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DOUBLE-EXPONENTIAL WHITTAKER CARDINAL FUNCTION APPROXIMATION SCHEME FOR BRATU-TYPE AND TROESCH'S PROBLEMS

Abstract. The authors attempt here to exercise an efficient approximation scheme for obtaining highly accurate approximate solutions to Bratu-type and Troesch's problems. The underlying mathematical ingredients of the scheme are the double exponential transformation followed by the finite Whittaker cardinal function approximation of functions in the basis generating Shannon–Kotelnikov multiresolution analysis of $L^2(\Omega)(\Omega = [a,b] \subset \mathbb{R})$. We provide a formula relating the exponent n in the desired order $(O(10^{-n}))$ of accuracy and the resolution J of the approximation space (Paley–Wiener space of bandwidth $[-2^J\pi, 2^J\pi]$) of multiresolution analysis of $L^2(\mathbb{R})$, the lower and upper limits in the finite sum in the approximation of the solution, and a formula for the *a posteriori* error. A comparison of the accuracy of the approximate solutions obtained with that of other results in the literature confirms the better efficiency of the present scheme.

1. Introduction. We consider the non-linear boundary value problem

$$\begin{cases} u''(x) + \lambda e^u = 0, \quad \lambda > 0,\, x \in [0,1], \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

This is the well known classical *Bratu problem* [10], named after the Romanian mathematician Vasile Bratu, also known as the Liouville–Bratu–Gelfand equation. It arises in multiple disciplines of science and engineering,

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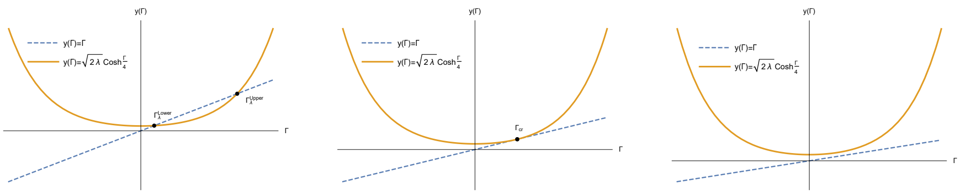
e.g., in the electrospinning process [69], the Chandrasekhar model for the expansion of the universe [15], nanotechnology [26], and the fuel ignition model of thermal combustion theory [41]. The exact solution to this boundary value problem is [32]

$$u(x) = -2\log\left(\frac{\cosh\left[\frac{\Gamma}{4}(2x-1)\right]}{\cosh\frac{\Gamma}{4}}\right), \tag{1.2}$$

where Γ satisfies the transcendental equation

$$\Gamma = \sqrt{2\lambda}\cosh\frac{\Gamma}{4}, \tag{1.3}$$

depicted in Figs. 1(a)–(c). From these figures it is evident that (1.3) for Γ admits a real solution for $\lambda \in (0, \lambda_{\mathrm{cr}}]$, where $\lambda_{\mathrm{cr}} = 3.51383071912516 12\ldots$. For $\lambda > \lambda_{\mathrm{cr}}$, there are no real Γ. On the other hand, for $\lambda < \lambda_{\mathrm{cr}}$ the solution Γ has two branches denoted by $\Gamma_\lambda^{\mathrm{Lower}}$ and $\Gamma_\lambda^{\mathrm{Upper}}$. So, for given $\lambda = \lambda' < \lambda_{\mathrm{cr}}$, Γ assumes two values, viz., $\Gamma_{\lambda'}^{\mathrm{Lower}}$ and $\Gamma_{\lambda'}^{\mathrm{Upper}}$. Some of their typical values are provided in Table 1. The solution of (1.1) provided in (1.2) with $\Gamma = \Gamma_\lambda^{\mathrm{Lower}}$ (resp. $\Gamma_\lambda^{\mathrm{Upper}}$) will be abbreviated as $u^{\mathrm{Lower}}(x)$ (resp. $u^{\mathrm{Upper}}(x)$).



(a) $\lambda < \lambda_{\mathrm{cr}}$ (b) $\lambda = \lambda_{\mathrm{cr}}$ (c) $\lambda > \lambda_{\mathrm{cr}}$

Fig. 1. Plots illustrating existence of solution (Γ) to (1.3) for different choices of the parameter λ in (1.1)

Table 1. Possible values of Γ in the solution (1.2) obtained by solving (1.3) for different choices of λ in (1.1)

λ	Γ	
	$\Gamma_\lambda^{\mathrm{Lower}}$	$\Gamma_\lambda^{\mathrm{Upper}}$
1	1.5171645990507542 . . .	10.9387027721221068 . . .
2	2.3575510538774020 . . .	8.5071995707130261 3 . . .
λ_{cr}	4.7987145610309353 . . .	

Various analytical and numerical methods have been applied to solve the Bratu problem [9, 36, 43, 27, 54, 65, 25, 47, 46, 2, 63, 51]. The limitation of those methods is that they only compute the lower branch solutions and also lose their accuracy when λ approaches $\lambda_{\rm cr}$. A thorough literature survey shows that very few studies have successfully obtained both branches of the solution to the Bratu problem.

In addition to the classical Bratu problem mentioned above, we consider other Bratu-type problems

$$\begin{cases} u''(x) - 2e^u = 0, \quad x \in [0,1], \\ u(0) = u'(0) = 0, \end{cases} \tag{1.4}$$

$$\begin{cases} u''(x) + e^{-2u} = 0, \quad x \in [0,1], \\ u'(0) = 1, \quad u'(1) = 1/2, \end{cases} \tag{1.5}$$

$$\begin{cases} u''(x) - \pi^2 e^u = 0, \quad x \in [0,1], \\ u(0) + 2u'(0) = -2\pi, \quad 2u(1) - u'(0) = -\pi. \end{cases} \tag{1.6}$$

These problems have been introduced and examined in [68, 55, 50]. Although these numerical methods may seem efficient, their implementation demands substantial computational effort.

Next, we consider the non-linear problem

$$u''(x) - \mu \sinh(\mu u) = 0, \qquad x \in [0,1], \tag{1.7}$$

satisfying the non-homogeneous Dirichlet boundary condition

$$u(0) = 0, \qquad u(1) = 1. \tag{1.8}$$

It is known as the *Troesch problem*, studied by B. A. Troesch in his seminal work [66, 67]. This problem arises in the investigation of the confinement of a plasma column by radiation pressure [23, 70] and in the theory of gas-porous electrodes [24, (7) and (8)]. A closed form solution to this BVP is [45]

$$u_\mu(x) = \frac{2}{\mu} \sinh^{-1}\left[\frac{u'_\mu(0)}{2} \operatorname{sc}\left(\mu x \,\middle|\, 1 - \frac{u'_\mu(0)^2}{4}\right)\right]. \tag{1.9}$$

The slope at the origin, $u'_\mu(0)$, that appears in the solution depends on (1.7) through the formula

$$u'_\mu(0) = 2\sqrt{1-m}, \tag{1.10}$$

where m is the root of the transcendental equation

$$\sqrt{1-m}\,\operatorname{sc}(\mu|m) = \sinh(\mu/2) \tag{1.11}$$

involving the parameter μ appearing in (1.7) and the intricate Jacobi elliptic function $\operatorname{sc}(\mu|m)$. Thus, it is apparent that the closed-form solution of the Troesch problem provided in (1.9) is not straightforward but requires a solution of a transcendental equation. Equation (1.11) has a unique solution because the monotonic growth of $\sinh(\mu/2)$ ensures that it intersects the modulated periodic function $\operatorname{sc}(\mu|m)$ in only one point for a fixed μ. For $\mu = 1$ and $\mu = 2$, the approximate values of m are $m_{\mu=1} =$

0.82140810518621196... and $m_{\mu=2} = 0.93275800773089574\ldots$. This problem is inherently unstable and difficult, especially when the sensitivity parameter μ is large. Troesch's problem has become a widely used test problem and has been studied extensively. Several numerical methods have also been developed in the last few decades to obtain approximate solutions to the problem [36, 54, 19, 31, 28, 22, 16, 17, 72, 33]. Although these methods can be efficient, their implementation often requires significant computational effort.

Whittaker cardinal function approximation (WCFA) within the Shannon–Kotelnikov multiresolution analysis (SKMRA) of $L^2(\mathbb{R})$ has proven effective for solving a wide range of linear, quasilinear, and non-linear problems in science and engineering, for example, in the process of heat transfer [18, 39], population growth [6], fluid mechanics [71], inverse problems [57], and medical imaging [62]. This method efficiently accommodates initial/boundary value problems (IVP/BVP) involving differential operators with Dirichlet and Neumann conditions [56, 58], and provides fast solvers for integral equations [38, 44, 61]. A combination of WCFA and a double exponential transformation offers several advantages, such as exponential decay in errors [29], adaptability for problems with singularities [60], stability due to rapid convergence [49] etc.

Motivated by these features, we develop a finite Whittaker cardinal function approximation (FWCFA) scheme to solve Bratu-type and Troesch problems accurately with reduced computational effort. Our approach involves two steps:

(i) splitting the unknown $u(x)$ into a known $u_0(x)$ and an unknown $v(x)$ part so that the new unknown $v(x)$ is a member of $H^1(\mathcal{D})$, the set of all analytic functions $F(z)$ in $\mathcal{D} \subset \mathbb{C}$ with $\mathcal{N}_1(F, \mathcal{D}) = \int_{\partial\mathcal{D}} |F(z)|\,|dz| < \infty$,

(ii) stretching the finite domain (Ω) to $\mathbb{R}$ using a double exponential (DE) transformation resulting in the unknown function $\bar{v}(\xi)$ becoming an element of the Paley–Wiener space (PWS) with a narrow support in $\mathbb{R}$, making it amenable to approximation with fewer terms in a FWCFA.

The proposed scheme transforms differential equations with initial/boundary conditions into a system of non-linear algebraic or transcendental equations (SNLATE). These equations can then be solved efficiently using iterative methods, such as the Jarratt-like method [52], or some library function available in computational software, e.g. FindRoot[·, ·] in MATHEMATICA. An estimate of *a posteriori* error in the approximate solution obtained here has been suggested and tested on the above-mentioned problems.

