f_i^{-1} for u_i. The inverse function extends in a natural way to any word v in $f_1, \ldots, f_n$. Let F be the element of V obtained as the free structure generated by the finite tuple $\bar{a}$. The group $G(V)$ has a natural action on F, taking $t \in G(V)$ and $b \in F$ to $t(b)$. Since $b = t'(a)$ for some t', the action takes $t'(a)$ to $t \circ t'(a)$.

Definition 4.21 (orbit under action of $G(V)$). For $\mathcal{A} \in V$ and $b \in \mathcal{A}$, the *orbit* of b under the action of $G(V)$ is the set of all x such that for some $t \in G(V)$, $t(b) = x$.

(2) The Gaifman group is generated formally by functions as described above. Some may prefer to think of its elements as equivalence classes of names for elements, which gives us a presentation.

(3) In terms of group presentation, this is the same as saying $G(V)$ is generated by the function symbols of V and has a relation $t = t'$ for every $T \vdash (\forall x)(t(x) = t'(x))$.

NOTE. For $\mathcal{A} \in V$ and $b \in \mathcal{A}$, the orbit of b under the action of $G(V)$ is just the set of elements of $\mathcal{A}$ generated by b. The *automorphism orbit* of b results from the action of the group of automorphisms.

LEMMA 4.22. *Let V be a generalized bijective variety, and let F be the free structure in V generated by the tuple $\bar{a}$. The action of $G(V)$ on F is well defined and simply transitive on the orbits.*

Proof. We first prove that the action is well defined. Suppose $t_1 = t_2$ in $G(V)$. Without loss of generality, we assume that t_1 is obtained from t_2 by applying an identity of $G(V)$, say $w = w'$. This means that $t_1 = uw(w')^{-1}v$ and $t_2 = uv$ for some words u, v. Then $(t_1, t'(a)) \mapsto t_1 \circ t'(a) = uw(w')^{-1}vt'(a)$. As $w = w'$ is an identity in $G(V)$, we have $T \vdash (\forall x)(w(x) = (w')(x))$ and for an element a of F, $F \models uw(w')^{-1}vt'(a) = uw'(w')^{-1}vt'(a) = uvt'(a) = t_2 t'(a)$. Thus, the action is well defined.

Recall that every element x of F has the form $t(a_i)$ for some generator a_i, and every such x is in the orbit of a_i. Thus, every orbit in F has the form $\{t(a_i) : t \in G(V)\}$ for some generator a_i. Now, take $x = t(a_i)$ in F and suppose that $F \models u \circ t(a_i) = v \circ t(a_i)$. Since F is free, we see that $T \models (\forall x)(ut(x) = vt(x))$. Therefore, $ut = vt$ holds in the group $G(V)$, so by cancellation, we have $u = v$. Thus, the action is simply transitive on its orbits. ∎

For our commutative generalized bijective variety with theory T, we have the following.

LEMMA 4.23.

(1) *For $u, v, w \in G(V)$, $T \vdash (\forall x)(u(w(x)) = v(w(x)) \leftrightarrow u(x) = v(x))$.*
(2) *For $\alpha \in C_{r,1}$ and $w \in G(V)$, $T \vdash (\forall x)(\alpha(w(x)) \leftrightarrow \alpha(x))$.*

For structures $\mathcal{A} \in V$ with a single generator a, all elements have the same local type. In fact, they are in the same automorphism orbit as well as the same orbit under the action of $G(V)$.

LEMMA 4.24. *Suppose $\mathcal{A} \in V$ is generated by a. For $\alpha \in C_{r,1}$,*

$$\mathcal{A} \models (\exists x)(\alpha(x)) \leftrightarrow (\forall x)(\alpha(x)).$$

Consider a local sentence ρ saying that there exists $\bar{x}$ with x_i satisfying $\alpha_i \in C_{r,1}$ and with $d(x_i, x_j) > 2r$ for $i < j$. For $\mathcal{A}$ generated by a single element a, ρ cannot be true unless the α_i's are all the same and $\mathcal{A}$ has a tuple of elements $\bar{x}$ such that $d(x_i, x_j) > 2r$ for $i < j$. Thus, the important local invariants are the sentences $(\exists x)(\alpha(x))$ for $\alpha \in C_{r,1}$ and the sentences saying that there are at least n elements at a distance at least $2r$. We will show that for these important sentences, the ones true in the free structure F have density 1.

We begin with some notation. For a string t of function symbols, we write t^n for the n-fold concatenation of t. We write $\langle t\rangle$ for the subgroup of $G(V)$ generated by the equivalence class of t—the elements are the equivalence classes of the strings t^n, $t^{-n} = (t^{-1})^n$.

We will need to understand truth in the structure $\mathcal{A}$ with presentation $a\,|\,R$, where R is a single identity. We note that any identity is equivalent over T to an identity of the form $t^*(a) = a$, where the length of t^* is bounded by a constant multiple of the length of R.

The next lemma will tell us a great deal about truth in $\mathcal{A}$.

LEMMA 4.25. *Let V be a generalized bijective variety, and consider presentations $a\,|\,R$, where R is an identity equivalent to one of the form $t^*(a) = a$. Then for $u, v \in G(V)$, $\langle a\,|\,R\rangle \models u(a) = v(a)$ iff u, v are in the same left coset of $\langle t^*\rangle$.*

Proof. Let $\mathcal{A} = \langle a\,|\,R\rangle$.

$\Leftarrow$: Without loss of generality, suppose $v = u(t^*)^n$. In $\mathcal{A}$, we have

$$v(a) = u(t^*)^n(a) = u(t^*)^{n-1}(a) = \cdots = u(a).$$

$\Rightarrow$: Now, suppose $u(a) = v(a)$ in $\mathcal{A}$. Then $T \cup \{t^*(a) = a\}$ must prove

$$u(a) = x_0(a) = x_1(a) = \cdots = x_\ell(a) = v(a)\ ,$$

where for each $i < \ell$, we have one of the following:

(i) $x_{i+1} = x_i t^*$,
(ii) $x_i = x_{i+1} t^*$, or
(iii) $x_i(a) = x_{i+1}(a)$.

In the first two cases, x_i and x_{i+1} are clearly in the same left coset of $\langle t^*\rangle$. In the third case, $x_i = x_{i+1}$ in $G(V)$, so again x_i and x_{i+1} are in the same left coset of $\langle t^*\rangle$. ∎

For a given identity $u(a) = v(a)$, we are interested in the identities R such that $\langle a\,|\,R\rangle \models u(a) = v(a)$. The lemma above lets us recognize these identities. We come to the theorem that gives conditions under which the sentences true in the free structure have limiting density 1.

For a string of symbols t, we write $\mathrm{len}(t)$ for the length of t.

THEOREM 4.26. *Let V be a commutative generalized bijective variety in the language $\{f_1, \ldots, f_n\}$, and consider presentations with a single generator a and a single identity. Let F be the free structure on a. If F is infinite, then the sentences true in F have limiting density* 1.

Proof. We show that for the important sentences α, if α is true in F, then it has density 1, and if α is false in F, then it has density 0. For structures in V with generator a, the important sentences say one of the following:

(1) $(\exists x)(\alpha(x))$ for $\alpha \in C_{r,1}$—this is equivalent to a finite conjunction of formulas of the form $u(a) = v(a)$ or $u(a) \neq v(a)$.
(2) $(\exists x_1, \dots x_n)(\bigwedge_{i<j} d(x_i, x_j) > 2r)$.

If F is infinite, then we can show that any sentence of the second form true in F is implied over T by a sentence of the first form true in F. A saturated model of the theory of F has infinitely many connected components, and the sentence $(\exists x_1, \dots x_n)(\bigwedge_{i<j} d(x_i, x_j) > 2r)$ is clearly true in this model. Therefore, it is true in F. Take witnesses $x_1, \dots, x_n$, where $x_i = w_i(a)$. Choose k such that all x_i are in $B_k(a)$, and take $\alpha \in C_{1,k}$ true of a in F. Then over T, $(\exists x)(\alpha(x))$ implies $(\exists x_1, \dots x_n)(\bigwedge_{i<j} d(x_i, x_j) > 2r)$.

