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Failure of the product and subgroup theorems for the covering dimension of topological groups

by

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Abstract. Strongly zero-dimensional topological groups G_1, G_2, and G such that $G_1 \times G_2$ has positive covering dimension and G contains a closed subgroup of positive covering dimension are constructed. Moreover, all finite powers of G_1 are Lindelöf and G_2 is second-countable. An example of a Lindelöf strongly zero-dimensional space X whose free, free Abelian, and free Boolean topological groups have positive covering dimension is also given.

Introduction. This paper is concerned with the covering dimension of topological groups. There are two definitions of covering dimension, in the sense of Čech (in terms of open covers) and in the sense of Katětov (in terms of cozero covers); these dimensions coincide for normal spaces. To differentiate them, following [7], we denote the former by dim and the latter by $\dim_0$; note that in most papers cited in what follows, the Katětov covering dimension is denoted by dim rather than by $\dim_0$.

For convenience, we assume all topological spaces and groups considered in this paper to be Tychonoff and often refer to topological spaces simply as spaces.

In the class of topological groups many topological properties behave better than in the class of topological spaces: the first axiom of countability implies pseudometrizability, pseudocompactness is multiplicative, local compactness implies normality, and so on. Dimension properties are no exception; for example, the three basic dimensions ind, Ind, and $\dim_0$ (and hence dim) of any locally compact topological group coincide [23], while those of a general compact space may differ (see, e.g., [13]). The present pa-

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per is concerned with three fundamental topics in the theory ofthe covering dimension of topological groups, namely, the behavior of covering dimension under multiplication, the covering dimension of free topological groups, and the monotonicity of covering dimension with respect to taking subgroups. We show that in all these respects things are not as good as might be expected: the covering dimension of topological groups neither is monotone with respect to subgroups nor satisfies the product inequality, and strong zero-dimensionality is not preserved by the free topological group functor.

It has long been known that the "logarithmic product law" $\dim_0(X \times Y) = \dim_0 X + \dim_0 Y$ may fail for very nice spaces: in the 1930 paper [25] Pontryagin constructed two-dimensional compact metrizable spaces K and L for which $\dim(K \times L) = 3 \neq \dim K + \dim L$. Even the inequality $\dim_0(X \times Y) \leq \dim_0 X + \dim_0 Y$, which is commonly referred to as the *product theorem for* $\dim_0$, is not generally true (see, e.g., [7, p. 205]), although it holds in some special cases (see [15, 19, 27]). However, in the class of topological groups, the situation is much better. First, the product theorem holds for all locally compact groups. This follows from the much more general identity $\dim_0 G = \dim_0 H + \dim_0 G/H$, where H is a closed subgroup of a topological group G and G/H denotes the coset space of G by H; it was proved for second-countable locally compact groups G by Yamanoshita [45] and for general locally compact groups G by Pasynkov [24]. For non-locally compact groups, the product theorem may fail: the *Erdős space* Q, that is, the subspace of all rational sequences in the Hilbert space ℓ^2, is a second-countable topological group for which $\dim Q = \dim Q^2 = 1 < 2 \dim Q$ (see [12]). However, no examples of the failure of the product theorem for topological groups have been known until now. Moreover, in [31, Theorem 3.3] Shakhmatov proved that $\dim_0(G \times H) \leq \dim_0 G + \dim_0 H$ for any totally bounded (i.e., precompact; see, e.g., [6, Corollary 3.7.17]) topological groups G and H. In the same paper he asked whether $\dim_0(G \times H) \leq \dim_0 G + \dim_0 H$ for arbitrary topological groups G and H. Various versions of this question can be found in [5]. For example, it is quite natural to ask whether the product theorem is true for ω-narrow groups G and H, because ω-narrowness is in a sense the closest generalization of total boundedness (a group is *ω-narrow* if it can be covered by countably many translates of any neighborhood of the identity element); this question can be found in [31] and [5, Question 6.14]. Note that both G and H are ω-narrow if and only if so is $G \times H$, so that the above question is equivalent to the question of whether the product theorem holds under the assumption that $G \times H$ is ω-narrow. Under the somewhat stronger assumption that $G \times H$ is Lindelöf, the answer is yes, since the product theorem is true for any spaces with strongly paracompact product [17]. However, the assumption that each of the groups G and H is Lindelöf is not sufficient. We construct two topological groups G and H

such that all finite powers of G are Lindelöf, H is second-countable, and $\dim_0(G \times H) > \dim_0 G + \dim_0 H = 0$, thereby answering (in the negative) Shakhmatov's question and [5, Questions 6.9 and 6.14].

A modification of this example gives a negative answer to Arkhangel'skii's old question of whether the free topological group $F(X)$ of any *strongly zero-dimensional space* X (that is, a space X with $\dim_0 X = 0$) is strongly zero-dimensional (see [4, p. 157/964], [5, Problem 8.17]). Arkhangel'skii himself showed in [2] that the answer is yes for metrizable spaces; in the same paper he proved that any topological group is a quotient of a strongly zero-dimensional group. Tkachenko supplemented this result with the theorem that every topological group G is a quotient of a *zero-dimensional group* H (that is, a group H with $\operatorname{ind} H = 0$) of the same weight as G [40, Theorem 1.1]. It is also known that $\operatorname{ind} F(X) = 0$ for any X with $\dim_0 X = 0$ [32, 33] (the same is true for free Abelian topological groups [39]). As mentioned, in what follows, we construct an example of a Lindelöf space X for which $\dim_0 X = 0$ but the covering dimensions of the free, free Abelian, and free Boolean topological groups are positive. It is appropriate to also mention here that a similar example for small inductive dimension exists as well; it was constructed by Shakhmatov in [29].

