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Hermitian geometry and holomorphic curves on C^*-algebras

by

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Abstract. A classical problem in Hermitian geometry, known as the equivalence problem, is to determine when two Hermitian holomorphic vector bundles are locally or globally equivalent. M. J. Cowen and R. G. Douglas related this problem to the issue of unitary equivalence of operators and introduced a broad and important class of operators denoted by $\mathcal{B}_n^m(\Omega)$. They showed that the curvature and its covariant derivatives of the universal complex vector bundle associated with an operator in $\mathcal{B}_n^m(\Omega)$ serve as geometric invariants. Subsequently, C. Apostol, M. Martin, and others developed a more extensive approach to study this problem, providing a complete characterization of the equivalence of holomorphic mappings related to Grassmann manifolds in the context of C^*-algebras. A natural question arises: in this broader non-commutative setting, what are the geometric quantities such as curvature and connection that correspond to these holomorphic mappings?

This paper aims to define the curvature and its covariant derivatives for extended holomorphic curves in the setting of C^*-algebras in the multivariable case, and to establish connections between these geometric quantities and their counterparts in complex geometry. As applications, the results obtained are used to study the unitary classification and similarity classification of the Cowen–Douglas class $\mathcal{B}_n^m(\Omega)$.

1. Introduction. Given an infinite-dimensional, complex, separable Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$, and let $\mathcal{G}r(n,\mathcal{H})$ represent the *Grassmann manifold*, consisting of all n-dimensional subspaces of $\mathcal{H}$. For a bounded domain Ω in m-dimensional complex space $\mathbb{C}^m$, a map $f : \Omega \to \mathcal{G}r(n,\mathcal{H})$ is called a *holomorphic curve* if there exist n holomorphic $\mathcal{H}$-valued functions $\gamma_1,\dots,\gamma_n$ on Ω such that for all $\lambda \in \Omega$,

$$f(\lambda) = \bigvee\{\gamma_1(\lambda),\dots,\gamma_n(\lambda)\},$$

where $\bigvee$ denotes the closed linear span. Such a map naturally induces an

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n-dimensional Hermitian holomorphic vector bundle E_f over Ω, defined as

$$E_f = \{(x, \lambda) \in \mathcal{H} \times \Omega : x \in f(\lambda)\},$$

equipped with the projection $\pi : E_f \to \Omega$ given by $\pi(x, \lambda) = \lambda$. Martin [27] proved that a smooth function $f : \Omega \to \mathcal{G}r(n, \mathcal{H})$ is analytic if and only if its corresponding self-adjoint projection $[f] : \Omega \to \mathcal{L}(\mathcal{H})$ satisfies $(1 - [f])\overline{\partial}_i[f] = 0$ for $1 \leq i \leq m$. Furthermore, Cowen and Douglas [5, 6] established that two holomorphic curves f and $\widetilde{f}$ are congruent if and only if their associated vector bundles E_f and $E_{\widetilde{f}}$ are locally equivalent as Hermitian holomorphic vector bundles. By employing the theory of involutive algebras [2] and the concept of osculation [9], Martin [26, 27] provided a simpler proof of this congruence theorem.

Apostol and Martin [2] employed C^*-algebraic methods to connect the Cowen–Douglas class, defined by Cowen and Douglas [5, 6], with C^*-algebra theory, thereby providing an algebraic characterization for these operators. Salinas [35] extended the classical Grassmann manifold and Hermitian holomorphic bundle to C^*-algebras. Martin and Salinas [29] presented the differential-geometric structure of generalized Grassmann manifolds in C^*-algebras and later [30] investigated the congruence problem for holomorphic curves into flag manifolds associated with C^*-algebras, subsequently applying their findings to the Cowen–Douglas theory and additionally studying the geometric properties of flag manifolds consisting of increasing n-tuples of projections in a C^*-algebra. Let $\mathcal{U}$ be a unital C^*-algebra, and let Ω be a bounded domain in $\mathbb{C}^m$. If an n-tuple $P = (P_1, \ldots, P_n)$ of idempotents in $\mathcal{U}$ satisfies $P_1 \leq \cdots \leq P_n$, then P is called an *extended* n*-flag* of $\mathcal{U}$. If an n-tuple $P = (P_1, \ldots, P_n)$ is an extended n-flag of $\mathcal{U}$ and every entry P_i, $1 \leq i \leq n$, is a projection in $\mathcal{U}$, then P is called an n*-flag* in $\mathcal{U}$. Let $\mathcal{I}_n(\mathcal{U})$ and $\mathcal{P}_n(\mathcal{U})$ be the set of all extended n-flags of $\mathcal{U}$ and all n-flags of $\mathcal{U}$, respectively. Moreover, for $n = 1$, the spaces $\mathcal{I}(\mathcal{U})$ and $\mathcal{P}(\mathcal{U})$ are alternatively called the *extended Grassmann manifold* and the *Grassmann manifold* of $\mathcal{U}$, respectively. Martin [28] shows that a smooth projection-valued mapping $P : \Omega \to \mathcal{P}(\mathcal{U})$ for $\mathcal{U} = \mathcal{L}(\mathcal{H})$ is holomorphic if and only if it satisfies the following equivalent Cauchy–Riemann equations:

$$\begin{aligned} &\partial P(\lambda) \cdot P(\lambda) = \partial P(\lambda), \quad P(\lambda) \cdot \partial P(\lambda) = 0, \\ &P(\lambda) \cdot \overline{\partial} P(\lambda) = \overline{\partial} P(\lambda), \quad \overline{\partial} P(\lambda) \cdot P(\lambda) = 0, \qquad \lambda \in \Omega, \end{aligned}$$

where $\partial = \sum_{i=1}^m \frac{\partial}{\partial \lambda_i}$ and $\overline{\partial} = \sum_{j=1}^m \frac{\partial}{\partial \overline{\lambda}_j}$. In particular, for $\Omega \subseteq \mathbb{C}$, a mapping $P : \Omega \to \mathcal{P}(\mathcal{U})$ is holomorphic if and only if $\overline{\partial} P(\lambda) P(\lambda) = 0$ for all $\lambda \in \Omega$ (see [30]).

Martin and Salinas [31] proved that a flag manifold admits a natural intrinsic complex structure, generalizing the result obtained by Wilkins [38] for Grassmann manifolds. Martin [28] established complete sets of invariants for holomorphic mappings with values in Grassmann manifolds of Hilbert spaces, Hermitian holomorphic vector bundles, and the Cowen–Douglas class by using the holomorphic spectra theory.

People were surprised to discover that the similarity classification of operators and associated holomorphic vector bundles has a profound and intrinsic connection with the K-theory of Banach algebras. Cao–Fang–Jiang [3], Jiang [18], Jiang–Guo–Ji [19], Jiang–Ji [20], and Ji [12] characterized the uniqueness of decomposition and the similarity classification of Cowen–Douglas operators by the K_0-group of the commutant, and generalized the result to holomorphic curves and extended holomorphic curves. Ji [13] defined the curvature and its covariant derivatives for holomorphic curves for a single variable on C^*-algebras, and utilized these formulas given by the congruence and similarity classification theorems to study holomorphic curves and extended holomorphic curves on C^*-algebras. These curvature formulas can be regarded as a generalization of their counterparts in complex geometry. Ji and Sarkar [17] studied the curvature, a projection formula, and a trace-curvature formula in terms of Hilbert–Schmidt norm by geometrical methods.

In the case where the C^*-algebra is $\mathcal{L}(\mathcal{H})$, the holomorphic mappings on it have a natural connection to the operator-valued Corona problem. The associated geometric quantities can be used to describe sufficient conditions for the Corona problem to hold. For Hilbert spaces E_1 and E_2, the well-known operator Corona problem aims to find a necessary and sufficient condition for a bounded analytic operator-valued function $F \in H^\infty_{E_1\to E_2}(\Omega)$ to have a left inverse in $H^\infty_{E_2\to E_1}(\Omega)$, that is, an operator-valued function $G \in H^\infty_{E_2\to E_1}(\Omega)$ such that $G(\lambda)F(\lambda) \equiv I$ for all $\lambda \in \Omega$, where I denotes the identity operator on E_1. Nikol'skiĭ [32, 33] solved this problem by utilizing idempotent-valued operators, i.e. extended holomorphic curves. Kwon–Treil [25] and Douglas–Kwon–Treil [8] characterized the similarity classification of certain Cowen–Douglas operators by using holomorphic curves built on model theorems [1, 36] and the operator Corona problem. Hou–Ji–Kwon [10] showed that the trace of the curvature of the Hermitian holomorphic vector bundle induced by the given Cowen–Douglas operator corresponds to the holomorphic curve mentioned earlier. The trace of the curvature serves as a similarity invariant for certain Cowen–Douglas operators. Furthermore, we note that the trace of the curvature of the holomorphic curve corresponding to this operator-valued function can be used to characterize the conditions under which the related operator-valued Corona problem holds.

In this paper, we successfully provide definitions for algebraically defined geometric quantities such as the curvature and the connection of holomorphic curves in Grassmann manifolds associated with C^*-algebras in the multivariable setting (see Definition 3.1). We demonstrate the compatibility of these geometric quantities with their classical counterparts and, simultaneously, offer an answer to the equivalence problem for extended holomorphic curves (see Theorems 3.5, 4.1 and Proposition 4.7). The organization of this paper is as follows. In Section 2 we introduce the notations and some preliminary knowledge. Section 3 is dedicated to defining the connection, the curvature and its covariant derivatives for multivariate extended holomorphic curves on C^*-algebras, and establishing the relationship between these algebraic constructs and their classical counterparts in complex geometry. In Section 4, we present the congruent and similarity classification of multivariate extended holomorphic curves by means of the curvature and its covariant derivatives, and implement the results obtained to the classification problem of the Cowen–Douglas class. Section 5 studies unitary equivalence in the class $\mathcal{FB}_n(\Omega)$ with flag structure by using the geometric properties of n-flags formed by holomorphic curves, and further provides a new characterization of unitary equivalence in this class.

2. Preliminaries. This section introduces the basic theory of extended holomorphic curves and compiles relevant definitions and lemmas required for subsequent discussions.

Throughout this paper, we fix a unital C^*-algebra $\mathcal{U}$ and an integer $m \geq 1$. An element $P \in \mathcal{U}$ is called an *idempotent* if $P^2 = P$. A self-adjoint idempotent is simply referred to as a *projection*. Let $\mathcal{I}(\mathcal{U})$ and $\mathcal{P}(\mathcal{U})$ denote the sets of all idempotents and all projections in $\mathcal{U}$, respectively. The spaces $\mathcal{I}(\mathcal{U})$ and $\mathcal{P}(\mathcal{U})$ are also called the *extended Grassmann manifold* and the *Grassmann manifold* of $\mathcal{U}$, respectively. Specific properties of these spaces are discussed in [4, 27, 34, 35, 38].

Definition 2.1 ([30]). Let $\mathcal{U}$ be a unital C^*-algebra, and let $n > 1$ be an integer.

(i) An n-tuple $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$ of idempotents in $\mathcal{U}$ is called an *extended n-flag* of $\mathcal{U}$ if it satisfies
$$\mathcal{I}_1 \leq \cdots \leq \mathcal{I}_n.$$
(ii) An n-tuple $P = (P_1, \dots, P_n)$ is called an *n-flag* of $\mathcal{U}$ if it is an extended n-flag of $\mathcal{U}$ and each entry P_i, $1 \leq i \leq n$, is a projection in $\mathcal{U}$.

We denote by $\mathcal{I}_n(\mathcal{U})$ the set of all extended n-flags of $\mathcal{U}$, and by $\mathcal{P}_n(\mathcal{U})$ the set of all n-flags of $\mathcal{U}$.

To investigate the congruence problem for holomorphic mappings into flag manifolds associated with C^*-algebras and the classification problem for the Cowen–Douglas class, Martin and Salinas [30] introduced the definition of holomorphic curves on C^*-algebras, which is more general than the Cowen–Douglas class. Let Ω be a bounded domain in $\mathbb{C}^m$, and let $\mathbb{Z}_+^m$

denote the set of all m-tuples of non-negative integers. For any multi-index $I = (i_1, \dots, i_m) \in \mathbb{Z}_+^m$, we set, as usual,

$$|I| = i_1 + \cdots + i_m, \quad I! = i_1! \cdots i_m!, \quad D^I = \partial_1^{i_1} \cdots \partial_m^{i_m}, \quad \overline{D}^I = \overline{\partial}_1^{i_1} \cdots \overline{\partial}_m^{i_m},$$

and

$$\partial^I = \partial_1^{i_1} + \cdots + \partial_m^{i_m}, \quad \overline{\partial}^I = \overline{\partial}_1^{i_1} + \cdots + \overline{\partial}_m^{i_m},$$

where $\partial_i = \frac{\partial}{\partial \lambda_i}$ and $\overline{\partial}_i = \frac{\partial}{\partial \overline{\lambda}_i}$, $1 \le i \le m$, denote the holomorphic and the anti-holomorphic partial differentiation operators, respectively. In particular, $\partial = \partial_1 + \cdots + \partial_m$ and $\overline{\partial} = \overline{\partial}_1 + \cdots + \overline{\partial}_m$. For notational simplicity, we write $0 = (0, \dots, 0)$ and $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}_+^m$, $1 \le i \le m$, with 1 in the i-th position.

Let $P : \Omega \to \mathcal{P}(\mathcal{U})$ be a smooth map. It follows from [28, 30] that P is holomorphic if and only if it satisfies (2.2) below. To generalize the results on holomorphic curves in the Grassmann manifold of C^*-algebras to the case of extended Grassmann manifolds, Ji introduced the concept of extended holomorphic curves in extended Grassmann manifolds for $\Omega \subset \mathbb{C}$ (see [13, Definition 2.12]). For the same purpose, we introduce the definition of extended holomorphic curves in extended Grassmann manifolds with respect to multiple variables.

Definition 2.2. Let Ω be a bounded domain in $\mathbb{C}^m$. A smooth map $\mathcal{I} : \Omega \to \mathcal{I}(\mathcal{U})$ is called an *extended holomorphic curve* if it satisfies the following conditions:

$$\begin{aligned} &\partial\mathcal{I}(\lambda) \cdot \mathcal{I}(\lambda) = \partial\mathcal{I}(\lambda), \quad \mathcal{I}(\lambda) \cdot \partial\mathcal{I}(\lambda) = 0, \\ &\mathcal{I}(\lambda) \cdot \overline{\partial}\mathcal{I}(\lambda) = \overline{\partial}\mathcal{I}(\lambda), \quad \overline{\partial}\mathcal{I}(\lambda) \cdot \mathcal{I}(\lambda) = 0, \qquad \lambda \in \Omega. \end{aligned} \tag{2.1}$$

Moreover, the above formulas are equivalent to

$$\begin{aligned} &\partial_i\mathcal{I}(\lambda) \cdot \mathcal{I}(\lambda) = \partial_i\mathcal{I}(\lambda), \quad \mathcal{I}(\lambda) \cdot \partial_i\mathcal{I}(\lambda) = 0, \\ &\mathcal{I}(\lambda) \cdot \overline{\partial}_i\mathcal{I}(\lambda) = \overline{\partial}_i\mathcal{I}(\lambda), \quad \overline{\partial}_i\mathcal{I}(\lambda) \cdot \mathcal{I}(\lambda) = 0, \qquad 1 \le i \le m, \lambda \in \Omega. \end{aligned}$$

Definition 2.3. Let Ω be a bounded domain in $\mathbb{C}^m$. A smooth map $P : \Omega \to \mathcal{P}(\mathcal{U})$ is called a *holomorphic curve* if it satisfies the following Cauchy–Riemann equations:

$$\begin{aligned} &\partial P(\lambda) \cdot P(\lambda) = \partial P(\lambda), \quad P(\lambda) \cdot \partial P(\lambda) = 0, \\ &P(\lambda) \cdot \overline{\partial} P(\lambda) = \overline{\partial} P(\lambda), \quad \overline{\partial} P(\lambda) \cdot P(\lambda) = 0, \qquad \lambda \in \Omega. \end{aligned} \tag{2.2}$$

From [30, Section 1.3] it follows that for any idempotent $\mathcal{I} \in \mathcal{I}(\mathcal{U})$, there exists a unique projection $P \in \mathcal{P}(\mathcal{U})$ satisfying $\mathcal{I}P = P$ and $P\mathcal{I} = \mathcal{I}$. Two extended holomorphic curves $\mathcal{I}_1$ and $\mathcal{I}_2$ are said to be *congruent* (denoted by $\mathcal{I}_1 \sim_u \mathcal{I}_2$) if there exists a unitary $U \in \mathcal{U}$ such that

$$U\mathcal{I}_1(\lambda) = \mathcal{I}_2(\lambda)U, \qquad \lambda \in \Omega.$$

Analogously, $\mathcal{I}_1$ and $\mathcal{I}_2$ are called *similarity equivalent* (denoted by $\mathcal{I}_1 \sim_s \mathcal{I}_2$) if there exists an invertible $X \in \mathcal{U}$ satisfying

$$X\mathcal{I}_1(\lambda) = \mathcal{I}_2(\lambda)X, \qquad \lambda \in \Omega.$$

The study of the congruence problem for holomorphic curves also originates from the Cowen–Douglas theory, pioneered by Cowen and Douglas [5, 6]. Let $\mathcal{H}$ be a separable complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For two m-tuples of commuting operators $\mathbf{T} = (T_1, \ldots, T_m)$ and $\mathbf{S} = (S_1, \ldots, S_m)$ in $\mathcal{L}(\mathcal{H})^m$, we recall two fundamental equivalence relations: $\mathbf{T}$ and $\mathbf{S}$ are *unitarily equivalent* (denoted by $\mathbf{T} \sim_u \mathbf{S}$) if there exists a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $UT_i = S_iU$ for all $1 \le i \le m$; further, $\mathbf{T}$ and $\mathbf{S}$ are *similar* (denoted by $\mathbf{T} \sim_s \mathbf{S}$) if there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ satisfying $XT_i = S_iX$ for all $1 \le i \le m$.

For any m-tuple of commuting operators $\mathbf{T} = (T_1, \ldots, T_m) \in \mathcal{L}(\mathcal{H})^m$, define the associated operator $\mathscr{D}_{\mathbf{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ by

$$\mathscr{D}_{\mathbf{T}} x = (T_1x, \ldots, T_mx), \qquad x \in \mathcal{H}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Omega$, the *joint kernel* of $\mathbf{T} - \lambda := (T_1 - \lambda_1, \ldots, T_m - \lambda_m)$ is $\ker \mathscr{D}_{\mathbf{T}-\lambda} = \bigcap_{i=1}^m \ker(T_i - \lambda_i)$.