Our investigation is organized as follows. A discussion of some essential properties of Shannon wavelet basis for understanding the FWCFA of functions is presented in Section 2. Section 3 explains the stretching of the compact domain Ω to $\mathbb{R}$ and outlines the error bounds in approximating the unknown function $\bar{v}(\xi)$ using FWCFA. In Section 4, we detail the method for transforming the second-order non-linear ODE with various initial and boundary conditions into a SNLATE. Section 5 analyses the method's convergence and discusses the relationship between the exponent n in the desired order ($O(10^{-n})$) of accuracy, the resolution J, the boundary ξ^B of the effective support of $\bar{v}(\xi)$, and the formula for the *a posteriori* error. Section 6 describes the implementation of the scheme on Bratu-type and Troesch problems, while Section 7 summarizes our key findings.

2. Shannon wavelet basis. The Shannon wavelet basis (SWB) represents a family of orthogonal classical functions with many exciting properties [38, 61, 14, 11], including the SKMRA [38] of $L^2(\mathbb{R})$ in nested Paley–Wiener spaces. Given that these are classical functions, their derivatives can be explicitly expressed [61, 12, 13].

The scale function $\phi^S(x)$ of the SWB in SKMRA is the well-known sinc function:

$$\phi^S(x) = \operatorname{sinc}(x) \equiv \begin{cases} \frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \tag{2.1}$$

which plays a crucial role in signal analysis and function approximation theory. Here, we restate some fundamental properties that are beneficial for a complete understanding of the FWCFA scheme within the framework of splitting $L^2(\mathbb{R})$ into Paley–Wiener spaces.

(i) Interpolating property [61]: The values of $\phi^S_{Jk}(x)$ at $x_l = \frac{l}{2^J}$, $J, k, l \in \mathbb{Z}$, are given by

$$\phi^S_{Jk}\left(\frac{l}{2^J}\right) = \delta_{kl} = \begin{cases} 1 & \text{for } k = l, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

(ii) One-point quadrature formula [61]: For any band-limited function $F \in L^2(\mathbb{R})$ with $\operatorname{supp}(\hat{F}) \subseteq [-2^J\pi, 2^J\pi]$ and $k \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} (\phi^S_{Jk})^*(x) F(x)\, dx = \frac{1}{2^J} F\left(\frac{k}{2^J}\right). \tag{2.3}$$

Here, $(\phi^S_{Jk})^*$ indicates the complex conjugate of ϕ^S_{Jk}.

(iii) Derivatives of the scale function ϕ^S [61]:

$$\phi^{S'}(x-k) = \phi^{S'}_{0k}(x) = \begin{cases} 0 & \text{if } x = k, \\ \frac{(-1)^{x-k}}{x-k} & \text{if } x \neq k \text{ and } x \in \mathbb{Z} - \{0\}, \end{cases} \tag{2.4a}$$

$$\phi^{S''}(x-k) = \phi^{S''}_{0k}(x) = \begin{cases} -\frac{\pi^2}{3} & \text{if } x = k, \\ \frac{-2(-1)^{x-k}}{(x-k)^2} & \text{if } x \neq k \text{ and } x \in \mathbb{Z} - \{0\}. \end{cases} \tag{2.4b}$$

These properties will be useful in transforming differential equations into SNLATE.

2.1. Whittaker cardinal function approximation of functions $g \in L^2(\mathbb{R})$ in SWB. For any $g \in L^2(\mathbb{R})$, the fact that $\hat{g} \in L^2(\mathbb{R})$ implies that the support of $\hat{g}(\omega)$ in the frequency domain can be divided into the following disjoint subspaces of finite or infinite bands:

$$[-2^{j_0}\pi, 2^{j_0}\pi] \cup \bigcup_{j=j_0}^{J} \{[-2^{j+1}\pi, -2^{j}\pi] \cup [2^{j}\pi, 2^{j+1}\pi]\}, \qquad j_0 \in \mathbb{Z},\ J \in \mathbb{N}.$$

The projection of a function in $L^2(\mathbb{R})$ onto Paley–Wiener subspaces with the bandwidths defined above is regarded as the SKMRA of $L^2(\mathbb{R})$. Any function g in the above class can be represented in the SWB with a finite number of terms defined as

$$\Phi_J^S(x) = \{\phi_{Jk}^S(x) : k \in \Lambda_J^{\phi}\} \tag{2.5}$$

so that $g(x)$ can be approximated by

$$g(x) \approx g_J^{\mathrm{FWCFA}}(x) = \sum_{k \in \Lambda_J^{\phi}} g\left(\frac{k}{2^J}\right) \phi_{Jk}^S(x). \tag{2.6}$$

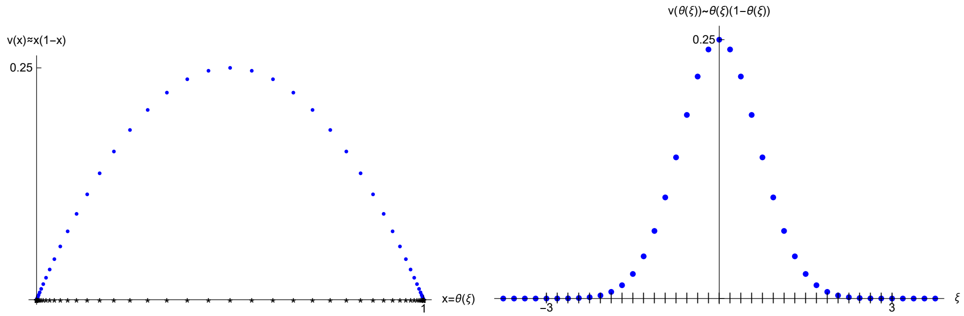
Here, $\Lambda_J^{\phi} \subset \mathbb{Z}$ is a (finite) index set. The number of its elements depends on the desired approximation accuracy and the effective support of $g(x)$. The approximation error is small because the coefficients $|g(k/2^J)|$, for $k \in \mathbb{Z} - \Lambda_J^{\phi}$, are sufficiently small. In [61], the series in (2.6) is referred to as the FWCFA for $g \in L^2(\mathbb{R})$.

3. Stretching a compact domain using a double exponential transformation. In the previous section, it was observed that any function $g \in L^2(\mathbb{R})$ can be efficiently approximated by FWCFA (e.g., (2.6)) in the SWB $\Phi_J^S(x)$. To extend the usefulness of that basis to approximate functions $g \in L^2(\Omega)$, $\Omega = [a, b] \subset \mathbb{R}$, it is convenient to adopt a non-uniform stretching of the domain Ω to $\mathbb{R}$ through an appropriate strictly monotonic mapping $x = \theta(\xi)$ so that $\xi \to -\infty$ as $x \to a$ and $\xi \to \infty$ when $x \to b$.

We apply the non-uniform stretching map (known as DE or the tanh-sinh transformation) [61]

$$x = \theta(\xi) = \frac{a+b}{2} + \frac{b-a}{2} \tanh\left(\frac{\pi}{2}\sinh(\xi)\right), \qquad \xi \in \mathbb{R}. \tag{3.1}$$

To exhibit the merit of this transformation, the plot of the image x_k for equally spaced discrete choices of $\xi_k = k/2^j$, $k \in \{-3 \times 2^j, \dots, 3 \times 2^j\}$, the graph of the polynomial $v(x) = x(1-x)$ and the function $\bar{v}(\xi)$ ($\equiv v(\theta(\xi))$) evaluated at x_k and ξ_k respectively are depicted in Figs. 2(a)–(b). From these figures it is apparent that (i) the images x_k $(= \theta(\xi_k))$ for ξ_k's beyond a finite domain containing $\xi = 0$ are dense around the boundaries $x = 0$ and $x = 1$, (ii) the function $\bar{v}(\xi)$ decays more rapidly as $|\xi| \to \infty$ in the ξ-space in comparison to algebraic decay around $x = 0$ and $x = 1$ in the x-space. The rate of decay of $\bar{v}(\xi)$ as $|\xi| \to \infty$ can be obtained using the following definition and lemma.



(a) (b)

Fig. 2. The image x_{jk} of $\theta(\xi)$ for equdistant nodes $\xi_{jk} = k/2^j$, $k = -3\times 2^j, \ldots, 3\times 2^j$ and the plot of $\bar{v}(\xi) = v(\theta(\xi))(v(x) \sim x(1-x))$ at those nodes (at the images $x_{jk} = \theta(\xi_{jk})$) for $j = 3$.

Definition 3.1 ([40]). Let $H^1(\mathcal{D})$ be the family of all functions g which are analytic in $\mathcal{D}$ and satisfy $\mathcal{N}_1(g, \mathcal{D}) = \int_{\partial\mathcal{D}} |g(z)|\,|dz| < \infty$ and let $L_\delta(\mathcal{D})$ be the set of all analytic functions in $H^1(\mathcal{D})$. Then there exists a positive constant C which satisfies

$$L_\delta(\mathcal{D}) = \{g \in H^1(\mathcal{D}) : |g(z)| \leq C|Q^\delta_{a,b}(z)| \text{ for all } z \in \mathcal{D}\},$$

where $Q^\delta_{a,b}(z) = [(z-a)(b-z)]^\delta$ and $\delta > 0$.