Note that if H is a closed subgroup of a locally compact group G, then both $\dim_0 H$ and $\dim_0 G/H$ do not exceed $\dim_0 G$, because $\dim_0 G = \dim_0 H + \dim_0 G/H$. However, it follows from Arkhangel'skii's and Tkachenko's theorems cited above that, in the general case, the covering and small inductive dimensions of topological groups are not monotone with respect to taking quotient groups. The question of whether the covering dimension is monotone with respect to taking subgroups is much more difficult to answer. Shakhmatov proved that the *subgroup theorem* $\dim_0 H \le \dim_0 G$ for the covering dimension $\dim_0$ of a topological group G and its subgroup H holds if G is locally pseudocompact [30, Corollary 3.13] or H is totally bounded [30, Theorem 3.5]. Tkachenko improved the latter result by proving that the subgroup theorem for $\dim_0$ also holds whenever H is $\mathbb{R}$-factorizable, in particular, whenever it is Lindelöf or embeds in a product of Lindelöf Σ-groups as a dense subgroup [42, Theorem 2.7]. However, the question of whether $\dim_0 H \le \dim_0 G$ for any topological group G and any subgroup H of G has remained unanswered until now. Shakhmatov asked this question in [30, Question 6.7] with a mention that it "was claimed to be an old one by Zambakhidze". In [31, Problem 2.1], he asked the same question for an arbitrary topological group G and its ω-narrow subgroup H. Furthermore, Tkachenko included this question for an arbitrary topological group G and its arbitrary or closed subgroup H in [43, Problem 6.9] as a problem "mentioned by many specialists"; it was also repeated in [5, Question 6.1]. In the present paper, we answer it by constructing a topological group G with $\dim_0 G = 0$

which contains a closed subgroup H of positive covering dimension $\dim_0$. Moreover, H is ω-narrow, being the product of two Lindelöf groups.

1. Preliminaries. Suppose we are given a set X and a family $\mathscr{F}$ of its subsets. If there exists an integer $n \geq -1$ such that every point of X belongs to at most $n+1$ elements of $\mathscr{F}$, then the smallest such n is called the *order* of $\mathscr{F}$; otherwise, the order of $\mathscr{F}$ is infinite. Given a topological space X, the *Čech covering dimension* $\dim X$ of X is the smallest integer n for which any finite open cover of X has a finite open refinement of order n, provided that such an integer exists; if it does not exist, then $\dim X$ is infinite. The definition of the *Katětov covering dimension* $\dim_0 X$ is similar but uses covers by cozero sets: $\dim_0 X$ equals the smallest integer n for which any finite cozero cover of X has a finite cozero refinement of order n if such an integer exists, and is infinite otherwise. For normal spaces, the dimensions dim and $\dim_0$ coincide (see, e.g., [7, Proposition 11.2]). A space X for which $\dim_0 X = 0$ is said to be *strongly zero-dimensional.*

In what follows, we occasionally mention the *small* and *large inductive dimensions* $\operatorname{ind} X$ and $\operatorname{Ind} X$ of a space X; we refer the reader to [7] or [11] for their definitions and properties. For our purposes, it is enough to know that $\operatorname{ind} X = 0$ if and only if X has a base consisting of clopen sets, $\operatorname{Ind} X = 0$ if and only if $\dim_0 X = 0$, and any X with $\operatorname{Ind} X < \infty$ is normal. A space X with $\operatorname{ind} X = 0$ is said to be *zero-dimensional.* Obviously, any strongly zero-dimensional space is zero-dimensional (recall that we consider only Tychonoff spaces).

A state-of-the-art presentation of the dimension theory of topological spaces is given in the highly recommended book [7] of Michael Charalambous.

Remark 1. There are many examples of spaces X for which $\dim X > \dim_0 X$ (see, e.g., [7]), but the author is unaware of any example of a space X for which $\dim_0 X > \dim X$. However, even if such examples exist, we have $\dim_0 X = 0$ whenever $\dim X = 0$, because any finite disjoint open cover of X consists of clopen sets, which are obviously cozero.

We use the notation $\mathbb{R}$ for the set of real numbers, $\mathbb{N}$ for the set of positive integers, and ω for the set of non-negative integers. By $\oplus$ we denote the topological sum of spaces and by $|A|$ the cardinality of a set A.

A subset Y of a space X is said to be *C-embedded* in X if any real-valued continuous function on Y has a continuous extension to X, and Y is *z-embedded* in X if every zero set of Y is the trace on Y of some zero set of X. A topological space admitting a coarser metrizable topology is said to be *submetrizable.*

To distinguish groups without topology from topological groups, we refer to the former as *abstract groups.*

The topology of a topological group G is *linear* if the open subgroups of G form a base of neighborhoods of the identity element in G. Clearly, all groups with linear topology are zero-dimensional, because any open subgroup of any topological group is closed.