The group $G(V)$ is Abelian and finitely generated, so it is a finite direct product of cyclic groups generated by some elements $b_1, \dots, b_k$. We write $\Pi_i(x)$ for the projection of an element x on the subgroup generated by b_i. Since $G(V)$ is infinite, some b_i must have infinite order. Without loss of generality, we suppose b_1 has infinite order and generates a copy of $\mathbb{Z}$. We focus on $\Pi_1(x)$, and we suppose that the values are integers.

Each identity R has the form $t(a) = t'(a)$, but this is equivalent to an identity of the form $t^*(a) = a$. Let $e_0 = \max_i |\Pi_1(f_i)|$. If $\mathrm{len}(t) \leq r$, then the projection $\Pi_1(t)$ is an integer bounded by $r \cdot e_0$. If $\mathrm{len}(t), \mathrm{len}(t') \leq r$, then $d(t, t') \leq 2r$. Then $|\Pi_1(t) - \Pi_1(t')| \leq 2r \cdot e_0$. To prove Theorem 4.26, it is enough to show that all statements of the form $t(a) = t'(a)$ or $t(a) \neq t'(a)$ true in F have limiting density 1. The proof consists of two steps.

1. The first step is to show that for a fixed k, the set of presentations $a \,|\, t(a) = t'(a)$ such that $|\Pi_1(t) - \Pi_1(t')| < k$ has limiting density 0.
2. The second step is to show that for a fixed k and a fixed identity R of the form $t(a) = t'(a)$, if $|\Pi_1(t) - \Pi_1(t'))| > e_0 k$, then for any u, v such that $d(u, v) \leq k$ in the Gaifman graph $G(F)$, we have $F \models u(a) = v(a)$ if and only if $\langle a \,|\, R\rangle \models u(a) = v(a)$.

Toward the first step, we prove some lemmas.

Lemma 4.27.

(1) *The number of identities of length m is $n^m(m+1)$. Furthermore, for every $0 \leq k \leq m+1$, there are exactly n^m identities of length m in which t (the string of function symbols on the left side) has length k.*
(2) $P_s = \dfrac{n^{s+1}(s+2)(n-1)+1}{(n-1)^2}$.

Proof. For (1), the number of strings of function symbols of length m is n^m. To determine an identity $t(a) = t'(a)$, we choose one of the $m+1$

initial segments to serve as the left-hand side. For (2), we simply note that

$$P_s = \sum_{0\le m\le s} (m+1)n^m = (1+2n+\cdots+(s+1)n^s)$$
$$= \frac{(s+2)n^{s+2}-(s+2)n^{s+1}+1}{(n-1)^2} = \frac{n^{s+1}(s+2)(n-1)+1}{(n-1)^2}. \blacksquare$$

The next lemma may be interpreted as saying that a random identity of length $\le s$ has length $> \sqrt{s}$.

Lemma 4.28. $\lim_{s\to\infty} \frac{P_{s^2}-P_s}{P_{s^2}} = 1.$

Proof. Using Lemma 4.27, we get

$$\frac{P_s}{P_{s^2}} = \frac{n^{s+1}(s+2)(n-1)+1}{n^{s^2+1}(s^2(n-1)+2)+1}.$$

This clearly has limit 0, so $\frac{P_{s^2}-P_s}{P_{s^2}} = 1 - \frac{P_s}{P_{s^2}}$ has limit 1. $\blacksquare$

Let $P_{=m}$ be the number of identities of length exactly m, and let $P_{=m}(A)$ be the number of identities in A of length equal to m. Calculating the limit of $P_{=s}(A)/P_{=s}$ is often easier than calculating the limit of $P_s(A)/P_s$. The lemma below gives us permission to do that.

Lemma 4.29. *For any set A of identities of arbitrary length, if $P_{=s}(A)/P_{=s}$ has limit 0, then so does $P_s(A)/P_s$.*

Proof. We show that for $\epsilon > 0$, there is some m such that for $s \ge m$, $P_s(A)/P_s < \epsilon$. Take m_1 such that for all $s \ge m_1$, we have $P_{=s}(A)/P_{=s} < \epsilon/2$, and take m_2 such that for all s such that $s \ge m_2$, we have $P_{\sqrt{s}}/P_s < \epsilon/2$. Let $s \ge m_1, m_2$. Then

$$P_s(A) - P_{\sqrt{s}}(A) = \sum_{\sqrt{s}<m\le s} P_{=m}(A) < \frac{\epsilon}{2} \sum_{\sqrt{s}<m\le s} P_{=m} = \frac{\epsilon}{2}(P_s - P_{\sqrt{s}}).$$

This gives us

$$\frac{P_s(A)}{P_s} = \frac{P_{\sqrt{s}}(A)}{P_s} + \frac{P_s(A)-P_{\sqrt{s}}(A)}{P_s} < \frac{\epsilon}{2} + \frac{\epsilon}{2}\cdot\frac{P_s - P_{\sqrt{s}}}{P_s} < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon. \blacksquare$$

The next lemma will complete the first step of the proof of Theorem 4.26. We write t and t' for both strings of function symbols and elements of $G(V)$.

Lemma 4.30. *For every $k \in \mathbb{N}$, we have*

$$\lim_{s\to\infty} \frac{P_s(|\Pi_1(t) - \Pi_1(t')| < k)}{P_s} = 0.$$

Proof. By Lemma 4.29, it suffices to prove that

$$\lim_{s\to\infty} \frac{P_{=s}(|\Pi_1(t) - \Pi_1(t')| < k)}{P_{=s}} = 0.$$

Furthermore, since k is fixed, it is enough to prove that for every $k \in \mathbb{Z}$,

$$\lim_{s\to\infty} \frac{P_{=s}(\Pi_1(t) - \Pi_1(t') = k)}{P_{=s}} = 0.$$

Fix s. The identities of length s form a finite probability space, and the random variables $\Pi_1(t)$ and $\Pi_1(t')$ are not independent. By Lemma 4.27, we may consider $\Pi_1(t) - \Pi_1(t')$ conditioned on t having length ℓ. Then $\mathrm{len}(t') = s - \ell$. For each $\ell \leq s$, the number of identities with $\mathrm{len}(t) = \ell$ and $\mathrm{len}(t') = s-\ell$ is equal to the number of strings of length s, so the probability that $\mathrm{len}(t) = \ell$ is $\frac{1}{s+1}$. The probability that $\Pi_1(t) - \Pi_1(t') = k$ is the sum over $\ell \leq s$ of the probability that $\mathrm{len}(t) = \ell$ times the conditional probability that $\Pi_1(t) - \Pi_1(t') = k$ given $\mathrm{len}(t) = \ell$. We have

$$\frac{P_{=s}(\Pi_1(t) - \Pi_1(t') = k)}{P_{=s}} = \frac{1}{s+1} \sum_{\ell=0}^{s} \frac{P_{=s}(\Pi_1(t) - \Pi_1(t') = k \;\&\; \mathrm{len}(t) = \ell \;\&\; \mathrm{len}(t') = s - \ell)}{P_{=s}(\mathrm{len}(t) = \ell \;\&\; \mathrm{len}(t') = s - \ell)}.$$

We write X_ℓ for $\Pi_1(t)$ conditioned on t having length ℓ. Then, as a random variable, X_ℓ is a sum of ℓ i.i.d. random variables Y_{ℓ_k} whose value is equal to the projection of the kth symbol. All function symbols are equally likely. Thus, with probability $1/n$, Y will be $\Pi_1(f_i)$ for $1 \leq i \leq n$. As $s \to \infty$, we have $\ell \to \infty$. By the Central Limit Theorem, X_ℓ/ℓ converges to a normal distribution. This means in particular that, for every ϵ, there is some ℓ_ϵ such that for every $\ell > \ell_\epsilon$, the probability that $X_\ell = i$ is less than ϵ for all i, i.e.,

$$\frac{P_{=s}(\Pi_1(t) = i \;\&\; \mathrm{len}(t) = \ell)}{P_{=s}(\mathrm{len}(t) = \ell)} < \epsilon.$$

Without loss of generality, we will assume that $\mathrm{len}(t) \geq \mathrm{len}(t')$, so $\ell \geq s/2$. Thus, $\ell > \ell_\epsilon$ whenever $s > 2\ell_\epsilon$.