Definition 2.4 ([5, 6]). For $m, n \in \mathbb{N}$ and $\Omega \subset \mathbb{C}^m$, the *Cowen–Douglas class* $\mathcal{B}_n^m(\Omega)$ consists of m-tuples of commuting operators $\mathbf{T} = (T_1, \ldots, T_m) \in \mathcal{L}(\mathcal{H})^m$ satisfying the following conditions:

- $\operatorname{ran} \mathscr{D}_{\mathbf{T}-\lambda}$ is closed for all $\lambda \in \Omega$;
- $\dim \ker \mathscr{D}_{\mathbf{T}-\lambda} = n$ for all $\lambda \in \Omega$;
- $\bigvee_{\lambda\in\Omega} \ker \mathscr{D}_{\mathbf{T}-\lambda} = \mathcal{H}$, where $\bigvee$ stands for the closed linear span.

For any m-tuple of commuting operators $\mathbf{T} = (T_1, \ldots, T_m) \in \mathcal{B}_n^m(\Omega)$, Cowen and Douglas [5, 6] proved the existence of the associated Hermitian holomorphic vector bundle $E_{\mathbf{T}}$ of rank n over Ω, defined as

$$E_{\mathbf{T}} = \{(\lambda, x) \in \Omega \times \mathcal{H} : x \in \ker(\mathbf{T} - \lambda)\}, \qquad \pi(\lambda, x) = \lambda.$$

The induced map $P_{\mathbf{T}} : \Omega \to \mathcal{G}r(n, \mathcal{H})$ given by

$$P_{\mathbf{T}}(\lambda) = \ker(\mathbf{T} - \lambda), \qquad \lambda \in \Omega,$$

defines a holomorphic curve, as shown by Curto and Salinas [7]. Moreover, two m-tuples $\mathbf{T}$ and $\mathbf{S}$ in $\mathcal{B}_n^m(\Omega)$ are unitarily equivalent if and only if the complex bundles $E_{\mathbf{T}}$ and $E_{\mathbf{S}}$ are equivalent as Hermitian holomorphic vector bundles if and only if $P_{\mathbf{T}}$ and $P_{\mathbf{S}}$ are congruent [5, 6].

Definition 2.5 ([5, 6]). Let $\mathbf{T} \in \mathcal{B}_n^m(\Omega)$, let $\{F_i\}_{i=1}^n$ be a holomorphic frame of $E_{\mathbf{T}}$, and let the Gram matrix be $H(\lambda) = (\langle F_j(\lambda), F_i(\lambda)\rangle)_{i,j=1}^n$. The *classical connection* $\Theta(\lambda)$ and the *classical curvature* $\mathcal{K}_{\mathbf{T}}$ of $E_{\mathbf{T}}$ are defined by

(2.3)

$$\Theta(\lambda) = H^{-1}(\lambda)\sum_{i=1}^{m}\Big(\frac{\partial}{\partial\lambda_i}H(\lambda)\Big), \mathcal{K}_{\mathbf{T}}(\lambda) = -\sum_{i,j=1}^{m}\frac{\partial}{\partial\overline{\lambda}_j}\Big(H^{-1}(\lambda)\frac{\partial}{\partial\lambda_i}H(\lambda)\Big).$$

Moreover, the *covariant derivatives* $\mathcal{K}_{\mathbf{T},\lambda^I\overline{\lambda}^J}$ of $\mathcal{K}_{\mathbf{T}}$ are defined as follows:

(i) $\mathcal{K}_{\mathbf{T},\overline{\lambda}_j} = \frac{\partial}{\partial\overline{\lambda}_j}(\mathcal{K}_{\mathbf{T}})$,
(ii) $\mathcal{K}_{\mathbf{T},\lambda_i} = \frac{\partial}{\partial\lambda_i}(\mathcal{K}_{\mathbf{T}}) + \big[H^{-1}\frac{\partial}{\partial\lambda_i}H, \mathcal{K}_{\mathbf{T}}\big]$, $\lambda\in\Omega$.

Remark 2.6. In Definition 2.5, for any integer $m \geq 2$, we derive the following identity:

$$\begin{aligned}&((\mathcal{K}_{\mathcal{I}_{\mathbf{T}}})_{\overline{\lambda}_k})_{\lambda_l} - ((\mathcal{K}_{\mathcal{I}_{\mathbf{T}}})_{\lambda_l})_{\overline{\lambda}_k}\\ &= \overline{\partial}_k(H^{-1}\partial_l H)\overline{\partial}(h^{-1}\partial H) - \overline{\partial}(H^{-1}\partial H)\overline{\partial}_k(H^{-1}\partial_l H), \quad 1\leq l,k\leq m.\end{aligned}$$

This implies that the order of partial differentiation for the covariant derivatives $\mathcal{K}_{\mathcal{I}_{\mathbf{T}},\lambda^I,\overline{\lambda}^J}$ is in general non-commutative, that is, their values depend on the order of the multi-indices I and J.

In classical complex geometry, the curvature $\mathcal{K}_{\mathbf{T}}$ and its covariant derivatives $\mathcal{K}_{\mathbf{T},\lambda^I\overline{\lambda}^J}$ defined for $E_{\mathbf{T}}$ have been proven in [5, 6] to be unitary invariants of $\mathbf{T}\in\mathcal{B}_n^m(\Omega)$. Uchiyama [37], Kwon–Treil [25], Douglas–Kwon–Treil [8], Kwon [24], Hou–Ji–Kwon [10], and Ji–Ji–Kwon–Xu [14] have characterized the similarity classification of special Cowen–Douglas operators using model theorems and the Corona problem.

Definition 2.7. Let B be a unital C^*-algebra. The *Hilbert B-module* $l^2(\mathbb{N},B)$ is defined as

$$l^2(\mathbb{N},B) := \Big\{(a_i)_{i\in\mathbb{N}} : a_i\in B \text{ and } \sum_{i\in\mathbb{N}}\|a_i\|^2<\infty\Big\},$$

where $\|\cdot\|$ denotes the norm on B.

Let $\mathcal{U} := \mathscr{L}(l^2(\mathbb{N},B))$ denote the C^*-algebra of all bounded linear operators on $l^2(\mathbb{N},B)$. Consider the operator-valued holomorphic functions $F,G:\Omega\to\mathscr{L}(\mathbb{C}^n,l^2(\mathbb{N},B))$ defined by

$$F(\lambda)(w_1,\dots,w_n) = \sum_{i=1}^{n} w_iF_i(\lambda), \qquad G(\lambda)(w_1,\dots,w_n) = \sum_{i=1}^{n} w_iG_i(\lambda)$$

for $(w_1,\dots,w_n)\in\mathbb{C}^n$ and $\lambda\in\Omega$, where each function $F_i,G_i:\Omega\to l^2(\mathbb{N},B)$, $1\leq i\leq n$, is holomorphic.

The mapping $\mathcal{I}:\Omega\to\mathcal{I}(\mathcal{U})$ given by

$$\mathcal{I}(\lambda) = F(\lambda)(G^*(\lambda)F(\lambda))^{-1}G^*(\lambda) \tag{2.4}$$

defines an extended holomorphic curve in the sense of Definition 2.2 (as established in Proposition 2.8 below) and satisfies $\operatorname{ran} F(\lambda) = \operatorname{ran} \mathcal{I}(\lambda)$. We denote by $\mathcal{I}_n(\Omega,\mathcal{U})$ the set of all such mappings.

PROPOSITION 2.8. *The mapping* $\mathcal{I} = F(G^*F)^{-1}G^* : \Omega \to \mathcal{I}(\mathcal{U})$ *as defined in Definition* 2.7 *is an extended holomorphic curve.*

Proof. Since $\mathcal{I}^2 = [F(G^*F)^{-1}G^*][F(G^*F)^{-1}G^*] = F(G^*F)^{-1}G^* = \mathcal{I}$, it follows that $\mathcal{I}(\lambda)$ is an idempotent for all $\lambda \in \Omega$. Note that

$$\partial_i G^* = 0, \quad \overline{\partial}_i F = 0, \quad \partial_i (G^*F)^{-1} = -(G^*F)^{-1}\partial_i(G^*F)(G^*F)^{-1},$$

and $\overline{\partial}_i(G^*F)^{-1} = -(G^*F)^{-1}\overline{\partial}_i(G^*F)(G^*F)^{-1}$ for all $1 \le i \le m$. We further have

$$\begin{aligned}
\partial_i\mathcal{I}\mathcal{I} &= [\partial_i F(G^*F)^{-1}G^* + F\partial_i(G^*F)^{-1}G^*][F(G^*F)^{-1}G^*] \\
&= \partial_i F(G^*F)^{-1}G^* + F\partial_i(G^*F)^{-1}G^* = \partial_i\mathcal{I}, \\
\mathcal{I}\partial_i\mathcal{I} &= [F(G^*F)^{-1}G^*][\partial_i F(G^*F)^{-1}G^* + F\partial_i(G^*F)^{-1}G^*] \\
&= F(G^*F)^{-1}\partial_i(G^*F)(G^*F)^{-1}G^* + F\partial_i(G^*F)^{-1}G^* = 0, \\
\mathcal{I}\overline{\partial}_i\mathcal{I} &= [F(G^*F)^{-1}G^*][F\overline{\partial}_i(G^*F)^{-1}G^* + F(G^*F)^{-1}\overline{\partial}_i G^*] \\
&= F\overline{\partial}_i(G^*F)^{-1}G^* + F(G^*F)^{-1}\overline{\partial}_i G^* = \overline{\partial}_i\mathcal{I}, \\
\overline{\partial}_i\mathcal{I}\mathcal{I} &= [F\overline{\partial}_i(G^*F)^{-1}G^* + F(G^*F)^{-1}\overline{\partial}_i G^*][F(G^*F)^{-1}G^*] \\
&= F\overline{\partial}_i(G^*F)^{-1}G^* + F(G^*F)^{-1}\overline{\partial}_i(G^*F)(G^*F)^{-1}G^* = 0.
\end{aligned}$$

Thus, $\mathcal{I}$ satisfies all conditions of Definition 2.2, as expected. ∎

EXAMPLE 2.9. Let Ω be a bounded domain in $\mathbb{C}^2$, and let $\mathcal{U} = M_3(\mathbb{C}^2)$ denote the space of 3×3 matrices over $\mathbb{C}^2$. Define the mapping $\mathcal{I} : \Omega \to M_3(\mathbb{C}^2)$ by

$$\mathcal{I}(\lambda) = \frac{1}{1 + 2|\lambda_1|^2 + 4|\lambda_2|^2}\begin{pmatrix} 1 & \overline{\lambda}_1 & \overline{\lambda}_2 \\ 2\lambda_1 & 2|\lambda_1|^2 & 2\lambda_1\overline{\lambda}_2 \\ 4\lambda_2 & 4\overline{\lambda}_1\lambda_2 & 4|\lambda_2|^2 \end{pmatrix}, \qquad \lambda \in \Omega.$$

Set

$$F(\lambda) = \begin{pmatrix} 1 \\ 2\lambda_1 \\ 4\lambda_2 \end{pmatrix}, \ G(\lambda) = \begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \ F^*(\lambda) = (1, 2\overline{\lambda}_1, 4\overline{\lambda}_2), \ G^*(\lambda) = (1, \overline{\lambda}_1, \overline{\lambda}_2),$$

where $F^*(\lambda)$ and $G^*(\lambda)$ denote the adjoint vectors of $F(\lambda)$ and $G(\lambda)$, respectively. A direct computation shows that $G^*(\lambda)F(\lambda) = 1 + 2|\lambda_1|^2 + 4|\lambda_2|^2$, which is a non-zero complex scalar for all $\lambda \in \Omega$. Thus, we see that the inverse $(G^*(\lambda)F(\lambda))^{-1}$ is well-defined, and

$$\mathcal{I}(\lambda) = F(\lambda)(G^*(\lambda)F(\lambda))^{-1}G^*(\lambda), \qquad \lambda \in \Omega.$$

Furthermore, $\mathcal{I}$ is an extended holomorphic curve.

For any $\mathcal{I} = F(G^*F)^{-1}G^* \in \mathcal{I}_n(\Omega,\mathcal{U})$, the dimension of $\bigvee_{\lambda\in\Omega} \operatorname{ran}\mathcal{I}(\lambda)$ can be either finite or infinite. Example 2.9 provides a case in which this dimension is finite. Using the frame of Hermitian holomorphic vector bundles, we show that every Cowen–Douglas tuple $\mathbf{T} \in \mathcal{B}_n^m(\Omega)$ induces an extended holomorphic curve $\mathcal{I}_{\mathbf{T}}$ of the form given in Definition 2.7. Let $\mathbf{T} \in \mathcal{B}_n^m(\Omega) \cap \mathcal{L}(\mathcal{H})^m$, and let $\{F_i\}_{i=1}^n$ and $\{G_i\}_{i=1}^n$ be holomorphic frames of the Hermitian holomorphic vector bundle $E_{\mathbf{T}}$ and an auxiliary Hermitian holomorphic vector bundle E, respectively. Let $F = (F_1, \ldots, F_n)$ and $G = (G_1, \ldots, G_n)$, and define the mapping $\mathcal{I}_{\mathbf{T}} : \Omega \to \mathcal{I}(\mathcal{L}(\mathcal{H}))$ as follows:

$$\mathcal{I}_{\mathbf{T}}(\lambda) := F(\lambda)(G^*(\lambda)F(\lambda))^{-1}G^*(\lambda).$$

Then $\mathcal{I}_{\mathbf{T}} : \Omega \to \mathcal{I}(\mathcal{L}(\mathcal{H}))$ is an extended holomorphic curve, and $\mathcal{I}_{\mathbf{T}}(\lambda)$ is the matrix representation of an idempotent operator from $\mathcal{H}$ onto $\ker \mathscr{D}_{\mathbf{T}-\lambda}$. By Definition 2.4, we have

$$\bigvee_{\lambda\in\Omega} \operatorname{ran}\mathcal{I}_{\mathbf{T}}(\lambda) = \mathcal{H},$$

so the dimension of $\bigvee_{\lambda\in\Omega} \operatorname{ran}\mathcal{I}_{\mathbf{T}}(\lambda)$ is infinite.

3. The curvature of extended holomorphic curves on C^*-algebras. In this section, we mainly construct the curvature and its covariant derivatives for multivariate extended holomorphic curves on C^*-algebras, with a particular focus on establishing the connection between these algebraic constructs and their classical counterparts in complex geometry.

Definition 3.1. Let $\mathcal{I} = F(G^*F)^{-1}G^* \in \mathcal{I}_n(\Omega,\mathcal{U})$ be an extended holomorphic curve. We define the *connection* and the *curvature* of $\mathcal{I}$ as

$$\boldsymbol{\Theta} = (\partial F)H^{-1}G^* - \partial\mathcal{I} \quad \text{and} \quad \mathscr{K}(\mathcal{I}) = \overline{\partial}\mathcal{I}\partial\mathcal{I}, \tag{3.1}$$

respectively, where $H(\lambda) = G^*(\lambda)F(\lambda)$. The *covariant derivative* $\mathscr{K}_{I,J}(\mathcal{I})$ of $\mathscr{K}(\mathcal{I})$ is defined as follows:

$$\mathscr{K}_{I+e_l,J}(\mathcal{I}) = \mathscr{K}_{\sum_{p\neq l} i_p e_p,J}(\mathcal{I}) + \mathcal{I}\big(\partial_l(\mathscr{K}_{i_l e_l,J}(\mathcal{I}))\big), \qquad i_p \neq 0, \tag{3.2}$$

$$\mathscr{K}_{I,J+e_k}(\mathcal{I}) = \mathscr{K}_{I,\sum_{q\neq k} j_q e_q}(\mathcal{I}) + \big(\overline{\partial}_k(\mathscr{K}_{I,j_k e_k}(\mathcal{I}))\big)\mathcal{I}, \quad j_q \neq 0, \tag{3.3}$$

where $I = (i_1, \ldots, i_m)$, $J = (j_1, \ldots, j_m) \in \mathbb{Z}_+^m$ and $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}_+^m$, $1 \le i \le m$, with 1 in the i-th position.

Remark 3.2. For any extended holomorphic curve $\mathcal{I} = F(G^*F)^{-1}G^* \in \mathcal{I}_n(\Omega,\mathcal{U})$, from Definition 3.1 we have

$$\mathscr{K}_{0+e_l,e_k}(\mathcal{I}) = \mathcal{I}(\partial_l(\mathscr{K}_{0,e_k})) = \overline{\partial}_k\partial\mathcal{I}\partial_l\partial\mathcal{I} - \overline{\partial}\mathcal{I}\partial_l\mathcal{I}\overline{\partial}_k\mathcal{I}\partial\mathcal{I} - \overline{\partial}_k\mathcal{I}\partial_l\mathcal{I}\overline{\partial}\mathcal{I}\partial\mathcal{I},$$

$$\mathscr{K}_{e_l,0+e_k}(\mathcal{I}) = (\overline{\partial}_k(\mathscr{K}_{e_l,0}))\mathcal{I} = \overline{\partial}_k\partial\mathcal{I}\partial_l\partial\mathcal{I} - \overline{\partial}\mathcal{I}\partial_l\mathcal{I}\overline{\partial}_k\mathcal{I}\partial\mathcal{I} - \overline{\partial}\mathcal{I}\partial\mathcal{I}\overline{\partial}_k\mathcal{I}\partial_l\mathcal{I},$$

for $1 \le l, k \le m$. This shows that the covariant derivative $\mathscr{K}_{I,J}(\mathcal{I})$ of the curvature $\mathscr{K}(\mathcal{I})$ associated with $\mathcal{I} \in \mathcal{I}_n(\Omega,\mathcal{U})$ depends on the order of I and J.

To sum up Remarks 2.6 and 3.2, both the covariant derivative $\mathcal{K}_{\mathcal{I},\lambda^I\overline{\lambda}^J}$ (Definition 2.5) of the classical curvature $\mathcal{K}_{\mathcal{I}}$ in the complex geometry of extended holomorphic curves and the covariant derivative $\mathscr{K}_{I,J}(\mathcal{I})$ (Definition 3.1) of the curvature $\mathscr{K}(\mathcal{I})$ in the C^*-algebra depend on the order of I and J. Moreover, we find that the covariant derivatives of these two curvatures admit decompositions, as shown in Lemmas 3.3 and 3.4 below.