Lemma 3.2. *For $v \in L_\delta(\mathcal{D})$, let $\bar{v}(\xi) = v(x)|_{x=\theta(\xi)}$. Then there exists $C_1 > 0$ such that*

$$|\bar{v}(\xi)| \leq C_1 e^{-\frac{\delta\pi}{2} e^{|\xi|}}, \qquad |\xi| \to \infty. \tag{3.2}$$

Proof. If $v \in L_\delta(\mathcal{D})$, using Definition 3.1, we get

$$\begin{aligned} |\bar{v}(\xi)| = |v(\theta(\xi))| &\leq C|(\theta(\xi)-a)(b-\theta(\xi))|^\delta \\ &\leq C\left|\left\{\frac{a+b}{2} + \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(\xi)\right) - a\right\}\right. \\ &\quad \left.\times \left\{b - \frac{a+b}{2} - \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(\xi)\right)\right\}\right|^\delta \end{aligned}$$

$$
\begin{aligned}
&\le C\left(\frac{b-a}{2}\right)^{2\delta}\left|\left\{1+\tanh\left(\frac{\pi}{2}\sinh(\xi)\right)\right\}\right.\\
&\qquad\times\left.\left\{1-\tanh\left(\frac{\pi}{2}\sinh(\xi)\right)\right\}\right|^{\delta}\\
&\le C\left(\frac{b-a}{2}\right)^{2\delta}\left|\left\{1-\tanh^2\left(\frac{\pi}{2}\sinh(\xi)\right)\right\}\right|^{\delta}\\
&\le C\left(\frac{b-a}{2}\right)^{2\delta}\left|\operatorname{sech}^2\left(\frac{\pi}{2}\sinh(\xi)\right)\right|^{\delta}\\
&\le C\left(\frac{b-a}{2}\right)^{2\delta}\frac{2^{2\delta}}{|e^{\frac{\pi}{2}\sinh(\xi)}+e^{-\frac{\pi}{2}\sinh(\xi)}|^{2\delta}}\\
&\le C(b-a)^{2\delta}\begin{cases} e^{\delta\pi\sinh(\xi)} & \text{as } \xi\to-\infty,\\ e^{-\delta\pi\sinh(\xi)} & \text{as } \xi\to\infty\end{cases}\\
&\le C(b-a)^{2\delta}\begin{cases} e^{-\frac{\delta\pi}{2}e^{-\xi}} & \text{as } \xi\to-\infty,\\ e^{-\frac{\delta\pi}{2}e^{\xi}} & \text{as } \xi\to\infty.\end{cases}
\end{aligned}
$$

Thus,

$$
|\bar{v}(\xi)|\le C_1 e^{-\frac{\delta\pi}{2}e^{|\xi|}} \qquad \text{as } |\xi|\to\infty
$$

with $C_1 = C(b-a)^{2\delta}$. ■

The subtle role of the double exponential transformation is that it accelerates the decay rate of a function. The result of Lemma 3.2 suggests that the double exponential transformation accelerates the algebraic decay of the function $v(x)$ to double exponential decay rate as $|\xi|\to\infty$. Such rapid decay of $\bar{v}(\xi)$ not only provides its convergent Whittaker cardinal function representation in the basis generated by $\Phi^S(\xi)$ [35, Def. 2.1, p. 22] but can be truncated to a finite Whittaker cardinal function representation (2.6) with errors bounded in the following theorem.

Theorem 3.3 ([40, 64]). *For preassigned $J, M\in\mathbb{N}$, $\xi^B = M/2^J$, there exist constants $C_2, C_3\in\mathbb{R}^+$ depending on $r\in\mathbb{N}\cup\{0\}$ and the exponential decay rate d and δ such that*

$$
\begin{aligned}
\text{(3.3)}\qquad &\sup_{\xi\in\mathbb{R}}\left|\bar{v}^{(r)}(\xi)-\left(\frac{d^r}{d\xi^r}\right)\sum_{k=-M}^{M}\bar{v}\left(\frac{k}{2^J}\right)\phi^S_{Jk}(\xi)\right|\\
&\qquad\qquad\le C_2 2^{Jr}e^{-2^J\pi d}+C_3 2^{J(r+1)}\frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}}.
\end{aligned}
$$

Here $C_2 = (1+e)\frac{C_1}{\pi d}$ (e is the Euler number) and $C_3 = \frac{4C(b-a)^{2\delta}}{\pi\delta}$.

The RHS of (3.3) provides bounds for the residual errors in FWCFA of $\bar{v}(\xi)$ and its derivatives.

REMARK 3.4. For an analytic function $w(z)$ in $\mathcal{D}$ with the decay property

$$w(z) \sim \begin{cases} (z-a)^{\alpha} & \text{as } z \to a, \\ (b-z)^{\beta} & \text{as } z \to b, \end{cases} \tag{3.4}$$

$\delta = \min\{\alpha, \beta\}$ and one can choose $d \in (0, \pi/2)$ [59, Eqs. (1.7.11), (1.7.15), p. 67, and Fig. 1.7.4c, p. 68].

4. Methodology. In this section, we develop the scheme to solve non-linear problems in a generic form

$$u''(x) + f(x, u) = 0, \quad x \in \Omega \ (= [a, b]), \tag{4.1}$$

satisfying any one of the following initial/boundary conditions (IC/BC):

(i) Initial condition (IC):

$$u(a) = u_a, \quad u'(a) = u'_a,$$

(ii) Dirichlet's boundary condition (DBC):

$$u(a) = u_a, \quad u(b) = u_b,$$

(iii) Neumann's boundary condition (NBC):

$$u'(a) = u'_a, \quad u'(b) = u'_b,$$

(iv) Robin's boundary condition (RBC):

$$\alpha_a u(a) + \beta_a u'(a) = \gamma_a, \qquad \alpha_b u(b) + \beta_b u'(b) = \gamma_b.$$

Here, $a, b, u_a, u_b, u'_a, u'_b, \alpha_a, \beta_a, \gamma_a, \alpha_b, \beta_b, \gamma_b$ are all finite real constants, and the non-linear term f is in $C(\Omega \times \mathbb{R})$. To transform a non-homogeneous IC/BC into a homogeneous BC, we split the solution (unknown) $u(x)$ into

$$u(x) = u_0(x) + v(x), \tag{4.2}$$

involving a prescribed function $u_0(x)$ and a new dependent variable $v(x)$ in the PWS. The choice of $u_0(x)$ is suggested by the requirement that $v(x)$ satisfies homogeneous boundary conditions, viz.,

$$v(a) = v(b) = 0 \quad \text{in the case of DBC,} \tag{4.3a}$$

$$v(a) = v(b) = 0 = v'(a) = v'(b) \quad \text{for other conditions.} \tag{4.3b}$$

Substitution of $u(x)$ given in (4.2) and its derivatives into (4.1) gives the equation for the new unknown $v(x)$:

$$v''(x) + u''_0(x) + f(x, u_0 + v) = 0. \tag{4.4}$$

Using the transformation $x = \theta(\xi)$ of the independent variable and the relations

$$\frac{d}{dx} = \frac{1}{\theta'(\xi)}\frac{d}{d\xi}, \quad \frac{d^2}{dx^2} = \frac{1}{\theta'(\xi)^2}\frac{d^2}{d\xi^2} - \frac{\theta''(\xi)}{\theta'(\xi)^3}\frac{d}{d\xi} \tag{4.5}$$

in (4.4), one gets the equation for $\bar{v}(\xi)$:

$$\frac{1}{\theta'(\xi)^2}\bar{v}''(\xi) - \frac{\theta''(\xi)}{\theta'(\xi)^3}\bar{v}'(\xi) + \bar{u}_0''(\xi) + \bar{f}(\xi, \bar{u}_0 + \bar{v}) = 0, \qquad \xi \in \mathbb{R}. \tag{4.6}$$

The homogeneous BC (4.3a) and (4.3b) for $v(x)$ are transformed to

$$\bar{v}(\pm\infty) = 0 \tag{4.7a}$$

and

$$\bar{v}(\pm\infty) = \bar{v}'(\pm\infty) = 0 \tag{4.7b}$$

respectively. In (4.6) we have used the notations

$$\begin{aligned} \bar{v}(\xi) &= v(x)|_{x=\theta(\xi)}, \\ \bar{u}_0''(\xi) &= u_0''(x)|_{x=\theta(\xi)}, \\ \bar{f}(\xi, \bar{u}_0 + \bar{v}) &= f(x, u_0 + v)|_{x=\theta(\xi)}. \end{aligned} \tag{4.8}$$

The boundary condition in (4.7a) or (4.7b) implies $\bar{v} \in L^2(\mathbb{R}) \cap C^2(\mathbb{R})$ [61]. Following the discussion of Section 2.1, one may then approximate $\bar{v}(\xi)$ by a FWCFA (2.6) in the basis $\Phi_J^S(\xi) = \{\phi_{Jk}^S(\xi) : k \in \Lambda_J^\phi\}$ of the approximation space (PWS of bandwidth $[-2^J\pi, 2^J\pi]$) of the SKMRA as

$$\bar{v}(\xi) \simeq \bar{v}_J^{\mathrm{FWCFA}}(\xi) = \sum_{k\in\Lambda_J^\phi} C_{Jk}\phi_{Jk}^S(\xi). \tag{4.9}$$

Here, $\Lambda_J^\phi \subset \mathbb{Z}$ is an appropriate finite index set such that the error due to omission of $C_{Jk}\phi_{Jk}^S(\xi)$ for (the threshold value) $k \in \mathbb{Z} \setminus \Lambda_J^\phi$ is much lower than the desired order $O(10^{-n})$ of accuracy. Using (4.9) in conjunction with

$$\frac{d}{d\xi}(\bar{v}_J^{\mathrm{FWCFA}}(\xi)) = 2^J \sum_{k\in\Lambda_J^\phi} C_{Jk}\phi_{0k}^{S'}(\eta)|_{\eta=2^J\xi}, \tag{4.10a}$$

$$\frac{d^2}{d\xi^2}(\bar{v}_J^{\mathrm{FWCFA}}(\xi)) = 2^{2J} \sum_{k\in\Lambda_J^\phi} C_{Jk}\phi_{0k}^{S''}(\eta)|_{\eta=2^J\xi} \tag{4.10b}$$

in (4.6), one gets a relation involving elements of the basis $\Phi^S_J(\xi)$:

(4.11)
$$\frac{1}{\theta'(\xi)^2}2^{2J}\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^{S''}(2^J\xi-k)-2^J\frac{\theta''(\xi)}{\theta'(\xi)^3}\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^{S'}(2^J\xi-k)+\bar{u}_0''(\xi) + \bar{f}\Big(\xi,\bar{u}_0(\xi)+\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^S_{Jk}(\xi)\Big)=0.$$

Multiplying both sides with $\theta'(\xi)^2$, we get

(4.12)
$$2^{2J}\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^{S''}(2^J\xi-k)-2^J\frac{\theta''(\xi)}{\theta'(\xi)}\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^{S'}(2^J\xi-k)+\theta'(\xi)^2\bar{u}_0''(\xi) + \theta'(\xi)^2\bar{f}\Big(\xi,\bar{u}_0(\xi)+\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^S_{Jk}(\xi)\Big)=0.$$

To get the unknown coefficients $\{C_{Jk} : k\in\Lambda^\phi_J\}$, (4.12) is transformed to a SNLATE

(4.13)
$$2^{2J}\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^{S''}(l-k)-2^J\frac{\theta''\big(\frac{l}{2^J}\big)}{\theta'\big(\frac{l}{2^J}\big)}\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^{S'}(l-k)+\theta'\Big(\frac{l}{2^J}\Big)^2\bar{u}_0''\Big(\frac{l}{2^J}\Big) + \theta'\Big(\frac{l}{2^J}\Big)^2\bar{f}\Big(\frac{l}{2^J},\bar{u}_0\Big(\frac{l}{2^J}\Big)+\sum_{k\in\Lambda^\phi_J} C_{Jk}\phi^S(l-k)\Big)=0,$$

by taking the inner product of (4.12) with each element of the basis $\Phi^S_J(\xi)$ and exploiting the one-point quadrature rule (2.3) provided in Section 2.