Now, we have

$$\begin{aligned}
&\frac{P_{=s}(\Pi_1(t) - \Pi_1(t') = k \;\&\; \mathrm{len}(t) = \ell \;\&\; \mathrm{len}(t') = s - \ell)}{P_{=s}(\mathrm{len}(t) = \ell \;\&\; \mathrm{len}(t') = s - \ell)} \\
&\quad = \sum_i \frac{P_{=s}(\Pi_1(t) = i \;\&\; \mathrm{len}(t) = \ell)}{P_{=s}(\mathrm{len}(t) = \ell)} \cdot \frac{P_{=s}(\Pi_1(t') = i - k \;\&\; \mathrm{len}(t') = s - \ell)}{P_{=s}(\mathrm{len}(t') = s - \ell)} \\
&\quad < \sum_i \epsilon \cdot \frac{P_{=s}(\Pi_1(t') = i - k \;\&\; \mathrm{len}(t') = s - \ell)}{P_{=s}(\mathrm{len}(t') = s - \ell)} < \epsilon.
\end{aligned}$$

Combining these, we get

$$\lim_{s\to\infty} \frac{P_{=s}(\Pi_1(t) - \Pi_1(t') = k)}{P_{=s}}$$
$$= \lim_{s\to\infty} \frac{1}{s+1} \sum_{\ell=0}^{s} \frac{P_{=s}(\Pi_1(t) - \Pi_1(t') = k \;\&\; \mathrm{len}(t) = \ell \;\&\; \mathrm{len}(t') = s - \ell)}{P_{=s}(\mathrm{len}(t) = \ell \;\&\; \mathrm{len}(t') = s - \ell)}$$
$$= 0. \blacksquare$$

We proceed to the second step of the proof. Recall that $e_0 = \max_i |\Pi_1(f_i)|$.

Lemma 4.31. *Fix R of the form $t(a) = t'(a)$, and fix k such that $|\Pi_1(t) - \Pi_1(t'))| > e_0 k$. For any u, v at a distance $\leq k$ in the Gaifman graph of F, we have $F \models u(a) = v(a)$ if and only if $\langle a \mid R\rangle \models u(a) = v(a)$, where $\langle a \mid R\rangle$ is the structure given by the presentation $a \mid R$.*

Proof. Based on the discussion before Lemma 4.27, we can see that if $u(a)$ and $v(a)$ are adjacent in the Gaifman graph, then $|\Pi_1(u) - \Pi_1(v)| = |\Pi_1(f_i)| \leq e_0$ for some f_i. Thus, if $d(u,v) \leq k$ in the Gaifman graph, then $|\Pi_1(u) - \Pi_1(v)| \leq e_0 k$. We will also write $t^* = t^{-1} \circ t'$, where t^{-1} is the term that is the inverse of t in the theory of the commutative generalized bijective variety. Note that t^{-1} may be longer than t, but this does not affect the argument below.

It is easy to see that if $u(a) = v(a)$ holds in F, then it holds in the structure $\langle a \mid R\rangle$, where R is $t(a) = t'(a)$, which is equivalent to $t^*(a) = a$. Suppose $\langle a \mid R\rangle \models u(a) = v(a)$. By Lemma 4.25, this implies that u, v are in the same left coset of $\langle t^*\rangle$, i.e., $u^{-1}v \in \langle t^*\rangle$. Taking the projection Π_1, we see that $\Pi_1(u^{-1}v) \in \Pi_1(\langle t^*\rangle)$. For some integer k, we have $u^{-1}v = (t^*)^k \in \langle t^*\rangle$, and $\Pi_1((t^*)^k) = k \cdot \Pi_1(t^*)$. However, by assumption, $|\Pi_1(t^*)| = |\Pi_1(t^{-1}t')| > e_0|k|$, and we have $|\Pi_1(u^{-1}v)| \leq e_0|k|$. Therefore, we must have $k = 0$. It follows that $\Pi_1(u^{-1}v) = 0 \cdot \Pi_1(t^*) = 0$. Moreover, $u^{-1}v = (t^*)^0$. It follows that $u = v$ in $G(V)$, and $F \models u(a) = v(a)$. $\blacksquare$

We are ready to complete the proof of Theorem 4.26. We just need to show that the sentences of the form $u(a) = v(a)$ or $u(a) \neq v(a)$ true in F have limiting density 1. By Lemma 4.30, for any integer k, the set of identities $t(a) = t'(a)$ such that $|\Pi_1(t) - \Pi_1(t')| > \epsilon_0|k|$ has density 1. For a fixed sentence $u(a) = v(a)$, take k such that u, v both have length at most $k/2$, so that $u(a), v(a)$ are at distance at most k. Then by Lemma 4.31, the sentence $u(a) = v(a)$ holds in F iff it holds in the structures given by identities $t(a) = t'(a)$ such that $|\Pi_1(t) = \Pi_1(t')| > \epsilon_0 k$, where this set has density 1. $\blacksquare$

This theorem can be generalized to presentations with multiple generators.

THEOREM 4.32. *Let V be a commutative generalized bijective variety in the language $\{f_1, \ldots, f_n\}$ and suppose that the free structure on a is infinite. Then for the structures in V with an m-tuple $\overline{a}$ of generators and a single identity, the sentences true in the free structure on $\overline{a}$ have limiting density* 1.

To do so, we need the following lemma.

LEMMA 4.33. *Let V be a commutative generalized bijective variety, with theory T. Let F_m be the free structure on m generators. Suppose that F_1 is infinite. Then for all $m \geq 1$, F_m and F_1 satisfy the same theory.*

Proof. All elements of F_1 have the same local type. Now, F_1 has a saturated elementary extension F^* whose Gaifman graph has infinitely many connected components. Let A be the substructure of F^* extending F_1 and generated by an m-tuple $a_1, \ldots, a_m$ from different connected components. Clearly, F_1 and F^* satisfy the same special local sentences. Since the sentences are existential, any special local sentence true in F_1 is true in A, and any special local sentence true in A is true in F^*.

We may suppose that F_m has generators $a_1, \ldots, a_n$. The connected component of a_i in F_m and in A is generated by a_i—the elements are named by terms $t(a_i)$. The special r-local formula $\alpha(x) \in C_{r,1}$ true of the elements of F_1 is true of each a_i in F_m and in A. We have an isomorphism from F_m onto A that takes a_i to a_i and takes $B_r(a_i)$ in F_m to $B_r(a_i)$ in A. Then F_1 and F_m have the same theory. ■

Proof of Theorem 4.32. For presentations with m generators and a single identity, we consider separately the set M of presentations in which the identity involves a single generator and the complementary set $\neg M$ in which the identity involves two distinct generators. For a presentation $\bar{a}|t_1(a_i) = t_2(a_i)$ in M, the resulting structure is the disjoint union of the structure $\langle a_i \,|\, t_1(a_i) = t_2(a_i)\rangle$ (with generator a_i) and $m-1$ copies of F_1 (one for each of the other a_j's). The identities in $\neg M$ have the form $t_1(a_i) = t_2(a_j)$ for $i \neq j$. In the structure $\langle \overline{a} \,|\, t_1(a_i) = t_2(a_j)\rangle$, the connected component of a_i and the connected component of a_j are collapsed via the relation $t(a_i) = t_1^{-1}t_2t(a_j)$. Thus, the structure is a disjoint union of $m-1$ copies of the free structure on one generator.