A *Boolean group* is a group in which all elements are of order 2. All such groups are Abelian; moreover, all of them are vector spaces over the two-element field $\mathbb{F}_2$. Therefore, any Boolean group B has a *basis* X, that is, there exists a set $X \subset B$ such that X generates B as a vector space (or, equivalently, as a group) and any function from X to a vector space V over $\mathbb{F}_2$ (or, equivalently, to a Boolean group V) extends to a linear function (homomorphism) $B \to V$. In other words, B is the free group on X in the variety of all Boolean groups. Given a set X, the Boolean group with basis X is nothing but the set $[X]^{<\omega}$ of finite subsets of X endowed with the operation of symmetric difference, which plays the role of addition. The zero element is the empty set. Each point $x \in X$ is identified with the singleton $\{x\}$.

For a topological space X with topology τ, the *free topological group* $F(X)$ (the *free Abelian topological group* $A(X)$, the *free Boolean topological group* $B(X)$) is the topological (topological Abelian, topological Boolean) group containing X as a subspace, generated by X, and defined by the *universal property* that any continuous function from X to a topological group G (topological Abelian group G, topological Boolean group G) extends to a continuous homomorphism $F(X) \to G$ ($A(X) \to G$, $B(X) \to G$). In other words, this is the abstract free (free Abelian, free Boolean) group on the set X endowed with the finest group topology inducing the topology τ on X. Basic information about the groups $F(X)$ and $A(X)$ can be found in [34] and [6]; the groups $B(X)$ were studied in [36].

If X is a zero-dimensional space, then the *free linear topological group* $F^{\mathrm{lin}}(X)$, the *free Abelian linear topological group* $A^{\mathrm{lin}}(X)$, and the *free Boolean linear topological group* $B^{\mathrm{lin}}(X)$ are also defined [36, Theorem 2]. Their definitions are similar to those of the free, free Abelian, and free Boolean topological groups of X, the only difference being that their topologies are required to be linear and that continuous functions from X to G must extend to continuous homomorphisms only for G with linear topology. For any zero-dimensional X, the groups $F^{\mathrm{lin}}(X)$, $A^{\mathrm{lin}}(X)$, and $B^{\mathrm{lin}}(X)$ are Hausdorff and contain X as a closed subspace. For more details concerning free linear topological groups, see [36].

Remark 2. Thanks to the universal property, $A(X)$ is a continuous homomorphic image of $F(X)$, $B(X)$ is a continuous homomorphic image of $A(X)$, and $F^{\mathrm{lin}}(X)$, $A^{\mathrm{lin}}(X)$, and $B^{\mathrm{lin}}(X)$ are the images of $F(X)$, $A(X)$, and $B(X)$, respectively, under the continuous identity isomorphisms.

The topology of any topological group G with identity element 1 is induced by the natural *two-sided group uniformity* $\mathscr{V}_G$ with base

$$\{\{(g,h) \in G \times G : h \in gV \cap Vg\} : V \text{ is a neighborhood of } 1\}$$

(see [11, Example 8.1.17]). A topological group G is said to be *Raikov complete* if $\mathscr{V}_G$ is complete. It is well known that G is Raikov complete if and only if it is closed in any topological group containing G as a topological subgroup [28]. More details on Raikov complete groups can be found in [6, Section 3.6] (see also [28]).

A topological group G is said to be $\mathbb{R}$-*factorizable* if, for every continuous function $f\colon G \to \mathbb{R}$, there exists a continuous homomorphism $h\colon G \to H$ to a second-countable topological group H and a continuous function $g\colon H \to \mathbb{R}$ with $f = g \circ h$. This very useful notion was introduced by Tkachenko [41], who showed, among other things, that any Lindelöf group is $\mathbb{R}$-factorizable [41, Assertion 1.1] and that any $\mathbb{R}$-factorizable group G is ω-*narrow*, that is, for every neighborhood U of the identity element in G, there exists a countable set $A \subset G$ for which $A \cdot U = G$ (see [6, Proposition 8.1.3]).

In what follows, we repeatedly use the following known theorems.

Theorem A (see, e.g., [6, Corollary 7.1.18]). *The free topological group $F(X)$ of a space X is Lindelöf if and only if X^n is Lindelöf for each $n \in \mathbb{N}$.*

Therefore, if X^n is Lindelöf for each $n \in \mathbb{N}$, then $A(X)$, $B(X)$, $F^{\mathrm{lin}}(X)$, $A^{\mathrm{lin}}(X)$, and $B^{\mathrm{lin}}(X)$ are Lindelöf.

Theorem B ([30, Theorem 3.1]; see also [6, Theorem 8.8.4]). *Suppose that G is a zero-dimensional $\mathbb{R}$-factorizable group, H is a second-countable topological group, and $f\colon G \to H$ is a continuous homomorphism. Then there exists a zero-dimensional second-countable topological group G' and continuous surjective homomorphisms $g\colon G \to G'$ and $h\colon G' \to H$ such that $f = g \circ h$.*

Theorem C ([20, 21]; see also [7, Theorem 11.22]). *If Y is a z-embedded subspace of a topological space X, then $\dim_0 Y \le \dim_0 X$.*

The definitions and facts used in this paper without reference can be found in [11] or [10].