It is worth mentioning here that transforming (4.12) to a SNLATE like (4.13) involving collocation and the Galerkin method will provide the same set of equations whenever $\bar{f}(\xi,\bar{u}_0+\bar{v})$ is an element of PWS with bandwidth $[-2^J\pi,2^J\pi]$. The function $u_0(x)$ in (4.2) may contain some additional unknowns such as $u(a),u(b),\ldots$. To obtain these unknowns, we add a few additional equations, e.g., $\bar{v}'(\xi_a-1/2^J),\bar{v}'(\xi_b+1/2^J),\ldots=0$ appropriately. The advantage of using FWCFA of the unknown solution is the minimum appearance of unknowns $\tilde{C}_J=\tilde{C}^\phi_J\cup\{u(0),\ldots\}$ with $\tilde{C}^\phi_J=\{C_{Jk} : k\in\Lambda^\phi_J\}$ in the SNLATE

(4.14)
$$\tilde{F}(\tilde{C}_J)=0,$$

due to the one-point quadrature rule (2.3) for the elements in the basis. Here

$\tilde{F} : \mathfrak{R} \subset \mathbb{R}^n \to \mathbb{R}^n$ $(n = \#\tilde{C}_J)$ is defined by

$$\tilde{F}(\tilde{C}_J) = \mathfrak{A}\tilde{C}_J + \tilde{\mathscr{F}}(\tilde{C}_J) \tag{4.15}$$

involving the matrix $\mathfrak{A} = [a_{lk}]$ with elements

$$a_{lk} = 2^{2J}\phi^{S''}(l-k) - 2^J \frac{\theta''\left(\frac{l}{2^J}\right)}{\theta'\left(\frac{l}{2^J}\right)}\phi^{S'}(l-k) + \theta'\left(\frac{l}{2^J}\right)^2 \bar{u}_0''\left(\frac{l}{2^J}\right). \tag{4.16}$$

The other part

$$\tilde{\mathscr{F}}(\tilde{C}_J) = \{\mathscr{F}_l(\tilde{C}_J) : l \in \Lambda_J^\phi\}$$

involves the non-linear terms

$$\mathscr{F}_l(\tilde{C}_J) = \theta'\left(\frac{l}{2^J}\right)^2 \bar{f}\left(\frac{l}{2^J}, \bar{u}_0\left(\frac{l}{2^J}\right) + \sum_{k\in\Lambda_J^\phi} C_{Jk}\phi^S(l-k)\right). \tag{4.17}$$

The non-linear terms $f(x,u)$ appearing in (1.1), (1.4)–(1.6) and (1.7) are smooth functions, so $\tilde{F}(\tilde{C}_J)$ is continuously differentiable. Furthermore, from Fig. 2(b), it is apparent that $\bar{v}(\xi)$ behaves qualitatively like a Gaussian function. It may be further verified that the Jacobian matrix

$$J_{\tilde{F}}(\tilde{C}_J) = \left[\frac{\partial \tilde{F}}{\partial \tilde{C}_J}\right]$$

is non-singular in the neighborhood of $\tilde{C}_J^0 = \{e^{-(k/2^J)^2} \,/\, 3e^{-(k/2^J)^2}, k \in \Lambda_J^\phi\}$.

Consequently, FindRoot[·, ·] in MATHEMATICA/Jarratt-like iterative schemes [52] with input $\tilde{C}_J^0$ may be used to solve (4.14). The resulting values of unknowns C_{Jk}, $k \in \Lambda_J^\phi$, have been used in (4.9) to obtain $\bar{v}_J^{\text{FWCFA}}(\xi)$. Then use of FWCFA of $\bar{v}(\xi)$ followed by the inverse transformation

$$\xi = \Theta(x) \equiv \sinh^{-1}\left[\frac{2}{\pi}\tanh^{-1}\left(\frac{2x-(b+a)}{b-a}\right)\right] \tag{4.18}$$

in (4.2) gives FWCFA for $u(x)$:

$$\begin{aligned} u(x) \simeq u_J^{\text{FWCFA}}(x) &= u_0^{\text{Approx}}(x) + \bar{v}_J^{\text{FWCFA}}(\Theta(x)) \\ &= u_0^{\text{Approx}}(x) + \sum_{k\in\Lambda_J^\phi} C_{Jk}\phi_{Jk}^S(\Theta(x)). \end{aligned} \tag{4.19}$$

4.1. Asymptotic behavior of solution to (4.4)**.** The asymptotic behavior of the solution to (4.4) with given IC/BC can be derived through the following procedure:

Around $x = a$ in Ω, (4.4) leads to

$$v''(x) = -(u_0''(a) + f(a, u_0(a))). \tag{4.20}$$

Use of the BC (4.3a) or (4.3b) gives

$$v(x) \simeq -\tfrac{1}{2}(u_0''(a) + f(a, u_0(a)))w_a(x), \tag{4.21}$$

where

$$w_a(x) = \begin{cases} x-a & \text{for DBC,} \\ (x-a)^2 & \text{for IC/NBC/RBC.} \end{cases} \tag{4.22}$$

Similarly, around $x = b$, (4.4) leads to

$$v''(x) = -(u_0''(b) + f(b, u_0(b))). \tag{4.23}$$

Again, the BC (4.3a) or (4.3b) around $x = b$ implies

$$v(x) \simeq -\tfrac{1}{2}\big(u_0''(b) + f(b, u_0(b))\big) w_b(x), \tag{4.24}$$

where

$$w_b(x) = \begin{cases} b-x & \text{for DBC,} \\ (b-x)^2 & \text{for IC/NBC/RBC.} \end{cases} \tag{4.25}$$

Comparison of (4.22), (4.25) and (3.4) gives the exponents α, β of asymptotic decay of solution of (4.1) near the boundaries are $\alpha = \beta = 1$ in case of DBC and $\alpha = \beta = 2$ for IC/NBC/RBC, leading to $\delta = 1$ for DBC and $\delta = 2$ for the rest. Since $f(x, u) \in C(\Omega \times \mathbb{R})$, (4.21) with (4.22) and (4.24) with (4.25) imply that $v \in L_\delta(\Omega)$ with

$$C = \max\left\{ \left| \frac{u_0''(a) + f(a, u_0(a))}{2} \right|, \left| \frac{u_0''(b) + f(b, u_0(b))}{2} \right| \right\}. \tag{4.26}$$

Therefore, following [59, Lemma 1.3.6, p. 136],

$$v \in H^1(\Omega) \quad \text{with } N_1[v, \Omega] < \infty.$$

Moreover, the use of the DE transformation ($x = \theta(\xi)$), provided in (3.1), in (4.21), (4.22), (4.24) and (4.25) implies the asymptotic behavior

$$\bar{v}(\xi) \simeq -(b-a)^{2\delta} \begin{cases} \frac{u_0''(a)+f(a,u_0(a))}{2} e^{-\frac{\pi\delta}{2}e^{-\xi}} & \text{as } \xi \to -\infty, \\ \frac{u_0''(b)+f(b,u_0(b))}{2} e^{-\frac{\pi\delta}{2}e^{\xi}} & \text{as } \xi \to \infty, \end{cases} \tag{4.27}$$

which may be recast in compact form as

$$|\bar{v}(\xi)| \le (b-a)^{2\delta} C e^{-\frac{\pi\delta}{2}e^{|\xi|}} \qquad \text{as } |\xi| \to \infty. \tag{4.28}$$

Then use of $\mathcal{F}(e^{-\frac{\pi\delta}{2}e^{|\xi|}}) \sim E(1-i\omega, \pi\delta/2) + E(1+i\omega, \pi\delta/2)$ (obtained by applying FourierTransform[·, ·, ·] of MATHEMATICA) implies the asymptotic behavior of the Fourier transform $\hat{\bar{v}}(\omega)$,

$$|\hat{\bar{v}}(\omega)| \le C_1 e^{-d|\omega|} \qquad \text{as } |\omega| \to \infty, \tag{4.29}$$

where $d \in (0, \pi/2)$ and

$$C_1 = (b-a)^{2\delta} C. \tag{4.30}$$

The choice of δ is determined by the exponent of decay provided in (4.22) and (4.25), while C can be obtained by using (4.26). The values of unknowns (e.g., $u(a)$, $u(b)$ etc.) may be estimated by the corresponding values of the solution of the linear part of (4.1) with the same IC/BC. The estimate of C and C_1 involving the input $f(x,u)$ of the model and IC/BC derived in (4.26) and (4.30) will be useful for obtaining estimates of the resolution J and boundary ξ^B of the effective support of $\bar{v}(\xi)$ in the following section.

5. The convergence analysis

5.1. Estimate of residual error. For the convergence analysis of the approximate solution (4.9) in the effective support Ω_ξ of $\bar{v}(\xi)$ to the second-order non-linear ODE, we write (4.6) in operator form,

$$\mathcal{O}[\bar{v}](\xi) \equiv \bar{v}''(\xi) - \frac{\theta''(\xi)}{\theta'(\xi)}\bar{v}'(\xi) + \theta'(\xi)^2\bar{u}_0''(\xi) + \theta'(\xi)^2\bar{f}(\xi, \bar{u}_0 + \bar{v}). \tag{5.1}$$

It may be verified that

$$\theta'(\xi)^2 \in \mathbb{H}^\infty(\Omega_\xi) \equiv \Big\{g : \Omega_\xi \to \Omega_\xi : \sup_{\xi\in\Omega_\xi} |g| < \infty\Big\}.$$

It is further assumed that $\theta'(\xi)^2\bar{f}(\xi, \bar{u}_0 + \bar{v})$ is a Lipschitz function with respect to the second argument and satisfies the inequality

$$\begin{aligned}|\theta'(\xi)^2|\,|\bar{f}(\xi, \bar{u}_0^1 + \bar{v}_1) - \bar{f}(\xi, \bar{u}_0^2 + \bar{v}_2)| &\le K_L|\bar{u}_0^1 + \bar{v}_1 - (\bar{u}_0^2 + \bar{v}_2)| \\ &\le K_L|\bar{u}_0^1 - \bar{u}_0^2| + K_L|\bar{v}_1 - \bar{v}_2|\end{aligned} \tag{5.2}$$

for some Lipschitz constant $K_L \in \mathbb{R}$.