LEMMA 3.3. *Let $\mathcal{I} \in \mathcal{I}_n(\Omega,\mathcal{U})$ be an extended holomorphic curve. Then $\mathscr{K}_{I,J}(\mathcal{I})$ has the following properties:*

(i) $\mathscr{K}_{I,0}(\mathcal{I}) = \sum_{i_p\neq 0} \mathscr{K}_{i_pe_p,0}(\mathcal{I})$ *for any* $I = \sum_{p=1}^m i_pe_p \in \mathbb{Z}_+^m \setminus \{0\}$;
(ii) $\mathscr{K}_{0,J}(\mathcal{I}) = \sum_{j_q\neq 0} \mathscr{K}_{0,j_qe_q}(\mathcal{I})$ *for any* $J = \sum_{q=1}^m j_qe_q \in \mathbb{Z}_+^m \setminus \{0\}$;
(iii) $\mathscr{K}_{I,J}(\mathcal{I}) = \sum_{i_p,j_q\neq 0} \mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I})$ *for any* $I = \sum_{p=1}^m i_pe_p$, $J = \sum_{q=1}^m j_qe_q \in \mathbb{Z}_+^m \setminus \{0\}$, *where the order of partial differentiation on the right-hand side corresponds to that on the left-hand side, and* $e_p = (0,\dots,1,\dots,0) \in \mathbb{Z}_+^m$ *with* 1 *in the* p*-th position.*

Proof. By Remark 3.2, $\mathscr{K}_{I,J}(\mathcal{I})$ depends on the order of partial differentiation specified by I and J. Let $I = e_{p_1} + e_{p_2}$ with $1 \le p_1 \neq p_2 \le m$, where the partial differentiation order of $\mathscr{K}_{I,0}(\mathcal{I})$ is first with respect to λ_{p_1}, then $\overline{\lambda}_{p_2}$. By (3.2), we have

$$\mathscr{K}_{e_{p_1}+e_{p_2},0}(\mathcal{I}) = \mathscr{K}_{e_{p_1},0}(\mathcal{I}) + \mathcal{I}(\partial_{p_2}(\mathscr{K}(\mathcal{I}))) = \mathscr{K}_{e_{p_1},0}(\mathcal{I}) + \mathscr{K}_{e_{p_2},0}(\mathcal{I}).$$

Without losing generality, assume that for some $K \in \mathbb{Z}_+^m \setminus \{0\}$, property (i) holds for all I with $0 < I \le K$. For $I + e_l$, $1 \le l \le m$, by (3.2), we deduce that

$$\begin{aligned}\mathscr{K}_{I+e_l,0}(\mathcal{I}) &= \mathscr{K}_{\sum_{i_p\neq 0,\, p\neq l} i_pe_p,0}(\mathcal{I}) + \mathcal{I}(\partial_l(\mathscr{K}_{i_le_l,0}(\mathcal{I}))) \\ &= \sum_{\substack{i_p\neq 0\\ p\neq l}} \mathscr{K}_{i_pe_p,0}(\mathcal{I}) + \mathscr{K}_{(i_l+1)e_l,0}(\mathcal{I}),\end{aligned}$$

where $e_l = (0,\dots,1,\dots,0) \in \mathbb{Z}_+^m$ with 1 in the l-th position. By induction, property (i) holds for all $I \in \mathbb{Z}_+^m \setminus \{0\}$. Similarly, property (ii) holds.

For property (iii), let $I = e_{p_1} + e_{p_2}$, $J = e_{q_1} + e_{q_2}$, where $1 \le p_1, p_2, q_1, q_2 \le m$. The covariant derivative $\mathscr{K}_{I,J}(\mathcal{I})$ has the partial differentiation order $\lambda_{p_1}, \overline{\lambda}_{q_1}, \lambda_{p_2}, \overline{\lambda}_{q_2}$. By Definition 3.1,

$$\mathscr{K}_{0+e_{p_1},0}(\mathcal{I}) = \mathcal{I}(\partial_{p_1}(\mathscr{K}(\mathcal{I}))) \quad \text{and} \quad \mathscr{K}_{e_{p_1},0+e_{q_1}}(\mathcal{I}) = (\overline{\partial}_{q_1}(\mathscr{K}_{e_{p_1},0}(\mathcal{I})))\mathcal{I}.$$

If $p_1 \neq p_2$, we have

$$\begin{aligned}\mathscr{K}_{e_{p_1}+e_{p_2},e_{q_1}}(\mathcal{I}) &= \mathscr{K}_{e_{p_1},e_{q_1}}(\mathcal{I}) + \mathcal{I}(\partial_{p_2}(\mathscr{K}_{0,e_{q_1}}(\mathcal{I}))) \\ &= \mathscr{K}_{e_{p_1},e_{q_1}}(\mathcal{I}) + \mathscr{K}_{e_{p_2},e_{q_1}}(\mathcal{I}).\end{aligned}$$

Then, for $q_1 \neq q_2$,

$$\begin{aligned}\mathscr{K}_{I,e_{q_1}+e_{q_2}}(\mathcal{I}) &= \mathscr{K}_{I,e_{q_1}}(\mathcal{I}) + (\overline{\partial}_{q_2}(\mathscr{K}_{I,0}(\mathcal{I})))\mathcal{I} \\ &= \mathscr{K}_{e_{p_1},e_{q_1}}(\mathcal{I}) + \mathscr{K}_{e_{p_2},e_{q_1}}(\mathcal{I}) + \mathscr{K}_{e_{p_1},e_{q_2}}(\mathcal{I}) + \mathscr{K}_{e_{p_2},e_{q_2}}(\mathcal{I}),\end{aligned}$$

and for $q_1 = q_2$,

$$\begin{aligned}\mathscr{K}_{I,e_{q_1}+e_{q_2}}(\mathcal{I}) &= (\overline{\partial}_{q_2}(\mathscr{K}_{e_{p_1}+e_{p_2},e_{q_1}}(\mathcal{I})))\mathcal{I} \\ &= (\overline{\partial}_{q_2}(\mathscr{K}_{e_{p_1},e_{q_1}}(\mathcal{I}) + \mathscr{K}_{e_{p_2},e_{q_1}}(\mathcal{I})))\mathcal{I} \\ &= \mathscr{K}_{e_{p_1},2e_{q_1}}(\mathcal{I}) + \mathscr{K}_{e_{p_2},2e_{q_1}}(\mathcal{I}).\end{aligned}$$

If $p_1 = p_2$, we obtain $\mathscr{K}_{e_{p_1}+e_{p_2},e_{q_1}}(\mathcal{I}) = \mathscr{K}_{2e_{p_1},e_{q_1}}(\mathcal{I})$. Then, for $q_1 \neq q_2$,

$$\mathscr{K}_{I,e_{q_1}+e_{q_2}}(\mathcal{I}) = \mathscr{K}_{I,e_{q_1}}(\mathcal{I}) + (\overline{\partial}_{q_2}(\mathscr{K}_{I,0}(\mathcal{I})))\mathcal{I} = \mathscr{K}_{2e_{p_1},e_{q_1}}(\mathcal{I}) + \mathscr{K}_{2e_{p_1},e_{q_2}}(\mathcal{I}),$$

and for $q_1 = q_2$,

$$\begin{aligned}\mathscr{K}_{I,e_{q_1}+e_{q_2}}(\mathcal{I}) &= (\overline{\partial}_{q_2}(\mathscr{K}_{e_{p_1}+e_{p_2},e_{q_1}}(\mathcal{I})))\mathcal{I} \\ &= (\overline{\partial}_{q_2}(\mathscr{K}_{2e_{p_1},e_{q_1}}(\mathcal{I})))\mathcal{I} = \mathscr{K}_{2e_{p_1},2e_{q_1}}(\mathcal{I}).\end{aligned}$$

Without loss of generality, assume that $\mathscr{K}_{I,J}(\mathcal{I}) = \sum_{i_p,j_q\neq 0} \mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I})$ for some $K \in \mathbb{Z}_+^m \setminus \{0\}$ and any $I, J \leq K$, where the order of partial differentiation on the right-hand side corresponds to that on the left-hand side. Therefore, by Definition 3.1, for $\mathscr{K}_{I+e_l,J}(\mathcal{I})$ and $\mathscr{K}_{I,J+e_k}(\mathcal{I})$, $1 \leq l, k \leq m$, we have

$$\begin{aligned}\mathscr{K}_{I+e_l,J}(\mathcal{I}) &= \mathscr{K}_{\sum_{i_p\neq 0,\, p\neq l} i_pe_p,J}(\mathcal{I}) + \mathcal{I}(\partial_l(\mathscr{K}_{i_le_l,J}(\mathcal{I}))) \\ &= \sum_{\substack{i_p,j_q\neq 0\\ p\neq l}} \mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I}) + \mathcal{I}\Big(\partial_l\Big(\sum_{j_q\neq 0} \mathscr{K}_{i_le_l,j_qe_q}(\mathcal{I})\Big)\Big) \\ &= \sum_{\substack{i_p,j_q\neq 0\\ p\neq l}} \mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I}) + \sum_{j_q\neq 0} \mathscr{K}_{(i_l+1)e_l,j_qe_q}(\mathcal{I}),\end{aligned}$$

and

$$\begin{aligned}\mathscr{K}_{I,J+e_k}(\mathcal{I}) &= \mathscr{K}_{I,\sum_{\substack{j_q\neq 0\\ q\neq k}} j_qe_q}(\mathcal{I}) + (\overline{\partial}_k(\mathscr{K}_{I,j_ke_k}(\mathcal{I})))\mathcal{I} \\ &= \sum_{\substack{i_p,j_q\neq 0\\ q\neq k}} \mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I}) + \Big(\overline{\partial}_k\Big(\sum_{i_p\neq 0} \mathscr{K}_{i_pe_p,j_ke_k}(\mathcal{I})\Big)\Big)\mathcal{I} \\ &= \sum_{\substack{i_p,j_q\neq 0\\ q\neq k}} \mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I}) + \sum_{i_p\neq 0} \mathscr{K}_{i_pe_p,(j_k+1)e_k}(\mathcal{I}).\end{aligned}$$

Thus, property (iii) holds for all $I, J \in \mathbb{Z}_+^m \setminus \{0\}$. ∎

For an extended holomorphic curve $\mathcal{I} = F(G^*F)^{-1}G^* \in \mathcal{I}_n(\Omega,\mathcal{U})$, by Definition 2.5, the metric is given by $H(\lambda) = G^*(\lambda)F(\lambda)$. The *classical connection* $\Theta(\lambda)$ and the *classical curvature* $\mathcal{K}_{\mathcal{I}}$ of $\mathcal{I}$ are defined by

$$\Theta(\lambda) = H^{-1}(\lambda)\sum_{i=1}^{m}\Big(\frac{\partial}{\partial\lambda_i}H(\lambda)\Big),\ \mathcal{K}_{\mathcal{I}}(\lambda) = -\sum_{i,j=1}^{m}\frac{\partial}{\partial\overline{\lambda}_j}\Big(H^{-1}(\lambda)\frac{\partial}{\partial\lambda_i}H(\lambda)\Big).$$

Moreover, the covariant derivatives $\mathcal{K}_{\mathcal{I},\lambda^I,\overline{\lambda}^J}$ of $\mathcal{K}_{\mathcal{I}}$ are defined as follows:

(i) $\mathcal{K}_{\mathcal{I},\overline{\lambda}_j} = \frac{\partial}{\partial\overline{\lambda}_j}(\mathcal{K}_{\mathcal{I}})$,

(ii) $\mathcal{K}_{\mathcal{I},\lambda_i} = \frac{\partial}{\partial\lambda_i}(\mathcal{K}_{\mathcal{I}}) + \big[H^{-1}\frac{\partial}{\partial\lambda_i}H, \mathcal{K}_{\mathcal{I}}\big]$, $\lambda\in\Omega$.

According to this definition, $\mathcal{K}_{\mathcal{I},\lambda^I,\overline{\lambda}^J}$ for $\mathcal{I}\in\mathcal{I}_n(\Omega,\mathcal{U})$ also has a decomposition expression similar to Lemma 3.3. The proof is omitted here.

LEMMA 3.4. *Let $\mathcal{I}\in\mathcal{I}_n(\Omega,\mathcal{U})$ be an extended holomorphic curve. Then $\mathcal{K}_{\mathcal{I},\lambda^I,\overline{\lambda}^J}$ has the following properties:*

(i) $\mathcal{K}_{\mathcal{I},\lambda^I} = \sum_{i_p\neq 0}\mathcal{K}_{\mathcal{I},\lambda^{i_pe_p}}$ *for any* $I = \sum_{p=1}^{m} i_pe_p \in \mathbb{Z}_+^m\setminus\{0\}$*;*

(ii) $\mathcal{K}_{\mathcal{I},\overline{\lambda}^J} = \sum_{j_q\neq 0}\mathcal{K}_{\mathcal{I},\overline{\lambda}^{j_qe_q}}$ *for any* $J = \sum_{q=1}^{m} j_qe_q \in \mathbb{Z}_+^m\setminus\{0\}$*;*

(iii) $\mathcal{K}_{\mathcal{I},\lambda^I,\overline{\lambda}^J} = \sum_{i_p,j_q\neq 0}\mathcal{K}_{\mathcal{I},\lambda^{i_pe_p},\overline{\lambda}^{j_qe_q}}$ *for any* $I = \sum_{p=1}^{m} i_pe_p$, $J = \sum_{q=1}^{m} j_qe_q \in \mathbb{Z}_+^m\setminus\{0\}$, *where the order of partial differentiation on the right-hand side corresponds to that on the left-hand side, and* $e_p = (0,\dots,1,\dots,0) \in \mathbb{Z}_+^m$ *with* 1 *in the* p*-th position.*

The following theorem is the main result of this section. It establishes a relationship between the curvature and its covariant derivatives, and their classical counterparts for multivariate extended holomorphic curves on C^*-algebras.

THEOREM 3.5. *Let $\mathcal{I} = F(G^*F)^{-1}G^* \in \mathcal{I}_n(\Omega,\mathcal{U})$ be an extended holomorphic curve. Then for all $\lambda\in\Omega$ and $I,J\in\mathbb{Z}_+^m$,*

$$\mathscr{K}_{I,J}(\mathcal{I})(\lambda)F(\lambda) + F(\lambda)\mathcal{K}_{\mathcal{I},\lambda^I,\overline{\lambda}^J}(\lambda) = 0,$$

where $F,G:\Omega\to\mathscr{L}(\mathcal{H},l^2(\mathbb{N},B))$ are operator-valued holomorphic functions.

Proof. Let $H = G^*F$; then $\mathcal{I} = FH^{-1}G^*$. We compute

$$\begin{aligned}\overline{\partial}\mathcal{I}\partial\mathcal{I} &= \overline{\partial}(FH^{-1}G^*)\partial(FH^{-1}G^*)\\ &= F(\overline{\partial}H^{-1}G^* + H^{-1}\overline{\partial}G^*)(\partial FH^{-1} + F\partial H^{-1})G^*\\ &= F\overline{\partial}(H^{-1}\partial H)H^{-1}G^*.\end{aligned}\tag{3.4}$$

By (2.3), (3.1) and (3.4), we obtain $\mathscr{K}(\mathcal{I}) = F(-\mathcal{K}_{\mathcal{I}})H^{-1}G^*$, and then

$$\mathscr{K}(\mathcal{I})F + F\mathcal{K}_{\mathcal{I}} = 0.\tag{3.5}$$

Since $\mathcal{I}$ is an extended holomorphic curve, by Definition 2.2 we have

$$\partial_l\overline{\partial}\mathcal{I} = \partial_l\mathcal{I}\overline{\partial}\mathcal{I} - \overline{\partial}\mathcal{I}\partial_l\mathcal{I}.$$

Then

$$\begin{aligned}\mathscr{K}_{e_l,0}(\mathcal{I}) &= \mathcal{I}(\partial_l(\mathscr{K}(\mathcal{I}))) \\ &= \mathcal{I}(\partial_l\overline{\partial}\mathcal{I}\partial\mathcal{I} + \overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I}) \\ &= \mathcal{I}(\partial_l\mathcal{I}\overline{\partial}\mathcal{I}\partial\mathcal{I} - \overline{\partial}\mathcal{I}\partial_l\mathcal{I}\partial\mathcal{I}) + \mathcal{I}\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I} = \overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I}\end{aligned}$$

and

$$\begin{aligned}\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I} &= \overline{\partial}(FH^{-1}G^*)\partial_l(\partial(FH^{-1}G^*)) \\ &= F(\overline{\partial}H^{-1}G^* + H^{-1}\overline{\partial}G^*) \\ &\quad \times (\partial\partial_l FH^{-1} + \partial F\partial_l H^{-1} + \partial_l F\partial H^{-1} + F\partial\partial_l H^{-1})G^* \\ &= F(-\partial_l(\mathcal{K}_{\mathcal{I}}) + \overline{\partial}H^{-1}\partial H\partial_l H^{-1}H + H^{-1}\overline{\partial}\partial H\partial_l H^{-1}H \\ &\quad - \partial_l H^{-1}\partial\overline{\partial}H - \partial_l H^{-1}\overline{\partial}H\partial H^{-1}H)H^{-1}G^* \\ &= F(-\partial_l(\mathcal{K}_{\mathcal{I}}) - [H^{-1}\partial_l H, \mathcal{K}_{\mathcal{I}}])H^{-1}G^* = F(-\mathcal{K}_{\mathcal{I},\lambda^{e_l}})H^{-1}G^*\end{aligned}$$

for any $1 \leq l \leq m$. Thus,

$$\mathscr{K}_{e_l,0}(\mathcal{I})F + F\mathcal{K}_{\mathcal{I},\lambda^{e_l}} = 0, \qquad 1 \leq l \leq m. \tag{3.6}$$

Similarly,

$$\mathscr{K}_{0,e_k}(\mathcal{I})F + F\mathcal{K}_{\mathcal{I},\overline{\lambda}^{e_k}} = 0, \qquad 1 \leq k \leq m. \tag{3.7}$$

Note that

$$\begin{aligned}\mathscr{K}_{0,e_k}(\mathcal{I}) &= \big(\overline{\partial}_k(\mathscr{K}_{0,0}(\mathcal{I}))\big)\mathcal{I} = [\overline{\partial}_k\overline{\partial}\mathcal{I}\partial\mathcal{I} + \overline{\partial}\mathcal{I}(\partial\mathcal{I}\overline{\partial}_k\mathcal{I} - \overline{\partial}_k\mathcal{I}\partial\mathcal{I})]\mathcal{I} = \overline{\partial}_k\overline{\partial}\mathcal{I}\partial\mathcal{I}, \\ -\mathcal{K}_{\mathcal{I},\overline{\lambda}^{e_k}} &= \overline{\partial}_k(\overline{\partial}(H^{-1}\partial H)) \\ &= \overline{\partial}_k\overline{\partial}H^{-1}\partial H + \overline{\partial}H^{-1}\partial\overline{\partial}_k H + \overline{\partial}_k H^{-1}\partial\overline{\partial}H + H^{-1}\partial\overline{\partial}_k\overline{\partial}H.\end{aligned}$$

