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Complex Monge–Ampère equations on singular spaces

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Abstract. We investigate the complex Monge–Ampère operator on a bounded strongly pseudoconvex domain of a closed, connected, singular, and locally irreducible complex-analytic subvariety. We first examine the classes $\mathcal{E}^p$ for $p > 0$ and establish a characterization of their images under the complex Monge–Ampère operator. This result answers a question posed by N. Q. Dieu, T. V. Long. We then turn to the weighted energy classes $\mathcal{E}_\chi(\Omega)$, consisting of negative plurisubharmonic functions with finite χ-energy, and provide a precise characterization of their images under the complex Monge–Ampère operator, where χ is a convex increasing function satisfying $\chi(0) = 0$ and $\chi(-\infty) = -\infty$.

1. Introduction. In this article, we consider a connected, singular, closed complex-analytic subvariety $X \subset \mathbb{C}^N$ of pure dimension n with $2 \leqslant n < N$, which we also assume to be locally irreducible. These assumptions guarantee that all the usual notions of plurisubharmonic (psh) functions on X coincide (see [Dem85]).

Let $\Omega \Subset X$ be a domain which we assume to be strongly pseudoconvex in the sense that there exists a negative C^2 exhaustion function on Ω that extends to a strictly plurisubharmonic function on a neighborhood of $\overline{\Omega}$. We consider a volume form μ, possibly singular, semipositive, and of finite total mass. In general, μ will be smooth outside an analytic subset of Ω, and its singularities will reflect the geometry of the singularities of the variety X.

In this setting, E. Bedford [Bed82] extended the definition of the complex Monge–Ampère operator to locally bounded plurisubharmonic functions on Ω, showing that the fundamental properties obtained by E. Bedford and B. A. Taylor [BT76] in the case of a domain in $\mathbb{C}^n$ remain valid.

When $\mu = f\beta^n$, where β is a positive Hermitian form on X and $f \in L^p(\Omega, \beta^n)$ for some $p > 1$, the Dirichlet problem with continuous boundary

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data was resolved in [GGZ23], establishing the existence and uniqueness of a continuous solution on $\overline{\Omega}$. In contrast, when the volume form is more singular (e.g., $f \in L^1(\Omega)$), one cannot in general expect the existence of a bounded solution.

A central question in pluripotential theory is to understand the action of the complex Monge–Ampère operator on various classes of psh functions. To address this problem, U. Cegrell [Ceg98] defined, in the case of a bounded hyperconvex domain $\Omega \Subset \mathbb{C}^n$, the finite energy classes $\mathcal{E}^p(\Omega)$ for $p > 0$, and provided a characterization of the image of these classes under the complex Monge–Ampère operator. This work was extended to the singular setting by N. Q. Dieu and T. V. Long [DL22], who introduced the domain of definition $\mathcal{E}(\Omega)$ of the complex Monge–Ampère operator in the case of a bounded hyperconvex domain Ω in a singular variety. Building on the local regularization techniques from [Bed82] and the ideas of Cegrell, they established key properties of these classes and gave a precise characterization of the image of $\mathcal{E}^1(\Omega)$ under the complex Monge–Ampère operator.

However, their method did not allow for a full generalization of this characterization to the classes $\mathcal{E}^p(\Omega)$ for $p \neq 1$, thus raising the following open problem:

> *Characterize the image of $\mathcal{E}^p(\Omega)$ under the complex Monge–Ampère operator for $p > 0$.*

In this singular setting, an important question concerns the validity of Stokes' formula, which is essential for performing integration by parts. This validity is ensured by [Bun66, Theorem 4.2] (see also [Bed82, Lemma 3.2]), recalled in Section 2 below (see Theorem 2.7).

Our first theorem generalizes Cegrell's result to a singular variety by providing a complete characterization of the image of $\mathcal{E}^p(\Omega)$ under the complex Monge–Ampère operator for $p > 0$.

THEOREM 1.1. *Let $\Omega \Subset X$ be a strongly pseudoconvex domain, μ a positive Borel measure on Ω, and $p > 0$. Then the following three properties are equivalent:*

(1) *there exists $u \in \mathcal{E}^p(\Omega)$ such that $(dd^c u)^n = \mu$ on Ω,*
(2) *$\mathcal{E}^p(\Omega) \subset L^p(\Omega, \mu)$,*
(3) *there exists a constant $A > 0$ such that for all $\psi \in \mathcal{T}(\Omega)$,*

$$\int_{\Omega} (-\psi)^p \, d\mu \leqslant A E_p(\psi)^{\frac{p}{n+p}}. \tag{1.1}$$

This result unifies and extends known results from the smooth setting to a more general singular context.

We then introduce the weighted classes $\mathcal{E}_\chi(\Omega)$, defined using weight functions $\chi : \mathbb{R}^- \to \mathbb{R}^-$ that are convex, increasing, and satisfy $\chi(0) = 0$ and

$\chi(-\infty) = -\infty$. These classes provide a natural generalization of the classes $\mathcal{E}^p(\Omega)$ for $0 < p \leqslant 1$, for instance by choosing the weight function χ defined by $\chi(t) := -(-t)^p$.

In this setting, we establish the following theorem, which provides a characterization of the image of $\mathcal{E}_\chi(\Omega)$ under the complex Monge–Ampère operator:

THEOREM 1.2. *Let $\Omega \Subset X$ be a strongly pseudoconvex domain, μ a positive Borel measure on Ω, and $\chi : \mathbb{R}^- \to \mathbb{R}^-$ a convex increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) = 0$. Then the following three properties are equivalent:*

(1) *there exists $u \in \mathcal{E}_\chi(\Omega)$ such that $(dd^c u)^n = \mu$ on Ω,*
(2) $\mathcal{E}_\chi(\Omega) \subset L_\chi(\Omega, \mu)$,
(3) *there exists a constant $c_\mu > 0$ such that for all $\psi \in \mathcal{T}(\Omega)$,*

$$\int_\Omega -\chi(\psi)\, d\mu \leqslant c_\mu (E_\chi(\psi)^{\frac{1}{n+1}} + 1). \tag{1.2}$$

An analogous version of the equivalence (1)⇔(3) was obtained by D. D. Thai and D. V. Vu in [TV22] in the compact case.

This article is organized as follows. In Section 2, we recall the essential definitions: plurisubharmonic functions, the complex Monge–Ampère operator, Cegrell's classes, as well as some preliminary results needed for the rest of the work. Section 3 is devoted to the proof of a characterization theorem for the image of $\mathcal{E}^1(\Omega)$ under the complex Monge–Ampère operator. This result plays a central role in the proof of our two main theorems. In Section 4, we establish Theorem 1.1, while Section 5 is dedicated to the proof of Theorem 1.2

2. Preliminaries.

Let X be a closed singular complex-analytic space ($X \subset \mathbb{C}^N$) of pure dimension n with $2 \leqslant n < N$. We will denote by X_{reg} the complex manifold of regular points of X. The set

$$X_{\text{sing}} := X \setminus X_{\text{reg}}$$

of singular points is an analytic subset of X of complex codimension strictly greater than 1.

2.1. Plurisubharmonic functions

DEFINITION 2.1. Let Ω be an open subset of $X \hookrightarrow \mathbb{C}^N$.

(1) A function $u : \Omega \to [-\infty, +\infty[$ is said to be *plurisubharmonic* on Ω if for every $x \in \Omega$, there exists a neighborhood U of x in $\mathbb{C}^N$ and a plurisubharmonic function $\tilde{u}$ on U such that $u = \tilde{u}|_{X \cap U}$ on $X \cap U$.

(2) A function $u : \Omega \to [-\infty, +\infty[$ is said to be *weakly plurisubharmonic* (according to Fornæss–Narasimhan [FN80]) if u is upper semicontinuous on Ω and for every holomorphic function $f : \Delta \to \Omega$, the composition $u \circ f$ is subharmonic on Δ, where Δ is the unit disk of $\mathbb{C}$.

Definition 2.2. We say that X is a *Stein space* if it admits a C^2-smooth strongly plurisubharmonic exhaustion function on X.

Moreover, X is a Stein space because the function $\psi(z) := |z|^2$, restricted to X, where $|\cdot|$ denotes the Hermitian norm on $\mathbb{C}^N$, constitutes a strongly plurisubharmonic exhaustion function on X.

Theorem 2.3 ([FN80]). *If X is a Stein manifold, then u is plurisubharmonic on Ω, an open subset of X, if and only if it is weakly plurisubharmonic on Ω.*

Theorem 2.4 ([Dem85]). *Suppose that X is locally irreducible. If u is plurisubharmonic on X_{reg} and locally bounded from above on X, then the function u^* defined by*

$$u^*(x) := \limsup_{X_{\mathrm{reg}} \ni y \to x} u(y)$$

is plurisubharmonic on X.

Definition 2.5. A domain $\Omega \Subset X$ is said to be *strongly pseudoconvex* if it admits a negative, C^2-smooth strongly plurisubharmonic exhaustion function, i.e., a function ρ, strongly plurisubharmonic in a neighborhood Ω' of $\overline{\Omega}$, such that

$$\Omega := \{x \in \Omega' : \rho(x) < 0\}$$

and, for every $c < 0$,

$$\Omega_c := \{x \in \Omega' : \rho(x) < c\} \Subset \Omega$$

is relatively compact in Ω.

