

A note on the equivalence of Gromov boundary and metric boundary

by

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Abstract. We introduce the concept of quasihyperbolically visible spaces. As a tool, we study the connection between the Gromov boundary and the metric boundary.

1. Introduction. For a metric space (Ω, d) , its metric completion and metric boundary are denoted by $\overline{\Omega}^d$ and $\partial_d \Omega = \overline{\Omega}^d \setminus \Omega$, respectively. Let (Ω, d) be a rectifiably connected, locally compact, non-complete metric space. We say that (Ω, d) is *minimally nice* if the identity map from (Ω, d) to (Ω, λ_Ω) is continuous, where λ_Ω is the length metric on Ω with respect to d (see Definition 3.2). Note that every proper domain in $\mathbb{R}^n$ is minimally nice. Consider the metric space (Ω, k_Ω) , where k_Ω is the quasihyperbolic metric on the metric space (Ω, d) (see Section 3.3). If (Ω, k_Ω) is Gromov hyperbolic, then its Gromov boundary and Gromov compactification are denoted by $\partial_G \Omega$ and $\overline{\Omega}^G$, respectively (see Section 3.2 for the definitions). Bonk, Heinonen, and Koskela, in their seminal work [BHK01] on the uniformization of Gromov hyperbolic spaces, established a two-way correspondence between uniform metric spaces and Gromov hyperbolic spaces (see [BHK01, Theorem 1.1]). The following important result in [BHK01] says that uniform domains in $\mathbb{R}^n$ can be characterized in terms of Gromov hyperbolicity.

THEOREM 1.1 ([BHK01, Theorem 1.11]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the following are equivalent:*

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- (i) Ω is uniform.
- (ii) (Ω, k_Ω) is Gromov hyperbolic and the identity map $\text{id}_\Omega : (\Omega, k_\Omega) \rightarrow (\Omega, d_{\text{Euc}})$ extends as a homeomorphism $\Phi : \overline{\Omega}^G \rightarrow \overline{\Omega}^{\text{Euc}}$ such that Φ is a quasimöbius homeomorphism from $\partial_G \Omega$ onto $\partial_{\text{Euc}} \Omega$.

Similar results were obtained in the settings of Banach spaces and metric measure spaces by Väisälä [V05, Theorem 2.29] and by Herron, Shanmugalingam, and Xie [HSX08], respectively. A central issue in this context is to determine when, and in what sense, the identity map $\text{id}_\Omega : (\Omega, k_\Omega) \rightarrow (\Omega, d)$ extends to the boundary. Much of the existing literature (again see [BHK01, V05, HSX08]) seeks additional regularity of such an extension, for instance quasimöbius or quasimöbius behavior. In contrast, in this paper we focus only on the *existence* of a boundary extension with basic properties, namely continuity and surjectivity (and, under additional assumptions, that it is a homeomorphism). This leads to the following general question.

PROBLEM 1. Let (Ω, d) be a minimally nice space. Suppose (Ω, k_Ω) is Gromov hyperbolic. When is the following true?

The identity map $\text{id}_\Omega : (\Omega, k_\Omega) \rightarrow (\Omega, d)$ extends to a continuous surjective map, or to a homeomorphism, $\widehat{\text{id}_\Omega} : \partial_G \Omega \rightarrow \partial_d \Omega$.

This is an interesting problem in the field of metric space analysis. The work of Lammi [L11a, L11b] is the most general work with regard to Problem 1. Note that to have a continuous surjective, or a homeomorphic, extension, we need $\overline{\Omega}^d$ to be compact; hence $\partial_d \Omega$ must be compact, since the Gromov boundary $\partial_G \Omega$ is compact. But in general, the metric boundary $\partial_d \Omega$ need not be compact. For instance, let

$$\Omega = (0, 1) \times (0, 1) \setminus \bigcup_{j=1}^{\infty} \left(\left\{ \frac{1}{2^j} \right\} \times \left[0, \frac{1}{2} \right] \right)$$

and equip Ω with the inner metric λ_Ω (see Definition 3.2). Then the inner boundary is closed and bounded but not compact. Indeed, let $\{x_j\} \subset \partial_I \Omega$ be the sequence of midpoints

$$x_j = \left(\frac{3}{2^{j+2}}, 0 \right).$$

It is easy to see that $\lambda_\Omega(x_j, x_{j+1}) \geq 1$ for every $j \in \mathbb{N}$, and hence this sequence does not have a convergent subsequence. The following result of Lammi [L11b] says that, under the quasiconvexity assumption, compactness of the metric boundary is necessary and sufficient for the homeomorphic extension.

THEOREM 1.2 ([L11b, Theorem 3.4]). *Let (Ω, d) be a locally compact and non-complete quasiconvex space. Assume that (Ω, k_Ω) is Gromov hyperbolic*

and the Gehring–Hayman theorem holds in (Ω, d) . Then the following are equivalent:

- (i) $\partial_d \Omega$ is compact.
- (ii) $\overline{\Omega}^d = \Omega \cup \partial_d \Omega$ is compact.
- (iii) The identity map $\text{id}_\Omega : (\Omega, k_\Omega) \rightarrow (\Omega, d)$ extends as a homeomorphism from $\partial_G \Omega$ onto $\partial_d \Omega$.

Further, the following result of Lammi [L11a] gives an analytic condition for the homeomorphic extension.