We have

$$\begin{aligned}\mathscr{K}_{0+e_l,e_k}(\mathcal{I}) &= \mathcal{I}\big(\partial_l(\mathscr{K}_{0,e_k}(\mathcal{I}))\big) \\ &= -\overline{\partial}\mathcal{I}\partial_l\mathcal{I}\overline{\partial}_k\mathcal{I}\partial\mathcal{I} - \overline{\partial}_k\mathcal{I}\partial_l\mathcal{I}\overline{\partial}\mathcal{I}\partial\mathcal{I} + \overline{\partial}_k\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&-(\mathcal{K}_{\mathcal{I},\overline{\lambda}^{e_k}})_{\lambda^{e_l}} \\ &= -(\partial_l(\mathcal{K}_{\mathcal{I},\overline{\lambda}^{e_k}}) + [H^{-1}\partial_l H, \mathcal{K}_{\mathcal{I},\overline{\lambda}^{e_k}}]) \\ &= \partial_l(\overline{\partial}_k\overline{\partial}H^{-1}\partial H + \overline{\partial}H^{-1}\partial\overline{\partial}_k H + \overline{\partial}_k H^{-1}\partial\overline{\partial}H + H^{-1}\partial\overline{\partial}_k\overline{\partial}H) \\ &\quad + H^{-1}\partial_l H(\overline{\partial}_k\overline{\partial}H^{-1}\partial H + \overline{\partial}H^{-1}\partial\overline{\partial}_k H + \overline{\partial}_k H^{-1}\partial\overline{\partial}H + H^{-1}\partial\overline{\partial}_k\overline{\partial}H) \\ &\quad - (\overline{\partial}_k\overline{\partial}H^{-1}\partial H + \overline{\partial}H^{-1}\partial\overline{\partial}_k H + \overline{\partial}_k H^{-1}\partial\overline{\partial}H + H^{-1}\partial\overline{\partial}_k\overline{\partial}H)H^{-1}\partial_l H\end{aligned}$$

$$
\begin{aligned}
&= \partial_l\overline{\partial}_k\overline{\partial}H^{-1}\partial H + \overline{\partial}_k\overline{\partial}H^{-1}\partial_l\partial H + \partial_l\overline{\partial}H^{-1}\partial\overline{\partial}_k H + \overline{\partial}H^{-1}\partial_l\partial\overline{\partial}_k H\\
&\quad + \partial_l\overline{\partial}_k H^{-1}\partial\overline{\partial}H + \overline{\partial}_k H^{-1}\partial_l\partial\overline{\partial}H + H^{-1}\partial_l\partial\overline{\partial}_k\overline{\partial}H + H^{-1}\partial_l H\overline{\partial}_k\overline{\partial}H^{-1}\partial H\\
&\quad + H^{-1}\partial_l H\overline{\partial}H^{-1}\partial\overline{\partial}_k H + H^{-1}\partial_l H\overline{\partial}_k H^{-1}\partial\overline{\partial}H - \overline{\partial}_k\overline{\partial}H^{-1}\partial H H^{-1}\partial_l H\\
&\quad - \overline{\partial}H^{-1}\partial\overline{\partial}_k H H^{-1}\partial_l H - \overline{\partial}_k H^{-1}\partial\overline{\partial}H H^{-1}\partial_l H - H^{-1}\partial\overline{\partial}_k\overline{\partial}H H^{-1}\partial_l H,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.9)\qquad & F(-(\mathcal{K}_{\mathcal{I},\overline{\lambda}^{e_k}})_{\lambda^{e_l}})H^{-1}G^*\\
&= \overline{\partial}_k\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I} + F(\partial_l\overline{\partial}H^{-1}\partial\overline{\partial}_k H + \partial_l\overline{\partial}_k H^{-1}\partial\overline{\partial}H + H^{-1}\partial_l H\overline{\partial}H^{-1}\partial\overline{\partial}_k H\\
&\quad + H^{-1}\partial_l H\overline{\partial}_k H^{-1}\partial\overline{\partial}H - \partial_l\overline{\partial}_k H^{-1}\overline{\partial}H H^{-1}\partial H - H^{-1}\partial_l\overline{\partial}H\overline{\partial}_k H^{-1}\partial H\\
&\quad - \overline{\partial}H^{-1}\partial_l H\overline{\partial}_k H^{-1}\partial H + \partial_l H^{-1}\overline{\partial}_k H\overline{\partial}H^{-1}\partial H)H^{-1}G^*\\
&= \overline{\partial}_k\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I} + F(\partial_l\overline{\partial}H^{-1}\partial\overline{\partial}_k H + \partial_l\overline{\partial}_k H^{-1}\partial\overline{\partial}H + H^{-1}\partial_l H\overline{\partial}H^{-1}\partial\overline{\partial}_k H\\
&\quad + H^{-1}\partial_l H\overline{\partial}_k H^{-1}\partial\overline{\partial}H + \partial_l\overline{\partial}_k H^{-1}H\overline{\partial}H^{-1}\partial H + \partial_l H^{-1}\overline{\partial}H\overline{\partial}_k H^{-1}\partial H\\
&\quad + \partial_l\overline{\partial}H^{-1}H\overline{\partial}_k H^{-1}\partial H + \partial_l H^{-1}\overline{\partial}_k H\overline{\partial}H^{-1}\partial H)H^{-1}G^*\\
&= \overline{\partial}_k\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I} + F(\overline{\partial}(\partial_l H^{-1}H)\overline{\partial}_k(H^{-1}\partial H)\\
&\quad + \overline{\partial}_k(\partial_l H^{-1}H)\overline{\partial}(H^{-1}\partial H))H^{-1}G^*\\
&= \overline{\partial}_k\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I} + F(-\overline{\partial}(H^{-1}\partial_l H)\overline{\partial}_k(H^{-1}\partial H)\\
&\quad - \overline{\partial}_k(H^{-1}\partial_l H)\overline{\partial}(H^{-1}\partial H))H^{-1}G^*\\
&= \overline{\partial}_k\overline{\partial}\mathcal{I}\partial_l\partial\mathcal{I} - \overline{\partial}\mathcal{I}\partial_l\mathcal{I}\overline{\partial}_k\mathcal{I}\partial\mathcal{I} - \overline{\partial}_k\mathcal{I}\partial_l\mathcal{I}\overline{\partial}\mathcal{I}\partial\mathcal{I}.
\end{aligned}
$$

Therefore, by (3.8) and (3.9), we conclude that

$$\mathcal{I}\left(\partial_l(\mathscr{K}_{0,e_k}(\mathcal{I}))\right)F + F(\mathcal{K}_{\mathcal{I},\overline{\lambda}^{e_k}})_{\lambda^{e_l}} = 0.$$

Similarly,

$$(\overline{\partial}_k(\mathscr{K}_{e_l,0}(\mathcal{I})))\mathcal{I}F + F(\mathcal{K}_{\mathcal{I},\lambda^{e_l}})_{\overline{\lambda}^{e_k}} = 0.$$

Thus,

$$\mathscr{K}_{e_l,e_k}(\mathcal{I})F + F\mathcal{K}_{\mathcal{I},\lambda^{e_l},\overline{\lambda}^{e_k}} = 0, \qquad 1 \le l,k \le m. \tag{3.10}$$

By (3.5)–(3.7) and (3.10), we see that $\mathscr{K}_{i_l e_l, j_k e_k}(\mathcal{I})F + F\mathcal{K}_{\mathcal{I},\lambda^{i_l e_l},\overline{\lambda}^{j_k e_k}} = 0$ for all $0 \le i_l, j_k \le 1$. Proceeding by induction, assume that $\mathscr{K}_{i_l e_l, j_k e_k}(\mathcal{I})F + F\mathcal{K}_{\mathcal{I},\lambda^{i_l e_l},\overline{\lambda}^{j_k e_k}} = 0$ for certain $i_l, j_k \ge 0$. As $\mathscr{K}_{i_l e_l, j_k e_k}(\mathcal{I}) = \mathscr{K}_{i_l e_l, j_k e_k}(\mathcal{I})\mathcal{I}$, we have

$$
\begin{aligned}
&\mathscr{K}_{i_l e_l, j_k e_k + e_k}(\mathcal{I})\\
&\quad = (\overline{\partial}_k(\mathscr{K}_{i_l e_l, j_k e_k}(\mathcal{I})))\mathcal{I} = \overline{\partial}_k(-F\mathcal{K}_{\mathcal{I},\lambda^{i_l e_l},\overline{\lambda}^{j_k e_k}}H^{-1}G^*)\mathcal{I}\\
&\quad = -F(\overline{\partial}_k\mathcal{K}_{\mathcal{I},\lambda^{i_l e_l},\overline{\lambda}^{j_k e_k}}H^{-1}G^* + \mathcal{K}_{\mathcal{I},\lambda^{i_l e_l},\overline{\lambda}^{j_k e_k}}(\overline{\partial}_k H^{-1} + H^{-1}\overline{\partial}_k H H^{-1})G^*)\\
&\quad = -F\mathcal{K}_{\mathcal{I},\lambda^{i_l e_l},\overline{\lambda}^{j_k e_k + e_k}}H^{-1}G^*,
\end{aligned}
$$

and

$$\begin{aligned}\mathscr{K}_{i_le_l+e_l,j_ke_k}(\mathcal{I}) &= \mathcal{I}(\partial_l(-F\mathcal{K}_{\mathcal{I},\lambda^{i_le_l},\overline{\lambda}^{j_ke_k}}H^{-1}G^*))\\ &= -F(\partial_l\mathcal{K}_{\mathcal{I},\lambda^{i_le_l},\overline{\lambda}^{j_ke_k}} + [H^{-1}\partial_l H,\mathcal{K}_{\mathcal{I},\lambda^{i_le_l},\overline{\lambda}^{j_ke_k}}])H^{-1}G^*\\ &= -F\mathcal{K}_{\mathcal{I},\lambda^{i_le_l+e_l},\overline{\lambda}^{j_ke_k}}H^{-1}G^*.\end{aligned}$$

This means that $\mathscr{K}_{i_le_l,j_ke_k}(\mathcal{I})F+F\mathcal{K}_{\mathcal{I},\lambda^{i_le_l},\overline{\lambda}^{j_ke_k}}=0$ for any $i_l,j_k\geq 0$. Hence, by Lemmas 3.3 and 3.4, we obtain, $I,J\in\mathbb{Z}_+^m\setminus\{0\}$,

$$\mathscr{K}_{I,0}(\mathcal{I}) = \sum_{i_l\neq 0}\mathscr{K}_{i_le_l,0}(\mathcal{I}) = -\sum_{i_l\neq 0}F\mathcal{K}_{\mathcal{I},\lambda^{i_le_l}}H^{-1}G^* = -F\mathcal{K}_{\mathcal{I},\lambda^I}H^{-1}G^*,$$
$$\mathscr{K}_{0,J}(\mathcal{I}) = \sum_{j_k\neq 0}\mathscr{K}_{0,j_ke_k}(\mathcal{I}) = -\sum_{j_k\neq 0}F\mathcal{K}_{\mathcal{I},\overline{\lambda}^{j_ke_k}}H^{-1}G^* = -F\mathcal{K}_{\mathcal{I},\overline{\lambda}^J}H^{-1}G^*,$$

and

$$\begin{aligned}\mathscr{K}_{I,J}(\mathcal{I}) &= \sum_{i_l,j_k\neq 0}\mathscr{K}_{i_le_l,j_ke_k}(\mathcal{I}) = -\sum_{i_l,j_k\neq 0}F\mathcal{K}_{\mathcal{I},\lambda^{i_le_l},\overline{\lambda}^{j_ke_k}}H^{-1}G^*\\ &= -F\mathcal{K}_{\mathcal{I},\lambda^I,\overline{\lambda}^J}H^{-1}G^*.\end{aligned}$$

Hence, $\mathscr{K}_{I,J}(\mathcal{I})F+F\mathcal{K}_{\mathcal{I},\lambda^I,\overline{\lambda}^J}=0$ for all $I,J\in\mathbb{Z}_+^m$. ∎

COROLLARY 3.6. *Let* $\mathbf{T}=(T_1,\ldots,T_m)\in\mathcal{B}_n^m(\Omega)$. *Then for all* $\lambda\in\Omega$ *and* $I,J\in\mathbb{Z}_+^m$,

$$\operatorname{trace}\mathscr{K}_{I,J}(\mathcal{I}_{\mathbf{T}})(\lambda) = -\operatorname{trace}\mathcal{K}_{\mathcal{I}_{\mathbf{T}},\lambda^I,\overline{\lambda}^J}(\lambda).$$

COROLLARY 3.7. *Let* $\mathbf{T}=(T_1,\ldots,T_m)\in\mathcal{B}_n^m(\Omega)$. *The classical curvature* $\mathcal{K}_{\mathbf{T}}$ *of* $E_{\mathbf{T}}$ *and its covariant derivatives* $\mathcal{K}_{\mathbf{T},\lambda^I,\overline{\lambda}^J}$ *are similar across different holomorphic frames. Moreover, their traces* $\operatorname{trace}\mathcal{K}_{\mathbf{T},\lambda^I,\overline{\lambda}^J}$ *for all* $I,J\in\mathbb{Z}_+^m$ *are invariant, meaning they do not depend on the choice of holomorphic frames.*

The above results reveal a relationship between the curvature $\mathscr{K}(\mathcal{I})$ and the classical curvature $\mathcal{K}_{\mathcal{I}}$ for an extended holomorphic curve $\mathcal{I}\in\mathcal{I}_n(\Omega,\mathcal{U})$. Additionally, the corresponding connection $\boldsymbol{\Theta}$ and classical connection Θ exhibit a similar relationship.

PROPOSITION 3.8. *Let* $\mathcal{I}=F(G^*F)^{-1}G^*\in\mathcal{I}_n(\Omega,\mathcal{U})$. *The connection* $\boldsymbol{\Theta}$ *and the classical connection* Θ *of* $\mathcal{I}$ *satisfy*

$$\boldsymbol{\Theta}(\lambda)F(\lambda)=F(\lambda)\Theta(\lambda)\quad\textit{and}\quad\boldsymbol{\Theta}(\lambda)\mathcal{I}(\lambda)=\boldsymbol{\Theta}(\lambda),\qquad\lambda\in\Omega.$$

Proof. Since $(\partial\mathcal{I})\mathcal{I}=\partial\mathcal{I}$, we have

$$\boldsymbol{\Theta}\mathcal{I} = [(\partial F)H^{-1}G^*-\partial\mathcal{I}]\mathcal{I} = (\partial F)H^{-1}G^*-\partial\mathcal{I} = \boldsymbol{\Theta}$$

and

$$\boldsymbol{\Theta} F = [(\partial F)H^{-1}G^* - \partial \mathcal{I}]F = \partial F - [(\partial F)H^{-1}G^* + F(\partial H^{-1})G^*]F$$
$$= -F\partial H^{-1}H = F\Theta. \ \blacksquare$$

Intuitively, the covariant derivatives of the curvature as defined in Definition 3.1 on C^*-algebras exhibit greater symmetry compared to their counterparts in complex geometry (Definition 2.5). For Hermitian holomorphic vector bundles, the classical curvature and its covariant derivatives (Definition 2.5) form unitary invariants of the Cowen–Douglas class. Motivated by this result and Theorem 3.5, one may employ the curvature and its covariant derivatives (Definition 3.1) in the algebraic setting to investigate the congruence and similarity classification of more general multivariate extended holomorphic curves on C^*-algebras.

4. Classification of extended holomorphic curves over C^*-algebras. In this section, we focus on investigating the congruence and similarity classification of multivariate extended holomorphic curves on C^*-algebras. Our approach centers on the intrinsic structure of the curvature and its covariant derivatives as analytical tools, from which we derive the corresponding classification criteria. Subsequently, we apply these results to the study of the unitary and similarity classification of the Cowen–Douglas class.

THEOREM 4.1. *Let $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{I}_n(\Omega, \mathcal{U})$. If $\mathcal{I}_1 \sim_u \mathcal{I}_2$, then*

$$\mathscr{K}_{I,J}(\mathcal{I}_1) \sim_u \mathscr{K}_{I,J}(\mathcal{I}_2)$$

for all $I, J \in \mathbb{Z}_+^m$.

To prove the above theorem, which is the main theorem of this section, we need to establish the following three lemmas. They reveal the intrinsic structure of the curvature and its covariant derivatives for multivariate extended holomorphic curves on C^*-algebras. For $I = (i_1, \ldots, i_m), J = (j_1, \ldots, j_m) \in \mathbb{Z}_+^m$, we write $I \geq J$ if $i_k \geq j_k$ for all $1 \leq k \leq m$. For $I \geq J$, we define

$$\binom{I}{J} = \frac{I!}{J!(I-J)!} = \frac{i_1! i_2! \cdots i_m!}{j_1! j_2! \cdots j_m! (i_1 - j_1)!(i_2 - j_2)! \cdots (i_m - j_m)!}.$$

LEMMA 4.2. *Let $\mathcal{I} = F(G^*F)^{-1}G^* \in \mathcal{I}_n(\Omega, \mathcal{U})$ with $H = G^*F$. Then for any $1 \leq i, j \leq m$ and $I, J \in \mathbb{Z}_+^m$, the following properties hold:*

(i) $D^I \mathcal{I} = \sum\limits_{0 \leq I_1 \leq I} \binom{I}{I_1} D^{I_1} F D^{I - I_1} H^{-1} G^*$;

(ii) $\overline{D}^{e_j} D^I \mathcal{I} = D^I \mathcal{I} \overline{D}^{e_j} \mathcal{I} - \overline{D}^{e_j} \mathcal{I} D^I \mathcal{I} - \sum\limits_{p=1}^{m} \sum\limits_{e_p \leq I_1 \leq I - e_p} \binom{I}{I_1} D^{I - I_1} \mathcal{I} \overline{D}^{e_j} \mathcal{I} D^{I_1} \mathcal{I}$;

(iii) $\overline{D}^J\mathcal{I} = \sum\limits_{0\le J_1\le J}\binom{J}{J_1}F\overline{D}^{J_1}H^{-1}\overline{D}^{J-J_1}G^*$;

(iv) $\overline{D}^J D^{e_i}\mathcal{I} = D^{e_i}\mathcal{I}\overline{D}^J\mathcal{I} - \overline{D}^J\mathcal{I}D^{e_i}\mathcal{I} - \sum\limits_{p=1}^{m}\sum\limits_{e_p\le J_1\le J-e_p}\binom{J}{J_1}\overline{D}^{J-J_1}\mathcal{I}D^{e_i}\mathcal{I}\overline{D}^{J_1}\mathcal{I}$.