THEOREM 5.1. *If $\bar{v}_J^{\mathrm{FWCFA}}(\xi)$ in (4.9) is the FWCFA of solution of (5.1) in the approximation space (PWS of bandwidth $[-2^J\pi, 2^J\pi]$) of resolution J, then there exist constants C_2, C_3 (depending on δ and d, the asymptotic decay rate of $v(x)$ and $\hat{\bar{v}}(\omega)$, mentioned in Theorem 3.3), and C_4 such that the residual error of approximation is bounded by*

$$\begin{aligned}|\mathcal{O}[\bar{v}_J^{\mathrm{FWCFA}}](\xi)| \le (1 + C_4 + K_L)\left(2^{2J}C_2e^{-2^J\pi d} + 2^{3J}C_3\frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}}\right) \\ + \sup_{\xi\in\Omega_\xi} K_L|\bar{u}_0(\xi) - \bar{u}_0^{\mathrm{Approx}}(\xi)|.\end{aligned} \tag{5.3}$$

Proof. Set

$$0 < C_4 = \sup_{\xi\in\Omega_\xi}\left|\frac{\theta''(\xi)}{\theta'(\xi)}\right| < \infty.$$

Then the LHS of (5.3) can be written as

$$\begin{aligned}|\mathcal{O}[\bar{v}_J^{\mathrm{FWCFA}}](\xi)| &= |\mathcal{O}[\bar{v}](\xi) - \mathcal{O}[\bar{v}_J^{\mathrm{FWCFA}}](\xi)| \\ &\le |\bar{v}''(\xi) - \bar{v}_J^{FWCFA''}(\xi)| + \left|\frac{\theta''(\xi)}{\theta'(\xi)}(\bar{v}'(\xi) - \bar{v}_J^{FWCFA'}(\xi))\right| \\ &\quad + |\theta'(\xi)^2|\,|\bar{f}(\xi, \bar{u}_0(\xi) + \bar{v}(\xi)) - \bar{f}(\xi, \bar{u}_0^{\mathrm{Approx}}(\xi) + \bar{v}_J^{\mathrm{FWCFA}}(\xi))|.\end{aligned}$$

Use of (5.2) in the above relation leads to

$$\begin{aligned}|\mathcal{O}[\bar{v}_J^{\text{FWCFA}}](\xi)| &\le |\bar{v}''(\xi) - \bar{v}_J^{\text{FWCFA}''}(\xi)| \\ &\quad + \sup_{\xi\in\Omega_\xi}\left|\frac{\theta''(\xi)}{\theta'(\xi)}\right|\left|\bar{v}'(\xi) - \bar{v}_J^{FWCFA'}(\xi)\right| \\ &\quad + K_L|\bar{v}(\xi) - \bar{v}_J^{\text{FWCFA}}(\xi)| + K_L|\bar{u}_0(\xi) - \bar{u}_0^{\text{Approx}}(\xi)|.\end{aligned}$$

Using the results of Theorem 3.3 with $r = 0, 1$ and 2 and the bound C_4 stated at the beginning of the proof, we can recast the above inequality as

$$\begin{aligned}|\mathcal{O}[\bar{v}_J^{\text{FWCFA}}](\xi)| &\le 2^{2J}C_2e^{-2^J\pi d} + 2^{3J}C_3\frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}} \\ &\quad + C_4\left(2^JC_2e^{-2^J\pi d} + 2^{2J}C_3\frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}}\right) \\ &\quad + K_L\left(C_2e^{-2^J\pi d} + 2^JC_3\frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}}\right) \\ &\quad + \sup_{\xi\in\Omega_\xi} K_L|\bar{u}_0(\xi) - \bar{u}_0^{\text{Approx}}(\xi)| \\ &\le 2^{2J}C_2\left(1 + \frac{C_4}{2^J} + \frac{K_L}{2^{2J}}\right)e^{-2^J\pi d} \\ &\quad + 2^{3J}C_3\left(1 + \frac{C_4}{2^J} + \frac{K_L}{2^{2J}}\right)\frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}} \\ &\quad + \sup_{\xi\in\Omega_\xi} K_L|\bar{u}_0(\xi) - \bar{u}_0^{\text{Approx}}(\xi)| \\ &\le (1 + C_4 + K_L)\left(2^{2J}C_2e^{-2^J\pi d} + 2^{3J}C_3\frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}}\right) \\ &\quad + \sup_{\xi\in\Omega_\xi} K_L|\bar{u}_0(\xi) - \bar{u}_0^{\text{Approx}}(\xi)|. \ \blacksquare\end{aligned}$$

This inequality indicates that when the function $u_0(x)$ does not contain any additional unknowns among $\{u(a), u(b), u'(a), u'(b)\}$, the residual error is decaying exponentially for an appropriate choice of the resolution J and the boundary ξ^B of the effective support of $\bar{v}(\xi)$ provided in the following theorem. However, if $u_0(x)$ includes such unknowns, the residual error decay may slightly depend on the convergence of the iterative method used, such as a Newton-type method.

THEOREM 5.2 ([34]). *For the desired order $O(10^{-n})$ of accuracy in the approximation $\bar{v}_J^{\text{FWCFA}}(\xi)$ of $\bar{v}(\xi)$, an estimate for the resolution J of the approximation space of SKMRA and the boundary ξ^B of the effective support*

$\Omega_\xi = [-\xi^B, \xi^B]$ *of* $\bar{v}(\xi)$ *may be obtained by using the straightforward formulae*

$$J \geq \log_2 \left[\frac{1}{\pi d} \log \left(\frac{2(e+1)C_1 10^n}{\pi d} \right) \right] \tag{5.4}$$

and

$$\xi^B \geq \log \left(\frac{2 \log(2^{2+J} \times 10^n (b-a)^{2\delta} C)}{\pi \delta} \right). \tag{5.5}$$

5.2. Estimates of sup and *a posteriori* errors [34]. In case the exact solution to (4.1) is known, the L^∞-error of the approximate solution at resolution J may be obtained by using the formula

$$\mathrm{Err}_J^{L^\infty} = \sup_{x\in[a,b]} |u^{\mathrm{Exact}}(x) - u_J^{\mathrm{FWCFA}}(x)|. \tag{5.6}$$

When the exact solution is not known and $u_0(x)$ does not contain additional unknowns, an *a posteriori* error in $u_J^{\mathrm{FWCFA}}(x)$ is taken to be

$$\mathrm{Err}_J^{\mathrm{a\,post}} = \sup_{k\in\Lambda_J^\phi} |C_{J+1,2k} - C_{Jk}|. \tag{5.7}$$

However, in case $u_0(x)$ contains some additional unknowns among $u(a), u(b)$, $u'(a), u'(b)$, the *a posteriori* error incorporates errors in additional unknowns:

$$\begin{aligned} \mathrm{Err}_J^{\mathrm{a\,post}} = \sup \big\{ &|C_{J+1,2k} - C_{Jk}|, k \in \Lambda_J^\phi, |u_{J+1}^{\mathrm{Approx}}(a) - u_J^{\mathrm{Approx}}(a)|, \\ &|u_{J+1}^{\mathrm{Approx}}(b) - u_J^{\mathrm{Approx}}(b)|, |u_{J+1}'^{\mathrm{Approx}}(a) - u_J'^{\mathrm{Approx}}(a)|, \\ &|u_{J+1}'^{\mathrm{Approx}}(b) - u_J'^{\mathrm{Approx}}(b)| \big\}. \end{aligned} \tag{5.8}$$

6. Implementation of the scheme. The scheme developed in the previous section will be exercised here to obtain approximate solutions to the Bratu-type and Troesch problems. It has been compared with the results obtained by other methods to assess its efficiency.

6.1. Bratu-type problem

6.1.1. *Bratu problem.* We consider here the problem (1.1)

$$\begin{cases} u''(x) + \lambda e^u = 0, & \lambda > 0,\ x \in [0,1], \\ u(0) = u(1) = 0. \end{cases} \tag{6.1}$$

Comparison of (6.1) with (4.1) shows that the non-linear term is $f(x,u) = \lambda e^u$. The behavior of the exact solution for different values of λ and some approximation/numerical methods to obtain approximate solutions are available in [9, 43, 54, 65, 25, 47, 46, 2, 63, 51, 8, 1].

For DBC, the known part $u_0(x)$ is taken to be

$$u_0(x) = 0. \tag{6.2}$$

Then (4.4) for $v(x)$ is

$$v''(x) + \lambda e^v = 0, \quad x \in [0, 1], \tag{6.3}$$

satisfying the homogeneous DBC

$$v(0) = 0 = v(1). \tag{6.4}$$

Since, (6.3) satisfies the DBC, from the discussion in Section 4.1, the exponents α, β of asymptotic decay of $v(x)$ near the boundaries are $\alpha = \beta = 1$, leading to $\delta = 1$. The use of the expression $u_0(x)$ in (6.2) and $f(x, u)$ (mentioned above) in (4.26) and (4.30) gives $C = C_1 = \lambda/2$.

In the subsequent step, formulae (5.4) and (5.5) with $\delta = 1, d = \pi/4$ and $C = C_1 = \lambda/2$ are used to obtain the lower bounds for the resolution J and the boundary ξ^B involving the exponent n of the order $O(10^{-n})$ of accuracy. These limits suggest the choices for the resolution J and the bound ξ^B of the effective support of $\bar{v}(\xi)$ that result in the index set $\Lambda_J^\phi = \{-M, \dots, M\}$ with $M = 2^J \xi_J^B$ in the FWCFA (4.9).