For fixed s, we have a finite probability space. For a sentence φ, the probability that φ is true is $\frac{P_s(\varphi)}{P_s} = \frac{P_s(M \ \& \ \varphi)}{P_s} + \frac{P_s(\neg M \ \& \ \varphi)}{P_s}$. For presentations with a single generator, we write P'_s and $P'_s(\varphi)$. By Theorem 4.26,

$$\frac{P'_s(\varphi)}{P'_s} \to \begin{cases} 1 & \text{if } F_1 \models \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

Now, $P_s(M \ \& \ \varphi)/P_s$ is the probability of $M \ \& \ \varphi$. This is equal to the probability of M times the conditional probability of φ given M. The probability of M is $1/n$. The conditional probability of φ given M is the same as the probability of φ for presentations with a single generator; namely, $P'_s(\varphi)/P'_s$. Thus, $\frac{P_s(M\ \&\ \varphi)}{P_s} = \frac{1}{n} \cdot \frac{P'_s(\varphi)}{P'_s}$. As $s \to \infty$, this approaches $\frac{1}{n}$ if φ is true in the free structures, and 0 otherwise.

Similarly, the probability of $\neg M\ \&\ \varphi$ is the probability of $\neg M$ times the conditional probability of φ given $\neg M$. The probability of $\neg M$ is $\frac{n-1}{n}$. The conditional probability of φ given $\neg M$ is 1 if φ is true in F_{m-1}, and 0 otherwise. Thus,

$$\frac{P_s(\neg M\ \&\ \varphi)}{P_s} = \begin{cases} \frac{n-1}{n} & \text{if } F_{m-1} \models \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

In total, $P_s(\varphi)/P_s$ has limit $\frac{1}{n} + \frac{n-1}{n} = 1$ if φ is true in the free structures, and 0 otherwise. ∎

Remark. Using the multidimensional Central Limit Theorem [27], we can generalize the theorem and corollary above to any commutative generalized bijective variety V where $\mathbb{Z}^k$ embeds into $G(V)$. In this case, the random structures in V with a single generator and k identities satisfy the zero-one conjecture, and the limiting theory agrees with the theory of the free structure. However, without the condition that $\mathbb{Z}^k$ embeds in $G(V)$, the statement is false, as witnessed by the bijective structures with two identities considered in Section 3.3.

4.3.1. *Superstability.* We make a brief comment on the superstability of completions of the theory of generalized bijective varieties. Recall that for an infinite cardinal κ, a (complete) theory T is *κ-stable* if for every set A in a model of T, if A has cardinality κ, then the set of complete types over A has cardinality κ as well. A theory is *stable* if it is κ-stable for some κ, and it is *superstable* if it is κ-stable for all sufficiently large κ. If the language of T is countable, then $\kappa \geq 2^{\aleph_0}$ will suffice. For more on stable theories, see [19, Chapter 4].

Proposition 4.34. *All completions of the theory of a generalized bijective variety are superstable.*

Proof. Let T be a completion of this theory, and let X be a subset of some model of T of cardinality $\kappa \geq 2^{\aleph_0}$. We show that the number of 1-types over X is at most κ.

A type in a variable x over X will say one of the following:

(1) $x = t(a)$ for some $a \in X$.
(2) For every term t and every $a \in X$, $x \neq t(a)$, and x satisfies a certain quantifier-free 1-type $p(x)$.

From this, it follows that if κ is the cardinality of X, then the number of 1-types over X is at most $\kappa + 2^{\aleph_0}$. Thus, for $\kappa \geq 2^{\aleph_0}$, T is κ-stable. ∎

REMARK. If we drop the condition that T is a completion of the theory of a generalized bijective variety, then there are theories in a language with finitely many unary function symbols that are unstable. We will not give an example here, although one is easily obtained taking the theory of a structure in the variety that we will study in Section 6.3.

4.4. Failure of the zero-one law. The next result gives conditions under which the zero-one law fails.

THEOREM 4.35. *Let L be a language with n unary functions, including f, where $n \geq 2$. Let V be a variety such that for some term t involving a symbol apart from f, the theory T of V contains the sentence $(\forall x)(\forall y)(t(x) = t(y))$. Consider presentations with an m-tuple $\bar{a}$ of generators and a single identity, and suppose that in the free structure, $f(t(a)) \neq t(a)$. Then there is a sentence with limiting density neither 0 nor 1.*

REMARK. The sentence $(\forall x)(\forall y)(t(x) = t(y))$ says that t has a constant value. If t involved just the symbol f, then the free structure would satisfy the sentence $f(t(a)) = t(f(a)) = t(a)$.

Proof of Theorem 4.35. Let φ be a sentence saying that f fixes the constant given by t. For instance, we may take $\varphi = (\forall x)(f(t(x)) = t(x))$. We show that φ does not have limiting density 0 or 1. We consider presentations with a tuple $\bar{a}$ of m generators and a single identity. Let A be the set of identities of the form $u(a_i) = v(a_j)$, where $u(a_i) = t(u'(a_i))$ and $v(a_j) = f(t(v'(a_j)))$. In the resulting structures, f fixes the constant, so φ is true. Let B be the set of identities of the form $u(a_i) = v(a_j)$, where $u(a_i) = t(u'(a_i))$ and $v(a_j) = t(v'(a_j))$. The resulting structure is free and f does not fix the constant, so φ is false. We show that neither A nor B has density 0. It follows that neither φ nor $\neg\varphi$ has density 0.

The number of identities of length ℓ is $m^2 n^\ell(\ell+1)$. Then

$$P_s = m^2 \sum_{0 \leq \ell \leq s} (\ell+1)n^\ell = m^2(1 + 2n + \cdots + (s+1)n^s)$$
$$= m^2\left(\frac{(s+2)n^{s+2} - (s+2)n^{s+1} + 1}{(n-1)^2}\right) = m^2\left(\frac{n^{s+1}(s+2)(n-1) + 1}{(n-1)^2}\right).$$

Say that t has length r. Then the identities in A have length at least $2r+1$, and for $\ell = 2r+1+\ell'$, the number of identities in A of length ℓ is $m^2 n^{\ell'}(\ell'+1)$. Then

$$P_s(A) = m^2 \sum_{2r+1+\ell' \leq s} (\ell'+1)n^{\ell'} = m^2(1 + 2n + \cdots + (s-2r)n^{s-2r-1})$$
$$= m^2\left(\frac{n^{s-2r}(s-2r+1)(n-1)+1}{(n-1)^2}\right),$$

and
$$\frac{P_s(A)}{P_s} = \frac{1}{n^{2r+1}} \frac{(s-2r+1)(n-1)+1}{(s+2)(n-1)+1} \to \frac{1}{n^{2r+1}}.$$

The identities in B have length at least $2r$. For $\ell = 2r + \ell'$, the number of identities in B of length ℓ is $m^2 n^{\ell'}(\ell'+1)$. Then
$$P_s(B) = m^2 \sum_{2r+\ell' \leq s} (\ell'+1)n^{\ell'} = m^2(1 + 2n + \cdots + (s-2r+1)n^{s-2r})$$
$$= m^2\left(\frac{n^{s-2r+1}(s-2r+2)(n-1)+1}{(n-1)^2}\right),$$

and
$$\frac{P_s(B)}{P_s} = \frac{1}{n^{2r}} \frac{(s-2r+2)(n-1)+1}{(s+2)(n-1)+1} \to \frac{1}{n^{2r}}.$$

Since $n \geq 2$, both of these limits are strictly between 0 and 1. ∎

5. Naming the generators

5.1. A general result. In this subsection, we consider algebraic languages that may contain non-unary functions. Let V be a variety in a language L with axioms generating a theory T. We consider presentations with a fixed generating tuple $\bar{a}$ and k identities. Let L' be the result of adding to L constants for the generators. We show that, under certain conditions, the L'-sentences true in the free structure have limiting density 1.