Proof. For any $e_i = (0,\ldots,1,\ldots,0)\in\mathbb{Z}_+^m$, where $1\le i\le m$ and the non-zero entry 1 is in the p-th position, we have

$$D^{e_i}\mathcal{I} = D^{e_i}(FH^{-1}G^*) = D^{e_i}FH^{-1}G^* + FD^{e_i}H^{-1}G^*,$$

which implies that property (i) holds for $I = e_i$. Without loss of generality, assume property (i) holds for some $I\in\mathbb{Z}_+^m$, that is,

$$D^I\mathcal{I} = \sum_{0\le I_1\le I}\binom{I}{I_1}D^{I_1}FD^{I-I_1}H^{-1}G^*.$$

Then for any $1\le l\le m$, we have

$$\begin{aligned}
D^{I+e_l}\mathcal{I} &= D^{e_l}\left[\sum_{0\le I_1\le I}\binom{I}{I_1}D^{I_1}FD^{I-I_1}H^{-1}G^*\right]\\
&= \sum_{0\le I_1\le I}\binom{I}{I_1}(D^{I_1+e_l}FD^{I-I_1}H^{-1} + D^{I_1}FD^{I-I_1+e_l}H^{-1})G^*\\
&= FD^{I+e_l}H^{-1}G^*\\
&\quad + \sum_{0<I'<I+e_l}\left[\binom{I}{I'-e_l} + \binom{I}{I'}\right]D^{I'}FD^{I-I'+e_l}H^{-1}G^* + D^{I+e_l}FH^{-1}G^*\\
&= \sum_{0\le I'\le I+e_l}\binom{I+e_l}{I'}D^{I'}FD^{(I+e_l)-I'}H^{-1}G^*.
\end{aligned}$$

It follows that property (i) holds for all $I\in\mathbb{Z}_+^m$.

Since $\mathcal{I}$ is an extended holomorphic curve, we know that $D^{e_i}\mathcal{I}\cdot\mathcal{I} = D^{e_i}\mathcal{I}$ and $\overline{D}^{e_j}\mathcal{I}\cdot\mathcal{I} = 0$. Thus,

$$\begin{aligned}
\overline{D}^{e_j}(D^{e_i}\mathcal{I}) = \overline{D}^{e_j}(D^{e_i}\mathcal{I}\cdot\mathcal{I}) &= (\overline{D}^{e_j}D^{e_i}\mathcal{I})\mathcal{I} + (D^{e_i}\mathcal{I})(\overline{D}^{e_j}\mathcal{I})\\
&= (D^{e_i}\mathcal{I})(\overline{D}^{e_j}\mathcal{I}) - (\overline{D}^{e_j}\mathcal{I})(D^{e_i}\mathcal{I}).
\end{aligned}$$

Hence, property (ii) holds for $I = e_i$. Without loss of generality, assume property (ii) holds for some $I\in\mathbb{Z}_+^m$. For any $e_l, e_j, e_p\in\mathbb{Z}_+^m$ with $1\le l,j,p\le m$ and $p\ne l$, we have

$$\begin{aligned}
&D^I\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{e_l}\mathcal{I} + D^{e_l}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^I\mathcal{I}\\
&\quad + \sum_{e_l\le I_1\le I-e_l}\binom{I}{I_1}(D^{I+e_l-I_1}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I_1}\mathcal{I} + D^{I-I_1}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I_1+e_l}\mathcal{I})\\
&= \sum_{e_l\le I'\le I}\binom{I+e_l}{I'}(D^{I+e_l-I'}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I'}\mathcal{I})
\end{aligned}\tag{4.1}$$

and

$$\sum_{e_p \le I_1 \le I - e_p} \binom{I}{I_1} (D^{I+e_l-I_1}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I_1}\mathcal{I} + D^{I-I_1}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I_1+e_l}\mathcal{I}) \\ = \sum_{e_p \le I' \le I+e_l-e_p} \binom{I+e_l}{I'} (D^{I+e_l-I'}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I'}\mathcal{I}). \tag{4.2}$$

Therefore, by (4.1) and (4.2), we obtain

$$\begin{aligned}
&\overline{D}^{e_j}D^{I+e_l}\mathcal{I} \\
&\quad = D^{e_l}(\overline{D}^{e_j}D^{I}\mathcal{I}) \\
&\quad = D^{e_l}\Big(D^I\mathcal{I}\overline{D}^{e_j}\mathcal{I} - \overline{D}^{e_j}\mathcal{I}D^I\mathcal{I} - \sum_{p=1}^{m}\sum_{e_p \le I_1 \le I-e_p}\binom{I}{I_1}D^{I-I_1}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I_1}\mathcal{I}\Big) \\
&\quad = D^{I+e_l}\mathcal{I}\overline{D}^{e_j}\mathcal{I} - \overline{D}^{e_j}\mathcal{I}D^{I+e_l}\mathcal{I} - D^I\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{e_l}\mathcal{I} - D^{e_l}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^I\mathcal{I} \\
&\qquad - \sum_{p=1}^{m}\sum_{e_p \le I_1 \le I-e_p}\binom{I}{I_1}(D^{I+e_l-I_1}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I_1}\mathcal{I} + D^{I-I_1}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I_1+e_l}\mathcal{I}) \\
&\quad = D^{I+e_l}\mathcal{I}\overline{D}^{e_j}\mathcal{I} - \overline{D}^{e_j}\mathcal{I}D^{I+e_l}\mathcal{I} \\
&\qquad - \sum_{p=1}^{m}\sum_{e_p \le I' \le I+e_l-e_p}\binom{I+e_l}{I'}(D^{I+e_l-I'}\mathcal{I}\overline{D}^{e_j}\mathcal{I}D^{I'}\mathcal{I}).
\end{aligned}$$

This completes the proof of property (ii).

Properties (iii) and (iv) can be shown similarly. ∎

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain, and let $\mathcal{I} = F(G^*F)^{-1}G^* \in \mathcal{I}_n(\Omega, \mathcal{U})$. For any $\lambda_0 \in \Omega$, there exists an open neighborhood $\Omega_0 \subset \Omega$ of λ_0 such that

$$\mathcal{I}(\lambda) = \sum_{I,J \in \mathbb{Z}_+^m} \frac{D^I\overline{D}^J\mathcal{I}(\lambda_0)}{I!J!}(\lambda-\lambda_0)^I(\overline{\lambda}-\overline{\lambda}_0)^J, \qquad \lambda \in \Omega_0.$$

This implies that the set $\{D^I\overline{D}^J\mathcal{I} : I, J \in \mathbb{Z}_+^m\}$ uniquely determines the extended holomorphic curve $\mathcal{I}$. In [30, Section 2.10] it is shown that every derivative $D^I\overline{D}^J P$ of the holomorphic curve $P = F(F^*F)^{-1}F^*$ can be expressed as a sum of monomials of the form

$$\pm(\overline{D}^{J_1}P)(D^{I_1}P)\cdots(\overline{D}^{J_t}P)(D^{I_t}P), \qquad \pm(D^{I_1}P)(\overline{D}^{J_1}P)\cdots(D^{I_t}P)(\overline{D}^{J_t}P).$$

The following result demonstrates that this property also holds for $\mathcal{I}$.

Lemma 4.3. *Let $\mathcal{I} \in \mathcal{I}_n(\Omega, \mathcal{U})$. Then, for any $I, J \in \mathbb{Z}_+^m$, the derivative $D^I\overline{D}^J\mathcal{I}(\lambda)$ can be expressed as a sum of terms of the form*

$$\pm(D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I})\cdots(D^{I_t}\mathcal{I})(\overline{D}^{J_t}\mathcal{I}), \tag{4.3}$$

where $I_1 + \cdots + I_t = I$ and $J_1 + \cdots + J_t = J$.

Proof. From $\mathcal{I}\overline{D}^I\mathcal{I} = \overline{D}^I\mathcal{I}$ and $D^I\mathcal{I}\cdot\mathcal{I} = D^I\mathcal{I}$, together with properties (ii) and (iv) of Lemma 4.2, we conclude that the present lemma holds for $D^I\overline{D}^{e_j}\mathcal{I}$ and $D^{e_i}\overline{D}^J\mathcal{I}$, where $I, J \in \mathbb{Z}_+^m$ and $1 \leq i, j \leq m$. Without loss of generality, assume that for some $I, J \in \mathbb{Z}_+^m$,

$$D^I\overline{D}^J\mathcal{I} = (D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I})\cdots(D^{I_t}\mathcal{I})(\overline{D}^{J_t}\mathcal{I}),$$

where $I_1 + \cdots + I_t = I$ and $J_1 + \cdots + J_t = J$. If $I_1, J_t \neq 0$, by property (iv) of Lemma 4.2, for $1 \leq l \leq m$, we obtain

$$\begin{aligned}
&D^{I+e_l}\overline{D}^J\mathcal{I}\\
&= D^{e_l}(D^I\overline{D}^J\mathcal{I})\\
&= \sum_{r=1}^{t}[(D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I})\cdots(D^{I_r+e_l}\mathcal{I})(\overline{D}^{J_r}\mathcal{I})\cdots(D^{I_t}\mathcal{I})(\overline{D}^{J_t}\mathcal{I})]\\
&\quad+\sum_{r=1}^{t}[(D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I})\cdots(D^{I_r}\mathcal{I})(D^{e_l}\overline{D}^{J_r}\mathcal{I})\cdots(D^{I_t}\mathcal{I})(\overline{D}^{J_t}\mathcal{I})]\\
&= \sum_{r=1}^{t}[(D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I})\cdots(D^{I_r+e_l}\mathcal{I})(\overline{D}^{J_r}\mathcal{I})\cdots(D^{I_t}\mathcal{I})(\overline{D}^{J_t}\mathcal{I})]\\
&\quad-\sum_{r=1}^{t}\sum_{p=1}^{m}\sum_{e_p\leq J'_r\leq J_r-e_p}\binom{J_r}{J'_r}[(D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I})\cdots(D^{I_r}\mathcal{I})(\overline{D}^{J_r-J'_r}\mathcal{I}D^{e_l}\mathcal{I}\overline{D}^{J'_r}\mathcal{I})\\
&\qquad\qquad\cdots(D^{I_t}\mathcal{I})(\overline{D}^{J_t}\mathcal{I})]\\
&\quad-[(D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I})\cdots(D^{I_r}\mathcal{I})(\overline{D}^{J_r}\mathcal{I})\cdots(D^{I_t}\mathcal{I})(\overline{D}^{J_t}\mathcal{I})(D^{e_l}\mathcal{I})],
\end{aligned}$$

which satisfies the conclusion of the present lemma. If $I_1 = J_t = 0$ and $J_1, I_t \neq 0$, then

$$\begin{aligned}
&D^{I+e_l}\overline{D}^J\mathcal{I}\\
&= D^{e_l}[(\overline{D}^{J_1}\mathcal{I})(D^{I_2}\mathcal{I})\cdots(\overline{D}^{J_{t-1}}\mathcal{I})(D^{I_t}\mathcal{I})]\\
&= \sum_{r=2}^{t}[(\overline{D}^{J_1}\mathcal{I})(D^{I_2}\mathcal{I})\cdots(D^{I_r+e_l}\mathcal{I})(\overline{D}^{J_r}\mathcal{I})\cdots(D^{I_t}\mathcal{I})]\\
&\quad-\sum_{r=1}^{t-1}\sum_{p=1}^{m}\sum_{e_p\leq J'_r\leq J_r-e_p}\binom{J_r}{J'_r}[(\overline{D}^{J_1}\mathcal{I})(D^{I_2}\mathcal{I})\cdots(D^{I_r}\mathcal{I})(\overline{D}^{J_r-J'_r}\mathcal{I}D^{e_l}\mathcal{I}\overline{D}^{J'_r}\mathcal{I})\\
&\qquad\qquad\cdots(D^{I_t}\mathcal{I})],
\end{aligned}$$

which also satisfies the present lemma. Similarly, for the cases where $I_1 = 0$ and $J_t \neq 0$, or $I_1 \neq 0$ and $J_t = 0$, $D^{I+e_l}\overline{D}^J\mathcal{I}$ can also be expressed as a sum of terms of the form given in (4.3). The same argument works for $D^I\overline{D}^{J+e_l}\mathcal{I}$ with $1 \leq l \leq m$. ∎

Based on the formal decomposition of the covariant derivatives of the curvature for an extended holomorphic curve given in Lemma 3.3, together with Lemmas 4.2 and 4.3, the covariant derivative $\mathscr{K}_{I,J}(\mathcal{I})$ of the curvature $\mathscr{K}(\mathcal{I})$ of an extended holomorphic curve $\mathcal{I} \in \mathcal{I}_n(\Omega, \mathcal{U})$ can also be expressed as a sum of monomials of a certain type.

LEMMA 4.4. *Let $\mathcal{I} \in \mathcal{I}_n(\Omega, \mathcal{U})$. For any $I, J \in \mathbb{Z}_+^m$, the covariant derivative $\mathscr{K}_{I,J}(\mathcal{I})$ can be expressed as a sum of monomials of the form*

$$\pm (\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1} \cdots (\overline{D}^{J_r}\mathcal{I}D^{I_r}\mathcal{I})^{l_r} \cdots (\overline{D}^{J_t}\mathcal{I}D^{I_t}\mathcal{I})^{l_t}, \quad I_r, J_r \in \mathbb{Z}_+^m \setminus \{0\},\ 1 \le r \le t,$$

where $I_r = s_{r,p}e_p + n_{r,l}e_l$, $J_r = s_{r,q}e_q + n_{r,k}e_k$, $s_{r,p}, s_{r,q} \ge 0$, $n_{r,l}, n_{r,k} \in \{0,1\}$, $l_1|I_1| + \cdots + l_t|I_t| = |I|+1$, $l_1|J_1| + \cdots + l_t|J_t| = |J|+1$, and $1 \le p, q, l, k \le m$.

Proof. By Lemma 3.3, $\mathscr{K}_{I,J}(\mathcal{I})$ can be expressed as a sum of terms of the form $\mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I})$. Thus, it suffices to show that $\mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I})$ is a sum of terms of the form

$$\pm(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1} \cdots (\overline{D}^{J_r}\mathcal{I}D^{I_r}\mathcal{I})^{l_r} \cdots (\overline{D}^{J_t}\mathcal{I}D^{I_t}\mathcal{I})^{l_t}.$$

Since $\mathcal{I}\overline{\partial}_k\partial_l\mathcal{I} = -\overline{\partial}_k\mathcal{I}\partial_l\mathcal{I}$ and $\overline{\partial}_k\partial_l\mathcal{I} = \partial_l\mathcal{I}\overline{\partial}_k\mathcal{I} - \overline{\partial}_k\mathcal{I}\partial_l\mathcal{I}$, by the proof of Theorem 3.5 we have

$$\mathscr{K}(\mathcal{I}) = \overline{\partial}\mathcal{I}\partial\mathcal{I} = \sum_{1\le l,k\le m} (\overline{D}^{e_k}\mathcal{I}D^{e_l}\mathcal{I}),$$

$$\mathscr{K}_{e_p,0}(\mathcal{I}) = \overline{\partial}\mathcal{I}\partial_p\partial\mathcal{I} = \sum_{1\le l,k\le m} (\overline{D}^{e_k}\mathcal{I}D^{e_p+e_l}\mathcal{I}),$$

$$\mathscr{K}_{0,e_q}(\mathcal{I}) = \overline{\partial}_q\overline{\partial}\mathcal{I}\partial\mathcal{I} = \sum_{1\le l,k\le m} (\overline{D}^{e_q+e_k}\mathcal{I}D^{e_l}\mathcal{I})$$

and

$$\begin{aligned}
&\mathscr{K}_{0+e_p,e_q}(\mathcal{I}) = \mathcal{I}\big(\partial_p(\mathscr{K}_{0,e_q}(\mathcal{I}))\big) \\
&= \sum_{1\le l,k\le m} (\overline{D}^{e_q+e_k}\mathcal{I}D^{e_p+e_l}\mathcal{I} - \overline{D}^{e_k}\mathcal{I}D^{e_p}\overline{D}^{e_q}\mathcal{I}D^{e_l}\mathcal{I} - \overline{D}^{e_q}\mathcal{I}D^{e_p}\mathcal{I}\overline{D}^{e_k}\mathcal{I}D^{e_l}\mathcal{I}),\\
&\mathscr{K}_{e_p,0+e_q}(\mathcal{I}) = (\overline{\partial}_q(\mathscr{K}_{e_p,0}(\mathcal{I})))\mathcal{I} \\
&= \sum_{1\le l,k\le m} (\overline{D}^{e_q+e_k}\mathcal{I}D^{e_p+e_l}\mathcal{I} - \overline{D}^{e_k}\mathcal{I}D^{e_p}\overline{D}^{e_q}\mathcal{I}D^{e_l}\mathcal{I} - \overline{D}^{e_k}\mathcal{I}D^{e_l}\mathcal{I}\overline{D}^{e_q}\mathcal{I}D^{e_p}\mathcal{I}).
\end{aligned}$$

This implies the lemma for $\mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I})$, where $i_p, j_q \in \{0,1\}$.