The SNLATE (4.13) for the unknowns C_{Jk} (for $l \in \Lambda_J^\phi$) then becomes

$$2^{2J} \sum_{k \in \Lambda_J^\phi} C_{Jk} \phi^{S''}(l-k) - 2^J \frac{\theta''\left(\frac{l}{2^J}\right)}{\theta'\left(\frac{l}{2^J}\right)} \sum_{k \in \Lambda_J^\phi} C_{Jk} \phi^{S'}(l-k) + \lambda \theta'^2\left(\frac{l}{2^J}\right) e^{\sum_{k \in \Lambda_J^\phi} C_{Jk} \phi^S(l-k)} = 0. \tag{6.5}$$

The non-linear MATHEMATICA solver FindRoot[$\cdot, \cdot$] is sensitive to initial guess, conditioning and intricacy in non-linearity. So, at lower resolution J, the number of unknown coefficients in SNLATE (6.5) is smaller, and the system is less constrained. This results in the solution being less accurate. However, at higher resolution J, the system becomes more constrained, resulting in more accurate solutions.

Inputs $\{\{C_{Jk}, e^{-(k/2^J)^2} / 3e^{-(k/2^J)^2}\} : k \in \Lambda_J^\phi\}$ for unknowns C_{Jk} have been provided subsequently to obtain a solution of (6.5) by using FindRoot[$\cdot, \cdot$]. The values of $v(x_J) = (\bar{v}(\frac{k}{2^J}) \simeq C_{Jk})$ for $\lambda = 1, 2, \lambda_{\mathrm{cr}}$ with several resolution (J) choices and boundaries ξ^B are depicted in Fig. 3(a). The results obtained in the previous step have then been used in (4.2) to obtain

$$u_J^{\mathrm{FWCFA}}(x) = \bar{v}_J^{\mathrm{FWCFA}}(\Theta(x)). \tag{6.6}$$

While the qualitative behavior of the approximate (upper and lower) solutions $u_5^{\mathrm{FWCFA}}(x)$ for $\lambda = 1, 2, \lambda_{\mathrm{cr}}$ and the corresponding exact solution provided in (1.2) are presented in Fig. 3(b), their pointwise *a posteriori* errors for several resolutions ($J = 2, \dots, 5$) and fixed $\xi^B = \frac{7}{2}$ are displayed in Fig. 3(c). Close examination of exponents of errors, the number of unknowns in SNLATE, number of iterations and the computational time for

FindRoot[·, ·] (presented in Cols. 2−20 of Table 2) reveal that the present scheme can provide all possible solutions (both lower and upper solutions, whenever they exist) of Bratu's problem for different choices of λ including its critical value λ_{cr} with high accuracy in significantly less computational time. These observations tempted us to exercise the proposed scheme for other Bratu-type IVP/BVP.

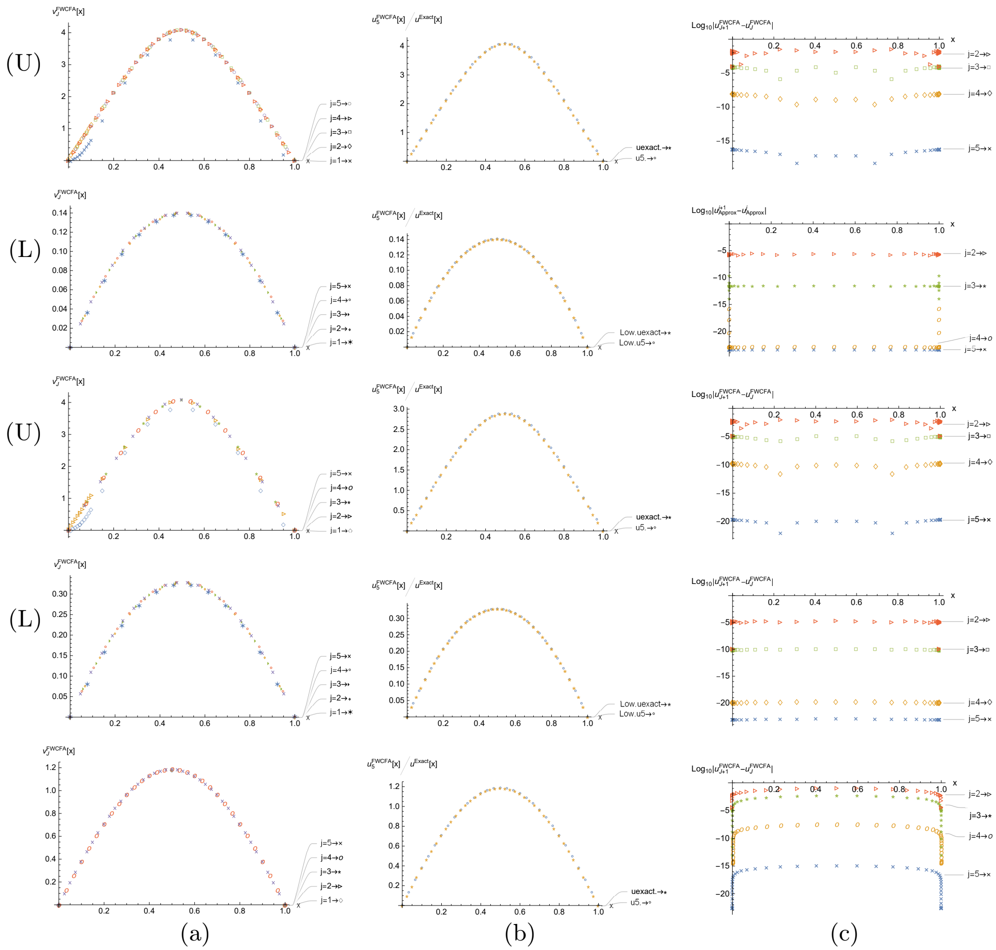


Fig. 3. Upper (U) and Lower (L) branch of (a) $v_J^{\text{FWCFA}}(x)$, $J = 1, \ldots, 5$ and (b) $u^{\text{Exact}}(x)$ and $u_5^{\text{FWCFA}}(x)$ given in (1.2) and (6.6) respectively and (c) pointwise error in $u_J^{\text{FWCFA}}(x)$ for $\lambda = 1$ (first two), $\lambda = 2$ (subsequent two) and $\lambda = \lambda_{\text{cr}}$ (last one) at different resolutions J

Table 2. For $\lambda = 1, 2, \lambda_{\text{cr}}$, the number of iterations for solving SNLATE, the computational time for solving SNLATE, the number of unknowns (in parentheses), n in $O(10^{-n})$ of *a posteriori* and L^∞-errors at $J = 4$ and $\xi^B = \frac{7}{2}$ in $u_J^{\text{FWCFA}}(x)$ and some other available methods for Bratu's problem (1.1)

λ	NI	CT	n																
			Present scheme		SGM	SSKCWA	OMMSLP	GA-ASM	TWM	INFCD	OIA	IFDM	FBCM	SLDCM	HOHWM	BCM	GWCM	ANN-SOS	SCA-HSCP
		in sec.	*a post.*	L^∞															
$u^{\text{Upper}}(x)(U)$ (boldface) and $u^{\text{Lower}}(x)(L)$ (italic in curly brackets) for $\lambda = 1$ and $\lambda = 2$.																			
1	**9**	**3**	**9**	**16**	**NA**	**NA**	**NA**	**NA**	**NA**	**11**	**NA**	**10**	**NA**	**NA**	**NA**	**NA**	**NA**	**NA**	**NA**
	{*6*}	{*1.70*}	{*23*}	{*23*}	{*13*}	{*16*}	{*16*}	{*8*}	{*12*}	{*14*}	{*7*}	{*10*}	{*13*}	{*14*}	{*15*}	{*7*}	{*7*}	{*6*}	{*NA*}
			(113)	(113)	(257)					(641)			(259)	(14)	(1024)				
2	**8**	**3.47**	**10**	**10**	**NA**	**NA**	**NA**	**NA**	**NA**	**10**	**NA**	**9**	**NA**	**NA**	**NA**	**NA**	**NA**	**NA**	**NA**
	{*7*}	{*1.70*}	{*20*}	{*20*}	{*12*}	{*12*}	{*16*}	{*8*}	{*9*}	{*13*}	{*4*}	{*9*}	{*12*}	{*15*}	{*14*}	{*5*}	{*4*}	{*5*}	{*16*}
			(113)	(113)	(257)					(641)			(259)	(14)	(1024)				
λ_{cr}	7	1.76	7	14	*NA*	*NA*	*NA*	10	*NA*	8	*NA*	*NA*	*NA*	*NA*	*NA*	*NA*	*NA*	*NA*	*NA*
			(113)	(113)						(641)									

NI: Number of iterations, CT: Computational time, NA: Not available, SGM: Sinc Galerkin method [43], SSKCWC: Spectral second kind Chebyshev wavelets algorithm [3], OMMSLP: Operational matrix method based on shifted Legendre polynomials [4], GA-ASM: Genetic algorithm and active-set method [42], TWM: Taylor wavelet method [30], INCFD: Iterative non-standard compact finite difference [25], OIA: Optimal iterative algorithm [46], IFDM: Iterative finite difference method [9], FBCM: Fourth order B-spline collocation method [47], SLDCM: Spectral Legendre's derivative collocation method [2], HOHWM: Higher order Haar wavelet method [63], BCM: Bernstein collocation method [51], GWCM: Gegenbauer wavelets collocation method [51], ANN-SOS: Artificial neutral network and symbolic organism search algorithm [5], SCA-HSCP: Spectral collocation algorithm via Hexic shifted Chebyshev polynomials [7].

6.1.2. *Bratu-type problems.* All the steps have also been exercised for problems (1.4)–(1.6) (discussed in the Introduction). To avoid the redundancy of steps, we present the expression of the SNLATE

(6.7)
$$2^{2J}\sum_{k\in\Lambda_J^\phi} C_{Jk}\phi^{S''}(l-k)-2^J\frac{\theta''\left(\frac{l}{2^J}\right)}{\theta'\left(\frac{l}{2^J}\right)}\sum_{k\in\Lambda_J^\phi} C_{Jk}\phi^{S'}(l-k)+\theta'\left(\frac{l}{2^J}\right)^2\bar{u}_0''\left(\frac{l}{2^J}\right) + \theta'\left(\frac{l}{2^J}\right)^2\bar{f}\left(\frac{l}{2^J},\bar{u}_0\left(\frac{l}{2^J}\right)+\sum_{k\in\Lambda_J^\phi} C_{Jk}\phi^{S}(l-k)\right)=0, \quad l\in\Lambda_J^\phi,$$

for the unknowns C_{Jk} and a few additional unknowns (listed in Col. 6 of Table 3) in $u_0(x)$. The inputs for the model equation, namely, IC/BC, exact solutions $u^{\text{Exact}}(x)$, and the known part $u_0(x)$ have been collected in Cols. 3−5 of Table 3 respectively. The equations for $v(x)$ are provided in Col. 7 of the table. These data were used to choose the input to solve the SNLATE mentioned above.