PROPOSITION 5.1. *Let T_F be the set of L'-sentences true in the free structure F generated by $\bar{a}$, and let S be the set of L'-sentences with limiting density* 1. *Then the following are equivalent:*

(1) *$T_F \subseteq S$,*
(2) *$T_F = S$,*
(3) *S has the following two properties:*
 (a) *S includes the sentences from T_F of the form $t(\bar{a}) \neq t'(\bar{a})$,*
 (b) *for any L'-formula $\varphi(x)$ with just x free, if $\varphi(t(\bar{a})) \in S$ for all closed terms $t(\bar{a})$, then $(\forall x)(\varphi(x)) \in S$.*

Proof. We will prove (1)⇒(2)⇒(3)⇒(1). First, we assume (1) and prove (2). We must show that $S \subseteq T_F$. Take $\varphi \in S$. If $\varphi \notin T_F$, then $\neg\varphi$ must be in T_F, so it is in S. Then φ has limiting density 0, and we have a contradiction. Next, we assume (2) and prove (3). We can see that T_F has properties (a) and (b), so S does as well. Finally, we assume (3) and prove (1). The set S has properties (a) and (b). Sentences that are logically equivalent have the same limiting density as well as the same truth value in the free structure F. We show by induction on $\varphi(\bar{a})$ that if $\varphi(\bar{a}) \in T_F$, then $\varphi(\bar{a}) \in S$. We suppose that the negations in our formulas are brought inside, next to the atomic formulas.

(1) Suppose φ has the form $t(\bar{a}) = t'(\bar{a})$. If $F \models \varphi$, then $T_F \vdash \varphi$, so the limiting density is 1.
(2) Suppose φ has the form $t(\bar{a}) \neq t'(\bar{a})$. By (a), if $F \models \varphi$, then φ has limiting density 1.
(3) Suppose $\varphi = \varphi_1 \ \& \ \varphi_2$. If $F \models \varphi$, then both conjuncts are true, so both have limiting density 1. Then φ also has limiting density 1.
(4) Suppose $\varphi = \varphi_1 \vee \varphi_2$. If $F \models \varphi$, then at least one disjunct is true, so it has limiting density 1. Then φ also has limiting density 1.
(5) Suppose $\varphi = (\exists x)(\psi(x))$. If $F \models \varphi$, then $F \models \psi(t(\bar{a}))$ for some $t(\bar{x})$. Then this sentence has limiting density 1, so φ also has limiting density 1.
(6) Suppose $\varphi = (\forall x)(\psi(x))$. If $F \models \varphi$, then $F \models \psi(t(\bar{a}))$ for all closed terms $t(\bar{a})$. Then the sentence $\psi(t(\bar{a}))$ has limiting density 1 for all $t(\bar{a})$, and by (b), $(\forall x)(\psi(x)) \in S$. ∎

Consider the following further property.

Property (c): If $(\exists x)(\psi(x)) \in S$, then $\psi(t(\bar{a})) \in S$ for some $t(\bar{a})$.

Lemma 5.2. *If S is complete (i.e., we have the zero-one law), then* (b) *and* (c) *are equivalent.*

Proof. First, suppose that (b) holds and that $(\exists x)(\psi(x)) \in S$. If there is no $t(\bar{a})$ such that $\psi(t(\bar{a})) \in S$, then $\neg\psi(t(\bar{a})) \in S$ for all $t(\bar{a})$, and $(\forall x)(\neg\psi(x)) \in S$ for a contradiction. Now, suppose (c) holds and that $\psi(t(\bar{a})) \in S$ for all $t(\bar{a})$. If $\neg(\forall x)(\psi(x)) \in S$, then $(\exists x)(\neg\psi(x)) \in S$. By (c), $\neg\psi(t(\bar{a})) \in S$ for some $t(\bar{a})$ for a contradiction, so $(\forall x)(\psi(x)) \in S$. ∎

Lemma 5.3. *Suppose S satisfies* (a) *and* (b). *Then for all formulas $\varphi(x, y)$ with free variables x, y, if $\varphi(t(\bar{a}), t'(\bar{a})) \in S$ for all terms $t(\bar{a}), t'(\bar{a})$, then $(\forall x)(\forall y)(\varphi(x, y)) \in S$.*

Proof. For a fixed term $t(\bar{a})$, suppose $\varphi(t(\bar{a}), t'(\bar{a})) \in S$ for all $t'(\bar{a})$. By (b), $(\forall y)(\varphi(t(\bar{a}), y)) \in S$ for all $t(\bar{a})$. So, by (b), $(\forall x)(\forall y)(\varphi(x, y)) \in S$. ∎

If the orbit of $\bar{a}$ in $\mathcal{A}$ is defined by an L-formula $\psi(\bar{x})$, then for each L'-sentence φ, we have $\mathcal{A} \models \varphi(\bar{a})$ iff $\mathcal{A}$ satisfies the L-sentences $(\exists \bar{x})(\psi(\bar{x}) \ \& \ \varphi(\bar{x}))$ and $(\forall \bar{x})(\psi(\bar{x}) \rightarrow \varphi(\bar{x}))$.

Proposition 5.4. *Let F be the free structure generated by $\bar{a}$. Suppose that the automorphism orbit of $\bar{a}$ in F (that is, the orbit of $\bar{a}$ under the automorphism group of F) is defined by the L-formula $\psi(\bar{x})$, and the L'-sentence $\psi(\bar{a})$ has limiting density* 1. *Suppose also that for all L-sentences*

φ, φ is true in F if and only if it has limiting density 1. *Then the same is true for all L'-sentences.*

Proof. Take an L'-sentence $\varphi(\bar{a})$ that is true in F. In F, this is equivalent to the L-sentence $\varphi^* = (\forall \bar{x})(\psi(\bar{x}) \rightarrow \varphi(\bar{x}))$. The sentence φ^* is true in F, so it has limiting density 1. The set of sentences with limiting density 1 is closed under logical consequence, so since $\psi(\bar{a})$ has limiting density 1, it follows that $\varphi(\bar{a})$ has limiting density 1. ∎

5.2. Generalized bijective structures and sentences with constants. In Theorem 4.32, we have seen that for the basic bijective variety and for the broader class of commutative generalized bijective varieties, when we consider presentations with finitely many generators and a single identity, the sentences (in the language of the variety) true in the free structure have density 1. We can apply Proposition 5.1 to extend this to sentences with a constant naming the generator.

Example 5.5. Let V be a commutative generalized bijective variety in the language L. Consider presentations with a single generator a and a single identity, and let L' be the extension of L with a constant for the generator. Suppose that the free structure F generated by a is infinite. In F, all elements are automorphic. In particular, a and $t(a)$ are automorphic via the automorphism $x \mapsto t(x)$. Preparing to apply Proposition 5.1, we take $\psi(x)$ to be $x = x$. Clearly, $\psi(a)$ has limiting density 1. By Theorem 4.26, the L-sentences true in the free structure have limiting density 1. Then Proposition 5.1 says that this holds also for the L'-sentences (involving a).

For a generating tuple $\bar{a}$, the sentences $\varphi(\bar{a})$ true in the free structure on $\bar{a}$ have density 1. To establish this, we need to take a closer look at the formulas $C_{r,n}$ from Section 4.2 in the expanded language L' with constants naming the generators. We consider the unary functions f_i as binary relations and the constants a_i as unary relations. Thus, we have atomic formulas with the meanings $x = a_i$ and $f_i(x) = y$.

Lemma 5.6. *Let V be a commutative generalized bijective variety, and consider presentations with a generating m-tuple $\bar{a}$ and a single identity. Let F be the free structure. If F is infinite, then for every $r \in \omega$, there is a set S of presentations such that*

(1) *S has limiting density* 1*,*
(2) *for $\alpha(\bar{x}) \in C_{r,m}$, the following are equivalent:*
   (a) *$\alpha(\bar{a})$ holds in F,*
   (b) *$\alpha(\bar{a})$ holds in some structure given by a presentation in S,*
   (c) *$\alpha(\bar{a})$ holds in all structures given by a presentation in S.*

If $\mathcal{A}$ is the structure given by the identity $t_1(a_i) = t_2(a_j)$, then for each r, we get an isomorphism p from $B_{2r}(\bar{a})$ in F to $B_{2r}(\bar{a})$ in $\mathcal{A}$, given by $p(u(a_k)) = u(a)$.