THEOREM 1.3 ([L11a, Theorem 1.1]). *Let (Ω, d) be a locally compact and non-complete quasiconvex space. Assume that (Ω, k_Ω) is Gromov hyperbolic and the Gehring–Hayman theorem holds in (Ω, d) . Let $x_0 \in \Omega$ be a base point and suppose that the quasihyperbolic metric satisfies*

$$k_\Omega(x_0, x) \leq \phi\left(\frac{\delta_\Omega(x_0)}{\delta_\Omega(x)}\right),$$

where $\phi : (0, \infty) \rightarrow (0, \infty)$ is an increasing function such that

$$\sum_{j=1}^{\infty} \frac{1}{\phi^{-1}(j)} < \infty.$$

Then the identification map $\partial_G \Omega \rightarrow \partial_d \Omega$ is a homeomorphism; moreover, $\partial_d \Omega$ is compact.

Throughout, for $z \in \Omega$ we write

$$\delta_\Omega(z) := d(z, \partial_d \Omega) = \inf \{d(z, w) : w \in \partial_d \Omega\}.$$

It is important to mention that Lammi’s proof of Theorem 1.3 does not use a compactness argument. Indeed, the compactness of $\partial_d \Omega$ comes as a consequence of the homeomorphic extension.

REMARK 1.1 (Equivalence of Gromov boundary and inner boundary). Let Ω be a bounded domain in $\mathbb{R}^n$. Equip it with the inner metric λ_Ω , and let $\partial_I \Omega$ be the inner boundary, i.e. the metric boundary of the metric space (Ω, λ_Ω) . Assume that (Ω, k_Ω) is Gromov hyperbolic. Consider the following problem.

PROBLEM 2. For which bounded domains $\Omega \subset \mathbb{R}^n$ does the identity map $\text{id}_\Omega : (\Omega, k_\Omega) \rightarrow (\Omega, \lambda_\Omega)$ extend to a homeomorphism from $\partial_G \Omega$ onto $\partial_I \Omega$?

Lammi’s work [L11a, L11b] is very helpful for Problem 2. This is because, as a length space, (Ω, λ_Ω) is a quasiconvex space, and hence Theorem 1.2 tells us that the compactness of the inner boundary is sufficient for the homeomorphic extension. Moreover, Theorem 1.3 gives us an analytic condition for the inner boundary to be compact.

The growth condition in Theorem 1.3 implies that (Ω, d) is bounded. Recently, Zhou et al. [ZLL22, Theorem 1.8] studied this problem for unbounded Gromov hyperbolic spaces. Further, Zhou et al. [ZLR21] studied Problem 1 in the case where Ω is a generalized John domain in $\mathbb{R}^n$ and d is the inner metric.

The main aim of the present work is to study Problem 1. This study extends our previous work [AP24], in which we examined the Euclidean variant of Problem 1, namely Ω as a bounded domain in $\mathbb{R}^n$ and d as the Euclidean metric. In that setting, we introduced the concept of QH-visibility domains and provided a complete solution (see [AP24, Theorems 2.2 and 2.3]). Motivated by this, we study Problem 1 via the visibility property and introduce the concept of quasihyperbolically visible spaces (see Definition 1.1 below). Our approach is also motivated by the recent work of Bharali and Zimmer [BZ17] and of Bracci, Nikolov, and Thomas [BNT22], where they studied a suitable form of a notion called “visibility”. The concept of visibility goes back to Eberlein and O’Neill [EO73], who introduced a general construction of compactification of Hadamard manifolds by attaching the boundary at infinity with the cone topology. Using the notion of boundary at infinity, Eberlein and O’Neill defined the term “visibility manifold” for a complete Riemannian manifold with non-positive sectional curvature. Bharali and Zimmer [BZ17] introduced Goldilocks domains in $\mathbb{C}^n$, which satisfy a visibility condition for the Kobayashi distance. This led Bharali and Maitra [BM21] to the notion of visibility domains in the Kobayashi setting. Moreover, our use of quasihyperbolic visibility to control the cluster set of a quasihyperbolic geodesic ray at infinity is inspired by the method of Bharali–Maitra [BM21], where cluster sets of dynamical orbits are controlled via the visibility property.

DEFINITION 1.1. Let (Ω, d) be a minimally nice space. We say that:

- (1) (Ω, k_Ω) has the *visibility property* if the following holds:
 - (*) For every pair of distinct points $p, q \in \partial_d \Omega$, there exists a compact set $K \subset \Omega$ such that for any sequences $(x_n), (y_n)$ in Ω with $x_n \xrightarrow{d} p$ and $y_n \xrightarrow{d} q$, every quasihyperbolic geodesic γ_n joining x_n and y_n intersects K .
- (2) (Ω, d) has the *quasihyperbolic visibility property* if (Ω, k_Ω) has the visibility property.
- (3) (Ω, d) is *quasihyperbolically visible* if (Ω, d) has the quasihyperbolic visibility property.

2. Results. We now state our main results. Throughout, we assume that $\overline{\Omega}^d$ is compact.

THEOREM 2.1. *Let (Ω, d) be a bounded minimally nice space. Assume that $\overline{\Omega}^d$ is compact and (Ω, k_Ω) is Gromov hyperbolic. Then the following are equivalent:*

- (i) *(Ω, d) is quasihyperbolically visible.*
- (ii) *The identity map id_Ω extends to a continuous surjective map $\Phi : \partial_G \Omega \rightarrow \partial_d \Omega$.*

In addition, if (Ω, d) is quasiconvex and the Gehring–Hayman theorem holds in (Ω, d) , then the extension map $\Phi : \partial_G \Omega \rightarrow \partial_d \Omega$ is a homeomorphism.