We denote by $\mathrm{PSH}(X)$ the set of plurisubharmonic functions on X.

2.2. Stokes' formula. We present two preliminary results that will be useful in the proofs of the main theorems.

To apply Stokes' formula, we begin by recalling a few properties of real-analytic subvarieties (see [Bed82, Bun66]).

Definition 2.6. Let V be a singular real analytic subvariety of an open subset of $\mathbb{R}^n$, with pure dimension $k < n$. Then a relatively compact open subset $D \subset V$ is called an *analytic domain* if ∂D is contained in a proper subvariety of a neighborhood of ∂D in V.

Theorem 2.7. *Let D be an oriented analytic domain of a singular subvariety V in an open subset of $\mathbb{R}^n$, with pure dimension k. Let α be a C^1*

differential form of degree $k-1$ in $\mathbb{R}^n$. Then

$$\int_{\partial D} \alpha = \int_D d\alpha, \tag{2.1}$$

i.e., Stokes' formula holds for smooth forms on analytic spaces.

REMARK 2.8. In our context, since X is a singular analytic variety of dimension $n < N$ in $\mathbb{C}^N$, and $\Omega \Subset X$ is a relatively compact domain with smooth boundary, Theorem 2.7 ensures that Stokes' formula remains valid.

2.3. Complex Monge–Ampère operator on singular spaces. The complex Monge–Ampère measure $(dd^c u)^n$ of a smooth psh function in a domain of $\mathbb{C}^n$ is given by the Radon measure

$$(dd^c u)^n = c_n \cdot \det\left(\frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}\right) dV_{\mathrm{eucl}},$$

where $c_n > 0$ is a constant. This definition was extended to all bounded psh functions by E. Bedford and B. A. Taylor, who laid the foundations of pluripotential theory in [BT76, BT82].

The complex Monge–Ampère operator has been defined and studied on singular complex varieties by E. Bedford [Bed82], and later by J.-P. Demailly [Dem85]. They showed that the operator

$$(dd^c \cdot)^n : \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X) \to \mathcal{M}^{n,n}(X),$$

where $\mathcal{M}^{n,n}(X)$ denotes the space of Radon measures on X, can be defined in the usual way on the regular part X_{reg} of X. That is, for every compact set $K \Subset X$, one has

$$\int_{K \setminus X_{\mathrm{sing}}} (dd^c u)^n < +\infty \quad \text{for all } u \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X).$$

Thus, the complex Monge–Ampère operator can be extended to X as a positive Borel measure with zero mass on the singular set X_{sing}.

As a consequence, all classical properties of the complex Monge–Ampère operator acting on $\mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ extend to the singular setting (see [Bed82, Dem85]). In particular:

PROPOSITION 2.9 ([Bed82]). *Let $\Omega \Subset X$ be a relatively compact open subset, and let $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$. Assume that*

$$\liminf_{z \to \zeta} (u(z) - v(z)) \geqslant 0 \quad \textit{for all } \zeta \in \partial\Omega.$$

Then

$$\int_{\{u<v\}} (dd^c v)^n \leqslant \int_{\{u<v\}} (dd^c u)^n.$$

In particular, if $(dd^c u)^n \leqslant (dd^c v)^n$ in the sense of Radon measures on Ω, then $v \leqslant u$ on Ω.

2.4. Cegrell's classes. We recall here the finite energy classes introduced by Cegrell [Ceg98, Ceg04], adopting the notation of [BGZ09] for consistency. In the rest of the article, unless otherwise mentioned, we consider a connected, singular, closed complex-analytic subvariety $X \subset \mathbb{C}^N$ of pure dimension n with $2 \leqslant n < N$, which we also assume to be locally irreducible, and we let $\Omega \Subset X$ be a strongly pseudoconvex domain. Then we define

$$\mathcal{T}(\Omega) = \Big\{ u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega) : u = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega (dd^c u)^n < +\infty \Big\}$$

and

$$\mathcal{E}(\Omega) = \Big\{ u \in \mathrm{PSH}(\Omega) : \forall z \in \Omega,\ \exists V \in \mathcal{V}(z) \text{ and } u_j \in \mathcal{T}(\Omega),\ u_j \downarrow u \text{ on } V \text{ with } \sup_j \int_\Omega (dd^c u_j)^n < +\infty \Big\},$$

where $\mathcal{V}(z)$ denotes a neighborhood of z, and

$$\mathcal{F}(\Omega) = \Big\{ u \in \mathrm{PSH}(\Omega) : \exists u_j \in \mathcal{T}(\Omega),\ u_j \downarrow u \text{ and } \sup_j \int_\Omega (dd^c u_j)^n < +\infty \Big\}.$$

Further, for any $p > 0$, we define

$$\mathcal{E}^p(\Omega) = \Big\{ u \in \mathrm{PSH}(\Omega) : \exists u_j \in \mathcal{T}(\Omega),\ u_j \downarrow u \text{ and } \sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < +\infty \Big\}$$

and

$$\mathcal{F}^p(\Omega) = \Big\{ u \in \mathrm{PSH}(\Omega) : \exists u_j \in \mathcal{T}(\Omega),\ u_j \downarrow u \text{ and } \sup_j \int_\Omega [1 + (-u_j)^p](dd^c u_j)^n < +\infty \Big\}.$$

We also recall the definition of the class $\mathcal{E}\chi(\Omega)$, originally introduced in [GZ07] for compact Kähler manifolds and later extended to domains in $\mathbb{C}^n$ in [BGZ09]. This formulation provides a unified framework that includes all the classes defined above. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function such that $\chi(-\infty) = -\infty$. The class $\mathcal{E}\chi(\Omega)$ is then defined as follows:

$$\mathcal{E}_\chi(\Omega) = \Big\{ u \in \mathrm{PSH}(\Omega) : \exists u_j \in \mathcal{T}(\Omega),\ u_j \downarrow u \text{ and } \sup_j \int_\Omega -\chi \circ (u_j)(dd^c u_j)^n < +\infty \Big\}.$$

For $u \in \mathcal{E}_\chi(\Omega)$, we define the *weighted Monge–Ampère energy* as

$$E_\chi(u) = \int_\Omega -\chi(u)(dd^c u)^n.$$

If $u \in \mathcal{E}^p(\Omega)$, the *p-energy* is defined by

$$E_p(u) = \int_\Omega (-u)^p (dd^c u)^n, \qquad u \in \mathcal{E}^p(\Omega).$$

If $\chi(t) = -1 - (-t)^p$, then $\mathcal{E}_\chi(\Omega) = \mathcal{F}^p(\Omega)$.

EXAMPLE 2.10. (1) For $0 < p \leqslant 1$, the function $\chi(t) := -(-t)^p$, $t \in \mathbb{R}^-$, is convex, increasing, satisfies $\chi(0) = 0$ and $\chi(-\infty) = -\infty$. In this case, $\mathcal{E}_\chi(\Omega) = \mathcal{E}^p(\Omega)$.

(2) The function $\chi(t) := -\log(1-t)$, $t \in \mathbb{R}^-$, is also convex and increasing with $\chi(0) = 0$ and $\chi(-\infty) = -\infty$.

THEOREM 2.11 ([DL22, Theorem A]). *Let $\Omega \Subset X$ be a strongly pseudoconvex domain. Then for any $u \in \mathrm{PSH}^-(\Omega)$, there exists a sequence $u_j \in \mathcal{T}(\Omega)$ such that $u_j \downarrow u$ on Ω.*

THEOREM 2.12 ([DL22]). *Let $\Omega \Subset X$ be a strongly pseudoconvex domain. Assume that $u, v \in \mathcal{E}^p(\Omega)$ for some $p \geqslant 1$, and that*

$$(dd^c u)^n \leqslant (dd^c v)^n \quad \text{on } \Omega.$$

Then $u \leqslant v$ on Ω.

3. Characterizing the image of $\mathcal{E}^1(\Omega)$. The image of $\mathcal{E}^1(\Omega)$ under the complex Monge–Ampère operator has been characterized by N. Q. Dieu and T. V. Long [DL22]. In this part of the present work, we pursue an alternative approach, relying on the variational method introduced by R. J. Berman, S. Boucksom, V. Guedj, and A. Zeriahi [BBGZ13] (see also [Lu15] for related developments).

LEMMA 3.1.

(1) *If $u \in \mathcal{E}^1(\Omega)$ and $v \in \mathrm{PSH}^-(\Omega)$ are such that $u \leqslant v$ then $v \in \mathcal{E}^1(\Omega)$.*

(2) (a) *If (u_j) is a decreasing sequence in $\mathcal{E}^1(\Omega)$ and if $\sup_j E_1(u_j) < +\infty$ then $u = \lim_{j\to+\infty} u_j \in \mathcal{E}^1(\Omega)$ and*

$$E_1(u) = \lim_{j\to+\infty} E_1(u_j).$$

(b) *If $u_j \to u$ in $\mathrm{PSH}(\Omega)$ for the L^1_{loc}-topology and if $\sup_j E_1(u_j) < +\infty$ then $u \in \mathcal{E}^1(\Omega)$ and*

$$E_1(u) \leqslant \liminf_{j\to+\infty} E_1(u_j).$$

(3) *For every $c > 0$,*

$$\mathcal{E}^1_c(\Omega) := \{u \in \mathcal{E}^1(\Omega) : 0 \leqslant E_1(u) \leqslant c\}$$

is convex and compact in $\mathcal{E}^1(\Omega)$.