2. Failure of the product theorem for the covering dimension of topological groups

Theorem 1. *There exist Boolean* (*and hence Abelian*) *topological groups G_1 and G_2 with the following properties:*

(1) *G_1^n is Lindelöf and submetrizable for every $n \in \mathbb{N}$;*
(2) *G_2 is second-countable;*

(3) *the topologies of* G_1 *and* G_2 *are linear;*
(4) $\dim_0 G_1 = \dim G_1 = 0$ *and* $\dim_0 G_2 = \dim G_2 = 0$*;*
(5) $\dim_0(G_1 \times G_2) > 0$ *and* $\dim(G_1 \times G_2) > 0$.

To prove the theorem, we need two lemmas.

LEMMA 1. *If a space* X *is a retract of a topological group* G, *then it is a retract of the free topological group* $F(X)$. *If* G *is Abelian* (*Boolean*), *then* X *is a retract of the free Abelian topological group* $A(X)$ (*of the free Boolean topological group* $B(X)$). *If the topology of* G *is linear, then* X *is a retract of the free linear topological group* $F^{\mathrm{lin}}(X)$*; if, in addition,* G *is Abelian* (*Boolean*), *then* X *is a retract of* $A^{\mathrm{lin}}(X)$ (*of* $B^{\mathrm{lin}}(X)$).

Proof. Let $r\colon G \to X$ be a retraction. Then the restriction $r|_{\langle X\rangle}$ of r to the subgroup $\langle X\rangle$ of G generated by X is a retraction as well, because $X \subset \langle X\rangle$. By the definition of $F(X)$ the identity map $\mathrm{id}_X\colon X \to X \subset \langle X\rangle$ extends to a continuous homomorphism $h\colon F(X) \to \langle X\rangle$. Clearly, $r|_{\langle X\rangle} \circ h$ is a retraction.

In the cases where G is Abelian or Boolean and where the topology of G is linear, the argument is similar. ■

LEMMA 2. *Every second-countable space* X *which is a retract of a topological group* G *is a retract of a topological group* H *with the following properties:*

(1) H *is second-countable;*
(2) *if* G *is Abelian or Boolean, then so is* H*;*
(3) *if* X *is zero-dimensional, then so is* H*;*
(4) *if* G *is Abelian* (1) *and its topology is linear, then so is the topology of* H.

Proof. Suppose that a second-countable space X is a retract of a topological group G. By Lemma 1, X is a retract of the free topological group $F(X)$; let r be a retraction $F(X) \to X$. According to Theorem A, $F(X)$ is Lindelöf and hence $\mathbb{R}$-factorizable. By [41, Assertion 1.1] there exists a second-countable group H, a continuous surjective homomorphism $h\colon F(X) \to H$, and a continuous map $f\colon H \to X$ for which $r = f \circ h$. Let $Y = h(X)$. Since the restriction $r|_X$ is a homeomorphism, it follows that so are both restrictions $h|_X$ and $f|_Y$. Clearly, $(f|_Y)^{-1} \circ f\colon H \to Y$ is a retraction. Thus, X is homeomorphic to a retract of H.

If G is Abelian or Boolean, then we render H Abelian or Boolean by replacing $F(X)$ with $A(X)$ or $B(X)$, respectively, in the above argument. The groups $A(X)$ and $B(X)$ are Lindelöf by Theorem A.

If X is zero-dimensional (and hence strongly zero-dimensional, being second-countable), then so are $F(X)$ [1, Proposition 1] (see also [6, The-

(1) This assumption is made for simplicity, it can be dropped.

orem 7.6.16]), $A(X)$ [39], and $B(X)$ [36, Theorem 8]. Thus, in this case, Theorem B applies, according to which the group H can be made zero-dimensional.

Suppose that G is Abelian and its topology is linear. Then X is a retract of the free Abelian linear topological group $A^{\mathrm{lin}}(X)$ (by Lemma 1). Let $r\colon A^{\mathrm{lin}}(X) \to X$ be a retraction. By Theorem A, $A^{\mathrm{lin}}(X)$ is Lindelöf. As above, applying [41, Assertion 1.1], we find a second-countable group $\widetilde{H}$, a continuous surjective homomorphism $\tilde{h}\colon A^{\mathrm{lin}}(X) \to \widetilde{H}$, and a continuous map $\tilde{f}\colon \widetilde{H} \to X$ for which $r = \tilde{f}\circ\tilde{h}$. Note that $\widetilde{H}$ is Abelian. Let $\{U_n : n \in \omega\}$ be a base of neighborhoods of zero in $\widetilde{H}$. For each n, $\tilde{h}^{-1}(U_n)$ contains an open subgroup A_n of $A^{\mathrm{lin}}(X)$; we set $H_n = \tilde{h}(A_n)$. The subgroups H_n form a subbase of neighborhoods of zero for some group topology on $\widetilde{H}$; let H be $\widetilde{H}$ with this new topology. The new topology is finer than the old one. Therefore, the map $f\colon H \to X$ coinciding with $\tilde{f}$ as a map of sets is continuous. The surjective homomorphism $h\colon A^{\mathrm{lin}}(X) \to H$ coinciding with $\tilde{h}$ as a map of abstract groups is continuous as well, because the preimage of any basic neighborhood of zero in H contains an open neighborhood of zero in $A^{\mathrm{lin}}(X)$. The group H is second-countable, because it is metrizable (being first-countable) and Lindelöf (being a continuous image of the Lindelöf group $A^{\mathrm{lin}}(X)$), and X is a retract of H by the same considerations as in the second paragraph of this proof.