Assume the lemma holds for some $\mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I})$. We need to show it also holds for $\mathscr{K}_{i_pe_p+e_p,j_qe_q}(\mathcal{I})$ and $\mathscr{K}_{i_pe_p,j_qe_q+e_q}(\mathcal{I})$. Without loss of generality, let

$$\mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I}) := (\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1},$$

where $I_1 = s_{1,p}e_p + n_{1,l}e_l$, $J_1 = s_{1,q}e_q + n_{1,k}e_k$, $n_{1,l}, n_{1,k} \in \{0,1\}$ and $l_1|I_1| = i_p + 1$, $l_1|J_1| = j_q + 1$. By property (iv) of Lemma 4.2, we have

$$
\begin{aligned}
(4.4)\quad & \mathscr{K}_{i_pe_p+e_p,j_qe_q}(\mathcal{I}) \\
&= \mathcal{I}(\partial_p(\mathscr{K}_{i_pe_p,j_qe_q}(\mathcal{I}))) = \mathcal{I}(D^{e_p}(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1}) \\
&= \mathcal{I}\sum_{r=0}^{l_1-1}[(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^r(D^{e_p}\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I} + \overline{D}^{J_1}\mathcal{I}D^{I_1+e_p}\mathcal{I})(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1-1-r}] \\
&= \sum_{r=0}^{l_1-1}[(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^r(\overline{D}^{J_1}\mathcal{I}D^{I_1+e_p}\mathcal{I})(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1-1-r}] \\
&\quad - \sum_{r=0}^{l_1-1}\Bigg[(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^r \sum_{e_q\le J_1'\le J_1-e_q}\frac{J_1!}{J_1'!(J_1-J_1')!} \\
&\qquad\qquad \times (\overline{D}^{J_1-J_1'}\mathcal{I}D^{e_p}\mathcal{I}\overline{D}^{J_1'}\mathcal{I}D^{I_1}\mathcal{I})(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1-1-r}\Bigg].
\end{aligned}
$$

It follows that the lemma holds for $\mathscr{K}_{i_pe_p+e_p,j_qe_q}(\mathcal{I})$. Similarly, it holds for $\mathscr{K}_{i_pe_p,j_qe_q+e_q}(\mathcal{I})$ as well. ∎

Proof of Theorem 4.1. Since $\mathcal{I}_1 \sim_u \mathcal{I}_2$, there exists a unitary $U \in \mathcal{U}$ such that $U\mathcal{I}_1(\lambda) = \mathcal{I}_2(\lambda)U$ for all $\lambda \in \Omega$. It follows that

$$
\begin{aligned}
UD^{i_le_l+k_pe_p}\mathcal{I}_1(\lambda) &= D^{i_le_l+k_pe_p}\mathcal{I}_2(\lambda)U, \\
U\overline{D}^{j_ke_k+n_qe_q}\mathcal{I}_1(\lambda) &= \overline{D}^{j_ke_k+n_qe_q}\mathcal{I}_2(\lambda)U,
\end{aligned}
$$

for all non-negative integers i_l, k_p, j_k and n_q. This implies that

$$U\overline{D}^{j_ke_k+n_qe_q}\mathcal{I}_1(\lambda)D^{i_le_l+k_pe_p}\mathcal{I}_1(\lambda) = \overline{D}^{j_ke_k+n_qe_q}\mathcal{I}_2(\lambda)D^{i_le_l+k_pe_p}\mathcal{I}_2(\lambda)U.$$

By Lemma 4.4, for any $I, J \in \mathbb{Z}_+^m$, the covariant derivative $\mathscr{K}_{I,J}(\mathcal{I})$ is a sum of monomials of the form

$$\pm(\overline{D}^{J_1}\mathcal{I}D^{I_1}\mathcal{I})^{l_1}\cdots(\overline{D}^{J_r}\mathcal{I}D^{I_r}\mathcal{I})^{l_r}\cdots(\overline{D}^{J_t}\mathcal{I}D^{I_t}\mathcal{I})^{l_t},$$

where $I_r = s_{r,p}e_p + n_{r,l}e_l$, $J_r = s_{r,q}e_q + n_{r,k}e_k$ and $1 \le r \le t$. Thus,

$$U\mathscr{K}_{I,J}(\mathcal{I}_1)(\lambda) = \mathscr{K}_{I,J}(\mathcal{I}_2)(\lambda)U, \qquad \lambda \in \Omega. \ ∎$$

As established in Theorem 4.1, the congruence of extended holomorphic curves in $\mathcal{I}_n(\Omega,\mathcal{U})$ implies the unitary equivalence of the covariant derivatives of the curvature on C^*-algebras. In complex geometry, however, the covariant derivatives of the classical curvature do not exhibit unitary equivalence, but instead satisfy the following pointwise similarity equivalence.

Corollary 4.5. *Let $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{I}_n(\Omega,\mathcal{U})$. If $\mathcal{I}_1 \sim_u \mathcal{I}_2$, then there exists an invertible operator Y_λ depending on λ such that*

$$Y_\lambda\mathcal{K}_{\mathcal{I}_1,\lambda^I,\overline{\lambda}^J}(\lambda) = \mathcal{K}_{\mathcal{I}_2,\lambda^I,\overline{\lambda}^J}(\lambda)Y_\lambda.$$

Proof. If $\mathcal{I}_1 \sim_u \mathcal{I}_2$, there exists a unitary operator $U \in \mathcal{U}$ such that $U\mathcal{I}_1(\lambda) = \mathcal{I}_2(\lambda)U$ for all $\lambda \in \Omega$. By Theorem 4.1, it follows that $U\mathscr{K}_{I,J}(\mathcal{I}_1)(\lambda) = \mathscr{K}_{I,J}(\mathcal{I}_2)(\lambda)U$ for all $I, J \in \mathbb{Z}_+^m$. Set

$$\mathcal{I}_1(\lambda) = F_1(\lambda)H_1^{-1}(\lambda)G_1^*(\lambda) \quad \text{and} \quad \mathcal{I}_2(\lambda) = F_2(\lambda)H_2^{-1}(\lambda)G_2^*(\lambda),$$

where $H_i(\lambda) = G_i^*(\lambda)F_i(\lambda)$ for $i = 1, 2$. From Theorem 3.5, for any $\lambda \in \Omega$ and $I, J \in \mathbb{Z}_+^m$, we have

$$U(F_1(\lambda)\mathcal{K}_{\mathcal{I}_1,\lambda^I,\overline{\lambda}^J}(\lambda)H_1^{-1}(\lambda)G_1^*(\lambda)) = (F_2(\lambda)\mathcal{K}_{\mathcal{I}_2,\lambda^I,\overline{\lambda}^J}(\lambda)H_2^{-1}(\lambda)G_2^*(\lambda))U.$$

Therefore,

$$H_2^{-1}(\lambda)G_2^*(\lambda)UF_1(\lambda)\mathcal{K}_{\mathcal{I}_1,\lambda^I,\overline{\lambda}^J}(\lambda) = \mathcal{K}_{\mathcal{I}_2,\lambda^I,\overline{\lambda}^J}(\lambda)H_2^{-1}(\lambda)G_2^*(\lambda)UF_1(\lambda).$$

Define

$$Y_\lambda := H_2^{-1}(\lambda)G_2^*(\lambda)UF_1(\lambda) \quad \text{and} \quad Z_\lambda := H_1^{-1}(\lambda)G_1^*(\lambda)U^*F_2(\lambda),$$

so that $Y_\lambda Z_\lambda = Z_\lambda Y_\lambda = I_n$. Hence, Y_λ is an invertible operator depending on λ and satisfies $Y_\lambda \mathcal{K}_{\mathcal{I}_1,\lambda^I,\overline{\lambda}^J}(\lambda) = \mathcal{K}_{\mathcal{I}_2,\lambda^I,\overline{\lambda}^J}(\lambda)Y_\lambda$. ■

The following demonstrates that the covariant derivatives of the curvature for extended holomorphic curves on C^*-algebras can be used to describe the similarity classification of these curves, and can find specific applications in the similarity classification of the Cowen–Douglas class.

LEMMA 4.6. *Let $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{I}_n(\Omega, \mathcal{U})$. Then $\mathcal{I}_1 \sim_s \mathcal{I}_2$ if and only if there exists an invertible operator $X \in \mathcal{U}$ such that*

$$XD^I\mathcal{I}_1(\lambda)\overline{D}^J\mathcal{I}_1(\lambda) = D^I\mathcal{I}_2(\lambda)\overline{D}^J\mathcal{I}_2(\lambda)X, \qquad \lambda \in \Omega,\ I, J \in \mathbb{Z}_+^m.$$

Proof. On the one hand, if $\mathcal{I}_1 \sim_s \mathcal{I}_2$, there exists an invertible operator $X \in \mathcal{U}$ such that $X\mathcal{I}_1(\lambda) = \mathcal{I}_2(\lambda)X$ for all $\lambda \in \Omega$. For any $\lambda \in \Omega$ and $I, J \in \mathbb{Z}_+^m$, it follows that

$$XD^I\mathcal{I}_1(\lambda) = D^I\mathcal{I}_2(\lambda)X \quad \text{and} \quad X\overline{D}^J\mathcal{I}_1(\lambda) = \overline{D}^J\mathcal{I}_2(\lambda)X.$$

Thus, $XD^I\mathcal{I}_1(\lambda)\overline{D}^J\mathcal{I}_1(\lambda) = D^I\mathcal{I}_2(\lambda)\overline{D}^J\mathcal{I}_2(\lambda)X$.

On the other hand, suppose there exists an invertible operator $X \in \mathcal{U}$ such that for all $\lambda \in \Omega$ and $I, J \in \mathbb{Z}_+^m$,

$$XD^I\mathcal{I}_1(\lambda)\overline{D}^J\mathcal{I}_1(\lambda) = D^I\mathcal{I}_2(\lambda)\overline{D}^J\mathcal{I}_2(\lambda)X.$$

By Lemma 4.3, for each $k = 1, 2$ and any $I, J \in \mathbb{Z}_+^m$, the mixed derivative $D^I\overline{D}^J\mathcal{I}_k$ can be expressed as a sum of terms of the form

$$\pm(D^{I_1}\mathcal{I}_k)(\overline{D}^{J_1}\mathcal{I}_k)\cdots(D^{I_t}\mathcal{I}_k)(\overline{D}^{J_t}\mathcal{I}_k),$$

where $I_1 + \cdots + I_t = I$ and $J_1 + \cdots + J_t = J$. By induction on the order of derivatives, we conclude that $XD^I\overline{D}^J\mathcal{I}_1(\lambda) = D^I\overline{D}^J\mathcal{I}_2(\lambda)X$ for any $\lambda \in \Omega$ and $I, J \in \mathbb{Z}_+^m$. Now, fix $\lambda_0 \in \Omega$. Since $\mathcal{I}_k$ is holomorphic in Ω, there exists an open neighborhood $\Omega_0 \subset \Omega$ of λ_0 such that $\mathcal{I}_k$ admits the power series expansion

$$\mathcal{I}_k(\lambda) = \sum_{I,J\in\mathbb{Z}_+^m} \frac{D^I\overline{D}^J\mathcal{I}_k(\lambda_0)}{I!J!}(\lambda-\lambda_0)^I(\overline{\lambda}-\overline{\lambda}_0)^J$$

for all $\lambda \in \Omega_0$ and $k = 1, 2$. Applying X to both sides of the expansion for $\mathcal{I}_1$ and using the commutativity of X with all mixed derivatives of $\mathcal{I}_1$ and $\mathcal{I}_2$, we obtain that $X\mathcal{I}_1(\lambda) = \mathcal{I}_2(\lambda)X$ for all $\lambda \in \Omega_0$. Since Ω is connected and Ω_0 is open, this equality extends to all $\lambda \in \Omega$ by analytic continuation. Thus, $\mathcal{I}_1 \sim_s \mathcal{I}_2$. ∎

PROPOSITION 4.7. *Let* $\mathbf{T},\mathbf{S}\in B_n^m(\Omega)\cap\mathcal{L}(\mathcal{H})^m$. *If* $\mathcal{I}_{\mathbf{T}}\sim_s\mathcal{I}_{\mathbf{S}}$, *then* $\mathbf{T}\sim_s\mathbf{S}$.

Proof. If $\mathcal{I}_{\mathbf{T}} \sim_s \mathcal{I}_{\mathbf{S}}$, there exists an invertible operator X such that $X\mathcal{I}_{\mathbf{T}}(\lambda) = \mathcal{I}_{\mathbf{S}}(\lambda)X$ for all $\lambda \in \Omega$. Recall that $\mathcal{I}_{\mathbf{T}}(\lambda)$ and $\mathcal{I}_{\mathbf{S}}(\lambda)$ are idempotent operators, with

$$\mathcal{I}_{\mathbf{T}}(\lambda) : \mathcal{H} \to \ker \mathscr{D}_{\mathbf{T}-\lambda} \quad \text{and} \quad \mathcal{I}_{\mathbf{S}}(\lambda) : \mathcal{H} \to \ker \mathscr{D}_{\mathbf{S}-\lambda}.$$

By $X\mathcal{I}_{\mathbf{T}}(\lambda) = \mathcal{I}_{\mathbf{S}}(\lambda)X$, taking any $\xi \in \ker \mathscr{D}_{\mathbf{T}-\lambda}$, we have

$$X\xi = X\mathcal{I}_{\mathbf{T}}(\lambda)\xi = \mathcal{I}_{\mathbf{S}}(\lambda)(X\xi) \in \ker \mathscr{D}_{\mathbf{S}-\lambda}.$$

This implies $X\xi \in \ker \mathscr{D}_{\mathbf{S}-\lambda}$, so $X(\ker \mathscr{D}_{\mathbf{T}-\lambda}) \subseteq \ker \mathscr{D}_{\mathbf{S}-\lambda}$. Since X is invertible, its inverse X^{-1} satisfies $X^{-1}\mathcal{I}_{\mathbf{S}}(\lambda) = \mathcal{I}_{\mathbf{T}}(\lambda)X^{-1}$, which similarly implies $X^{-1}(\ker \mathscr{D}_{\mathbf{S}-\lambda}) \subseteq \ker \mathscr{D}_{\mathbf{T}-\lambda}$. Thus, $X^{-1}(\ker \mathscr{D}_{\mathbf{S}-\lambda}) = \ker \mathscr{D}_{\mathbf{T}-\lambda}$ for all $\lambda \in \Omega$. Note that

$$\bigvee_{\lambda\in\Omega} \ker \mathscr{D}_{\mathbf{T}-\lambda} = \mathcal{H} = \bigvee_{\lambda\in\Omega} \ker \mathscr{D}_{\mathbf{S}-\lambda},$$

where $\bigvee$ denotes the closed linear span. Hence, $X\mathbf{T} = \mathbf{S}X$. ∎

EXAMPLE 4.8. Let $\mathbf{T}, \mathbf{S} \in \mathcal{B}_1^m(\Omega)$ be the adjoints of the multiplication tuples on Hilbert spaces $\mathcal{H}_K$ and $\mathcal{H}_{\widehat{K}}$ with reproducing kernels K and $\widehat{K}$, respectively. Assume that

$$K(\omega,\lambda) = \sum_{I\in\mathbb{Z}_+^m} a_I^2\omega^I\overline{\lambda}^I \quad \text{and} \quad \widehat{K}(\omega,\lambda) = \sum_{I\in\mathbb{Z}_+^m} b_I^2\omega^I\overline{\lambda}^I,$$

where $a_I, b_I > 0$ and $\omega, \lambda \in \Omega$. If $\mathcal{I}_{\mathbf{T}} \sim_s \mathcal{I}_{\mathbf{S}}$, then there exist constants c and C such that

$$0 < c \le \frac{b_I}{a_I} \le C < \infty, \qquad I \in \mathbb{Z}_+^m.$$

Proof. Let $\alpha(\lambda) := K(\cdot,\overline{\lambda})$ and $\beta(\lambda) := \widehat{K}(\cdot,\overline{\lambda})$. Define

$$\mathcal{I}_{\mathbf{T}}(\lambda) = \alpha(\lambda)(\gamma^*(\lambda)\alpha(\lambda))^{-1}\gamma^*(\lambda), \qquad \mathcal{I}_{\mathbf{S}}(\lambda) = \beta(\lambda)(\beta^*(\lambda)\beta(\lambda))^{-1}\beta^*(\lambda),$$

where $H(\omega,\lambda) = \sum_{I\in\mathbb{Z}_+^m} \frac{b_I^4}{a_I^2}\omega^I\overline{\lambda}^I$ and $\gamma(\lambda) = H(\cdot,\overline{\lambda})$ for $\omega, \lambda \in \Omega$. Then $\mathcal{I}_{\mathbf{T}}$ and $\mathcal{I}_{\mathbf{S}}$ are extended holomorphic curves, and $\gamma^*(\lambda)\alpha(\lambda) = \beta^*(\lambda)\beta(\lambda)$. If

$\mathcal{I}_{\mathbf{T}} \sim_s \mathcal{I}_{\mathbf{S}}$, by Lemma 4.6, there exists an invertible operator X such that

$$XD^I\mathcal{I}_{\mathbf{T}}(\lambda)\overline{D}^J\mathcal{I}_{\mathbf{T}}(\lambda) = D^I\mathcal{I}_{\mathbf{S}}(\lambda)\overline{D}^J\mathcal{I}_{\mathbf{S}}(\lambda)X$$

for all $\lambda \in \Omega$ and $I, J \in \mathbb{Z}_+^m$. By Lemma 4.2, it follows that

$$XD^I\alpha(\lambda)\overline{D}^J\gamma^*(\lambda) = D^I\beta(\lambda)\overline{D}^J\beta^*(\lambda)X.$$

Taking $\lambda = 0$, we have

$$D^I\alpha(0)\overline{D}^J\gamma^*(0) = \frac{a_I b_J^2}{a_J} I!J!e_{I,J} \quad \text{and} \quad D^I\beta(0)\overline{D}^J\beta^*(0) = b_I b_J I!J!e_{I,J},$$

where $e_{I,J}$ denotes the infinite matrix with 1 in the (I,J)-th entry and 0 elsewhere. Let $((x_{I,J}))$ be the matrix form of the bounded invertible operator X. Without loss of generality, assume $x_{0,0} \neq 0$, then $((x_{I,J}))\frac{a_I b_J^2}{a_J}e_{I,J} = b_I b_J e_{I,J}((x_{I,J}))$. We conclude that

$$x_{I,I} = \frac{b_I a_{I+e_i}}{a_I b_{I+e_i}} x_{I+e_i,I+e_i}, \qquad I \in \mathbb{Z}_+^m,\ 1 \leq i \leq m,$$

which implies $x_{0,0} = \frac{b_0 a_I}{a_0 b_I} x_{I,I}$. Consequently, there exist constants c and C such that $0 < c \leq \frac{b_I}{a_I} \leq C$ for all $I \in \mathbb{Z}_+^m$. ∎

5. The flag structure of the class $\mathcal{FB}_n(\Omega)$. To investigate the geometric properties of increasing n-tuples of projections in a unital C^*-algebra $\mathcal{U}$, Martin and Salinas [30] introduced the notion of flag manifolds on $\mathcal{U}$. Ji, Jiang, Keshari, and Misra [15, 16] defined a subclass $\mathcal{FB}_n(\Omega)$ of $\mathcal{B}_n^1(\Omega)$ with a flag structure, proving that the curvature together with the second fundamental form constitutes a complete set of invariants. Furthermore, $\mathcal{FB}_n(\Omega)$ is norm dense in $\mathcal{B}_n^1(\Omega)$ (see [21]). Inspired by these results, in this section we decompose the holomorphic curve induced by operators in $\mathcal{FB}_n(\Omega)$, derive the flag structure on the holomorphic curve, and ultimately use this flag structure to characterize the unitary classification of $\mathcal{FB}_n(\Omega)$.