Proof. (1) Since $u \leqslant v$, we have $E_1(v) \leqslant E_1(u)$ (see [Ceg98] and [Lu15, Theorem 3.23]). Therefore, v belongs to $\mathcal{E}^1(\Omega)$.

(2) (a) Let (u_j) be a decreasing sequence in $\mathcal{E}^1(\Omega)$.

Let ρ be an exhaustion function for Ω, and define $\varphi_j := \max(u_j, j\rho) \in \mathcal{T}(\Omega)$. Indeed, note that $\varphi_j = 0$ on $\partial\Omega$. Since $u_j \leqslant \varphi_j$ and $j\rho \leqslant \varphi_j$, and $j\rho = \varphi_j$ on $\partial\Omega$, the comparison principle yields

$$\int_\Omega (dd^c\varphi_j)^n \leqslant \int_\Omega (dd^c(j\rho))^n = j^n \int_\Omega (dd^c\rho)^n < +\infty.$$

As $\varphi_j \downarrow u$ and $u_j \leqslant \varphi_j$, it follows that

$$\sup_j E_1(\varphi_j) \leqslant \sup_j E_1(u_j) < +\infty,$$

hence $u \in \mathcal{E}^1(\Omega)$.

To show $E_1(u) \leqslant \liminf_j E_1(u_j)$, note that since $u_j \downarrow u$, the measures $\mu_j := (dd^c u_j)^n$ converge weakly to $\mu := (dd^c u)^n$. Set $f_j := -u_j \uparrow -u =: f$; it follows by lower semicontinuity that

$$\liminf_j \int_\Omega f_j \, d\mu_j \geqslant \int_\Omega f \, d\mu,$$

which implies

$$\liminf_{j\to+\infty} \int_\Omega (-u_j)(dd^c u_j)^n \geqslant \int_\Omega (-u)(dd^c u)^n,$$

and thus

$$E_1(u) \leqslant \liminf_j E_1(u_j).$$

Since $u \leqslant u_j$, it follows that $E_1(u_j) \leqslant E_1(u)$ and hence

$$\lim_{j\to+\infty} E_1(u_j) \leqslant E_1(u).$$

Combining the inequalities, we obtain

$$\lim_{j\to+\infty} E_1(u_j) = E_1(u).$$

(b) Let (u_j) be a sequence in $\mathcal{E}^1(\Omega)$ converging to u in $L^1_{\mathrm{loc}}(\Omega)$ with $\sup_j E_1(u_j) < +\infty$.

Recall that for a function $v : \Omega \to \mathbb{R} \cup \{-\infty\}$ locally bounded from above, the upper semicontinuous regularization is defined by

$$v^\star(z) := \limsup_{z'\to z} v(z') < +\infty.$$

Thus, $v^\star(z)$ is upper semicontinuous and satisfies $v \leqslant v^\star$.

Define

$$v_j := \sup_{k\geqslant j} u_k, \qquad \tilde{u}_j := v_j^\star.$$

Then

- $u_j \leqslant \tilde{u}_j$,
- $\tilde{u}_j \in \mathcal{E}^1(\Omega)$ (since the u_k are uniformly bounded in energy, and $\tilde{u}_j$ is obtained by upper semicontinuous regularization),
- $\tilde{u}_j \downarrow u$ (the L^1_{loc}-convergence of u_j to u guarantees that $\tilde{u}_j(z) \downarrow u(z)$ for all $z \in \Omega$).

By part (1), we have $u \in \mathcal{E}^1(\Omega)$ and

$$E_1(u) \leqslant \liminf_{j\to+\infty} E_1(\tilde{u}_j) \leqslant \liminf_{j\to+\infty} E_1(u_j).$$

(3) Let (u_j) be a sequence in $\mathcal{E}^1_c(\Omega)$. Since $\sup_j E_1(u_j) \leqslant c$, the sequence (u_j) cannot go uniformly to $-\infty$ on Ω. Therefore, there exists a subsequence (still denoted by (u_j)) converging to $u \in \mathrm{PSH}(\Omega)$ in $L^1_{\mathrm{loc}}(\Omega)$. Define

$$\varphi_j := \Big(\sup_{k\geqslant j} u_k\Big)^{\star} \in \mathcal{E}^1(\Omega), \qquad \forall j.$$

Then $\varphi_j \downarrow u$ and $\sup_j E_1(\varphi_j) \leqslant c$. By (2)(a), we have $u \in \mathcal{E}^1(\Omega)$ and

$$c \geqslant \liminf_{j\to+\infty} E_1(\varphi_j) = E_1(u).$$

Hence, $u \in \mathcal{E}^1_c(\Omega)$. ∎

The following result is an immediate consequence of Lemma 3.1(2).

COROLLARY 3.2. *The functional $E_1 : \mathcal{E}^1(\Omega) \to \mathbb{R}$ defined for every $u \in \mathcal{E}^1(\Omega)$ by*

$$E_1(u) = \int_\Omega (-u)(dd^c u)^n$$

is lower semicontinuous in $\mathcal{E}^1_c(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for any $c > 0$.

DEFINITION 3.3. Let μ be a positive Borel measure on Ω. The functional $\mathcal{F}_\mu : \mathcal{E}^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{F}_\mu(u) = \frac{1}{n+1} E_1(u) + \int_\Omega u\, d\mu.$$

We say that $\mathcal{F}_\mu$ is *proper* (with respect to E_1) if $\mathcal{F}_\mu(u) \to +\infty$ whenever $E_1(u) \to +\infty$.

DEFINITION 3.4. Let $p > 0$. We denote by $\mathcal{M}_p$ the set of all positive Borel measures μ such that

$$\mathcal{E}^p(\Omega) \subset L^p(\Omega, \mu).$$

LEMMA 3.5. *Let $u, v \in \mathcal{E}^1(\Omega)$. Then*

$$E_1(u+v)^{\frac{1}{n+1}} \leqslant E_1(u)^{\frac{1}{n+1}} + E_1(v)^{\frac{1}{n+1}}.$$

Moreover, if $\mu \in \mathcal{M}_1$, then the functional $\mathcal{F}_\mu$ is convex and proper.

Proof (see [ÅCC12, Lemma 2.2]). Let $u, v \in \mathcal{E}^1(\Omega)$. We first observe that

$$\int_\Omega (-(u+v))(dd^c(u+v))^n = \int_\Omega (-u)(dd^c(u+v))^n + \int_\Omega (-v)(dd^c(u+v))^n.$$

According to point [DL22, Lemma 2.11(c)], we deduce the following estimates. For the first term we obtain

$$\int_\Omega (-u)(dd^c(u+v))^n \leqslant \Big(\int_\Omega (-u)(dd^c u)^n\Big)^{\frac{1}{n+1}} \Big(\int_\Omega (-(u+v))(dd^c(u+v))^n\Big)^{\frac{n}{n+1}},$$

while for the second term we have

$$\int_\Omega (-v)(dd^c(u+v))^n \leqslant \Big(\int_\Omega (-v)(dd^c v)^n\Big)^{\frac{1}{n+1}} \Big(\int_\Omega (-(u+v))(dd^c(u+v))^n\Big)^{\frac{n}{n+1}}.$$

It follows that

$$E_1(u+v) \leqslant E_1(u+v)^{\frac{n}{n+1}} E_1(u)^{\frac{1}{n+1}} + E_1(u+v)^{\frac{n}{n+1}} E_1(v)^{\frac{1}{n+1}},$$

that is,

$$E_1(u+v)^{1-\frac{n}{n+1}} \leqslant E_1(u)^{\frac{1}{n+1}} + E_1(v)^{\frac{1}{n+1}}.$$

Hence,

$$E_1(u+v)^{\frac{1}{n+1}} \leqslant E_1(u)^{\frac{1}{n+1}} + E_1(v)^{\frac{1}{n+1}}.$$

Since $E_1^{\frac{1}{n+1}}$ is homogeneous of degree 1, for $0 \leqslant t \leqslant 1$ we have

$$E_1(tu+(1-t)v)^{\frac{1}{n+1}} \leqslant tE_1(u)^{\frac{1}{n+1}} + (1-t)E_1(v)^{\frac{1}{n+1}}, \tag{3.1}$$

which shows that $E_1^{\frac{1}{n+1}}$ is convex. Similarly, E_1 is also convex. Indeed, as the function $s \mapsto s^{n+1}$ is convex on $[0, +\infty)$, relation (3.1) immediately implies

$$E_1(tu+(1-t)v) \leqslant tE_1(u) + (1-t)E_1(v).$$

We deduce that $\mathcal{F}_\mu$ is convex when $\mu \in \mathcal{M}_1$.