If G is Boolean, then H can be rendered Boolean by considering $B^{\mathrm{lin}}(X)$ instead of $A^{\mathrm{lin}}(X)$. ■

Proof of Theorem 1. Our construction of the topological groups G_1 and G_2 is based on Charalambous' modification of Przymusiński's construction in [26] of a strongly zero-dimensional Lindelöf space whose square is normal but not strongly zero-dimensional. Namely, in [7, proof of Theorem 27.5] subsets S, S_1 and S_2 of the Cantor set C and topologies τ_1 and τ_2 on C with certain properties were defined. We put $C_1 = (C, \tau_1)$ and denote the usual Euclidean topology of C by τ. The set S_2 is assumed to be endowed with the topology induced by τ_2, which coincides with that induced by τ.

We need the following properties of τ_1, C_1 and S_2:

(i) τ_1 is finer than τ;
(ii) τ_1 has a base consisting of sets closed in τ;
(iii) C_1 is first-countable and Lindelöf;
(iv) S_2 is second-countable;
(v) $\dim C_1 = \dim_0 C_1 = 0$;
(vi) $\dim S_2 = \dim_0 S_2 = 0$.

In [7, Example 27.8] it was shown that

(vii) $\dim_0(C_1 \times S_2) > 0$ (and hence $\dim(C_1 \times S_2) > 0$).

In [36] the construction was refined so as to satisfy the additional condition

(viii) C_1^n is Lindelöf for each $n \in \mathbb{N}$.

Recall that a topological space is said to be *non-Archimedean* if it has a base such that, given any two of its elements, either they are disjoint or one of them contains the other (see [22]). Note that the Cantor set C (with the usual topology), as well as its subspace S_2, is non-Archimedean. According to [14, Theorem 3 (version 2)], any space X admitting a coarser non-Archimedean topology σ and having a base consisting of σ-closed sets is a retract of a Boolean topological group with linear topology. By Lemma 1 any such X is a retract of $B^{\mathrm{lin}}(X)$ (in fact, the group constructed in [14] *is* $B^{\mathrm{lin}}(X)$). Thus, C_1 and S_2 are retracts of the zero-dimensional Boolean groups $B^{\mathrm{lin}}(C_1)$ and $B^{\mathrm{lin}}(S_2)$, respectively.

For each $n \in \mathbb{N}$, we denote the topological sum of n copies of C_1 by $\bigoplus_n C_1$. According to [36, Proposition 7], $(B(C_1))^n$ is topologically isomorphic to the group $B(\bigoplus_n C_1)$, which is Lindelöf by Theorem A. Since $B^{\mathrm{lin}}(C_1)$ is a continuous image of $B(C_1)$, it follows that all finite powers $(B^{\mathrm{lin}}(C_1))^n$ are Lindelöf.

Let us show that $B^{\mathrm{lin}}(C_1)$ is submetrizable. Since $C_1 = (C, \tau_1)$ and the topology τ_1 is finer than the Euclidean topology τ of the Cantor set C, it follows that the identity isomorphism $B^{\mathrm{lin}}(C_1) \to B^{\mathrm{lin}}(C)$ extending the identity map $C_1 \to C$ is continuous. On the other hand, $B^{\mathrm{lin}}(C)$ is a continuous image of $F(C)$ and $F(C)$ has a countable network [6, Theorem 5.2.13]; therefore, $B^{\mathrm{lin}}(C)$ has a countable network as well and hence admits a coarser metrizable group topology [6, Corollary 7.1.17], which immediately implies the submetrizability of $B^{\mathrm{lin}}(C_1)$.

We set $G_1 = B^{\mathrm{lin}}(C_1)$ and let r_1 be a retraction $G_1 \to C_1$.

By Lemma 2, S_2 is a retract of a Boolean second-countable topological group with linear topology. We denote this group by G_2 and let r_2 be a retraction $G_2 \to S_2$.

Since the dimension dim of a Lindelöf space does not exceed its small inductive dimension ind (see, e.g., [7, Proposition 5.3]) and the dimensions $\dim_0$ and dim coincide for normal spaces (see, e.g., [7, Proposition 11.2]), it follows that $\dim G_1 = \dim_0 G_1 = 0$ and $\dim G_2 = \dim_0 G_2 = 0$. However, $\dim_0(G_1 \times G_2) > 0$. Indeed, clearly, $r_1 \times r_2 \colon G_1 \times G_2 \to C_1 \times S_2$ is a retraction. Thus, $C_1 \times S_2$ is a retract and hence a z-embedded subspace of $G_1 \times G_2$. According to Theorem C, in view of (vii) and Remark 1 we have $\dim_0(G_1 \times G_2) > 0$ and $\dim(G_1 \times G_2) > 0$. ■

Remark 3. Note that Theorem 1 gives also an example of topological groups G_1 and G_2 with the following properties:

(1) G_1^n is Lindelöf and submetrizable for every $n \in \mathbb{N}$, and G_2 is second-countable;
(2) $G_1 \times G_2$ is not normal;
(3) $\operatorname{Ind} G_1 = \operatorname{Ind} G_2 = 0$ and $\operatorname{Ind}(G_1 \times G_2) = \infty$.