REMARK 2.1. By saying that the Gehring–Hayman theorem holds in (Ω, d) , we mean that there exists a constant $C_{\text{gh}} \geq 1$ such that for each pair of points $x, y \in \Omega$ and each quasihyperbolic geodesic γ joining x and y , we have

$$\ell_d(\gamma[x, y]) \leq C_{\text{gh}} \ell_d(\sigma),$$

where σ is any other curve joining x and y in Ω . In other words, quasihyperbolic geodesics are essentially the shortest curves in Ω . We refer to [KL13, KLM14] for conditions ensuring that (Ω, d) satisfies the Gehring–Hayman theorem.

Our next result gives a criterion for a minimally nice space to be a quasihyperbolically visible space.

THEOREM 2.2. *Let (Ω, d) be a bounded minimally nice space. Fix a base point $x_0 \in \Omega$ and let $\phi : (0, \infty) \rightarrow (0, \infty)$ be a strictly increasing function with $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$ such that*

$$(1) \quad \int_0^\infty \frac{dt}{\phi^{-1}(t)} < \infty.$$

If the quasihyperbolic metric satisfies

$$(2) \quad k_\Omega(x_0, x) \leq \phi\left(\frac{\delta_\Omega(x_0)}{\delta_\Omega(x)}\right),$$

then (Ω, d) is quasihyperbolically visible.

REMARK 2.2. It is easy to see that Theorem 2.1 together with Theorem 2.2 gives Theorem 1.3. Furthermore, Theorems 2.1 and 2.2 together imply a special case of [ZLL22, Theorem 1.8(1)], which we explain below.

Explanation. Let (Ω, d) be a rectifiably connected, locally compact, non-complete bounded metric space. It is known that if (Ω, d) is bounded and φ -uniform, then there is an increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that for all $x \in \Omega$,

$$k_\Omega(x_0, x) \leq \phi\left(\frac{\delta_\Omega(x_0)}{\delta_\Omega(x)}\right),$$

where $x_0 \in \Omega$ satisfies $\delta_\Omega(x_0) = \max_{x \in \Omega} \delta_\Omega(x)$. In particular, we can take

$$\phi(t) = \varphi\left(\frac{\text{diam}(\Omega)t}{\delta_\Omega(x_0)}\right).$$

Note that the conditions

$$(3) \quad \int_0^\infty \frac{dt}{\phi^{-1}(t)} < \infty \quad \text{and} \quad \int_0^\infty \frac{dt}{\varphi^{-1}(t)} < \infty$$

are mutually equivalent. Hence, by Theorem 2.2, a bounded φ -uniform domain satisfying (3) is quasihyperbolically visible. This, together with Theorem 2.1 and [KLM14, Theorem 5.1], gives [ZLL22, Theorem 1.8(1)]. ■

3. Preliminaries

3.1. Metric geometry. A *curve* is a continuous function $\gamma : [a, b] \rightarrow \Omega$. Let $\mathcal{P}$ denote the set of all partitions $a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$. The *length* of a curve γ in a metric space (Ω, d) is

$$\ell_d(\gamma) = \sup_{\mathcal{P}} \sum_{k=0}^{n-1} d(\gamma(t_k), \gamma(t_{k+1})).$$

A curve γ is said to be *rectifiable* if $\ell_d(\gamma) < \infty$. A metric space (Ω, d) is said to be *rectifiably connected* if every pair of points $x, y \in \Omega$ can be joined by a rectifiable curve. For a rectifiable curve γ , we define the *arclength function* $s : [a, b] \rightarrow [0, \ell_d(\gamma)]$ by $s(t) = \ell_d(\gamma|_{[a, t]})$. The arclength function is of bounded variation. For any rectifiable curve $\gamma : [a, b] \rightarrow \Omega$, there is a unique map $\gamma_s : [0, \ell_d(\gamma)] \rightarrow \Omega$ such that $\gamma = \gamma_s \circ s$, and the curve γ_s is called the *arclength parametrization* of γ .

DEFINITION 3.1. A curve $\gamma : [a, b] \rightarrow (\Omega, d)$ is said to be a *geodesic* if it is an *isometry*, i.e. for all $t, t' \in [a, b]$, $d(\gamma(t), \gamma(t')) = |t - t'|$. An isometry $\gamma : [0, \infty) \rightarrow (\Omega, d)$ is called a *geodesic ray*, and an isometry $\gamma : (-\infty, \infty) \rightarrow (\Omega, d)$ is called a *geodesic line*.

DEFINITION 3.2. A metric space (Ω, d) is said to be

- (1) a *geodesic space* if every pair of points $x, y \in \Omega$ can be joined by a geodesic;
- (2) *proper* if every closed ball is compact in Ω ;
- (3) *intrinsic* (or a *path metric space*, or a *length space*) if for all $x, y \in \Omega$ we have

$$d(x, y) = \lambda_\Omega(x, y) := \inf_{\gamma} \ell_d(\gamma),$$

where the infimum is taken over all rectifiable curves γ in Ω joining x and y ; the metric λ_Ω is known as the *inner length metric* associated to (Ω, d) ;

- (4) *quasiconvex* if there exists a constant $A \geq 1$ such that every pair of points $x, y \in \Omega$ can be joined by a curve γ with $\ell_d(\gamma) \leq Ad(x, y)$.