We now prove that the functional $\mathcal{F}_\mu$ is proper when $\mu \in \mathcal{M}_1$. If $\mu \in \mathcal{M}_1$, there exists a constant $A > 0$ such that

$$\|u\|_{L^1(\mu)} \leqslant AE_1(u)^{\frac{1}{n+1}} \quad \text{for all } u \in \mathcal{E}^1(\Omega).$$

(cf. the proof of (2)⇒(3) in Section 4 for $p = 1$). Then

$$\mathcal{F}_\mu(u) = \frac{1}{n+1}E_1(u) - \|u\|_{L^1(\mu)} \geqslant \frac{1}{n+1}E_1(u) - AE_1(u)^{\frac{1}{n+1}}.$$

Thus,

$$\lim_{E_1(u)\to+\infty} \mathcal{F}_\mu(u) = +\infty. \ \blacksquare$$

Let $\varphi \in \mathcal{E}^1(\Omega)$ and $\psi \in \mathcal{T}(\Omega)$. For $-1 \leqslant t \leqslant 1$ we define

$$P(\varphi + t\psi) = (\sup\{u \in \mathrm{PSH}^-(\Omega) \mid u \leqslant \varphi + t\psi\})^*.$$

Lemma 3.6. *Let $\varphi \in \mathcal{E}^1(\Omega)$, $\psi \in \mathcal{T}(\Omega)$ and assume that ψ is continuous. Then*

$$\frac{d}{dt} E_1(P(\varphi + t\psi)) = \int_\Omega (-\psi)(dd^c\varphi)^n.$$

Proof. Refer to [Lu15, Lemma 4.11]; see also [ÅCC12, BBGZ13] for related results. ■

Lemma 3.7. *Let μ be a positive Borel measure on Ω such that the functional $\mathcal{F}_\mu$ is proper and lower semicontinuous on $\mathcal{E}^1(\Omega)$. Then there exists $\varphi \in \mathcal{E}^1(\Omega)$ such that*

$$\mathcal{F}_\mu(\varphi) = \inf_{\psi \in \mathcal{E}^1(\Omega)} \mathcal{F}_\mu(\psi).$$

Proof. Let $(\varphi_j) \subset \mathcal{E}^1(\Omega)$ be a sequence such that

$$\lim_{j\to+\infty} \mathcal{F}_\mu(\varphi_j) = \inf_{\psi \in \mathcal{E}^1(\Omega)} \mathcal{F}_\mu(\psi) \leqslant 0.$$

Since $\mathcal{F}_\mu$ is proper, we have $\sup_j E_1(\varphi_j) < +\infty$. It follows that the sequence (φ_j) forms a compact subset of $\mathcal{E}^1(\Omega)$. Therefore, there exists a subsequence of (φ_j), still denoted by (φ_j), which converges to some φ in $L^1_{\mathrm{loc}}(\Omega)$.

Since $\mathcal{F}_\mu$ is lower semicontinuous, we have

$$\liminf_{j\to+\infty} \mathcal{F}_\mu(\varphi_j) \geqslant \mathcal{F}_\mu(\varphi).$$

Hence, we deduce that φ is a minimizer of $\mathcal{F}_\mu$ on $\mathcal{E}^1(\Omega)$. ■

Theorem 3.8. *Let $\varphi \in \mathcal{E}^1(\Omega)$ and $\mu \in \mathcal{M}_1$. Then*

$$(dd^c\varphi)^n = \mu \iff \mathcal{F}_\mu(\varphi) = \inf_{\psi \in \mathcal{E}^1(\Omega)} \mathcal{F}_\mu(\psi).$$

Proof. Let $\mu \in \mathcal{M}_1$ and $\varphi \in \mathcal{E}^1(\Omega)$.

Suppose $(dd^c\varphi)^n = \mu$, and let $\psi \in \mathcal{E}^1(\Omega)$. By [DL22, Lemma 2.11(c)] and Young's inequality, we have

$$\begin{aligned}\int_\Omega (-\psi)\,d\mu = \int_\Omega (-\psi)(dd^c\varphi)^n &\leqslant E_1(\psi)^{\frac{1}{n+1}} E_1(\varphi)^{\frac{n}{n+1}} \\ &\leqslant \frac{1}{n+1} E_1(\psi) + \frac{n}{n+1} E_1(\varphi).\end{aligned}$$

Therefore,

$$\mathcal{F}_\mu(\varphi) = -\frac{n}{n+1} E_1(\varphi) \leqslant \frac{1}{n+1} E_1(\psi) + \int_\Omega \psi\,d\mu = \mathcal{F}_\mu(\psi),$$

which shows that

$$\mathcal{F}_\mu(\varphi) \leqslant \mathcal{F}_\mu(\psi).$$

Thus, φ minimizes $\mathcal{F}_\mu$ on $\mathcal{E}^1(\Omega)$.

Conversely, assume that $\varphi \in \mathcal{E}^1(\Omega)$ is a minimizer of $\mathcal{F}_\mu$. For any $\phi \in \mathcal{T}(\Omega)$, define, for $-1 \leqslant t \leqslant 1$,

$$g(t) = E_1\big(P(\varphi + t\phi)\big) + \int_\Omega (\varphi + t\phi)\, d\mu.$$

When $0 \leqslant t \leqslant 1$, we have $P(\varphi + t\psi) = \varphi + t\psi$, which belongs to $\mathcal{E}^1(\Omega)$ by convexity. For $-1 \leqslant t < 0$, since $\varphi \leqslant \varphi + t\psi$, it follows that $\varphi \leqslant P(\varphi + t\psi)$. Hence, $P(\varphi + t\psi) \in \mathcal{E}^1(\Omega)$. Moreover, since $\varphi + t\psi \in \mathcal{E}^1(\Omega)$ and $\mu \in \mathcal{M}_1$, we have

$$\int_\Omega -(\varphi + t\phi)\, d\mu < +\infty.$$

It follows that the function g is well defined.

We now proceed to show that g achieves its minimum. Since $P(\varphi+t\phi) \leqslant \varphi + t\phi$, we have $g(t) \geqslant \mathcal{F}_\mu(\varphi) = g(0)$ for all t, so g attains its minimum at $t = 0$ and $g'(0) = 0$. Since $\mu \in \mathcal{M}_1$, the function $t \mapsto \int_\Omega (\varphi + t\phi)\, d\mu$ is differentiable, with derivative given by $\int_\Omega \phi\, d\mu$. From Lemma 3.6 it follows that

$$\int_\Omega \phi (dd^c\varphi)^n = \int_\Omega \phi\, d\mu, \qquad \forall \phi \in \mathcal{T}(\Omega).$$

Since any smooth test function χ can be written as $\chi = \phi_1 - \phi_2$ with $\phi_1, \phi_2 \in \mathcal{T}(\Omega)$ (see [Ceg04, Lemma 3.1] and [DL22, Lemma 4.1]), it follows that $(dd^c\varphi)^n = \mu$. ∎

THEOREM 3.9. *Let $\Omega \Subset X$ be a strongly pseudoconvex domain, and let μ be a positive Borel measure on Ω. The following three properties are equivalent:*

(1) *there exists $u \in \mathcal{E}^1(\Omega)$ such that $(dd^c u)^n = \mu$ on Ω,*
(2) *$\mathcal{E}^1(\Omega) \subset L^1(\Omega, \mu)$,*
(3) *there exists a constant $A > 0$ such that for all $\psi \in \mathcal{T}(\Omega)$,*

$$\int_\Omega (-\psi)\, d\mu \leqslant A E_1(\psi)^{\frac{1}{n+1}}. \tag{3.2}$$

Proof. (1)⇒(2)⇒(3) will be shown in Section 4.

(3)⇒(1). Assume that inequality (3.2) holds for all $\psi \in \mathcal{T}(\Omega)$. Then, for any such ψ, we have

$$\mathcal{F}_\mu(\psi) = \frac{1}{n+1} E_1(\psi) + \int_\Omega \psi\, d\mu.$$

Using (3.2), we see that

$$\mathcal{F}_\mu(\psi) \geqslant \frac{1}{n+1} E_1(\psi) - A E_1(\psi)^{\frac{1}{n+1}}.$$

As $E_1(\psi) \to +\infty$, we obtain $\mathcal{F}_\mu(\psi) \to +\infty$, so $\mathcal{F}_\mu$ is proper. Then, by Lemma 3.7 and Theorem 3.8, there exists $\varphi \in \mathcal{E}^1(\Omega)$ such that

$$(dd^c\varphi)^n = \mu. \ \blacksquare$$

As a consequence of this result, we obtain the following Subsolution Theorem, which was also established in [DL22, Theorem E].

COROLLARY 3.10. *Let μ be a positive Borel measure on Ω that does not charge pluripolar sets. If there exists $\varphi \in \mathcal{E}^1(\Omega)$ such that*

$$\mu \leqslant (dd^c\varphi)^n,$$

then there exists $\psi \in \mathcal{E}^1(\Omega)$ such that

$$(dd^c\psi)^n = \mu \quad \text{and} \quad \varphi \leqslant \psi \qquad \text{on } \Omega.$$

Moreover, if $\varphi \in \mathcal{T}(\Omega)$, then $\psi \in \mathcal{T}(\Omega)$.

4. Characterizing the image of $\mathcal{E}^p(\Omega)$. We begin this section with the following results, which will be useful later on.