Indeed, Kodama proved that the product theorem $\dim_0(X \times Y) \le \dim_0 X + \dim_0 Y$ holds whenever $X \times Y$ is normal and Y is metrizable [15]. Therefore, the product of the groups G_1 and G_2 provided by Theorem 1 cannot be normal and hence $\operatorname{Ind}(G_1 \times G_2) = \infty$. On the other hand, $\operatorname{Ind} G_1 = \operatorname{Ind} G_2 = 0$, because both G_1 and G_2 are strongly zero-dimensional.

However, the following questions remain open.

Question 1. Does there exist a topological group G with $\dim_0(G \times G) > 2 \dim_0 G$?

Question 2. What values can the gap between $\dim_0(G_1 \times G_2)$ and $\dim_0 G_1 + \dim_0 G_2$ take for topological groups G_1 and G_2?

Question 3. Do there exist a Lindelöf separable (or ccc) topological group G_1 and a second-countable topological group G_2 with $\dim_0(G_1 \times G_2) > \dim_0 G_1 + \dim_0 G_2$?

Question 4. Do there exist a Lindelöf separable (or ccc) topological group G_1 and a second-countable topological group G_2 for which $G_1 \times G_2$ is not $\mathbb{R}$-factorizable?

Question 5 ([43, p. 1114]). Is it true that $\operatorname{ind}(G_1 \times G_2) \le \operatorname{ind} G_1 + \operatorname{ind} G_2$ for any topological groups G_1 and G_2? What if G_1 and G_2 are totally bounded?

3. Covering dimension is not preserved by free topological groups

Theorem 2. *There exists a Lindelöf space X with the following properties:*

(1) $\dim X = \dim_0 X = 0$;
(2) $\dim F(X) > 0$ *and* $\dim_0 F(X) > 0$;
(3) $\dim A(X) > 0$ *and* $\dim_0 A(X) > 0$;
(4) $\dim B(X) > 0$ *and* $\dim_0 B(X) > 0$.

Proof. We keep the notation of the preceding section. Let us show that $X = C_1 \oplus S_2$ has the desired properties.

Property (1) obviously follows from properties (v) and (vi) of the spaces C_1 and S_2 (see the proof of Theorem 1). Properties (3) and (4) follow from property (vii) of $C_1 \times S_2$ and the fact that, according to [44, Proposition 4] and [36, Proposition 7], the groups $A(C_1) \times A(S_2)$ and $B(C_1) \times B(S_2)$ are topologically isomorphic to $A(C_1 \oplus S_2)$ and $B(C_1 \oplus S_2)$, respectively. Indeed, we know from the proof of Theorem 1 that C_1 and S_2 are retracts of Boolean topological groups. By Lemma 1 they are also retracts of $A(C_1)$ and $B(C_1)$ and of $A(S_2)$ and $B(S_2)$, respectively. Therefore, $C_1 \times S_2$ is a retract of $A(C_1) \times A(S_2)$ and of $B(C_1) \times B(S_2)$. Thus, the groups $A(C_1 \oplus S_2) \cong A(C_1) \times A(S_2)$ and $B(C_1 \oplus S_2) \cong B(C_1) \times B(S_2)$ cannot be strongly zero-dimensional, because they contain the space $C_1 \times S_2$ with $\dim_0(C_1 \times S_2) > 0$ as a z-embedded subspace. By Remark 1 their Čech covering dimension dim cannot be zero either.

Let us prove (2). The natural multiplication map $i_2 \colon X \times X \to F(X)$ defined by $(x, y) \mapsto xy$ is a topological embedding (see, e.g., [6, Theorem 7.1.13]). Therefore, $C_1 \times S_2$ is topologically embedded in $F(X)$ as the subspace $Y = i_2(C_1 \times S_2)$ consisting of two-letter words of the form xy, where $x \in C_1$ and $y \in S_2$. Let us show that Y is a retract of $F(X)$.

The group $A(X) = A(C_1 \oplus S_2)$ is the topological quotient of $F(X) = F(C_1 \oplus S_2)$ by the commutator subgroup (see, e.g., [6, Theorem 7.1.11]). Let $h \colon F(C_1 \oplus S_2) \to A(C_1 \oplus S_2)$ be the canonical quotient homomorphism. Note that

$$h(Y) = \{x + y \in A(C_1 \oplus S_2) : x \in C_1,\ y \in S_2\}.$$

The isomorphism $i \colon A(C_1 \oplus S_2) \to A(C_1) \times A(S_2)$ constructed in [44] takes each point $x \in C_1 \oplus S_2$ to $(x, 0_2)$ if $x \in C_1$ and to $(0_1, x)$ if $x \in S_2$ (by 0_1 and 0_2 we denote the zero elements of $A(C_1)$ and $A(S_2)$, respectively), so that

$$i(x + y) = (x, y) \in C_1 \times S_2 \subset A(C_1) \times A(S_2)$$

for any $x \in C_1$ and $y \in S_2$. Obviously,

$$i(h(Y)) = C_1 \times S_2 \subset A(C_1) \times A(S_2).$$

As we mentioned above, $C_1 \times S_2$ is a retract of $A(C_1) \times A(S_2)$. Let $r \colon A(C_1) \times A(S_2) \to C_1 \times S_2$ be a retraction. Then the composition $r \circ i \circ h : F(C_1 \oplus S_2) \to C_1 \times S_2$ is surjective and continuous, and it takes every $xy \in Y$ to $(x, y) \in C_1 \times S_2$. To obtain the desired retraction $F(X) \to Y$, it remains to add the homeomorphism $i_2|_{C_1 \times S_2} \colon C_1 \times S_2 \to Y$ to this composition.