Definition 5.1 ([15, 16]). Let $\mathcal{FB}_n(\Omega)$ be the set of all operators $T \in \mathcal{B}_n^1(\Omega)$ of the form

$$T = \begin{pmatrix} T_0 & S_{0,1} & \cdots & S_{0,n-1} \\ 0 & T_1 & \cdots & S_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{n-1} \end{pmatrix},$$

where $\{T_i\}_{i=0}^{n-1} \subseteq \mathcal{B}_1^1(\Omega)$, $S_{i,i+1}$ are non-zero operators and $T_iS_{i,i+1} = S_{i,i+1}T_{i+1}$ for $0 \leq i \leq n-2$.

LEMMA 5.2 ([16]). *Operators T and $\widetilde{T}$ in $\mathcal{FB}_n(\Omega)$ are unitarily equivalent if and only if there exists a unitary operator $U = \operatorname{diag}\{U_0, \ldots, U_{n-1}\}$ such that $UT = \widetilde{T}U$.*

LEMMA 5.3 ([5]). *Operators T and $\widetilde{T}$ in $\mathcal{B}_1^1(\Omega)$ are unitarily equivalent if and only if $\mathcal{K}_T = \mathcal{K}_{\widetilde{T}}$.*

Let $T \in \mathcal{B}_n^1(\Omega) \cap \mathcal{L}(\mathcal{H})$, and let $\{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$ be a holomorphic frame of the Hermitian holomorphic vector bundle E_T. Set $\alpha = (\gamma_0, \gamma_1, \ldots, \gamma_{n-1})$ and $P_T(\lambda) = \alpha(\lambda) h^{-1}(\lambda) \alpha^*(\lambda)$, where $h(\lambda) = \alpha^*(\lambda)\alpha(\lambda)$ is the metric of E_T. Then $P_T : \Omega \to \mathcal{P}(\mathcal{L}(\mathcal{H}))$ is a holomorphic curve, and $P_T(\lambda)$ is the projection operator from $\mathcal{H}$ onto $\ker(T - \lambda)$.

LEMMA 5.4. *Let $T, S \in \mathcal{B}_n^1(\Omega) \cap \mathcal{L}(\mathcal{H})$. Then $T \sim_u S$ if and only if there exists a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $UP_T(\lambda) = P_S(\lambda)U$ for all $\lambda \in \Omega$.*

Proof. If $T \sim_u S$, there exists a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $UT = SU$. For any $\lambda \in \Omega$ and $x \in \ker(T - \lambda)$, we have

$$SUx = UTx = U(\lambda x) = \lambda Ux,$$

which shows $Ux \in \ker(S - \lambda)$. Thus,

$$U \ker(T - \lambda) \subseteq \ker(S - \lambda).$$

Since $\dim \ker(T - \lambda) = \dim \ker(S - \lambda) = n$ and U is unitary, it follows that

$$U \ker(T - \lambda) = \ker(S - \lambda).$$

Now, let $x = x_1 \oplus x_2 \in \mathcal{H}$, where $x_1 \in \ker(T - \lambda)$ and $x_2 \in [\ker(T - \lambda)]^{\perp}$. Then

$$UP_T(\lambda)x = Ux_1 \in \ker(S - \lambda).$$

For any $y \in \ker(S - \lambda)$, there exists $\widehat{x} \in \ker(T - \lambda)$ such that $y = U\widehat{x}$. Hence,

$$\langle y, Ux_2 \rangle = \langle U\widehat{x}, Ux_2 \rangle = \langle U^*U\widehat{x}, x_2 \rangle = \langle \widehat{x}, x_2 \rangle = 0,$$

which implies $Ux_2 \in [\ker(S - \lambda)]^{\perp}$. Therefore,

$$P_S(\lambda)Ux = P_S(\lambda)\left(Ux_1 \oplus Ux_2\right) = Ux_1 \oplus 0 = UP_T(\lambda)x$$

for any $x \in \mathcal{H}$. We conclude that $UP_T(\lambda) = P_S(\lambda)U$ for all $\lambda \in \Omega$.

Conversely, suppose there exists a unitary operator U such that $UP_T(\lambda) = P_S(\lambda)U$ for all $\lambda \in \Omega$. For any $x \in \ker(T - \lambda)$, we have

$$P_S(\lambda)Ux = UP_T(\lambda)x = Ux,$$

which implies $Ux \in \ker(S - \lambda)$. Thus, $U \ker(T - \lambda) \subseteq \ker(S - \lambda)$. Since $\dim \ker(T - \lambda) = \dim \ker(S - \lambda) = n$ and U is unitary, it follows that $U \ker(T - \lambda) = \ker(S - \lambda)$, and consequently $UT = SU$. ∎

Let E be an Hermitian holomorphic line bundle over a bounded domain $\Omega \subset \mathbb{C}$, and let γ be a non-vanishing holomorphic cross-section of E. Suppose that the derivatives $\gamma(\lambda), \gamma^{(1)}(\lambda), \ldots, \gamma^{(k)}(\lambda)$ are linearly independent for each $\lambda \in \Omega$. Then the associated *k-jet bundle* $\mathcal{J}_k(E)$ is the holomorphic vector bundle of rank $k+1$ with holomorphic frame $\{\gamma, \gamma^{(1)}, \ldots, \gamma^{(k)}\}$ (see [5, 23] for details). The following presents a decomposition formula for the holomorphic curve $P_T : \Omega \to \mathcal{P}(\mathcal{L}(\mathcal{H}))$ induced by the operator $T = \left(\begin{smallmatrix} T_0 & S \\ 0 & T_1 \end{smallmatrix}\right) \in \mathcal{FB}_2(\Omega)$, thus yielding the 2-flag $P = (P_{T_0}, P_T)$ as described in Definition 2.1.

LEMMA 5.5. *Let $T = \left(\begin{smallmatrix} T_0 & S \\ 0 & T_1 \end{smallmatrix}\right) \in \mathcal{FB}_2(\Omega) \cap \mathcal{L}(\mathcal{H})$. Then the projection operator $P_T(\lambda) : \mathcal{H} \to \ker(T-\lambda)$ admits the following decomposition:*

$$P_T(\lambda) = (I - \theta(\lambda))\begin{pmatrix} P_{T_0}(\lambda) & 0 \\ 0 & P_{T_1}(\lambda) \end{pmatrix} + \theta(\lambda)\begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix}.$$

where $P_{T_0}(\lambda)$ and $P_{T_1}(\lambda)$ are projections onto $\ker(T_0 - \lambda)$ *and* $\ker(T_1 - \lambda)$, *respectively.*

Moreover, the operator $\left(\begin{smallmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{smallmatrix}\right) : \mathcal{J}_1(t_0)(\lambda) \to \ker(T-\lambda)$ is an isomorphism and satisfies

$$\begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix}\begin{pmatrix} t_0(\lambda) \\ 0 \end{pmatrix} = \begin{pmatrix} t_0(\lambda) \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix}\begin{pmatrix} t_0'(\lambda) \\ 0 \end{pmatrix} = \begin{pmatrix} t_0'(\lambda) \\ t_1(\lambda) \end{pmatrix},$$

where t_1 is a non-vanishing holomorphic cross-section of E_{T_1}, $t_0 = -St_1$, $\mathcal{J}_1(t_0)(\lambda) = \bigvee\{t_0(\lambda), t_0'(\lambda)\}$, $\theta\|t_1\|^2 + (1+\theta)\mathcal{K}_{T_0}\|t_0\|^2 = 0$, and θ is the second fundamental form (*see* [16]).

Proof. Let t_1 be a non-vanishing holomorphic cross-section of E_{T_1} and let $t_0 = -St_1$. By [16, Proposition 2.4], $\{\gamma_0 = t_0,\ \gamma_1 = t_0' - t_1\}$ forms a holomorphic frame of E_T, which allows us to express the projection $P_T(\lambda)$ as

$$\begin{aligned} P_T(\lambda) &= (\gamma_0(\lambda) \quad \gamma_1(\lambda))\begin{pmatrix} \langle \gamma_0(\lambda), \gamma_0(\lambda)\rangle & \langle \gamma_1(\lambda), \gamma_0(\lambda)\rangle \\ \langle \gamma_0(\lambda), \gamma_1(\lambda)\rangle & \langle \gamma_1(\lambda), \gamma_1(\lambda)\rangle \end{pmatrix}^{-1}\begin{pmatrix} \gamma_0^*(\lambda) \\ \gamma_1^*(\lambda) \end{pmatrix} \\ &= \frac{h_0(\lambda)h_1(\lambda)}{h_0(\lambda)h_1(\lambda) - \mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\begin{pmatrix} t_0(\lambda)h_0^{-1}(\lambda)t_0^*(\lambda) & 0 \\ 0 & t_1(\lambda)h_1^{-1}(\lambda)t_1^*(\lambda) \end{pmatrix} \\ &\quad - \frac{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}{h_0(\lambda)h_1(\lambda) - \mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\begin{pmatrix} \mathsf{v}_{11} & \mathsf{v}_{12} \\ \mathsf{v}_{21} & \mathsf{v}_{22} \end{pmatrix} \\ &:= (1 - \theta(\lambda))\begin{pmatrix} P_{T_0}(\lambda) & 0 \\ 0 & P_{T_1}(\lambda) \end{pmatrix} + \theta(\lambda)\begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix}, \end{aligned}$$

where $h_0(\lambda)=\|t_0(\lambda)\|^2$, $h_1(\lambda)=\|t_1(\lambda)\|^2$, $\theta(\lambda)=-\frac{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}{h_0(\lambda)h_1(\lambda)-\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}$, and

$$\begin{aligned}
\mathsf{v}_{11}&=\frac{-t_0(\lambda)\frac{\partial^2}{\partial\lambda\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)+t_0(\lambda)\frac{\partial}{\partial\lambda}h_0(\lambda)t_0'^*(\lambda)}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)},\\
&\quad+\frac{t_0'(\lambda)\frac{\partial}{\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)-t_0'(\lambda)h_0(\lambda)t_0'^*(\lambda)}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)},\\
\mathsf{v}_{12}&=\frac{t_0(\lambda)\frac{\partial}{\partial\lambda}h_0(\lambda)t_1^*(\lambda)-t_0'(\lambda)h_0(\lambda)t_1^*(\lambda)}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)},\\
\mathsf{v}_{21}&=\frac{t_1(\lambda)\frac{\partial}{\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)-t_1(\lambda)h_0(\lambda)t_0'^*(\lambda)}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)},\\
\mathsf{v}_{22}&=0.
\end{aligned}$$

Direct computation shows that

$$\begin{aligned}
&\begin{pmatrix}F_{P_T}(\lambda) & S_{P_T}(\lambda)\\ S_{P_T}^*(\lambda) & 0\end{pmatrix}\begin{pmatrix}t_0(\lambda)\\ 0\end{pmatrix}\\
&=\begin{pmatrix}\frac{t_0(\lambda)\left[-\frac{\partial^2}{\partial\lambda\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)t_0(\lambda)+\frac{\partial}{\partial\lambda}h_0(\lambda)t_0'^*(\lambda)t_0(\lambda)\right]}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\\ \qquad+\frac{t_0'(\lambda)\left[\frac{\partial}{\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)t_0(\lambda)-h_0(\lambda)t_0'^*(\lambda)t_0(\lambda)\right]}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\\ \frac{t_1(\lambda)\left[\frac{\partial}{\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)t_0(\lambda)-h_0(\lambda)t_0'^*(\lambda)t_0(\lambda)\right]}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\end{pmatrix}\\
&=\begin{pmatrix}t_0(\lambda)\\ 0\end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix}F_{P_T}(\lambda) & S_{P_T}(\lambda)\\ S_{P_T}^*(\lambda) & 0\end{pmatrix}\begin{pmatrix}t_0'(\lambda)\\ 0\end{pmatrix}\\
&=\begin{pmatrix}\frac{t_0(\lambda)\left[-\frac{\partial}{\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)t_0'(\lambda)+\frac{\partial}{\partial\lambda}h_0(\lambda)t_0'^*(\lambda)t_0'(\lambda)\right]}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\\ \qquad+\frac{t_0'(\lambda)\left[\frac{\partial}{\partial\lambda}h_0(\lambda)t_0^*(\lambda)t_0'(\lambda)-h_0(\lambda)t_0'^*(\lambda)t_0'(\lambda)\right]}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\\ \frac{t_1(\lambda)\left[\frac{\partial}{\partial\overline{\lambda}}h_0(\lambda)t_0^*(\lambda)t_0'(\lambda)-h_0(\lambda)t_0'^*(\lambda)t_0'(\lambda)\right]}{\mathcal{K}_{T_0}(\lambda)h_0^2(\lambda)}\end{pmatrix}\\
&=\begin{pmatrix}t_0'(\lambda)\\ t_1(\lambda)\end{pmatrix},
\end{aligned}$$

and $\theta(\lambda)\|t_1(\lambda)\|^2+(1-\theta(\lambda))\mathcal{K}_{T_0}(\lambda)\|t_0(\lambda)\|^2=0$ for all $\lambda\in\Omega$. ∎

In Lemma 5.5, the operator $F_{P_T}(\lambda)$ acts as the identity on $\mathcal{J}_1(t_0)(\lambda) = \bigvee\{t_0(\lambda), t_0'(\lambda)\}$, satisfying

$$F_{P_T}(\lambda)t_0(\lambda) = t_0(\lambda) \quad \text{and} \quad F_{P_T}(\lambda)t_0'(\lambda) = t_0'(\lambda)$$

for all $\lambda \in \Omega$. Consequently, $F_{P_T}(\lambda)$ acts as the identity on $\mathcal{J}_1(t_0)(\lambda)$, and the projection $P_T(\lambda)$ admits the refined decomposition

$$P_T(\lambda) = (I - \theta(\lambda))\begin{pmatrix} P_{T_0}(\lambda) & 0 \\ 0 & P_{T_1}(\lambda) \end{pmatrix} + \theta(\lambda)\begin{pmatrix} I_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix},$$

where $I_{P_T}(\lambda)$ denotes the identity operator on $\mathcal{J}_1(t_0)(\lambda)$. For the class $\mathcal{FB}_2(\Omega)$, Theorems 2.10 and 2.11 in [16] establish a complete classification criterion for the unitary equivalence of such operators via the curvature and the second fundamental form. Theorem 4.8 in [11] describes the similarity classification of holomorphic vector bundles induced by Cowen–Douglas operators using holomorphic curves. Building upon Lemmas 5.4 and 5.5 above, we have further obtained several equivalent characterizations of unitary equivalence for operators in $\mathcal{FB}_2(\Omega)$ in terms of holomorphic curves and 2-flags on C^*-algebras.

Theorem 5.6. *Let* $T = \left(\begin{smallmatrix} T_0 & S \\ 0 & T_1 \end{smallmatrix}\right)$, $\widetilde{T} = \left(\begin{smallmatrix} \widetilde{T}_0 & \widetilde{S} \\ 0 & \widetilde{T}_1 \end{smallmatrix}\right) \in \mathcal{FB}_2(\Omega) \cap \mathcal{L}(\mathcal{H})$. *Suppose* t_1 *and* $\widetilde{t}_1$ *are non-vanishing holomorphic cross-sections of the line bundles* E_{T_1} *and* $E_{\widetilde{T}_1}$, *respectively, with* $t_0 := -St_1$, $\widetilde{t}_0 := -\widetilde{S}\widetilde{t}_1$. *Then the following statements are equivalent:*

(i) *the holomorphic curves* P_T *and* $P_{\widetilde{T}}$ *are congruent;*
(ii) $T \sim_u S$;
(iii) $\mathcal{K}_{T_0} = \mathcal{K}_{\widetilde{T}_0}$ (*or* $\mathcal{K}_{T_1} = \mathcal{K}_{\widetilde{T}_1}$) *and* $\frac{\|St_1\|^2}{\|t_1\|^2} = \frac{\|\widetilde{S}\widetilde{t}_1\|^2}{\|\widetilde{t}_1\|^2}$;
(iv) *there exists a unitary operator* $U = \operatorname{diag}\{U_{00}, U_{11}\} \in \mathcal{L}(\mathcal{H})$ *such that*

$$\begin{pmatrix} F_{P_T} & S_{P_T} \\ S^*_{P_T} & 0 \end{pmatrix} = U^* \begin{pmatrix} F_{P_{\widetilde{T}}} & S_{P_{\widetilde{T}}} \\ S^*_{P_{\widetilde{T}}} & 0 \end{pmatrix} U.$$

Proof. The equivalences (i)⇔(ii) and (ii)⇔(iii) follow from the $n = 2$ case of Lemma 5.4 and from [16, Theorem 2.10], respectively.