LEMMA 4.1. *Let $p > 0$, and let (ψ_j) be a decreasing sequence in $\mathcal{E}^p(\Omega)$ such that*

$$\sup_{j\geqslant 1} \int_\Omega (-\psi_j)^p (dd^c\psi_j)^n \leqslant M < +\infty.$$

Then $\psi = \lim_{j\to+\infty} \psi_j \in \mathcal{E}^p(\Omega)$ and

$$E_p(\psi) \leqslant \liminf_{j\to+\infty} E_p(\psi_j).$$

Proof. The argument is the same as in the case of $\mathcal{E}^1(\Omega)$. ■

PROPOSITION 4.2. *Let $p > 0$, and let (ψ_j) be a sequence in $\mathcal{T}(\Omega)$ such that the energies $E_p(\psi_j)$ are bounded. Then*

$$\psi = \sum_{j=1}^{+\infty} 4^{-j}\psi_j \in \mathcal{E}^p(\Omega).$$

Proof. See Remark 5.7(1) below. ■

We proceed to prove the characterization of the image of $\mathcal{E}^p(\Omega)$ via the complex Monge–Ampère operator.

THEOREM 4.3. *Let $\Omega \Subset X$ be a strongly pseudoconvex domain, μ a positive Borel measure on Ω, and $p > 0$. Then the following three properties are equivalent:*

(1) *there exists $u \in \mathcal{E}^p(\Omega)$ such that $(dd^c u)^n = \mu$ on Ω,*
(2) *$\mathcal{E}^p(\Omega) \subset L^p(\Omega, \mu)$,*

(3) *there exists a constant $A > 0$ such that for all $\psi \in \mathcal{T}(\Omega)$,*

$$\int_\Omega (-\psi)^p \, d\mu \leqslant A E_p(\psi)^{\frac{p}{n+p}}. \tag{4.1}$$

Proof. (1)⇒(2). Suppose there exists $u \in \mathcal{E}^p(\Omega)$ such that $(dd^c u)^n = \mu$ on Ω, and let $\psi \in \mathcal{E}^p(\Omega)$. Then

$$\begin{aligned}\int_\Omega (-\psi)^p (dd^c u)^n &= \int_{-\infty}^0 p s^{p-1}\Big[\int_{\{\psi<s\}} (dd^c u)^n\Big] ds \\ &= 2^p \int_{-\infty}^0 p t^{p-1}\Big[\int_{\{\psi<2t\}} (dd^c u)^n\Big] dt.\end{aligned}$$

Since

$$\{\psi < 2t\} \subset \{\psi < u + t\} \cup \{u < t\},$$

we have

$$\int_{\{\psi<2t\}} (dd^c u)^n \leqslant \int_{\{\psi<u+t\}} (dd^c u)^n + \int_{\{u<t\}} (dd^c u)^n.$$

By the comparison principle (Proposition 2.9),

$$\int_{\{\psi<u+t\}} (dd^c u)^n = \int_{\{\psi<u+t\}} (dd^c (u+t))^n \leqslant \int_{\{\psi<u+t\}} (dd^c \psi)^n.$$

Since $\{\psi < u + t\} \subset \{\psi < t\}$, it follows that

$$\int_{\{\psi<u+t\}} (dd^c \psi)^n \leqslant \int_{\{\psi<t\}} (dd^c \psi)^n.$$

Hence,

$$\begin{aligned}&\int_{-\infty}^0 p t^{p-1}\Big[\int_{\{\psi<2t\}} (dd^c u)^n\Big] dt \\ &\qquad \leqslant 2^p \int_{-\infty}^0 p t^{p-1} \int_{\{\psi<t\}} (dd^c \psi)^n \, dt + 2^p \int_{-\infty}^0 p t^{p-1} \int_{\{u<t\}} (dd^c u)^n \, dt.\end{aligned}$$

Consequently,

$$\int_\Omega (-\psi)^p (dd^c u)^n \leqslant 2^p [E_p(\psi) + E_p(u)] < +\infty.$$

This implies

$$\mathcal{E}^p(\Omega) \subset L^p(\Omega, \mu).$$

(2)⇒(3). We argue by contraposition, assuming that for every $A > 0$, there exists $\psi \in \mathcal{T}(\Omega)$ such that

$$\int_{\Omega} (-\psi)^p \, d\mu > A E_p(\psi)^{\frac{p}{n+p}}. \tag{4.2}$$

Since both sides of (4.2) are homogeneous of degree p, we may replace ψ by $\varphi = \psi/E_p(\psi)$, thus assuming $E_p(\psi) = 1$.

Inequality (4.2) implies that for each $j \in \mathbb{N}$, there exists $\psi_j \in \mathcal{T}(\Omega)$ such that (4.2) holds with $A_j = 8^{jp}$ and $E_p(\psi_j) = 1$ for all j.

Consider the function

$$\psi = \sum_{j=1}^{\infty} 4^{-j} \psi_j.$$

By Proposition 4.2, $\psi \in \mathcal{E}^p(\Omega)$.

Since $-\psi \geqslant 4^{-j}(-\psi_j)$ for all j, it follows that

$$(-\psi)^p \geqslant 4^{-jp}(-\psi_j)^p,$$

and thus

$$\int_{\Omega} (-\psi)^p \, d\mu \geqslant \int_{\Omega} 4^{-jp}(-\psi_j)^p \, d\mu = 4^{-jp} \int_{\Omega} (-\psi_j)^p \, d\mu \geqslant 4^{jp}, \qquad \forall j \in \mathbb{N}^*.$$

Therefore,

$$\int_{\Omega} (-\psi)^p \, d\mu \geqslant 4^{jp},$$

and $4^{jp} \to +\infty$ as $j \to +\infty$, which implies $\psi \notin L^p(\Omega, \mu)$. Hence,

$$\mathcal{E}^p(\Omega) \not\subset L^p(\Omega, \mu).$$

Thus, condition (4.1) follows from property (2).

(3)⇒(1). To prove this implication, we follow the argument of [BGZ09, Theorem 5.2] by defining

$$F(t) = A t^{\frac{p}{n+p}}.$$

Note that

$$\lim_{t \to +\infty} \frac{F(t)}{t} = 0.$$

First, inequality (4.1) extends to all $\psi \in \mathcal{E}^p(\Omega)$ by approximation (see Theorem 2.11).

Moreover, according to [DL22, Lemma 2.1], every pluripolar set is contained in the polar set of some function $\psi \in \mathcal{E}^p(\Omega)$.

Since (4.1) holds for functions in $\mathcal{E}^p(\Omega)$, it follows that $(-\psi)^p$ is μ-integrable on Ω, so the set where it is infinite has zero μ-measure. Hence, (4.1) implies that μ does not charge pluripolar sets.

We proceed in two steps:

Step 1: Assume μ has compact support $K \Subset \Omega$. By the generalized Radon–Nikodym theorem (see [Ra69] and [Ceg98, Theorem 6.3]), there exist $f \in L^1((dd^c\psi)^n)$ and $\psi \in \mathcal{T}(\Omega)$ such that

$$\mu = f(dd^c\psi)^n.$$

Consider the measures

$$\mu_j := \min(f, j^n)(dd^c\psi)^n \leqslant (dd^c(j\psi))^n,$$

where $j\psi \in \mathcal{T}(\Omega)$.

Since $\mu_j \leqslant (dd^c(j\psi))^n$ and $j\psi \in \mathcal{T}(\Omega)$, Corollary 3.10 ensures the existence of $u_j \in \mathcal{T}(\Omega) \subset \mathcal{E}^p(\Omega)$ such that

$$(dd^c u_j)^n = \mu_j.$$

Using $\mu_j \leqslant \mu$ and applying (4.1) to u_j, we get

$$\int_\Omega (-u_j)^p (dd^c u_j)^n = \int_\Omega (-u_j)^p \, d\mu_j \leqslant \int_\Omega (-u_j)^p \, d\mu \leqslant A E_p(u_j)^{\frac{p}{n+p}}.$$

Therefore,

$$E_p(u_j)^{\frac{n}{n+p}} \leqslant A.$$

By Lemma 4.1, the decreasing sequence (u_j) converges to some $u \in \mathcal{E}^p(\Omega)$. By continuity of the Monge–Ampère operator under decreasing limits,

$$(dd^c u_j)^n \to \mu \quad \text{and} \quad (dd^c u)^n = \mu.$$

Step 2: General case. Let (K_j) be an increasing exhaustion of Ω by compacts, and define

$$\mu_j = \mathbb{1}_{K_j}\mu.$$

Each μ_j has compact support; hence, by Step 1, there exist $u_j \in \mathcal{T}(\Omega)$ such that

$$(dd^c u_j)^n = \mu_j.$$

Since (μ_j) increases to μ, the comparison principle implies (u_j) is decreasing. Let

$$u = \lim_{j \to +\infty} u_j.$$

From (4.1), we have $E_p(u_j) \leqslant A^{\frac{n+p}{n}}$, so $\sup_j E_p(u_j) < +\infty$. By Lemma 4.1, $u \in \mathcal{E}^p(\Omega)$. Taking the limit and using the continuity of the Monge–Ampère operator for decreasing sequences, we conclude that

$$(dd^c u)^n = \mu. \ \blacksquare$$

5. Characterizing the image of $\mathcal{E}_\chi(\Omega)$. This section is devoted to the proof of our second main result. We begin with some preliminary notions.