3.2. Gromov hyperbolic spaces. Let (Ω, d) be a geodesic metric space. A *geodesic triangle* is the union of geodesics $\gamma_i : [a_i, b_i] \rightarrow \Omega$, $i = 1, 2, 3$, such that $a_i < b_i$ for each i and $\gamma_1(b_1) = \gamma_2(a_2)$, $\gamma_2(b_2) = \gamma_3(a_3)$, and $\gamma_3(b_3) = \gamma_1(a_1)$. The geodesics γ_1 , γ_2 , and γ_3 are called the *sides* of the geodesic triangle.

DEFINITION 3.3. Let $\delta \geq 0$. A geodesic metric space (Ω, d) is said to be *δ -hyperbolic* if every geodesic triangle is *δ -thin*, i.e. each side of the geodesic triangle is contained in the δ -neighborhood of the union of the other two sides. A geodesic metric space (Ω, d) is said to be *Gromov hyperbolic* if it is δ -hyperbolic for some $\delta \geq 0$.

Let (Ω, d) be a proper geodesic Gromov hyperbolic space and let $p \in \Omega$. Let $\mathcal{G}_p$ be the space of all geodesic rays $\gamma : [0, \infty) \rightarrow \Omega$ with $\gamma(0) = p$, endowed with the topology of uniform convergence on compact subsets of $[0, \infty)$. Define an equivalence relation $\sim$ on $\mathcal{G}_p$ as follows: for $\gamma, \sigma \in \mathcal{G}_p$,

$$\gamma \sim \sigma \iff \sup_{t \geq 0} d(\gamma(t), \sigma(t)) < \infty.$$

DEFINITION 3.4.

- (1) The *Gromov boundary* of Ω is denoted by $\partial_G \Omega$ and is defined as the quotient space $\mathcal{G}_p / \sim$ endowed with the quotient topology.
- (2) The *Gromov closure* of Ω is denoted by $\overline{\Omega}^G$ and defined as $\Omega \cup \partial_G \Omega$.

The set $\overline{\Omega}^G$ admits a natural topology, and with this topology $\overline{\Omega}^G$ is a compactification of Ω , known as the *Gromov compactification* of Ω (see, for instance, [BH99, Section III.H.3, Proposition 3.7]). To describe this topology, we need some additional notation. A *generalized ray* is either a *geodesic segment* or a geodesic ray.

- For a geodesic ray $\gamma \in \mathcal{G}_p$, define $\text{End}(\gamma) := [\gamma]$.
- For a geodesic segment $\gamma : [0, R] \rightarrow \Omega$ with $\gamma(0) = p$ (where $R > 0$), define $\text{End}(\gamma) := \gamma(R)$.

DEFINITION 3.5 (Gromov convergence). Let (Ω, d) be a proper geodesic Gromov hyperbolic space. Fix a base point $p \in \Omega$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\overline{\Omega}^G$ converges to $x \in \overline{\Omega}^G$ (denoted by $x_n \xrightarrow{\text{Gromov}} x$) if there exist generalized rays γ_n with $\gamma_n(0) = p$ and $\text{End}(\gamma_n) = x_n$ such that every subsequence of (γ_n) contains a further subsequence that converges uniformly on compact subsets to a generalized ray γ with $\gamma(0) = p$ and $\text{End}(\gamma) = x$.

In view of the Gromov topology of $\overline{\Omega}^G$, we have the following observations for geodesic rays and geodesic lines:

- For any geodesic ray $\gamma : [0, \infty) \rightarrow \Omega$, $\lim_{t \rightarrow \infty} \gamma(t) = \text{End}(\gamma) = [\gamma] \in \partial_G \Omega$. For any geodesic line $\gamma : (-\infty, \infty) \rightarrow \Omega$, both limits exist and $\lim_{t \rightarrow -\infty} \gamma(t) \neq \lim_{t \rightarrow \infty} \gamma(t)$.
- For a geodesic line γ , let us denote $\gamma(-\infty) = \lim_{t \rightarrow -\infty} \gamma(t)$ and $\gamma(\infty) = \lim_{t \rightarrow \infty} \gamma(t)$. Then for each $p \in \Omega$ and $\xi \in \partial_G \Omega$ there exists a geodesic ray $\gamma : [0, \infty) \rightarrow \Omega$ such that $\text{End}(\gamma) = \xi$.
- For each pair of distinct points $\xi, \eta \in \partial_G \Omega$ there exists a geodesic line γ with $\gamma(-\infty) = \xi$ and $\gamma(\infty) = \eta$ (see [BH99, Section III.H.3, Lemmas 3.1 and 3.2]). This property is called the *visibility property of Gromov hyperbolic spaces*.

3.3. The quasihyperbolic metric. Gehring and Palka [GP76] introduced the quasihyperbolic metric for domains $\Omega \subsetneq \mathbb{R}^n$ as a tool to study quasiconformal homogeneity. The same definition naturally generalizes in the minimally nice metric space setting.

DEFINITION 3.6. Let (Ω, d) be a rectifiably connected, locally compact, non-complete metric space. Then the metric boundary $\partial_d \Omega$ is non-empty, and we define the *quasihyperbolic metric* k_Ω in Ω by the length element $|dz|/\delta_\Omega(z)$, where $\delta_\Omega(z) = d(z, \partial_d \Omega)$ and $|dz|$ denotes the length element for the metric d . Thus, $k_\Omega(x, y) = \inf_\gamma \ell_k(\gamma)$, where the infimum is taken over all rectifiable curves γ in Ω joining x and y , and

$$\ell_k(\gamma) = \int_\gamma \frac{|dz|}{\delta_\Omega(z)}$$

is called the *quasihyperbolic length* of γ .