Proof. We use the notation from the proof of Theorem 4.26. Recall that $\varPi_1$ is the projection onto the copy of $\mathbb{Z}$ generated by b_1, where b_1 is an element of infinite order in the Abelian group $G(V)$ associated with the variety V. Let S be the set of presentations in which the identity $t_1(a_i) = t_2(a_j)$ satisfies $|\varPi_1(t_1) - \varPi_1(t_2)| > e_0 r$, where $e_0 = \max_i |\varPi_1(f_i)| + 4$. The fact that S has limiting density 1 follows from the proof of Theorem 4.26.

Since the formulas in $C_{r,m}$ uniquely describe the isomorphism type of $B_r(\bar{x})$, it suffices to show that p is an isomorphism. We know that p is surjective since $\mathcal{A}$ is a quotient of F. As in the proof of Theorem 4.26, it is also injective. Indeed, if $i = j$, then the projection from F to $\mathcal{A}$ is injective on the substructure generated by $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m$. On the substructure generated by a_i, if $u(a_i) = u'(a_i)$, then we can apply Lemma 4.25, and we see that (the equivalence class of) $u^{-1}u'$ is in $\langle t_1 t_2^{-1} \rangle$ as an element of $G(V)$. Since $|\varPi_1(t_1) - \varPi_1(t_2)| > e_0 r$ and the length of $u^{-1}u'$ is at most $4r$, this is not possible. If $i \neq j$, then, as in Theorem 4.32, the substructure generated by a_i and that generated by a_j are identified via $a_j = t_1 t_2^{-1}(a_i)$, while the projection map is injective on the substructure generated by the further generators a_k. Thus, if $u(a_i) = u'(a_j)$, then we must have $uu'^{-1} = t_1 t_2^{-1}$, but since $|\varPi_1(t_1) - \varPi_1(t_2)| > e_0 r$ and the length of $u^{-1}u'$ is at most $4r$, this is again impossible.

Recall that we are thinking of the language as relational and we have atomic formulas with the meanings $x = a_i$ and $f_i(x) = y$. The formula saying $x = a_i$ holds exactly on a_i in either $B_{2r}(\bar{a}; F)$ or $B_{2r}(\bar{a}; \mathcal{A})$. If the formula saying $f_i(x) = y$ holds in $B_{2r}(\bar{a}; F)$, then it holds of $p(x)$ and $p(y)$ in $B_{2r}(\bar{a}; \mathcal{A})$ because $\mathcal{A}$ is a quotient of F. Thus, we have $f_i(p(x)) = p(y)$. Finally, suppose that in $B_{2r}(\bar{a}; \mathcal{A})$, $f_i(p(x)) = p(y)$. Then $p(x) = u(a_j)$ for some j and $p(y) = f_i(u(a_j))$. However, the map p is bijective, so $F \models x = u(a_j)$ and $F \models y = f_i(u(a_j))$ as well. Then $f_i(x) = y$ holds in $B_{2r}(\bar{a}; F)$. This shows that p is an isomorphism, completing the proof. ∎

Theorem 5.7. *Let V be a commutative generalized bijective variety in the language L, and suppose that the free structure on one generator is infinite. Consider presentations with a fixed generating m-tuple $\bar{a}$ and a single identity. Let F be the free structure on $\bar{a}$. Let L' be the result of adding constants for the elements of $\bar{a}$ to L. Then an L'-sentence is true in F iff it has limiting density* 1.

Proof. Let φ' be an L'-sentence that is true in F, so $\varphi' = \varphi(\bar{a})$ for some L-formula φ. By Theorem 4.10, $\varphi(\bar{x})$ can be expressed as a finite disjunction $\bigvee_i \varphi_i(\bar{x})$ for $\varphi_i(\bar{x})$ of the form $\rho_i(\bar{x})$ & χ_i, where $\rho_i(\bar{x}) \in C_{r,m}$ and χ_i is a conjunction of special sentences and negations of special sentences. Recall that the special sentences have the form

$$(\exists v_1, \ldots, v_s)\Big(\bigwedge_i \alpha_i(v_i) \ \& \ \bigwedge_{i<j} d^{>2r}(v_i, v_j)\Big),$$

where $\alpha_i(v_i) \in C_{r,1}$ and $d^{>2r}(v_i, v_j)$ is the formula saying that the distance between v_i and v_j in the Gaifman graph is greater than $2r$. Since $F \models \varphi(\bar{a})$, we have $F \models \varphi_i(\bar{a})$ for all i, i.e., $F \models \rho_i(\bar{a})$ & χ_i. Since χ_i is an L-sentence true in F, it has limiting density 1 by Theorem 4.26. On the other hand, $\rho_i(\bar{x}) \in C_{r,m}$, and by Lemma 5.6, $\rho_i(\bar{a})$ also has limiting density 1. Thus, φ' has limiting density 1. ∎

6. More examples. In Section 3, we gave some examples illustrating different possible behaviors of limiting density. We considered sentences with no constants. In Section 4, we gave conditions guaranteeing that the sentences with limiting density 1 are those true in the free structure. In Section 5, we gave some results for sentences with constants naming the generators. In the current section, we look again at some of the examples from Section 3 in light of the results from Sections 4 and 5. We also give some further examples, illustrating more subtle points suggested by these results.

6.1. Examples of Proposition 5.1. Let V be a variety in the language L. Consider presentations with a fixed tuple $\bar{a}$ of generators and some number of identities, and let L' be the result of adding to L constants for the generators. Here, for reference, is the statement of Proposition 5.1.

PROPOSITION 5.1. *Let T_F be the set of L'-sentences true in the free structure F generated by $\bar{a}$, and let S be the set of L'-sentences with limiting density* 1. *Then the following are equivalent:*

(1) $T_F \subseteq S$,
(2) $T_F = S$,
(3) *S has the following two properties:*
  (a) *S includes the sentences from T_F of the form $t(\bar{a}) \neq t'(\bar{a})$,*
  (b) *for any L'-formula $\varphi(x)$ with just x free, if $\varphi(t(\bar{a})) \in S$ for all closed terms $t(\bar{a})$, then $(\forall x)(\varphi(x)) \in S$.*

The proposition says that conditions (a) and (b) are necessary and sufficient for the L'-sentences true in F to have density 1. We revisit some examples and see what the result says about them.

6.1.1. *Generalized bijective structures.* In Theorem 5.7, we saw that for the variety of generalized bijective structures and presentations with a single generator a and a single identity, any sentence, possibly involving the constant a, has limiting density 1 iff it is true in the free structure on a. Hence, we must have both properties (a) and (b) from Proposition 5.1.

6.1.2. *Abelian groups.* In Section 3.4, we saw that for Abelian groups and presentations with a single generator and a single relator, the zero-one law fails.

PROPOSITION 6.1. *For Abelian groups and presentations with a single generator and a single relator, Property* (a) *holds and Property* (b) *fails, witnessed by the formulas $\varphi(a, x)$ saying that $p^{n+1}x \neq p^n a$, where p is an odd prime.*

Proof. The free structure is $\mathbb{Z}$. Take a sentence of the form $ma \neq 0$. This is true in $\mathbb{Z}$, and the sentence is in S since all relators longer than $|m|$ make it true. Thus, Property (a) holds. Now fix n and an odd prime p. For all closed terms $t(a)$, the sentence $p^{n+1}t(a) \neq p^n(a)$ is in S. We have $p^{n+1}t(a) \neq p^n a$ for all terms $t(a) = ma$. By Property (a), the sentences $p^{n+1}t(a) \neq p^n a$ are all in S. If we had Property (b), then the sentence $(\forall x)(p^{n+1}x \neq p^n a)$ would be in S. However, recall from Section 3.4 that the sentence $\beta(p, n, 1)$ says that there is an element divisible by p^n and not by p^{n+1}. This is true in $\mathbb{Z}$ but not in S—by Lemmas 3.29 and 3.33, the limiting density is $1/p^{n+1}$. Since $p^n a = p^n a$ is logically valid, $p^n a$ is divisible by p^n in all models. Thus, if the sentence $(\forall x)(p^{n+1}x \neq p^n a)$ is in S, then $\beta(p, n, 1)$ is in S—in the models satisfying $(\forall x)(p^{n+1}x \neq p^n a)$, $p^n a$ is not divisible by p^{n+1}. This is a contradiction. ■

6.1.3. *Structures with a single unary function, one generator, and one identity.* The next example is from Section 3.2. The variety of unary functions has a single unary function symbol f and no axioms.