Definition 5.1. Let $K \subset \Omega$ be a compact subset. Its *Monge–Ampère capacity* relative to Ω is defined by

$$\mathrm{Cap}_\Omega(K) := \sup\Big\{\int_K (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \leqslant u \leqslant 0\Big\}.$$

Proposition 5.2 ([Koł96]). *Let $u \in \mathcal{T}(\Omega)$. Then, for every $s > 0$ and $t > 0$,*

$$t^n \,\mathrm{Cap}_\Omega(\{u < -s - t\}) \leqslant \int_{\{u<-s\}} (dd^c u)^n \leqslant s^n \,\mathrm{Cap}_\Omega(\{u < -s\}). \tag{5.1}$$

Definition 5.3. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function such that $\chi(0) = 0$. The class $\widehat{\mathcal{E}}_\chi(\Omega)$ is defined by

$$\widehat{\mathcal{E}}_\chi(\Omega) := \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \int_{t_\chi}^{+\infty} t^n \chi'(-t)\,\mathrm{Cap}_\Omega(\{\varphi < -t\})\,dt < +\infty\Big\}.$$

Definition 5.4. Let μ be a positive Borel measure on Ω and let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a convex increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) = 0$. We define

$$L_\chi(\Omega, \mu) := \Big\{u\ \mu\text{-measurable in } \Omega : \int_\Omega |\chi(u)|\,d\mu < +\infty\Big\}.$$

Following the approach in [BGZ09, Ben15], we obtain the proposition below:

Proposition 5.5. *We have the inclusion $\widehat{\mathcal{E}}_\chi(\Omega) \subset \mathcal{E}_\chi(\Omega)$, and conversely $\mathcal{E}_\chi(\Omega) \subset \widehat{\mathcal{E}}_{\widehat{\chi}}(\Omega)$, where $\widehat{\chi}(t) = \chi(t/2)$. Moreover, if $\chi : \,]-\infty, -t_\chi[\, \to \mathbb{R}^-$ is convex, then*

$$\mathcal{E}_\chi(\Omega) = \widehat{\mathcal{E}}_\chi(\Omega).$$

Here, t_χ is a real number such that $\chi(t) < 0$ for all $t < -t_\chi$, and $\chi(t) = 0$ for all $t \geqslant -t_\chi$.

Proposition 5.6. *Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a convex, increasing function satisfying $\chi(-\infty) = -\infty$ and $\chi(0) = 0$. Then*

(1) *for every $\alpha \geqslant 0$ and $u \in \mathcal{E}_\chi(\Omega)$, we have $\alpha u \in \mathcal{E}_\chi(\Omega)$; in other words, $\mathcal{E}_\chi(\Omega)$ is a positive cone,*
(2) *the set $\mathcal{E}_\chi(\Omega)$ is convex,*
(3) *if $(\psi_j) \subset \mathcal{T}(\Omega)$ is a sequence with $\sup_j E_\chi(\psi_j) < +\infty$, then*

$$\psi = \sum_{j=1}^{\infty} 4^{-j}\psi_j \ \textit{belongs to}\ \mathcal{E}_\chi(\Omega).$$

Proof. (1) Let $\alpha \geqslant 0$ and $u \in \mathcal{E}_\chi(\Omega)$. There exists a decreasing sequence $(u_j) \subset \mathcal{T}(\Omega)$ converging to u, hence $\alpha u_j \downarrow \alpha u$.

For $0 \leqslant \alpha \leqslant 1$, since $\alpha u_j \geqslant u_j$ and χ is increasing, we have $-\chi(\alpha u_j) \leqslant -\chi(u_j)$. Thus

$$\sup_j \int_\Omega -\chi(\alpha u_j)(dd^c \alpha u_j)^n \leqslant \sup_j \int_\Omega -\chi(\alpha u_j)(dd^c u_j)^n \leqslant \sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n < +\infty,$$

which implies that $\alpha u \in \mathcal{E}_\chi(\Omega)$.

For $\alpha > 1$, define $f(t) := -\chi(\alpha t) + \alpha\chi(t)$. Since χ' is increasing and $\alpha > 1$, we see that f is nondecreasing with $f(0) = 0$, hence $f(t) \leqslant 0$. This yields

$$-\chi(\alpha t) \leqslant -\alpha\chi(t),$$

and consequently

$$\sup_j \int_\Omega -\chi(\alpha u_j)(dd^c \alpha u_j)^n \leqslant \alpha^{n+1} \sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n < +\infty,$$

which shows $\alpha u \in \mathcal{E}_\chi(\Omega)$.

Therefore, $\mathcal{E}_\chi(\Omega)$ is a positive cone.

(2) For $a \in [0, 1]$ and $\varphi, \psi \in \mathcal{E}_\chi(\Omega)$, let $\Phi := a\varphi + (1 - a)\psi$. Since

$$\{\Phi < -t\} \subset \{\varphi < -t\} \cup \{\psi < -t\},$$

by subadditivity of capacity we have

$$\mathrm{Cap}_\Omega(\{\Phi < -t\}) \leqslant \mathrm{Cap}_\Omega(\{\varphi < -t\}) + \mathrm{Cap}_\Omega(\{\psi < -t\}).$$

It follows that

$$\begin{aligned}\int_\Omega -\chi(\Phi)(dd^c\Phi)^n &= \int_0^{+\infty} \chi'(-t)(dd^c\Phi)^n(\{\Phi < -t\})\,dt \\ &\leqslant \int_0^{+\infty} \chi'(-t)t^n\,\mathrm{Cap}_\Omega(\{\Phi < -t\})\,dt \\ &\leqslant \int_0^{+\infty} \chi'(-t)t^n\,\mathrm{Cap}_\Omega(\{\varphi < -t\})\,dt \\ &\quad + \int_0^{+\infty} \chi'(-t)t^n\,\mathrm{Cap}_\Omega(\{\psi < -t\})\,dt \\ &< +\infty,\end{aligned}$$

hence $\Phi \in \mathcal{E}_\chi(\Omega)$ by Proposition 5.5. This proves convexity.

(3) As in the proof of [Ben15, Theorem 3], let $(\psi_j)_j \subset \mathcal{T}(\Omega)$ and set

$$\psi := \sum_{j=1}^{+\infty} 4^{-j}\psi_j.$$

Since

$$\{\psi < -t\} \subset \bigcup_{j=1}^{\infty} \{2^{-j}\psi_j < -t\},$$

the subadditivity of capacity implies

$$\mathrm{Cap}_\Omega(\{\psi < -t\}) \leqslant \sum_{j=1}^{\infty} \mathrm{Cap}_\Omega(\{2^{-j}\psi_j < -t\}).$$

Therefore,

$$\begin{aligned}\int_\Omega -\chi(\psi)(dd^c\psi)^n &= \int_0^{+\infty} \chi'(-t)(dd^c\psi)^n(\{\psi < -t\})\,dt\\ &\leqslant \int_0^{+\infty} \chi'(-t)t^n\,\mathrm{Cap}_\Omega(\{\psi < -t\})\,dt\\ &\leqslant \sum_{j=1}^{\infty}\int_0^{+\infty} \chi'(-t)t^n\,\mathrm{Cap}_\Omega(\{2^{-j}\psi_j < -t\})\,dt\\ &= \sum_{j=1}^{\infty} 2^{-n(j-1)}\int_0^{+\infty} \chi'(-t)(2^{j-1}t)^n\,\mathrm{Cap}_\Omega(\{\psi_j < -2^jt\})\,dt.\end{aligned}$$

Using the first inequality of (5.1) with $2^{j-1}t$ for both variables, we have

$$\begin{aligned}(2^{j-1}t)^n\,\mathrm{Cap}_\Omega(\{\psi_j < -2^jt\}) &\leqslant (dd^c\psi_j)^n(\{\psi_j < -2^{j-1}t\})\\ &\leqslant (dd^c\psi)^n(\{\psi_j < -t\}),\end{aligned}$$

since $-2^{j-1}t \leqslant -t$. As $\chi'(-t) \geqslant 0$, it follows that

$$\int_0^{+\infty} \chi'(-t)(2^{j-1}t)^n\,\mathrm{Cap}_\Omega(\{\psi_j < -2^jt\})\,dt \leqslant \int_0^{+\infty} \chi'(-t)(dd^c\psi_j)^n(\{\psi_j < -t\})\,dt.$$

Since the $E_\chi(\psi_j)$ are uniformly bounded, there exists a constant $C > 0$ such that

$$2^{-n(j-1)}\int_0^{+\infty} \chi'(-t)(dd^c\psi_j)^n(\{\psi_j < -t\})\,dt \leqslant 2^n2^{-nj}C.$$

Consequently,

$$\int_{\Omega} -\chi(\psi)(dd^c\psi)^n \leqslant \frac{C}{2^{n+1}-1},$$

which shows that $\psi \in \mathcal{E}_\chi(\Omega)$. ∎

Remark 5.7. (1) It is worth noting that in the proof of point (3) in the proposition, the passage leading from

$$\sum_{j=1}^{\infty} 2^{-n(j-1)} \int_0^{+\infty} \chi'(-t)(2^{j-1}t)^n \operatorname{Cap}_\Omega(\{\psi_j < -2^j t\})\, dt$$

to the geometric series bound $C\sum_{j=1}^{\infty} 2^{-j(n+1)} < \infty$ does not rely on the convexity of χ. Indeed, it only uses the first inequality of (5.1) and the uniform bound on the integrals. Therefore, this argument remains valid in the setting of Proposition 4.2.