In view of [BHK01, Proposition 2.8], it is well known that if (Ω, d) is minimally nice, then (Ω, k_Ω) is a complete, proper, and geodesic metric space.

We recall some basic estimates for the quasihyperbolic metric which were first established for domains in $\mathbb{R}^n$ (see [GP76, Lemma 2.1]) and also hold in minimally nice spaces (see [BHK01, Chapter 2]). For all $x, y \in \Omega$, we have

$$\begin{aligned} (4) \quad k_\Omega(x, y) &\geq \log \left(1 + \frac{\lambda_\Omega(x, y)}{\min \{\delta_\Omega(x), \delta_\Omega(y)\}} \right) \\ &\geq \log \left(1 + \frac{d(x, y)}{\min \{\delta_\Omega(x), \delta_\Omega(y)\}} \right) \geq \left| \log \frac{\delta_\Omega(y)}{\delta_\Omega(x)} \right|, \end{aligned}$$

and if γ is a rectifiable curve in Ω joining x and y , then

$$(5) \quad \ell_k(\gamma) \geq \log \left(1 + \frac{\ell_d(\gamma)}{\min \{\delta_\Omega(x), \delta_\Omega(y)\}} \right).$$

In view of (4), we have the following observation.

OBSERVATION 1. If sequences $\{z_n\}, \{w_n\} \subset \Omega$ converge to two different points in $\partial_d \Omega$, then $\lim_{n \rightarrow \infty} k_\Omega(z_n, w_n) = \infty$.

At this stage, we mention the following important characterization of Gromov hyperbolicity of the quasihyperbolic metric, due to Bonk, Heinonen, and Koskela, and to Balogh and Buckley.

THEOREM 3.1 ([BHK01, Propositions 7.12, 7.14]; [BB03, Theorem 0.1]). *Let $\Omega \subset \mathbb{R}^n$ be a proper subdomain. Then (Ω, k_Ω) is Gromov hyperbolic if and only if Ω satisfies both a Gehring–Hayman condition and a ball separation condition with respect to the inner metric.*

4. Proof of Theorem 2.1. In this section, (Ω, d) is a bounded minimally nice space and we assume that $\overline{\Omega}^d$ is compact. Throughout, we parametrize quasihyperbolic geodesics (and geodesic rays) by k_Ω -arclength; in particular, for a quasihyperbolic geodesic γ we have $k_\Omega(\gamma(s), \gamma(t)) = |s - t|$. We divide the proof of Theorem 2.1 into several lemmas.

LEMMA 1. *Let (Ω, d) be a bounded minimally nice space and let γ be a quasihyperbolic geodesic ray in Ω . If for any sequence $t_n \rightarrow \infty$ the sequence $\{\gamma(t_n)\}$ d -converges to a point $x \in \overline{\Omega}^d$, then $x \in \partial_d \Omega$.*

Proof. Assume for a contradiction that $x \notin \partial_d \Omega$. Then $x \in \Omega$. Since γ is a quasihyperbolic geodesic ray parametrized by k_Ω -arclength,

$$k_\Omega(\gamma(0), \gamma(t_n)) = t_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

On the other hand, $\gamma(t_n) \rightarrow x$ in the metric d with $x \in \Omega$, so $\delta_\Omega(\gamma(t_n)) \rightarrow \delta_\Omega(x) > 0$. Hence, for n large we have $\delta_\Omega(\gamma(t_n)) \geq \frac{1}{2}\delta_\Omega(x)$, and therefore

$$k_\Omega(\gamma(t_n), x) \leq \frac{2}{\delta_\Omega(x)} \lambda_\Omega(\gamma(t_n), x) \xrightarrow{n \rightarrow \infty} 0.$$

By the triangle inequality,

$$|k_\Omega(\gamma(0), \gamma(t_n)) - k_\Omega(\gamma(0), x)| \leq k_\Omega(\gamma(t_n), x) \rightarrow 0,$$

and consequently $k_\Omega(\gamma(0), \gamma(t_n)) \rightarrow k_\Omega(\gamma(0), x) < \infty$, a contradiction. Therefore $x \in \partial_d \Omega$. ■

The limit in the next lemma belongs to $\partial_d \Omega$ by Lemma 1.

LEMMA 2. *Let (Ω, d) be a bounded minimally nice space. Suppose (Ω, d) is quasihyperbolically visible. If γ is a quasihyperbolic geodesic ray in Ω , then for any sequence $t_n \rightarrow \infty$ the limit $\lim_{n \rightarrow \infty} \gamma(t_n)$ exists in the metric d .*

Proof. Since $\overline{\Omega}^d$ is compact, the sequence $\{\gamma(t_n)\}$ has a convergent subsequence. By Lemma 1, every subsequential limit lies in $\partial_d \Omega$. Thus, it suffices to show that $\{\gamma(t_n)\}$ has a unique limit point. Assume for a contradiction

that $\{\gamma(t_n)\}$ has two distinct limit points $p, q \in \partial_d \Omega$, i.e. there are two subsequences $\{\gamma(t_{n_k})\}_k$ and $\{\gamma(t_{s_k})\}_k$ such that

$$\lim_{k \rightarrow \infty} \gamma(t_{n_k}) = p \neq q = \lim_{k \rightarrow \infty} \gamma(t_{s_k}).$$