PROPOSITION 6.2. *For the variety of unary functions and presentations of the form $a \mid f^m(a) = f^n(a)$, Property* (a) *holds and Property* (b) *fails.*

Proof. To show that Property (a) holds, consider a sentence of the form $f^i(a) \neq f^j(a)$. Note that the set of presentations with identity $f^n(a) = f^m(a)$ for $n, m > i+j+1$ has limiting density 1. Moreover, for any such presentation, we recall from Section 3.2 that the resulting structure is a finite chain leading to a finite cycle where the chain is longer than both $i + 1$ and $j + 1$. Then we have $f^i(a) \neq f^j(a)$ in the structure. Thus, $f^i(a) \neq f^j(a)$ is in S.

To show that Property (b) fails, let $\varphi(x) = (\forall y)(x \neq y \rightarrow f(x) \neq f(y))$. We will show that this witnesses the failure of (b). For any fixed $x = f^i(a)$, note that the set of presentations with identity $f^n(a) = f^m(a)$ for $n, m > i+1$ has limiting density 1. In any such presentation, we have $f^{j+1}(a) \neq f(y)$

unless $y = f^j(a)$. Thus, the sentence saying that the formula

$$\varphi(x) = (\forall y)(x \neq y \to f(x) \neq f(y))$$

holds for $x = f^i(a)$ is in S for any closed term $f^i(a)$. On the other hand, the sentence $(\forall x)(\forall y)(x \neq y \to f(x) \neq f(y))$ saying that f is injective has limiting density 0, as shown in Section 3.2. Thus Property (b) fails in this variety. ∎

6.1.4. *A new example.* In the next example, we modify the variety of bijective structures to obtain an example in which Property (b) holds but Property (a) fails.

Example 6.3. Let L_c be the language that consists of unary function symbols S, S^{-1} and a constant c, and let V be the variety with axioms saying that S and S^{-1} are inverse functions and $S^3(c) = c$. Consider presentations with a single generator a and one identity. For the resulting structure A, let A_a be the cycle generated by a and let A_c be the cycle generated by c. We describe the structures obtained from all possible identities, and we give some limiting densities.

(1) Let S_1 be the set of identities of the form $S^n(a) = S^m(a)$. This has density $\frac{1}{4}$. If $k = |m - n|$, then A_a is a k-cycle if $k > 0$ and a $\mathbb{Z}$-chain if $k = 0$. A_c is always a 3-cycle in this case.
(2) Let S_2 be the set of identities of the form $S^n(c) = S^m(c)$. This has density $\frac{1}{4}$. Then A_a is a $\mathbb{Z}$-chain (always the same), and A_c is a 3-cycle or a 1-cycle.
(3) Let S_3 be the set of identities of the form $S^n(a) = S^m(c)$ or $S^n(c) = S^m(a)$. This set has density $\frac{1}{2}$. In the resulting structure, $A_a = A_c$ is a 3-cycle.

To see that Property (a) fails, consider the sentence $c \neq S(c)$. This is true in the free structure but fails exactly in the subset of S_2 where A_c is a 1-cycle, which has limiting density $\frac{1}{6}$.

To show that Property (b) holds, assume for some $\varphi(x)$, $\varphi(t)$ has limiting density 1 for all closed terms $t = t(c, a)$. We will show that for $i = 1, 2, 3$, the set of identities in S_i for which the resulting structure satisfies $(\forall x)(\varphi(x))$ has the same density as S_i. For any finite set σ of closed terms $t = t(c)$ or $t = t(a)$, the sentence $\psi(c, a) = \bigwedge_{t \in \sigma} \varphi(t)$ has density 1. This makes the case S_3 easy. For the structures given by identities in S_3, all x are named by terms c, $S(c)$, or $S^2(c)$.

For the remaining cases, we use Gaifman's Locality Theorem. Consider a formula $\varphi'(u, v, x)$ in the language of bijective structures such that $\varphi(x) = \varphi'(c, a, x)$. By Gaifman, $\varphi'(u, v, x)$ is equivalent in bijective structures to a formula $\bigvee_i (\alpha_i(u, v, x) \ \& \ \beta_i)$, where for some r, β_i is a conjunction of local sentences and negations, each r'-local for $r' \leq r$, and $\alpha_i(u, v, x)$ is an r-local formula that describes the union of the r-balls around u, v, x.

For identities in S_2, A_c may have one element or three, and A_a is fixed. Let σ be a finite set with closed terms naming the elements of A_c and the elements of A_a that are not far from a, with $d(x, a) \leq 2r$, plus one more element $x = t^*(a)$ where $d(x, a) > 2r$. The sentence $\psi(a, c)$ saying that $\varphi(t)$ holds for all of these terms has density 1. For the other $x \in A_a$, the ones far from a, the balls $B_r(x)$ are isomorphic. If $t^*(a)$ satisfies $\alpha_i(c, a, x)$, then all elements do, so $\varphi(x)$ holds. Then $(\forall x)(\varphi(x))$ has density 1.

Finally, for an identity in S_1, A_c is a fixed 3-cycle, while A_a varies with the identity. Consider the disjuncts $\alpha_i(c, a, x)$ & β_i that might be satisfied by some $x \in A_a$. The same identities also yield plain bijective structures A_a. Let $\alpha'_i(a, x)$ be the part of $\alpha_i(c, a, x)$ describing the r-balls around a and x. For $x = t(a)$, x satisfies $\alpha_i(c, a, x)$ & β_i iff β_i holds in $A_a \cup Z_3$ and $\alpha'_i(a, x)$ holds in A_a. For each β_i, there is a sentence β'_i such that for the structures A given by an identity in S_1, $A \models \beta_i$ iff $A_a \models \beta'_i$. We may take β'_i to be a finite disjunction of conjunctions of sentences that, in the setting of bijective structures, are r'-local sentences or negations. We can see this by using the Feferman–Vaught Theorem or, less formally, just by thinking about what β_i says. Let $\varphi'(a, x) = \bigvee_i \alpha'_i(a, x)$ & β'_i. For $x = t(a)$, the formula $\varphi'(a, x)$ has density 1 in the bijective structure A_a. Then by our earlier result, $(\forall x)(\varphi'(x))$ has density 1. For each bijective structure generated by a in which $(\forall x)(\varphi'(x))$ is true, we consider the structure to be A_a, and in the variety we are currently considering, $A = A_a \cup Z_3$ satisfies $(\forall x)(\varphi(x))$, so this has density 1.

This example also shows that the zero-one law may fail if we allow constants in the language, giving an obstacle for generalizing Theorem 4.26 (on the zero-one law for generalized bijective structures) to varieties in a language with constants. Note that in Section 5, we did add constants naming a tuple of generators. However, these constants were not part of the language of the variety—they did not appear in the axioms.

6.2. Structures with a single unary function and more generators and identities. For the language with a single unary function symbol f and the variety with no axioms, we saw in Section 3.2 that for presentations with a single generator and a single identity, the zero-one law holds, but the limiting theory is not that of the free structure. In particular, the sentence φ saying that f is not injective has density 1, but it is false in the free structure. We now consider presentations with multiple generators and identities.

PROPOSITION 6.4. *Let L be the language with a single unary function symbol f, and let V be the variety with no axioms. For presentations with m*

generators and k identities, the sentence $\varphi = (\exists x, y)(f(x) = f(y)\ \&\ x \neq y)$ has density 1.