(2) Let $u \in \mathcal{E}_\chi(\Omega)$ and $v \in \mathcal{E}_\chi(\Omega)$. Then

$$u + v \in \mathcal{E}_\chi(\Omega).$$

Indeed, observe that

$$u + v = 2\big(\tfrac{1}{2}u + \tfrac{1}{2}v\big).$$

Since $\mathcal{E}_\chi(\Omega)$ is a positive convex cone, it follows that

$$\tfrac{1}{2}u + \tfrac{1}{2}v \in \mathcal{E}_\chi(\Omega),$$

and hence $u + v \in \mathcal{E}_\chi(\Omega)$.

Lemma 5.8. *Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing function such that $\chi(-\infty) = -\infty$. Let (ψ_j) be a decreasing sequence in $\mathcal{E}_\chi(\Omega)$ satisfying*

$$\sup_{j\geqslant 1} \int_{\Omega} -\chi(\psi_j)(dd^c\psi_j)^n \leqslant M < +\infty.$$

Then $\psi := \lim_{j\to+\infty} \psi_j \in \mathcal{E}_\chi(\Omega)$, and

$$E_\chi(\psi) \leqslant \liminf_{j\to+\infty} E_\chi(\psi_j).$$

Proof. The argument is the same as in the case of $\mathcal{E}^1(\Omega)$. ∎

We now state the following result, which provides a characterization of the image of $\mathcal{E}_\chi(\Omega)$ (offering a unified treatment of all classes $\mathcal{E}^p(\Omega)$ with $0 < p \leqslant 1$) under the complex Monge–Ampère operator. However, the main difficulty is that, unlike the energy E_p, the weighted energy E_χ is not homogeneous.

Theorem 5.9. *Let $\Omega \Subset X$ be a strongly pseudoconvex domain, let μ be a positive Borel measure on Ω, and let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a convex increasing function such that $\chi(-\infty) = -\infty$ and $\chi(0) = 0$. Then the following properties are equivalent:*

(1) *there exists a function* $u \in \mathcal{E}_\chi(\Omega)$ *such that* $(dd^c u)^n = \mu$ *in* Ω,
(2) $\mathcal{E}_\chi(\Omega) \subset L_\chi(\Omega, \mu)$,
(3) *there exists a constant* $c_\mu > 0$ *such that for all* $\psi \in \mathcal{T}(\Omega)$,

$$\int_\Omega -\chi(\psi)\, d\mu \leqslant c_\mu \big(E_\chi(\psi)^{\frac{1}{n+1}} + 1\big), \tag{5.2}$$

(3′) *there exists a constant* $c_\mu > 0$ *such that for every* $\psi \in \mathcal{T}(\Omega)$,

$$\int_\Omega -\chi(\psi)\, d\mu \leqslant c_\mu \big(E_\chi(\psi)^{1/n} + 1\big). \tag{5.3}$$

Proof. We will show the following chain of implications:

$$(1)\Rightarrow(3)\Rightarrow(3')\Rightarrow(1)\Rightarrow(2)\Rightarrow(3').$$

(1)⇒(3). Assume that there exists $u \in \mathcal{E}_\chi(\Omega)$ such that $(dd^c u)^n = \mu$ in Ω. By approximation, it suffices to consider $\psi \in \mathcal{T}(\Omega)$. Define $h(t) := -\chi(-t)$, which is increasing and concave. Since $h(0) = 0$, we have

$$h(t) \leqslant 2h(t/2).$$

Therefore,

$$\int_\Omega h(-\psi)\, d\mu \leqslant 2 \int_\Omega h(-\psi/2)\, d\mu = \int_0^{+\infty} h'(t)\, \mu(\{\psi/2 < -t\})\, dt.$$

Let $c \geqslant 1$. Then

$$\{\psi < -2t\} \subset \{\psi < cu - t\} \cup \{cu < -t\},$$

so

$$\begin{aligned}
\int_{\{\psi<-2t\}} (dd^c u)^n &\leqslant \int_{\{\psi<cu-t\}} (dd^c u)^n + \int_{\{cu<-t\}} (dd^c u)^n \\
&= c^{-n} \int_{\{\psi<cu-t\}} (dd^c (cu-t))^n + \int_{\{u<-t/c\}} (dd^c u)^n \\
&\leqslant c^{-n} \int_{\{\psi<cu-t\}} (dd^c \psi)^n + \int_{\{u<-t/c\}} (dd^c u)^n \\
&\leqslant c^{-n} \int_{\{\psi<-t\}} (dd^c \psi)^n + \int_{\{u<-t/c\}} (dd^c u)^n,
\end{aligned}$$

where the second-to-last inequality follows from the comparison principle, and the last one comes from the fact that $\{\psi < cu - t\} \subset \{\psi < -t\}$.

Hence,

$$\int_\Omega h(-\psi)\,d\mu \leqslant 2\int_0^{+\infty} h'(t)\Big[c^{-n}\int_{\{\psi<-t\}}(dd^c\psi)^n + \int_{\{u<-t/c\}}(dd^cu)^n\Big]\,dt$$
$$= 2c^{-n}\int_0^{+\infty} h'(t)\int_{\{\psi<-t\}}(dd^c\psi)^n\,dt$$
$$+ 2c\int_0^{+\infty} h'(cs)\int_{\{u<-s\}}(dd^cu)^n\,ds$$
$$\leqslant 2c^{-n}E_\chi(\psi) + 2cE_\chi(u),$$

since h' is decreasing and $c \geqslant 1$. We thus obtain

$$\int_\Omega h(-\psi)\,d\mu \leqslant 2[c^{-n}E_\chi(\psi) + cE_\chi(u)] \tag{5.4}$$

for every $c \geqslant 1$.

Set

$$c_0 := \left(\frac{E_\chi(\psi)}{E_\chi(u)}\right)^{\frac{1}{n+1}}.$$

If $c_0 \geqslant 1$ (i.e., $E_\chi(\psi) \geqslant E_\chi(u)$), then from (5.4) we get

$$\int_\Omega h(-\psi)\,d\mu \leqslant 2[c_0^{-n}E_\chi(\psi) + c_0E_\chi(u)]$$
$$= 2\left(\left(\frac{E_\chi(\psi)}{E_\chi(u)}\right)^{-\frac{n}{n+1}}E_\chi(\psi) + \left(\frac{E_\chi(\psi)}{E_\chi(u)}\right)^{\frac{1}{n+1}}E_\chi(u)\right)$$
$$= 2E_\chi(\psi)^{\frac{1}{n+1}}(E_\chi(u)^{\frac{n}{n+1}} + E_\chi(u)^{\frac{n}{n+1}}) = 4E_\chi(u)^{\frac{n}{n+1}}E_\chi(\psi)^{\frac{1}{n+1}}.$$

So, by setting $c_\mu = 4E_\chi(u)^{\frac{n}{n+1}}$, we obtain

$$\int_\Omega h(-\psi)\,d\mu \leqslant c_\mu E_\chi(\psi)^{\frac{1}{n+1}}.$$

If $c_0 < 1$, we use (5.4) with $c = 1$:

$$\int_\Omega h(-\psi)\,d\mu \leqslant 2[E_\chi(\psi) + E_\chi(u)] \leqslant 2[E_\chi(\psi)^{\frac{1}{n+1}}E_\chi(u)^{\frac{n}{n+1}} + E_\chi(u)]$$
$$\leqslant c_\mu(E_\chi(\psi)^{\frac{1}{n+1}} + 1),$$

where

$$c_\mu := \max(2E_\chi(u)^{\frac{n}{n+1}}, 2E_\chi(u)).$$

(3)⇒(3′). Assume (5.2). Let $\psi \in \mathcal{T}(\Omega)$.

If $E_\chi(\psi) \geqslant 1$, then

$$E_\chi(\psi)^{\frac{1}{n+1}} \leqslant E_\chi(\psi)^{1/n},$$

so

$$c_\mu(E_\chi(\psi)^{\frac{1}{n+1}}+1)\leqslant c_\mu(E_\chi(\psi)^{1/n}+1).$$

If $E_\chi(\psi)\leqslant 1$, then

$$E_\chi(\psi)^{\frac{1}{n+1}}\leqslant 1+E_\chi(\psi)^{1/n},$$

hence

$$c_\mu(E_\chi(\psi)^{1/n}+2)\leqslant 2c_\mu(E_\chi(\psi)^{1/n}+1).$$

Therefore, for every $\psi\in\mathcal{T}(\Omega)$, inequality (5.2) leads to (5.3).

(3′)⇒(1). Define $F:\mathbb{R}^+\to\mathbb{R}^+$ by

$$F(t)=c_\mu(t^{1/n}+1).$$

Then F is locally integrable on $\mathbb{R}^+$. Moreover, for $n\geqslant 2$,

$$\lim_{t\to+\infty}\frac{F(t)}{t}=\lim_{t\to+\infty}c_\mu(t^{\frac{1-n}{n}}+t^{-1})=0.$$

Using inequality (5.3), we have

$$\int_\Omega -\chi(\psi)\,d\mu\leqslant F(E_\chi(\psi))\leqslant c_\mu(E_\chi(\psi)^{1/n}+1)$$

for all $\psi\in\mathcal{T}(\Omega)$. Thus, arguing as in the case of $\mathcal{E}^p(\Omega)$, we see that μ does not charge pluripolar sets.