The function FindRoot[·, ·] with "WorkingPrecision → 80" has been exercised to obtain numerical solutions of SNLATE (6.7) with input $\{\{C_{Jk},0\}: k\in\Lambda_J^\phi\}$ for unknowns C_{Jk} for examples (1.4)−(1.6) uniformly. However, the known function $u_0(x)$ contains additional unknowns (listed in Col. 6 of Table 3). Consequently, the equations $\bar{v}'(-2^J\xi^B-1)=0$ and $\bar{v}'(2^J\xi^B+1)=0$ have been added to the SNLATE mentioned above with the corresponding inputs

$$\{\{u(1),\tfrac{1}{10}\},\{u'(1),\tfrac{1}{10}\}\}/\{\{u(0),\tfrac{1}{10}\},\{u(1),\tfrac{1}{10}\}\}/\{\{u(0),\tfrac{1}{10}\},\{u(1),\tfrac{1}{10}\}\}$$

for (1.4)/(1.5)/(1.6). We have provided the number of iterations and the computational cost for FindRoot[·, ·] in Cols. 4 and 5 of Table 4, respectively.

Subsequently, the approximate solutions of SNLATE just obtained have been used in (4.9), in the expression for $u_0(x)$ (Col. 5 of Table 3) and (4.19) to obtain $v_J(x)\equiv v_J^{\text{FWCFA}}(x)$, $u_0^{\text{Approx}}(x)$, and $u_J^{\text{FWCFA}}(x)$, respectively. The qualitative behavior of $v_J^{\text{FWCFA}}(x)$, $u_J^{\text{FWCFA}}(x)$ and $u^{\text{Exact}}(x)$ and pointwise *a posteriori* error in $u_J^{\text{FWCFA}}(x)$ ($J=2,\ldots,5$) (for the problems listed in Table 3) are presented in Figs. 4(a)–(c).

To examine the reliability of the estimate of *a posteriori* error in the FWCFA of the solution, n in the order $(O(10^{-n}))$ of *a posteriori* error evaluated by using solutions of SNLATE (6.7) in (5.8) and the L^∞-error obtained by using $u_J^{\text{FWCFA}}(x)$ and $u^{\text{Exact}}(x)$ (provided in Col. 4 of Table 3) in (5.6), are provided in Cols. 6 and 7 of corresponding rows of Table 4, respectively. Their comparison reveals that the estimate of *a posteriori* error suggested in (5.8) is precise even in the case of IC/NBC/RBC where additional unknowns appear in $u_0(x)$ and SNLATE.

A close examination of the computational time provided in Col. 5 and n of the *a posteriori* error provided in Col. 6 of Table 4 reveals uniformity of efficiency of the scheme for Bratu-type IVP/BVP.

We have also collected the n in approximate solutions obtained by other methods in Cols. 8−14 of the table. Comparison of exponents n establishes the better efficiency of the FWCFA scheme proposed here over the other approximation schemes for IVP (1.4) and BVP (1.5). The comparison of n in Cols. 6 and 13 indicates that the MS2CPSM achieves higher accuracy than our scheme at lower values of J. However, the accuracy of our scheme is better for $J \geq 5$ onward. Moreover, the FWCFA scheme has significantly less computational cost since the non-linear part in each element (6.7) of SNLATE of the present scheme contains a single unknown coefficient

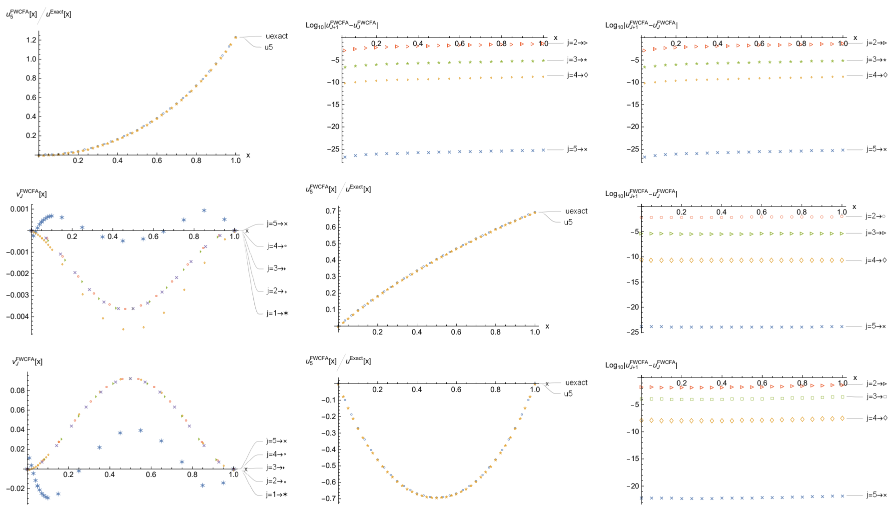
$$e^{\sum_{k\in\Lambda_J^{\phi}} C_{Jk}\phi^S(l-k)} = e^{C_{Jl}},$$

due to interpolating property (i), Section 2, instead of $M+1$ terms in $e^{\sum_{i=0}^{M} k_i \mathfrak{N}_i(x_l)}$ ((42), (47) and (48)) in MS2CPM.

(1.4)

(1.5)

(1.6)



(a) (b) (c)

Fig. 4. Plots of (a) $v_J^{\text{FWCFA}}(x)$ for $J = 1, \ldots, 5$, (b) $u_5^{\text{FWCFA}}(x)$ from (6.13) and the exact solutions, and (c) pointwise *a posteriori* errors in $u_J^{\text{FWCFA}}(x)$ at various resolution J for (1.4)–(1.6) (listed in Table 3).

Table 3. Equations for $u(x)$ for Bratu-type IVP/BVP (1.4)–(1.6), IC/BC, the exact solution, the known part $u_0(x)$, additional variables in SNLATE, and the equation for $v(x)$.

Ex.	Eq. for $u(x)$	IC/BC	$u^{\text{Exact}}(x)$	$u_0(x)$	Unknowns	Eq. for $v(x)$
(1.4)	$u'' - 2e^u = 0$	$u(0) = 0$, $u'(0) = 0$	$-2\log(\cos x)$	$u(1)(3x^2 - 2x^3) + u'(1)(x^3 - x^2)$	$u(1)$, $u'(1)$	$v''(x) + u(1)(6 - 12x) + u'(1)(6x - 2) - 2e^{(u_0+v)} = 0$
(1.5)	$u'' + e^{-2u} = 0$	$u'(0) = 1$, $u'(1) = \frac{1}{2}$	$\log(1 + x)$	$u(0)(1 - 3x^2 + 2x^3) + u(1)(3x^2 - 2x^3) + (x - 2x^2 + x^3) + \frac{1}{2}(x^3 - x^2)$	$u(0)$, $u(1)$	$v''(x) + u(0)(-6 + 12x) + u(1)(6 - 12x) + (6x - 4) + \frac{1}{2}(6x - 2) + e^{-2(u_0+v)} = 0$
(1.6)	$u'' - \pi^2 e^u = 0$	$u(0) + 2u'(0) = -2\pi$, $2u(1) - u'(1) = -\pi$	$-\log[2\{\cos\pi(\frac{x}{2} - \frac{1}{2})\}^2]$	$\frac{u(0)}{2}(2 - x - 4x^2 + 3x^3) + u(1)x^2 + \pi x(x - 1)$	$u(0)$, $u(1)$	$v''(x) + u(0)(-8 + 18x) + 2(\pi + u(1)) - \pi^2 e^{(u_0+v)} = 0$

Table 4. For (1.4)–(1.6) at different resolutions J, boundary ξ^B, the number of iterations (NI), computational time (CT), exponents n in $O(10^{-n})$ of *a posteriori* and L^∞-errors in $u_J^{\text{FWCFA}}(x)$ and those of some other available methods are cited here.

Ex.	ξ^B	J	NI	CT in sec.	n								
					Present scheme		LWM	ADM	GA-ASM	TWM	ANN-SOS	MS2CPSM	SCA-HSCP
					a post.	L^∞							
(1.4)	$\frac{4}{4}$	2	6	0.06	2	2	2	NA	7	8	5	NA	14
	$\frac{8}{4}$	3	6	0.29	6	6							
	$\frac{10}{4}$	4	6	1.92	9	9							
	$\frac{14}{4}$	5	6	17.78	23	23							
(1.5)	$\frac{6}{4}$	2	8	0.06	2	2	NA	5	NA	NA	NA	NA	NA
	$\frac{8}{4}$	3	7	0.34	7	7							
	$\frac{10}{4}$	4	7	2.81	10	10							
	$\frac{14}{4}$	5	7	20.76	25	25							
(1.6)	$\frac{6}{4}$	2	6	0.09	2	2	NA	NA	NA	NA	NA		NA
	$\frac{8}{4}$	3	6	0.33	5	5						7	
	$\frac{10}{4}$	4	6	2.53	9	9						10	
	$\frac{14}{4}$	5	6	21.93	22	22						13	

LWM: Legendre wavelet methods [68], ADM: Adomian decomposition method [55],
GA-ASM: Genetic algorithm and active-set method [42], TWM: Taylor wavelet methods [30],
ANN-SOS: Artificial neutral network and symbolic organism search algorithm [5],
MS2CPSM: Modified second-kind shifted Chebyshev polynomials spectral method [50],
SCA-HSCP: Spectral collocation algorithm via Hexic shifted Chebyshev polynomials [7].