Thus, Y is a retract and hence a z-embedded subspace of $F(X)$. Since Y is homeomorphic to $C_1 \times S_2$, we have $\dim_0 Y > 0$ (see property (vii) of $C_1 \times S_1$). Therefore, $\dim_0 F(X) > 0$ by Theorem C and $\dim F(X) > 0$ by Remark 1. ∎

Remark 4. For the space $X = C_1 \oplus S_2$ considered in the proof of Theorem 2, we have $\operatorname{Ind} X = 0$ and $\operatorname{Ind} F(X) = \operatorname{Ind} A(X) = \operatorname{Ind} B(X) = \infty$, because the groups $F(X)$, $A(X)$, and $B(X)$ are not normal. Indeed, it was shown in the proof of Theorem 2 that the product $C_1 \times S_2$ is a retract and hence a closed subspace of each of these groups, and this product is not normal (otherwise, according to [15], the relations $\dim_0 C_1 = \dim_0 S_2 = 0$ would imply $\dim_0(C_1 \times S_2) = 0$, because S_2 is metrizable).

As mentioned in the introduction, Theorem 2 answers an old question of Arkhangel'skii. Another old question of his still remains open.

Question 6 ([3, Question 9]; see also [5, Problem 8.18]). *Is it true that* $\operatorname{ind} F(X) = 0$ ($\operatorname{ind} A(X) = 0$) *for any metrizable space* X *with* $\operatorname{ind} X = 0$?

It is worth mentioning that, for every positive integer n, there exists an n-dimensional compact metrizable space such that all dimensions of any topological group containing it are greater than n and that all dimensions of any topological group containing the hedgehog space $J(\omega_1)$ with ω_1 spines are infinite, while all dimensions of $J(\omega_1)$ equal 1. These results were obtained by Kulesza in [16]. It is also known that there exists a zero-dimensional metrizable space X which does not embed in any zero-dimensional metrizable topological group [34].

The answer to the following question is not known either.

Question 7. *What values can* $\dim_0 F(X)$ ($\dim_0 A(X)$, $\dim_0 B(X)$) *take for* X *with* $\dim_0 X = 0$?

4. Failure of the subgroup theorem for the covering dimension of topological groups

Theorem 3. *There exists a strongly zero-dimensional Boolean* (*and hence Abelian*) *topological group* G *which contains a closed* ω*-narrow subgroup* H *with* $\dim_0 H > 0$.

The proof of this theorem uses the following lemma.

Lemma 3. *Any zero-dimensional* $\mathbb{R}$*-factorizable group* G *embeds in a product* P *of zero-dimensional second-countable groups as a subgroup. Moreover, if* G *is Abelian or Boolean, then so is* P*, and if in addition* G *has linear topology, then so does* P.

Proof. Let G be a zero-dimensional $\mathbb{R}$-factorizable group, and let $f_\alpha\colon G \to \mathbb{R}$, $\alpha \in A$, be all continuous real-valued functions on G (here A is some index set). It follows from the $\mathbb{R}$-factorizability of G and Theorem B that, for each $\alpha \in A$, there exists a zero-dimensional second-countable group H_α, a continuous surjective homomorphism $h_\alpha\colon G \to H_\alpha$, and a continuous function $g_\alpha\colon H_\alpha \to \mathbb{R}$ such that $f_\alpha = g_\alpha \circ h_\alpha$. Note that if G is Abelian or Boolean, then so are all H_α. Since G is Tychonoff, it follows that the family $\{f_\alpha : \alpha \in A\}$ separates points and closed sets and hence so does $\{h_\alpha : \alpha \in A\}$. Therefore, the diagonal

$$\mathop{\triangle}_{\alpha\in A} h_\alpha\colon G \to \prod_{\alpha\in A} H_\alpha$$

is a homeomorphic embedding. Clearly, this is a homomorphism. We set $P = \prod_{\alpha\in A} H_\alpha$.

In the case where G is Abelian and has linear topology, we can render the topologies of all H_α linear in the same manner as in the proof of Lemma 2: for each $\alpha \in A$, we fix a base $\{U_n : n \in \omega\}$ of neighborhoods of zero in H_α, choose a subgroup of H_α with open preimage under h_α in each U_n, and define the new group topology on H_α for which the chosen subgroups form a subbase of neighborhoods of zero. The maps h_α and f_α remain continuous with respect to the new topology, and the group H with this topology is first-countable and ω-narrow, because G is ω-narrow, being $\mathbb{R}$-factorizable, and continuous homomorphisms preserve ω-narrowness [6, Proposition 3.4.2]. Therefore, it is second-countable [6, Proposition 3.4.5]. Clearly, the topology of any product of groups with linear topology is linear. Therefore, the product P of the second-countable groups H_α with the new linear topologies is linear. ∎