We can proceed in two steps:

STEP 1: Assume that μ has compact support $K\Subset\Omega$. According to the generalized Radon–Nikodym theorem (see [Ra69] and [Ceg98, Theorem 6.3]) we have

$$\mu=f(dd^c\psi)^n\quad\text{ with } f\in L^1((dd^c\psi)^n)\text{ and }\psi\in\mathcal{T}(\Omega).$$

Consider the sequence

$$\mu_j:=\min(f,j^n)(dd^c\psi)^n\leqslant(dd^cj\psi)^n,$$

where $j\psi\in\mathcal{T}(\Omega)\subset\mathcal{E}^1(\Omega)$.

Since $\mu_j\leqslant(dd^cj\psi)^n$, and $j\psi\in\mathcal{T}(\Omega)$, it follows from Corollary 3.10 that μ_j is the Monge–Ampère measure of a function in $\mathcal{T}(\Omega)$, i.e., there exists $u_j\in\mathcal{T}(\Omega)$ such that $(dd^cu_j)^n=\mu_j$.

Since $\mu_j\leqslant\mu$, applying assumption (5.3) to $u_j\in\mathcal{T}(\Omega)$ yields

$$\int_\Omega -\chi(u_j)(dd^cu_j)^n=\int_\Omega -\chi(u_j)\,d\mu_j\leqslant\int_\Omega -\chi(u_j)\,d\mu\leqslant c_\mu(E_\chi(u_j)^{1/n}+1).$$

It follows that $E_\chi(u_j)\leqslant\max(2c_\mu,(2c_\mu)^{\frac{1}{n+1}})$. By Lemma 5.8, we have $u_j\downarrow u\in\mathcal{E}_\chi(\Omega)$. By the continuity of the Monge–Ampère operator for decreasing sequences, we obtain

$$(dd^cu_j)^n\to\mu\quad\text{and}\quad(dd^cu)^n=\mu.$$

STEP 2: General case. Let (K_j) be an increasing exhaustive sequence of compact subsets of Ω, and define

$$\mu_j = \mathbb{1}_{K_j}\mu.$$

Then μ_j has compact support, and from Step 1, there exists $u_j \in \mathcal{T}(\Omega)$ such that $(dd^c u_j)^n = \mu_j$. Since (μ_j) is increasing, by the comparison principle, (u_j) is decreasing, and $\sup_j E_\chi(u_j) < +\infty$. Therefore, $\lim_j u_j = u \in \mathcal{E}_\chi(\Omega)$. By the continuity of the Monge–Ampère operator for decreasing sequences, we conclude that $(dd^c u)^n = \mu$.

(1)⇒(2). Proving property (2) is equivalent to proving that $\chi(\mathcal{E}_\chi(\Omega)) \subset L^1(\Omega, d\mu)$. Now assume that there exists $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $\mu = (dd^c\varphi)^n$, and let us show that $\chi(\mathcal{E}_\chi(\Omega)) \subset L^1(\Omega, \mu)$. Let $u \in \mathcal{E}_\chi(\Omega)$. According to Remark 5.7, we have $u + \varphi \in \mathcal{E}_\chi(\Omega)$. Hence,

$$\int_\Omega -\chi(u)\, d\mu = \int_\Omega -\chi(u)(dd^c\varphi)^n \leqslant \int_\Omega -\chi(u+\varphi)(dd^c(u+\varphi))^n < +\infty.$$

Therefore, $\chi(u) \in L^1(\Omega, d\mu)$, i.e., $u \in L_\chi(\Omega, \mu)$.

(2)⇒(3′). We argue by contraposition. Suppose that for every $j \in \mathbb{N}$, there exists a function $\psi_j \in \mathcal{T}(\Omega)$ such that

$$\int_\Omega -\chi(\psi_j)\, d\mu > 8^j (E_\chi(\psi_j)^{1/n} + 1). \tag{5.5}$$

If the sequence of χ-energies $(E_\chi(\psi_j))_j$ is bounded, then by Proposition 5.6, we obtain

$$\psi = \sum_{j=1}^\infty 4^{-j}\psi_j \in \mathcal{E}_\chi(\Omega).$$

Since each ψ_j is negative, we have $\psi \leqslant 4^{-j}\psi_j$ for each j. As $-\chi$ is decreasing, it follows that

$$\int_\Omega -\chi(\psi)\, d\mu \geqslant \int_\Omega -\chi(4^{-j}\psi_j)\, d\mu \geqslant 4^j (E_\chi(\psi_j)^{1/n} + 1) \geqslant 4^j.$$

Since $4^j \to +\infty$ as $j \to +\infty$, we have $\psi \notin L_\chi(\Omega, \mu)$, so $\mathcal{E}_\chi(\Omega) \not\subset L_\chi(\Omega, \mu)$.

The same argument applies if there is a bounded subsequence $(E_\chi(\psi_{j_k}))$ of $(E_\chi(\psi_j))_j$.

Now, assume instead that $E_\chi(\psi_j) \to +\infty$ as $j \to +\infty$. We may suppose that $E_\chi(\psi_j) > 1$ for all j. Consider

$$\psi = \sum_{j=1}^\infty 4^{-j}\varepsilon_j\psi_j \quad \text{with } \varepsilon_j = E_\chi(\psi_j)^{-1/n}.$$

Then

$$E_\chi(\varepsilon_j\psi_j) = \int_\Omega -\chi(\varepsilon_j\psi_j)(dd^c(\varepsilon_j\psi_j))^n = \varepsilon_j^n \int_\Omega -\chi(\varepsilon_j\psi_j)(dd^c\psi_j)^n$$
$$\leqslant \varepsilon_j^n \int_\Omega -\chi(\psi_j)(dd^c\psi_j)^n = 1.$$

Hence,

$$\psi = \sum_{j=1}^{\infty} 4^{-j}\varepsilon_j\psi_j \in \mathcal{E}_\chi(\Omega).$$

Since the ψ_j are negative, we have $\psi \leqslant 4^{-j}\varepsilon_j\psi_j$, and thus

$$\int_\Omega -\chi(\psi)\,d\mu \geqslant \int_\Omega -\chi(4^{-j}\varepsilon_j\psi_j)\,d\mu \geqslant 4^{-j}\varepsilon_j \int_\Omega -\chi(\psi_j)\,d\mu$$
$$\geqslant 4^j\varepsilon_j(E_\chi(\psi_j)^{1/n} + 1) \geqslant 4^j \to +\infty.$$

Since $4^j \to +\infty$ as $j \to +\infty$, we have $\psi \notin L_\chi(\Omega,\mu)$, so $\mathcal{E}_\chi(\Omega) \not\subset L_\chi(\Omega,\mu)$.

Therefore, property (2) implies property (3′). ∎

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References

[ÅCC12] P. Åhag, U. Cegrell and R. Czyż, *On Dirichlet's principle and problem*, Math. Scand. 110 (2012), 235–250.

[Bed82] E. Bedford, *The operator $(dd^c)^n$ on complex spaces*, in: Séminaire Pierre Lelong – Henri Skoda (Analyse) Années 1980/81, Lecture Notes in Math. 919, Springer, New York, 1982, 294–323.

[BT76] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation*, Invent. Math. 37 (1976), 1–44.

[BT82] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[Ben15] S. Benelkourchi, *Weighted pluricomplex energy II*, Int. J. Partial Differ. Equ. 2015, art. 947819, 8 pp.

[BGZ09] S. Benelkourchi, V. Guedj and A. Zeriahi, *Plurisubharmonic functions with weak singularities*, in: Complex Analysis and Digital Geometry, Uppsala Univ., Uppsala, 2009, 57–74.

[BBGZ13] R. J. Berman, S. Boucksom, V. Guedj, and A. Zeriahi, *A variational approach to complex Monge–Ampère equations*, Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179–245.

[Bun66] L. Bungart, *Integration on real analytic varieties II. Stokes formula*, J. Math. Mech. 15 (1966), 1047–1054.

[Ceg98] U. Cegrell, *Pluricomplex energy*, Acta Math. 180 (1998), 187–217.

[Ceg04] U. Cegrell, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[Dem85] J.-P. Demailly, *Mesures de Monge–Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France 19 (1985), 124 pp.

[DL22] N. Q. Dieu and T. V. Long, *Plurisubharmonic functions and Monge–Ampère operators on complex varieties in bounded domains of* $\mathbb{C}^n$, J. Math. Anal. Appl. 505 (2022), no. 1, art. 125477, 40 pp.

[FN80] J. E. Fornæss and R. Narasimhan, *The Levi problem on complex spaces with singularities*, Math. Ann. 228 (1980), 47–72.

[GGZ23] V. Guedj, H. Guenancia, and A. Zeriahi, *Continuity of singular Kähler–Einstein potentials*, Int. Math. Res. Notices 2023, 1355–1377.

[GZ07] V. Guedj and A. Zeriahi, *The weighted Monge–Ampère energy of quasiplurisubharmonic functions*, J. Funct. Anal. 250 (2007), 442–482.

[Koł96] S. Kołodziej, *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge–Ampère operator*, Ann. Polon. Math. 65 (1996), 11–21.

[Lu15] C. H. Lu, *A variational approach to complex Hessian equations in* $\mathbb{C}^n$, J. Math. Anal. Appl. 431 (2015), 228–259.

[Ra69] J. Rainwater, *A note on the preceding paper*, Duke Math. J. 36 (1969), 798–800.

[TV22] D. D. Thai and D. V. Vu, *Complex Monge–Ampère equations with solutions in finite energy classes*, Math. Res. Lett. 29 (2022), 1659–1683.

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