Passing to further subsequences if necessary, we may assume that $t_{n_k} < t_{s_k} < t_{n_{k+1}}$ for all k . Consider the quasihyperbolic geodesic segments $\gamma_k = \gamma|_{[t_{n_k}, t_{s_k}]}$. Since (Ω, d) is a quasihyperbolically visible space, there exists a compact set $K \subset \Omega$ such that, for all sufficiently large k , the segment γ_k meets K . Thus, for each such k we can pick $t'_k \in (t_{n_k}, t_{s_k})$ with $\gamma(t'_k) \in K$. Since $t_{n_k} \rightarrow \infty$ and $t'_k \in (t_{n_k}, t_{s_k})$, we have $t'_k \rightarrow \infty$. Moreover,

$$t'_k = k_\Omega(\gamma(0), \gamma(t'_k)) \leq \max_{x \in K} k_\Omega(\gamma(0), x) < \infty,$$

a contradiction. Hence $\{\gamma(t_n)\}$ has a unique limit point, and therefore converges in d . ■

In particular, if $t_n \rightarrow \infty$, $s_n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \gamma(t_n) = \lim_{n \rightarrow \infty} \gamma(s_n)$. Therefore, we have the following lemma.

LEMMA 3. *Let (Ω, d) be a bounded minimally nice space. Suppose (Ω, d) is quasihyperbolically visible. If γ is a quasihyperbolic geodesic ray in Ω , then $\lim_{t \rightarrow \infty} \gamma(t)$ exists in $\partial_d \Omega$.*

Proof. Let $t_n \rightarrow \infty$. By Lemma 2, the limit $\lim_{n \rightarrow \infty} \gamma(t_n)$ exists in the metric d , and by Lemma 1 it belongs to $\partial_d \Omega$. Moreover, if $s_n \rightarrow \infty$ is another sequence, then the proof of Lemma 2 yields $\lim_{n \rightarrow \infty} \gamma(t_n) = \lim_{n \rightarrow \infty} \gamma(s_n)$. Hence $\lim_{t \rightarrow \infty} \gamma(t)$ exists and lies in $\partial_d \Omega$. ■

Further, we prove that equivalent quasihyperbolic geodesic rays land at the same point.

LEMMA 4. *Let (Ω, d) be quasihyperbolically visible. Fix $x_0 \in \Omega$ and let γ and σ be two quasihyperbolic geodesic rays with $\gamma(0) = x_0 = \sigma(0)$ such that $\gamma \sim \sigma$. Then*

$$\lim_{t \rightarrow \infty} \gamma(t) = \lim_{t \rightarrow \infty} \sigma(t).$$

Proof. In view of Lemma 3, both limits exist in $\partial_d \Omega$. Assume for a contradiction that

$$\lim_{t \rightarrow \infty} \gamma(t) = p \neq q = \lim_{t \rightarrow \infty} \sigma(t).$$

Choose a sequence $s_n \rightarrow \infty$ such that $\gamma(s_n) \rightarrow p$ and $\sigma(s_n) \rightarrow q$. Let $x_n = \gamma(s_n)$ and $y_n = \sigma(s_n)$; then $x_n \rightarrow p$ and $y_n \rightarrow q$. By Observation 1, $\lim_{n \rightarrow \infty} k_\Omega(x_n, y_n) = \infty$, contradicting $\gamma \sim \sigma$. ■

The following lemma will be useful to show that the extension map is continuous.

LEMMA 5. Let (Ω, d) be a bounded minimally nice space. Suppose (Ω, d) is quasihyperbolically visible. Fix $x_0 \in \Omega$.

- (i) Let $\gamma_n : [0, \infty) \rightarrow (\Omega, k_\Omega)$ be quasihyperbolic geodesic rays with $\gamma_n(0) = x_0$ for all n . If (γ_n) converges uniformly on compact subsets to a geodesic ray $\gamma : [0, \infty) \rightarrow (\Omega, k_\Omega)$, then

$$\lim_{t \rightarrow \infty} \gamma(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \gamma_n(t).$$

- (ii) Let $\gamma_n : [0, T_n] \rightarrow (\Omega, k_\Omega)$ be quasihyperbolic geodesics with $\gamma_n(0) = x_0$ for all n , where $T_n \rightarrow \infty$. If (γ_n) converges uniformly on compact subsets to a geodesic ray $\gamma : [0, \infty) \rightarrow (\Omega, k_\Omega)$, then

$$\lim_{t \rightarrow \infty} \gamma(t) = \lim_{n \rightarrow \infty} \gamma_n(T_n).$$

Proof. Since $\overline{\Omega}^d$ is compact, it suffices to prove each claim along an arbitrary convergent subsequence of the boundary values.