6.2. Troesch's problem. We consider here the Troesch problem, that is, (1.7) with non-homogeneous DBC

$$u(0) = 0, \quad u(1) = 1. \tag{6.8}$$

Despite the availability of several approximation schemes and numerical methods [36, 19, 31, 22, 16, 17, 72, 33] to obtain an approximate solution to this problem, the proposed scheme has been exercised here in order to examine its efficiency further. Following the steps described in Section 4, the known part $u_0(x)$ is taken to be

$$u_0(x) = x, \tag{6.9}$$

free from additional unknowns so that (4.4) for $v(x)$ becomes

$$v''(x) - \mu \sinh(\mu(x + v)) = 0, \quad x \in [0, 1], \tag{6.10}$$

with homogeneous DBC

$$v(0) = 0 = v(1). \tag{6.11}$$

Then the SNLATE for the unknowns C_{Jk} is

(6.12)

$$2^{2J} \sum_{k \in \Lambda_J^\phi} C_{Jk} \phi^{S''}(l-k) - 2^J \frac{\theta''(\frac{l}{2^J})}{\theta'(\frac{l}{2^J})} \sum_{k \in \Lambda_J^\phi} C_{Jk} \phi^{S'}(l-k) - \mu \theta'^2\left(\frac{l}{2^J}\right) \times \sinh\left(\mu\left(\theta\left(\frac{l}{2^J}\right) + \sum_{k \in \Lambda_J^\phi} C_{Jk} \phi^S(l-k)\right)\right) = 0, \quad l \in \Lambda_J^\phi.$$

The inputs $\{\{C_{Jk}, e^{-(\frac{k}{2^J})^2}\} : k \in \Lambda_j^\phi\}$ for the unknowns C_{Jk} have been used to obtain their approximate solution by using FindRoot[·, ·]. The values of $v(x_J)$ $(= \bar{v}(\frac{k}{2^J}) \simeq C_{Jk})$ obtained here for several resolutions J of the approximation space are depicted in Fig. 5(a) for the parameters $\mu = 1, 2$ in (6.10). Then, a FWCFA solution for Troesch's problem can be found as

$$u_J^{\text{FWCFA}}(x) = x + \bar{v}_J^{\text{FWCFA}}(\Theta(x)). \tag{6.13}$$

The plots of $u_5^{\text{FWCFA}}(x)$ and $u^{\text{Exact}}(x)$ (provided in (1.9)) for $\mu = 1$ and 2 at the resolution $J = 5$ are depicted in Fig. 5(b); the pointwise *a posteriori* errors in the approximate solutions at the nodes $l/2^J, l \in \Lambda_J^\phi$ for $J = 2, \ldots, 5$ and fixed $\xi_J^B = \frac{7}{2}$ are presented in Fig. 5(c). Close inspection of the plots provided in Fig. 5(c) reveals that the *a posteriori* error of the FWCFA of the solution begins with $O(10^{-5})$ and rapidly reduces to $O(10^{-24})$ as the resolution of the approximation space increases from 2 to 5. A comparison of the number of unknowns appearing in SNLATE (in parentheses in Table 5) and the global error $(\text{Err}_J^{\text{a post}})$ and the pointwise *a posteriori* errors with the errors in the approximate solutions obtained by other methods provided in Table 5 shows that the FWCFA scheme is capable of providing a highly accurate approximate solution of the Troesch problem at insignificant computational effort, as is evident from the values provided in Col. 2 of Table 5.

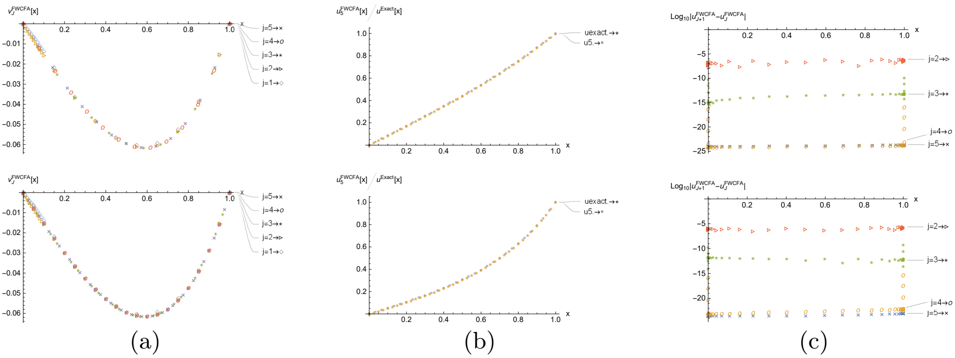


Fig. 5. Curves for (a) $v_J^{\text{FWCFA}}(x), J = 1, \ldots, 5$, (b) $u_5^{\text{FWCFA}}(x)$ given in (6.13) and its exact solution in (1.9) and (c) pointwise *a posteriori* errors in $u_J^{\text{FWCFA}}(x)$ at different resolution J for $\mu = 1$ (row 1) and $\mu = 2$ (row 2)

Table 5. Iteration counts, computational time (in FindRoot[·, ·]), number of nodes, and error exponents n in $O(10^{-n})$ for *a posteriori* and L^∞-errors in $u_J^{\text{FWCFA}}(x)$ with $\mu = 1$ (bold) and $\mu = 2$ (italic in braces), along with results for other methods (ordered chronologically) for the Troesch problem.

NI	CT	n						
		Present scheme		SGM	VSM	DESGM	TWM	CTBSM
	in sec.	*a post.*	L^∞					
1{*2*}	**1**{*2*}	**1**{*2*}	**1**{*2*}			$\mu = 1$		
6{*7*}	**0.06**{*0.05*}	**7**{*6*}	**6**{*7*}	5		7		
		(21)	(21)	(21)	5	(21)	5	7
6{*7*}	**0.27**{*0.28*}	**12**{*11*}	**14**{*10*}	6		10		
		(49)	(49)	(41)	(65)	(31)	(7)	(23)
6{*7*}	**1.72**{*1.75*}	**24**{*23*}	**25**{*20*}					
		(113)	(113)					

SGM: Sinc Galerkin method [72], VSM: Variational spline method [33],
DESGM: Double exponential Sinc Galerkin method [37], TWM: Taylor wavelet method [53],
CTBSCM: Cubic trigonometric B-spline method [21].

7. Conclusion. A finite Whittaker cardinal function approximation scheme in the sinc basis in the set up of Shannon–Kotelnikov multiresolution analysis of $L^2(\mathbb{R})$ has been applied here to obtain accurate approximate solutions to Bratu-type and Troesch-type problems involving various initial or boundary conditions appearing in the studies of several challenging scientific and engineering problems. The advantage of this scheme is its simplicity and low computational cost. This has been achieved through (i) a judicious splitting of the unknown solution $u(x)$ into a known polynomial $u_0(x)$ accommodating an initial or boundary condition plus an unknown part $v(x)$ in the Paley–Wiener class of Shannon–Kotelnikov multiresolution analysis and (ii) stretching the compact domain Ω ($\equiv [a, b] \subset \mathbb{R}$) to $\mathbb{R}$ with the help of a double exponential map $\theta(\xi)$ (in (3.1)), narrowing the width of effective support of the unknown function $\bar{v}(\xi)$ ($\equiv v(\theta(\xi))$). Bounds on the resolution (J) of the approximation space and the boundary of the effective support (of $\bar{v}(\xi)$) have also been provided to suggest their inputs for obtaining limits of the finite sum in the finite Whittaker cardinal function approximation accurate up to a desired order of accuracy ($O(10^{-n_d})$). A formula ((5.7)/(5.8)) for the *a posteriori* error is also provided to assess the precision of the approximate solutions whenever the exact solution to the model is not readily available.

The results presented here demonstrate that the finite Whittaker cardinal function approximation scheme is highly efficient for obtaining approximate solutions with less computational effort. It may be worth mentioning that another important aspect of the present approximation scheme is that this method provides some additional information, e.g., values/slopes of the solution at the boundary depending on the data provided in the initial or boundary conditions simultaneously, which may be directly used to obtain approximate values of physical observations whenever relations among those are available. The performance of the finite Whittaker cardinal function approximation scheme for the problems considered here encouraged us to try its extension to other mathematical models involving quasilinear ordinary differential equations, e.g., non-local boundary value problems, boundary value problems involving systems of non-linear ordinary differential equations of second and higher orders, non-linear partial differential equations [48, 20]. This work in progress will be reported elsewhere.

Appendix A. Outline of the proof of Theorem 5.2. We recall here formula (3.3) (for $r = 0$)

$$\left|\bar{v}(\xi) - \sum_{k=-M}^{M} \bar{v}\left(\frac{k}{2^J}\right)\phi^S_{Jk}(\xi)\right| \leq C_2 e^{-2^J \pi d} + C_3 2^J \frac{e^{-\frac{\delta\pi}{2}e^{\xi^B}}}{e^{\xi^B}}, \tag{A1}$$

involving the constants $C_2 = (1+e)\frac{C_1}{\pi d}, C_3 = \frac{4C(b-a)^{2\delta}}{\pi\delta}$, where C and C_1 appear in Definition 3.1 and Lemma 3.2 respectively. (A1) suggests that the desired order $O(10^{-n})$ of accuracy may be achieved if

$$C_2 e^{-2^J \pi d} + C_3 2^J \frac{e^{-\frac{\pi\delta}{2}e^{\xi^B}}}{e^{\xi^B}} \leq 10^{-n}. \tag{A2}$$

To obtain n-dependence of two parameters, the resolution J of the approximation space and the boundary ξ^B of the effective support of $\bar{v}(\xi)$, we split the above inequality into

(A3a) $$C_2 e^{-2^J \pi d} \leq \frac{10^{-n}}{2}$$

and

(A3b) $$C_3 2^J \frac{e^{-\frac{\pi\delta}{2} e^{\xi^B}}}{e^{\xi^B}} \leq \frac{10^{-n}}{2}.$$

Inequality in (A3a) can be rearranged as

$$J \geq \log_2 \left[\frac{1}{\pi d} \log \left(\frac{2(e+1)C_1 10^n}{\pi d} \right) \right].$$

On the other hand, a rearrangement of (A3b) followed by the substitution of $C_3 = \frac{4C(b-a)^{2\delta}}{\pi\delta}$ yields

$$\xi^B \geq \log \left(\frac{2\log(2^{2+J} \times 10^n (b-a)^{2\delta} C)}{\pi\delta} \right).$$

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