(iii)⇒(iv). If $\mathcal{K}_{T_0} = \mathcal{K}_{\widetilde{T}_0}$, and for non-vanishing holomorphic cross-sections t_1 and $\widetilde{t}_1$ of E_{T_1} and $E_{\widetilde{T}_1}$, respectively, the equality

$$\frac{\|St_1\|^2}{\|t_1\|^2} = \frac{\|\widetilde{S}\widetilde{t}_1\|^2}{\|\widetilde{t}_1\|^2}$$

holds, then we have $\frac{\|t_0(\lambda)\|}{\|t_1(\lambda)\|} = \frac{\|\widetilde{t}_0(\lambda)\|}{\|\widetilde{t}_1(\lambda)\|}$ for all $\lambda \in \Omega$. Therefore,

$$\frac{h_0(\lambda)h_1(\lambda)}{h_0(\lambda)h_1(\lambda) - h_0^2(\lambda)\mathcal{K}_{T_0}(\lambda)} = \frac{\widetilde{h}_0(\lambda)\widetilde{h}_1(\lambda)}{\widetilde{h}_0(\lambda)\widetilde{h}_1(\lambda) - \widetilde{h}_0^2(\lambda)\mathcal{K}_{\widetilde{T}_0}(\lambda)},$$
$$\frac{h_0^2(\lambda)\mathcal{K}_{T_0}(\lambda)}{h_0(\lambda)h_1(\lambda) - h_0^2(\lambda)\mathcal{K}_{T_0}(\lambda)} = \frac{\widetilde{h}_0^2(\lambda)\mathcal{K}_{\widetilde{T}_0}(\lambda)}{\widetilde{h}_0(\lambda)\widetilde{h}_1(\lambda) - \widetilde{h}_0^2(\lambda)\mathcal{K}_{\widetilde{T}_0}(\lambda)},$$

where $h_i(\lambda) = \|t_i(\lambda)\|^2$ and $\widetilde{h}_i(\lambda) = \|\widetilde{t}_i(\lambda)\|^2$ for $i = 0, 1$. Since $T \sim_u S$, by Lemmas 5.2 and 5.4, there exists a unitary operator $U = \operatorname{diag}\{U_{00}, U_{11}\}$ such that $UP_T(\lambda) = P_{\widetilde{T}}(\lambda)U$ for all $\lambda \in \Omega$, that is,

$$\begin{aligned}
&\begin{pmatrix} U_{00} & 0 \\ 0 & U_{11} \end{pmatrix} \Bigg[\frac{h_0(\lambda)h_1(\lambda)}{h_0(\lambda)h_1(\lambda) - h_0^2(\lambda)\mathcal{K}_{T_0}(\lambda)} \begin{pmatrix} P_{T_0}(\lambda) & 0 \\ 0 & P_{T_1}(\lambda) \end{pmatrix} \\
&\qquad\qquad - \frac{h_0^2(\lambda)\mathcal{K}_{T_0}(\lambda)}{h_0(\lambda)h_1(\lambda) - h_0^2(\lambda)\mathcal{K}_{T_0}(\lambda)} \begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix} \Bigg] \\
&= \Bigg[\frac{\widetilde{h}_0(\lambda)\widetilde{h}_1(\lambda)}{\widetilde{h}_0(\lambda)\widetilde{h}_1(\lambda) - \widetilde{h}_0^2(\lambda)\mathcal{K}_{\widetilde{T}_0}(\lambda)} \begin{pmatrix} P_{\widetilde{T}_0}(\lambda) & 0 \\ 0 & P_{\widetilde{T}_1}(\lambda) \end{pmatrix} \\
&\qquad - \frac{\widetilde{h}_0^2(\lambda)\mathcal{K}_{\widetilde{T}_0}(\lambda)}{\widetilde{h}_0(\lambda)\widetilde{h}_1(\lambda) - \widetilde{h}_0^2(\lambda)\mathcal{K}_{\widetilde{T}_0}(\lambda)} \begin{pmatrix} F_{P_{\widetilde{T}}}(\lambda) & S_{P_{\widetilde{T}}}(\lambda) \\ S^*_{P_{\widetilde{T}}}(\lambda) & 0 \end{pmatrix} \Bigg] \begin{pmatrix} U_{00} & 0 \\ 0 & U_{11} \end{pmatrix},
\end{aligned}$$

where $U_{00}P_{T_0}(\lambda) = P_{\widetilde{T}_0}(\lambda)U_{00}$ and $U_{11}P_{T_1}(\lambda) = P_{\widetilde{T}_1}(\lambda)U_{11}$. Hence, for $\lambda \in \Omega$,

$$\begin{pmatrix} U_{00} & 0 \\ 0 & U_{11} \end{pmatrix} \begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix} = \begin{pmatrix} F_{P_{\widetilde{T}}}(\lambda) & S_{P_{\widetilde{T}}}(\lambda) \\ S^*_{P_{\widetilde{T}}}(\lambda) & 0 \end{pmatrix} \begin{pmatrix} U_{00} & 0 \\ 0 & U_{11} \end{pmatrix}.$$

(iv)⇒(iii). Let $U = \operatorname{diag}\{U_{00}, U_{11}\}$ be a unitary operator satisfying

$$U \begin{pmatrix} F_{P_T} & S_{P_T} \\ S^*_{P_T} & 0 \end{pmatrix} = \begin{pmatrix} F_{P_{\widetilde{T}}} & S_{P_{\widetilde{T}}} \\ S^*_{P_{\widetilde{T}}} & 0 \end{pmatrix} U.$$

Then, for any $\lambda \in \Omega$, we have

$$\begin{aligned}
&\begin{pmatrix} U_{00} & 0 \\ 0 & U_{11} \end{pmatrix} \begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix} \begin{pmatrix} t_0(\lambda) \\ 0 \end{pmatrix} \\
&\qquad = \begin{pmatrix} F_{P_{\widetilde{T}}}(\lambda) & S_{P_{\widetilde{T}}}(\lambda) \\ S^*_{P_{\widetilde{T}}}(\lambda) & 0 \end{pmatrix} \begin{pmatrix} U_{00} & 0 \\ 0 & U_{11} \end{pmatrix} \begin{pmatrix} t_0(\lambda) \\ 0 \end{pmatrix}.
\end{aligned} \tag{5.1}$$

Assume that $U\begin{pmatrix} t_0(\lambda) \\ 0 \end{pmatrix} = \phi(\lambda)\begin{pmatrix} \widetilde{t}_0(\lambda) \\ 0 \end{pmatrix} + \psi(\lambda)\begin{pmatrix} \widetilde{t}'_0(\lambda) \\ \widetilde{t}_1(\lambda) \end{pmatrix}$ for some holomorphic functions $\phi, \psi \in \mathrm{Hol}(\Omega)$. By Lemma 5.5, we obtain

$$\begin{aligned}\begin{pmatrix} U_{00}t_0(\lambda) \\ 0 \end{pmatrix} &= U\begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix}\begin{pmatrix} t_0(\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{P_{\widetilde{T}}}(\lambda) & S_{P_{\widetilde{T}}}(\lambda) \\ S^*_{P_{\widetilde{T}}}(\lambda) & 0 \end{pmatrix} U \begin{pmatrix} t_0(\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi(\lambda)\widetilde{t}_0(\lambda) \\ 0 \end{pmatrix} + \begin{pmatrix} \psi(\lambda)\widetilde{t}'_0(\lambda) + \psi(\lambda)S_{P_{\widetilde{T}}}(\lambda)\widetilde{t}_1(\lambda) \\ \psi(\lambda)\widetilde{t}_1(\lambda) \end{pmatrix}.\end{aligned}$$

This implies $\psi \equiv 0$, and thus $U_{00}t_0(\lambda) = \phi(\lambda)\widetilde{t}_0(\lambda)$. By (5.1) and Lemma 5.5, we have

$$\begin{aligned}U\begin{pmatrix} t'_0(\lambda) \\ t_1(\lambda) \end{pmatrix} &= U\begin{pmatrix} F_{P_T}(\lambda) & S_{P_T}(\lambda) \\ S^*_{P_T}(\lambda) & 0 \end{pmatrix}\begin{pmatrix} t'_0(\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{P_{\widetilde{T}}}(\lambda) & S_{P_{\widetilde{T}}}(\lambda) \\ S^*_{P_{\widetilde{T}}}(\lambda) & 0 \end{pmatrix}\begin{pmatrix} \phi'(\lambda)\widetilde{t}_0(\lambda) + \phi(\lambda)\widetilde{t}'_0(\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi'(\lambda)\widetilde{t}_0(\lambda) \\ 0 \end{pmatrix} + \begin{pmatrix} \phi(\lambda)\widetilde{t}'_0(\lambda) \\ \phi(\lambda)\widetilde{t}_1(\lambda) \end{pmatrix},\end{aligned}$$

which implies $U_{11}t_1(\lambda) = \phi(\lambda)\widetilde{t}_1(\lambda)$. Therefore,

$$\begin{aligned}\frac{\|St_1(\lambda)\|^2}{\|t_1(\lambda)\|^2} = \frac{\|t_0(\lambda)\|^2}{\|t_1(\lambda)\|^2} &= \frac{\|U_{00}t_0(\lambda)\|^2}{\|U_{11}t_1(\lambda)\|^2} = \frac{\|\phi(\lambda)\widetilde{t}_0(\lambda)\|^2}{\|\phi(\lambda)\widetilde{t}_1(\lambda)\|^2} \\ &= \frac{|\phi(\lambda)|^2\|\widetilde{t}_0(\lambda)\|^2}{|\phi(\lambda)|^2\|\widetilde{t}_1(\lambda)\|^2} = \frac{\|\widetilde{t}_0(\lambda)\|^2}{\|\widetilde{t}_1(\lambda)\|^2} \\ &= \frac{\|\widetilde{S}\widetilde{t}_1(\lambda)\|^2}{\|\widetilde{t}_1(\lambda)\|^2}.\end{aligned}$$

Moreover, $U_{00}\ker(T_0 - \lambda) = \ker(\widetilde{T}_0 - \lambda)$ and $U_{11}\ker(T_1 - \lambda) = \ker(\widetilde{T}_1 - \lambda)$ for all $\lambda \in \Omega$. Since $T_i, \widetilde{T}_i \in \mathcal{B}_1(\Omega)$ for $i = 0, 1$, it follows that $T_i \sim_u \widetilde{T}_i$. By Lemma 5.3, we conclude that $\mathcal{K}_{T_0} = \mathcal{K}_{\widetilde{T}_0}$ and $\mathcal{K}_{T_1} = \mathcal{K}_{\widetilde{T}_1}$. ∎

Let

$$T = \begin{pmatrix} T_0 & S_{0,1} & \cdots & S_{0,n-1} \\ 0 & T_1 & \cdots & S_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{n-1} \end{pmatrix} \in \mathcal{FB}_n(\Omega).$$

For $i = 0, 1, \ldots, n-1$, define

$$T(i) = \begin{pmatrix} T_0 & S_{0,1} & \cdots & S_{0,i} \\ 0 & T_1 & \cdots & S_{1,i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_i \end{pmatrix}.$$

From the holomorphic curves $P_{T(i)}$ induced by the operators $T(i)$, we then construct the n-tuple $P(T) = (P_{T(0)}, \ldots, P_{T(n-1)})$. By Theorem 5.6, for operators $T, \widetilde{T} \in \mathcal{FB}_n(\Omega)$, we have $T \sim_u \widetilde{T}$ if and only if $P(T) \sim_u P(\widetilde{T})$. Let $\{\gamma_0, \ldots, \gamma_{n-1}\}$ be a holomorphic frame of E_T. An operator $T \in \mathcal{FB}_n(\Omega)$ is *quasi-homogeneous* as defined in [22] if there exist complex numbers $\phi_{i,j}$, $0 \leq i < j \leq n-1$, and positive real numbers $\{u_i\}_{i=0}^{n-1}$, Λ, α such that $\{\gamma_0, \ldots, \gamma_{n-1}\}$ has the form

$$\begin{aligned}
\gamma_0 &= t_0, \qquad \gamma_1 = \phi_{0,1}\frac{\partial}{\partial w}t_0 + t_1, \\
&\vdots \\
\gamma_j &= \phi_{0,j}\frac{\partial^j}{\partial w^j}t_0 + \phi_{1,j}\frac{\partial^{j-1}}{\partial w^{j-1}}t_1 + \cdots \\
&\quad + \phi_{i,j}\frac{\partial^{j-i}}{\partial w^{j-i}}t_i + \cdots + \phi_{j-1,j}\frac{\partial}{\partial w}t_{j-1} + t_j, \\
&\vdots \\
\gamma_{n-1} &= \phi_{0,n-1}\frac{\partial^{n-1}}{\partial w^{n-1}}t_0 + \phi_{1,n-1}\frac{\partial^{n-2}}{\partial w^{n-2}}t_1 + \cdots \\
&\quad + \phi_{i,n-1}\frac{\partial^{n-1-i}}{\partial w^{n-1-i}}t_i + \cdots + \phi_{n-2,n-1}\frac{\partial}{\partial w}t_{n-2} + t_{n-1},
\end{aligned}$$

where $t_i(w) \in \ker(T_{i,i} - w)$ and $\|t_i(w)\|^2 = \frac{\mu_i}{(1-|w|^2)^{\alpha+i\Lambda}}$ for $0 \leq i \leq n-1$. Let $\mathcal{J}_k(E_{T_0})$ denote the k-jet bundle of E_{T_0}. We thus obtain two holomorphic vector bundles equipped with flag structures:

$$\mathcal{J}_0(E_{T_0})(\lambda) \subseteq \mathcal{J}_1(E_{T_0})(\lambda) \subseteq \cdots \subseteq \mathcal{J}_l(E_{T_0})(\lambda) \subseteq \cdots \subseteq \mathcal{J}_{n-1}(E_{T_0})(\lambda)$$

and

$$E_{T(0)}(\lambda) \subseteq E_{T(1)}(\lambda) \subseteq \cdots \subset E_{T(l)}(\lambda) \subseteq \cdots \subseteq E_{T(n-1)}(\lambda).$$

By Theorem 5.6, we define an n-tuple $J = (J_0, \ldots, J_l, \ldots, J_{n-1})$ of bundle maps, where each map $J_l : \mathcal{J}_l(E_{T_0}) \to E_{T(l)}$ satisfies the condition

$$J_l(\lambda)(t_0^{(k)}(\lambda)) = \phi_{0,k}t_0^{(k)}(\lambda) + \phi_{1,k}t_1^{(k-1)}(\lambda) + \cdots + \phi_{k-1,k}t'_{k-1}(\lambda) + t_k(\lambda)$$

for $0 \leq k \leq l \leq n-1$ and makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{J}_0(E_{T_0}) \arrow[r, "I"] \arrow[d, "J_0"'] & \mathcal{J}_1(E_{T_0}) \arrow[r, "I"] \arrow[d, "J_1"'] & \cdots \arrow[r, "I"] & \mathcal{J}_l(E_{T_0}) \arrow[r, "I"] \arrow[d, "J_l"'] & \cdots \arrow[r, "I"] & \mathcal{J}_{n-1}(E_{T_0}) \arrow[d, "J_{n-1}"] \\
E_{T(0)} \arrow[r, "I"] & E_{T(1)} \arrow[r, "I"] & \cdots \arrow[r, "I"] & E_{T(l)} \arrow[r, "I"] & \cdots \arrow[r, "I"] & E_{T(n-1)}
\end{tikzcd}$$

where I denotes the identity operator.

PROPOSITION 5.7. *Let $T, \widetilde{T} \in \mathcal{FB}_n(\Omega) \cap \mathcal{L}(\mathcal{H})$ be quasi-homogeneous operators. Then $E_T \sim_u E_{\widetilde{T}}$ if and only if there exist bundle maps J_l and $\widetilde{J}_l$ such that the diagram*

$$\begin{tikzcd}
\mathcal{J}_l(E_{T_0}) \arrow[r, "J_l"] \arrow[d, "U(0)"'] & E_{T(l)} \arrow[d, "U(l)"] \\
\mathcal{J}_l(E_{\widetilde{T}_0}) \arrow[r, "\widetilde{J}_l"] & E_{\widetilde{T}(l)}
\end{tikzcd} \tag{5.2}$$

commutes for all $l = 0, \ldots, n-1$, where $U(l) = U_0 \oplus \cdots \oplus U_l$ is a diagonal unitary operator on $\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_l$ and $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$.

Proof. For sufficiency, it is enough to consider the case $l = n-1$ in the commutative diagram (5.2). For necessity, assume $E_T \sim_u E_{\widetilde{T}}$. By [5, Theorem 1.14], this implies $T \sim_u \widetilde{T}$. Since T and $\widetilde{T}$ belong to $\mathcal{FB}_n(\Omega)$, by Lemma 5.2 there exists a diagonal unitary operator $U = U_0 \oplus \cdots \oplus U_{n-1} \in \mathcal{L}(\mathcal{H})$ such that $UT = \widetilde{T}U$. Thus, $T(i) \sim_u \widetilde{T}(i)$ and $U_i T_i = \widetilde{T}_i U_i$ for all $i = 0, \ldots, n-1$. Moreover, there exist non-vanishing holomorphic cross-sections t_i of E_{T_i} and $\widetilde{t}_i$ of $E_{\widetilde{T}_i}$ such that $U_i t_i(\lambda) = \widetilde{t}_i(\lambda)$, and thus $U_i t_i^{(j)}(\lambda) = \widetilde{t}_i^{(j)}(\lambda)$ for all $j \geq 1$ and $\lambda \in \Omega$. Since T and $\widetilde{T}$ are quasi-homogeneous, there exist complex numbers $\{\phi_{i,j}\}_{i,j=0}^{n-1}$ such that holomorphic frames for E_T and $E_{\widetilde{T}}$ can be represented as

$$\{\gamma_j = \phi_{0,j} t_0^{(j)} + \cdots + \phi_{i,j} t_i^{(j-i)} + \cdots + \phi_{j-1,j} t'_{i-1} + t_j\}_{j=0}^{n-1}$$

and

$$\{\widetilde{\gamma}_j = \phi_{0,j} \widetilde{t}_0^{(j)} + \cdots + \phi_{i,j} \widetilde{t}_i^{(j-i)} + \cdots + \phi_{j-1,j} \widetilde{t}'_{i-1} + \widetilde{t}_j\}_{j=0}^{n-1},$$

respectively. We therefore define the bundle mappings $J_l : \mathcal{J}_l(E_{T_0}) \to E_{T(l)}$ and $\widetilde{J}_l : \mathcal{J}_l(E_{\widetilde{T}_0}) \to E_{\widetilde{T}(l)}$ by

$$\begin{aligned}
J_l(t_0^{(k)}) &= \phi_{0,k} t_0^{(k)} + \phi_{1,k} t_1^{(k-1)} + \cdots + \phi_{k-1,k} t'_{k-1} + t_k, \\
\widetilde{J}_l(\widetilde{t}_0^{(k)}) &= \phi_{0,k} \widetilde{t}_0^{(k)} + \phi_{1,k} \widetilde{t}_1^{(k-1)} + \cdots + \phi_{k-1,k} \widetilde{t}'_{k-1} + \widetilde{t}_k,
\end{aligned}$$

respectively, for $0 \leq k \leq l \leq n-1$. Setting $U(l) = U_0 \oplus \cdots \oplus U_l$, we have

$$\begin{aligned} U(l)J_l(t_0^{(k)}) &= U(l)[\phi_{0,k}t_0^{(k)} + \phi_{1,k}t_1^{(k-1)} + \cdots + \phi_{k-1,k}t'_{k-1} + t_k] \\ &= U_0(\phi_{0,k}t_0^{(k)}) + U_1(\phi_{1,k}t_1^{(k-1)}) + \cdots + U_{k-1}(\phi_{k-1,k}t'_{k-1}) + U_k t_k \\ &= \phi_{0,k}\widetilde{t}_0^{(k)} + \phi_{1,k}\widetilde{t}_1^{(k-1)} + \cdots + \phi_{k-1,k}\widetilde{t}'_{k-1} + \widetilde{t}_k \\ &= \widetilde{J}_l(\widetilde{t}_0^{(k)}) = \widetilde{J}_l U(l)(t_0^{(k)}), \end{aligned}$$

then $U(l)J_l(t_0^{(k)}) = \widetilde{J}_l U(l)(t_0^{(k)})$ for any $0 \leq k \leq l \leq n-1$. It follows that $U(l)J_l = \widetilde{J}_l U(0)$ for all $0 \leq l \leq n-1$. ∎

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