(i) By Lemma 3, for each n , let $p_n := \lim_{t \rightarrow \infty} \gamma_n(t) \in \partial_d \Omega$ and let $q := \lim_{t \rightarrow \infty} \gamma(t) \in \partial_d \Omega$. Passing to a subsequence, we may assume $p_n \rightarrow p \in \partial_d \Omega$. We show that $p = q$. Assume for a contradiction that $p \neq q$. Since γ lands at q , for every $j \in \mathbb{N}$, there exists $s_j \rightarrow \infty$ such that for every $t \geq s_j$, $d(\gamma(t), q) < 1/j$. In particular, $d(\gamma(s_j), q) < 1/j$. Since $\gamma_n \rightarrow \gamma$ uniformly on $[0, s_j]$, there exists $N(j)$ such that $d(\gamma_n(s_j), \gamma(s_j)) < 1/j$ for all $n \geq N(j)$, and hence

$$d(\gamma_n(s_j), q) < 2/j \quad \text{for all } n \geq N(j).$$

Now choose an increasing sequence of indices $n_j \geq N(j)$ such that $p_{n_j} \xrightarrow{d} p$ and $d(p_{n_j}, p) < 1/j$ for each j . For each j , since $\gamma_{n_j}(t) \rightarrow p_{n_j}$ as $t \rightarrow \infty$, pick $t_j > s_j$ such that $d(\gamma_{n_j}(t_j), p_{n_j}) < 1/j$. Then

$$d(\gamma_{n_j}(t_j), p) \leq d(\gamma_{n_j}(t_j), p_{n_j}) + d(p_{n_j}, p) < 2/j.$$

Set $y_j := \gamma_{n_j}(s_j)$ and $x_j := \gamma_{n_j}(t_j)$. Then $y_j \rightarrow q$ and $x_j \rightarrow p$. By quasihyperbolic visibility, there exists a compact set $K \subset \Omega$ such that every quasihyperbolic geodesic joining x_j to y_j intersects K . In particular, the geodesic segment $\gamma_{n_j}|_{[s_j, t_j]}$ intersects K , so we may choose $c_j \in [s_j, t_j]$ with $\gamma_{n_j}(c_j) \in K$. Since γ_{n_j} is parametrized by k_Ω -arclength and $\gamma_{n_j}(0) = x_0$,

$$c_j = k_\Omega(x_0, \gamma_{n_j}(c_j)) \leq \max_{x \in K} k_\Omega(x_0, x) =: M < \infty.$$

But $c_j \geq s_j \rightarrow \infty$, a contradiction. Hence $p = q$, proving (i).

(ii) Let $x_n := \gamma_n(T_n) \in \Omega$. Since $\overline{\Omega}^d$ is compact, passing to a subsequence, we may assume that $x_n \rightarrow p \in \overline{\Omega}^d$. Since $T_n = k_\Omega(x_0, x_n) \rightarrow \infty$ and k_Ω induces the original topology on Ω , we have $p \in \partial_d \Omega$. Let $q := \lim_{t \rightarrow \infty} \gamma(t) \in \partial_d \Omega$. Our aim is to show that $p = q$. If not, choose $s_j \rightarrow \infty$ such that $d(\gamma(s_j), q) < 1/j$ for each j . Since $\gamma_n \rightarrow \gamma$ uniformly on $[0, s_j]$,

for each j there exists $N(j)$ such that $d(\gamma_n(s_j), q) < 2/j$ for all $n \geq N(j)$. Also, since $x_n \rightarrow p$, for each j there exists $N'(j)$ such that $d(x_n, p) < 1/j$ for all $n \geq N'(j)$. Pick an increasing sequence $n_j \geq \max\{N(j), N'(j)\}$ such that $T_{n_j} > s_j$ (this is possible since $T_n \rightarrow \infty$). Set $y_j := \gamma_{n_j}(s_j)$ and $x_j := \gamma_{n_j}(T_{n_j})$. Then $y_j \rightarrow q$ and $x_j \rightarrow p$. By quasihyperbolic visibility, there exists a compact set $K \subset \Omega$ such that every quasihyperbolic geodesic joining x_j to y_j intersects K . In particular, the segment $\gamma_{n_j}|_{[s_j, T_{n_j}]}$ intersects K , so choose $c_j \in [s_j, T_{n_j}]$ with $\gamma_{n_j}(c_j) \in K$. As above,

$$c_j = k_\Omega(x_0, \gamma_{n_j}(c_j)) \leq M < \infty,$$

but $c_j \geq s_j \rightarrow \infty$, a contradiction. Hence $p = q$ and (ii) follows. ■

Proof of Theorem 2.1. The implication (i) $\Rightarrow$ (ii) follows from [AP24, Theorem 2.2]. Conversely, assume that Ω is quasihyperbolically visible. Fix $z_0 \in \Omega$. For $\xi \in \partial_G \Omega$, choose a representative quasihyperbolic geodesic ray γ_ξ with $\gamma_\xi(0) = z_0$ and set

$$\Phi(\xi) := \lim_{t \rightarrow \infty} \gamma_\xi(t) \in \partial_d \Omega,$$

which exists by Lemma 3 and is independent of the representative by Lemma 4. For $z \in \Omega$ set $\Phi(z) = z$.

To prove continuity of $\Phi : \overline{\Omega}^G \rightarrow \overline{\Omega}^d$, let $x_n \xrightarrow{\text{Gromov}} x$ in $\overline{\Omega}^G$. By definition of the Gromov topology (see Definition 3.5), we can represent x_n by generalized quasihyperbolic geodesics γ_n based at z_0 whose endpoints are x_n , such that every subsequence admits a further subsequence converging uniformly on compact subsets to a generalized geodesic γ ending at x . If $x \in \Omega$, then γ is a finite geodesic and $\Phi(x_n) \rightarrow x$ follows immediately. If $x \in \partial_G \Omega$, then γ is a geodesic ray in the class x , and Lemma 5 yields $\Phi(x_n) \rightarrow \Phi(x)$. Hence Φ is continuous.

Since $\overline{\Omega}^G$ is compact and $\overline{\Omega}^d$ is Hausdorff, $\Phi(\overline{\Omega}^G)$ is compact and therefore closed in $\overline{\Omega}^d$. Moreover, $\Phi(\Omega) = \Omega$, which is dense in $\overline{\Omega}^d$, so Φ is surjective.