Proof of Theorem 3. We use the same spaces C_1 and S_2 as in the proof of Theorem 1. By Lemma 1, C_1 and S_2 are retracts of the free Boolean groups $B(C_1)$ and $B(S_2)$, respectively. These groups are Lindelöf by Theorem A. According to [36, Theorem 8], they are zero-dimensional, and according to [41], they are $\mathbb{R}$-factorizable. By Lemma 3, $B(C_1)$ and $B(S_2)$ are embedded in products P_1 and P_2 of zero-dimensional second-countable Boolean groups as subgroups. By [18, Theorem 3] any product of zero-dimensional second-countable spaces is strongly zero-dimensional. Therefore, the group $G = P_1 \times P_2$ is strongly zero-dimensional, and it contains $H = B(C_1) \times B(S_2)$ as a subgroup. By Theorem C, $\dim_0 H > 0$, because $\dim_0(C_1 \times S_2) > 0$ and $C_1 \times S_2$ is a retract and hence a z-embedded subspace of H.

The subgroup H is closed in G, because it is Raikov complete. Indeed, C_1 and S_2 are paracompact, being Lindelöf, and therefore Dieudonné complete [8]. It follows from [38, Theorem 2.1] that the free Boolean topological group of any Dieudonné complete space is Raikov complete. It remains to recall that Raikov completeness is preserved by products (see, e.g., [6, Theorem 3.6.22]).

Finally, since any product of Lindelöf groups is ω-narrow [6, Propositions 5.1.1 and 5.1.3], it follows that so is H. ∎

It is easy to construct a similar example for the Čech covering dimension dim; however, the subgroup H cannot be made closed in this case, because the inequality $\dim Y \le \dim X$ holds for any space X and its closed subspace Y [7, Proposition 2.11].

THEOREM 4. *There exists a Boolean topological group G with linear topology such that $\dim G = \operatorname{Ind} G = 0$ and G contains a subgroup H with $\dim H = \operatorname{Ind} H = \infty$.*

Proof. Consider the Sorgenfrey plane $S \times S$. It is zero-dimensional and has weight 2^ω. Hence it embeds in the Cantor cube $K = \{0,1\}^{2^\omega}$ [11, Theorem 6.2.16]. The compact space K is zero-dimensional, being a product of zero-dimensional spaces. Therefore, the free Boolean linear topological group $B^{\mathrm{lin}}(K)$ is defined. Since $B^{\mathrm{lin}}(K)$ is zero-dimensional and Lindelöf (by Theorem A), it follows that $\dim B^{\mathrm{lin}}(K) = 0$ (and $\operatorname{Ind} B^{\mathrm{lin}}(K) = 0$), because the covering dimension of a Lindelöf space does not exceed its small inductive dimension [7, Proposition 5.3]. Let H denote the subgroup of $B^{\mathrm{lin}}(K)$ generated by $S \times S \subset K$. Clearly, $H \cap K = S \times S$ and hence $S \times S$ is closed in H. We have $\operatorname{Ind}(S \times S) = \infty$ (because $S \times S$ is not normal) and $\dim(S \times S) = \infty$ (see [9]). Thus, $\operatorname{Ind} H = \dim H = \infty$, since large inductive and Čech covering dimensions are monotone with respect to closed subspaces [7, Propositions 2.6 and 2.11]. ∎

Remark 5. Applying the argument in the proof of Theorem 3 to the free Boolean linear topological groups $B^{\mathrm{lin}}(C_1)$ and $B^{\mathrm{lin}}(S_2)$ instead of $B(C_1)$ and $B(S_2)$, we obtain an example of a strongly zero-dimensional Boolean group G with linear topology which contains a subgroup H with $\dim_0 H > 0$. However, it is unclear whether H can be made closed, because the free Boolean linear group of a Dieudonné complete (and even compact) space is not necessarily complete. For example, a base of neighborhoods of zero in the group $B^{\mathrm{lin}}(\xi) = [\xi]^{<\omega}$ for the usual convergent sequence $\xi = \mathbb{N} \cup \{\infty\}$ is formed by the subgroups

$$H_n = \{F \subset \xi \setminus \{1, \dots, n\} : |F| \text{ is even}\}$$

(see the description of the topology of $B^{\mathrm{lin}}(X)$ in [36, p. 497]). It is easy to see that $B^{\mathrm{lin}}(\xi)$ is topologically isomorphic to the σ-product of countably many copies of the discrete group $\mathbb{Z}_2 = \{0, 1\}$: the isomorphism takes every element $F \in B^{\mathrm{lin}}(\xi)$ (which is a finite subset of ξ) to the point $(x_n)_{n\in\omega} \in \{0, 1\}$ in which $x_0 = 1$ if and only if $\infty \in F$ and $x_n = 1$ for $n > 0$ if and only if $n \in F$. This σ-product is a dense and hence non-closed subgroup of $\mathbb{Z}_2^\omega$. Therefore, it is not complete.

This remark suggests the following questions.

Question 8. Is it true that any closed subgroup of a strongly zero-dimensional Boolean topological group with linear topology is strongly zero-dimensional?

Question 9. Is it true that $\dim_0 H \le \dim_0 G$ for any closed subgroup H of a Boolean topological group G with linear topology?

The following question is also of interest.

Question 10. Given a topological group G and its (closed) subgroup H, what values can the gap between $\dim_0 G$ and $\dim_0 H$ take?

This question is interesting in both cases $\dim_0 H < \dim_0 G$ and $\dim_0 H > \dim_0 G$.

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