Let the generators be $a_1, \ldots, a_m$. The identities have the form $f^p(a_i) = f^q(a_j)$. As before, the sentence φ is true if the chosen identities all satisfy that p, q are both non-zero and $p \neq q$. Indeed, in this case, without loss of generality, we may take n, i such that $f^n(a_i)$ appears as one side of some identity and there is no $m < n$ such that $f^m(a_i)$ appears as one side of some identity. Suppose $f^n(a_i) = f^q(a_j)$ is one of the identities. Then $x = f^{n-1}(a_i)$ and $y = f^{q-1}(a_j)$ witness φ.

We can show that φ has density 1. The number of identities of length r is $m^2(r+1)$. The number of length at most s is $m^2(1 + 2 + \cdots + (s+1)) = m^2(s+2)(s+1)/2$. The number of unordered sets of k identities of length at most s is

$$P_s = \binom{m^2\frac{(s+2)(s+1)}{2}}{k}.$$

We count the identities of length r such that $p, q \neq 0$ and $p \neq q$. If r is even, then there are at most $3m^2$ identities of length r for which the condition fails, namely $f^r(a_i) = a_j$, $a_i = f^r(a_j)$, and $f^{r/2}(a_i) = f^{r/2}(a_j)$. (If r is odd, then the number is at most $2m^2$.)

Thus, there are at least $m^2(r-2)$ identities of length r satisfying the condition, and there are at least $m^2\frac{(s-1)(s-2)}{2}$ identities of length at most s satisfying the condition. Let A be the set of presentations with all identities satisfying the condition. Then

$$P_s(A) \geq \binom{\frac{(s-1)(s-2)}{2}}{k}.$$

It is now a calculus exercise to show that $P_s(A)/P_S \to 1$, and the proposition follows.

Remark. We saw that when $m = k = 1$, the zero-one law holds. However, it does not hold in the case where $m = 1$ and $k = 2$. Suppose the two identities are $f^p(a) = f^q(a)$, $f^{p'}(a) = f^{q'}(a)$, and consider the sentence $\psi = (\exists x)(f(x) = x)$. This case is similar to the case of bijective structures with two identities in Section 3.3. The sentence ψ is true if and only if $\mathrm{GCD}(p - q, p' - q') = 1$. An argument like that in Section 3.3 shows that ψ has density strictly between 0 and 1. We omit the proof here.

6.3. Structures with multiple unary functions. We turn our attention to a more complicated case. Take the language with n function symbols $f_1, \ldots, f_n$ and the variety with no axioms, and consider presentations with m generators and k identities. We begin with the case where $k = 1$.

PROPOSITION 6.5. *Let φ be the sentence*

$$(\exists x)(\exists y)\Big(x \neq y \;\&\; \bigvee_{1\leq i,j\leq n} f_i(x) = f_j(y)\Big).$$

This sentence is false in the free structure, but it has limiting density 1 *among structures given by presentations with generators $a_1, \ldots, a_m$ and a single identity of the form $t(a_i) = t'(a_j)$.*

Proof. For m generators $a_1, \ldots, a_m$, the free structure F is the join of disjoint substructures generated by the separate a_j. In F, each element is uniquely expressed as $t(a_i)$, where the term t is built up out of the functions f_j. The terms are all distinct, and the sentence φ is false. For an identity $t(a_i) = t'(a_j)$, the length is the sum of the lengths of t, t'. The number of identities of length ℓ is $m^2 n^\ell(\ell+1)$, so the number of identities of length at most s, or P_s, is $m^2\sum_{0\leq\ell\leq s} n^\ell(\ell+1)$, which is equal to

$$m^2\frac{(n-1)(s+2)n^{s+1} - (n^{s+2}-1)}{(n-1)^2}.$$

Let A be the set of identities $t(a_i) = t'(a_j)$ such that t has length 0. We show that A has limiting density 0. The number of identities in A of length ℓ is $m^2 n^\ell$, so the number of length at most s is $m^2(\sum_{0\leq\ell\leq s} n^\ell)$, or $m^2\frac{n^{s+1}-1}{n-1}$. This is $P_s(A)$. It is not difficult to verify that $\lim_{s\to\infty} P_s(A)/P_s = 0$. Similarly, let B be the set of identities $t(a_i) = t'(a_j)$ such that t' has length 0. Then B also has limiting density 0. Therefore, the limiting density of $A\cup B$ is 0. Let C be the set of identities not in $A\cup B$. This will have limiting density 1. The identities in C have the form $t(a_i) = t'(a_j)$ where t, t' both have length at least 1. Say that $t(a_i) = f_{i'}(t^*(a_i))$ and $t'(a_j) = f_{j'}(t'^*(a_j))$ for terms t^* and t'^*. In the model given by the identity $t(a_i) = t'(a_j)$, we have $t^*(a_i) \neq t'^*(a_j)$. The elements $x = t^*(a_i)$ and $y = t'^*(a_j)$ witness that the sentence φ is true. ∎

Now, we consider presentations with more than one identity. We let φ be as in Proposition 6.5.

PROPOSITION 6.6. *For the language with n unary function symbols $f_1, \ldots, f_n$, let V be the variety with no axioms. For presentations with a fixed m-tuple of generators and k identities, where $k \geq 2$, the sentence φ has limiting density* 1.

Proof. Let I_s be the number of identities of length at most s. Then the number of presentations in which all identities have length at most s is $P_s = \binom{I_s}{k}$. Consider the identities $t(a_i) = t'(a_j)$ in which neither side has length 0. The number of these identities of length ℓ, where $\ell \geq 2$, is $m^2 n^\ell(\ell-1)$, so the number of length at most s is $m^2\sum_{2\leq\ell\leq s} n^\ell(\ell-1) =$

$m^2\frac{(n-1)sn^{s-1}-(n^s-1))}{(n-1)^2}$. For convenience, we call this J_s. Let C be the set of presentations with k identities in which neither side has length 0. Then $P_s(C) = \binom{J_s}{k}$.

We show by induction on k that $\lim_{s\to\infty} P_s(C)/P_s = 1$. We write P_s^k and $P_s^k(C)$ to indicate the value of k under consideration. For $k = 2$,

$$\frac{P_s^2(C)}{P_s^2} = \frac{J_s(J_s-1)}{I_s(I_s-1)}.$$

We know that $J_s/I_s \to 1$. For the expression $P_s^2(C)/P_s^2$, we divide top and bottom both by I_s and get a new numerator $J_s/I_s - 1/I_s$ that goes to 1 and a new denominator $I_s/I_s - 1/I_s$ that also goes to 1. Now, supposing that the statement holds for k, we show that it holds for $k+1$. We have

$$\frac{P_s^{k+1}(C)}{P_s^k} = \frac{P_s^k(C)}{P_s^k} \cdot \frac{J_s-k}{I_s-k}.$$

By the induction hypothesis, the first factor goes to 1. For the second factor, we again divide top and bottom by I_s. The new numerator is $J_s/I_s - k/I_s$, which has limit 1. The new denominator is $I_s/I_s - k/I_s$, which also has limit 1.

We claim that the sentence φ is true in all structures obtained from presentations in C. Take any presentation in C and consider the resulting model $\mathcal{A}$. No a_i is in the range of any function in any model of this sort. The given identities all take us from a non-trivial term in some a_i to a non-trivial term in some a_j and do not force us to assign values a_i, so we can fill out the rest of the function values without ever using these values a_i. Thus, all nontrivial identities true in $\mathcal{A}$ are true in all structures with presentations in C. Take an identity of shortest length, say $t(a_i) = t'(a_j)$, and proceed as for a single identity. Say that $t(a_i) = f_{i'}(t^*(a_i))$ and $t'(a_j) = f_{j'}(t'^*(a_j))$ for terms t^* and t'^*. By the minimality of the length of $t(a_i) = t'(a_j)$, we have $t^*(a_i) \neq t'^*(a_j)$. This witnesses the truth of φ. ∎

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