Now, we only need to show that Φ is a homeomorphism. For this, it remains to prove that Φ is injective. Let $\xi, \eta \in \partial_G \Omega$ with $\xi \neq \eta$. Choose a geodesic line $\Gamma : (-\infty, \infty) \rightarrow (\Omega, k_\Omega)$ with $\Gamma(-\infty) = \xi$ and $\Gamma(\infty) = \eta$. Set

$$p := \lim_{t \rightarrow \infty} \Gamma(t) \in \partial_d \Omega, \quad q := \lim_{t \rightarrow -\infty} \Gamma(t) \in \partial_d \Omega,$$

which exist by applying Lemma 3 to the rays $t \mapsto \Gamma(t)$ and $t \mapsto \Gamma(-t)$. We claim that $p \neq q$. Assume for a contradiction that $p = q$. Choose sequences $t_n \rightarrow \infty$ and $s_n \rightarrow -\infty$ and set $x_n := \Gamma(t_n)$ and $y_n := \Gamma(s_n)$. Then $x_n \rightarrow p$ and $y_n \rightarrow p$ in d , hence $d(x_n, y_n) \rightarrow 0$. By quasiconvexity, for each n there exists a curve σ_n joining x_n to y_n with $\ell_d(\sigma_n) \leq Ad(x_n, y_n)$. By the Gehring–

Hayman theorem, the quasihyperbolic geodesic segment $\Gamma[x_n, y_n]$ satisfies

$$\ell_d(\Gamma[x_n, y_n]) \leq C_{\text{gh}} \ell_d(\sigma_n) \leq C_{\text{gh}} \text{Ad}(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0.$$

Fix $R > 0$. For n large enough we have $s_n \leq -R \leq R \leq t_n$, hence the segment $\Gamma([-R, R])$ is contained in $\Gamma[x_n, y_n]$, and therefore

$$\ell_d(\Gamma([-R, R])) \leq \ell_d(\Gamma[x_n, y_n]) \rightarrow 0.$$

Thus $\ell_d(\Gamma([-R, R])) = 0$ for every $R > 0$, which forces Γ to be constant, a contradiction. Hence $p \neq q$, and therefore $\Phi(\xi) \neq \Phi(\eta)$. Thus Φ is a continuous bijection from the compact space $\overline{\Omega}^G$ to the Hausdorff space $\overline{\Omega}^d$, and hence Φ is a homeomorphism. This completes the proof of Theorem 2.1. ■

5. Proof of Theorem 2.2. For this proof, we need the following important remark.

REMARK 5.1. It is often convenient to parametrize a curve γ in (Ω, d) by quasihyperbolic length. We say that $g : [0, r] \rightarrow \Omega$ is a quasihyperbolic parametrization of a curve $\gamma : [a, b] \rightarrow (\Omega, d)$ if $\ell_k(g|_{[0, t]}) = t$ for all $t \in [0, r]$. Every rectifiable curve γ in (Ω, d) has a *quasihyperbolic parametrization* $g : [0, r] \rightarrow \Omega$ with image $\gamma([a, b])$, and g satisfies the Lipschitz condition

$$d(g(s), g(t)) \leq M|s - t|,$$

where $M = \max\{\delta_\Omega(x) : x \in \gamma([a, b])\}$. Since the metric derivative for Lipschitz curves exists a.e. (for details we refer to [A90], [AGS08, p. 2 and p. 24], and [AK00, Theorem 3.2]), we see that

- (i) $r = \ell_k(g)$,
- (ii) for a.e. $t \in [0, r]$ we have $|g'|(t) = \delta_\Omega(g(t))$, and for all $0 \leq s < t \leq r$,

$$d(g(s), g(t)) \leq \int_s^t |g'|(\tau) d\tau.$$

Here $|g'|(t)$ denotes the metric derivative in (Ω, d) . If γ is a quasihyperbolic geodesic, then $g : [0, r] \rightarrow (\Omega, k_\Omega)$ is an isometry and we say that g is a *quasihyperbolic geodesic* from $g(0)$ to $g(r)$.

Proof of Theorem 2.2. Suppose (Ω, d) is not quasihyperbolically visible. Then there exist two distinct points $p, q \in \partial_d \Omega$ and sequences $\{x_n\}, \{y_n\} \subset \Omega$ such that $x_n \xrightarrow{d} p$ and $y_n \xrightarrow{d} q$ as $n \rightarrow \infty$, and a sequence of quasihyperbolic geodesics $\gamma_n : [a_n, b_n] \rightarrow \Omega$ joining x_n and y_n such that

$$(6) \quad \max_{t \in [a_n, b_n]} \delta_\Omega(\gamma_n(t)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By a suitable quasihyperbolic parametrization $g_n : [a_n, b_n] \rightarrow \Omega$ of γ_n with $g_n(a_n) = x_n$, $g_n(b_n) = y_n$ and $|b_n - a_n| = \ell_k(\gamma_n)$, we can assume that for

all n we have $a_n \leq 0 \leq b_n$ and

$$(7) \quad \max_{t \in [a_n, b_n]} \delta_\Omega(g_n(t)) = \delta_\Omega(g_n(0)).$$

Furthermore, by Remark 5.1 each g_n satisfies the Lipschitz condition

$$d(g_n(s), g_n(t)) \leq M_n |s - t|, \quad s, t \in [a_n, b_n],$$

where $M_n = \max\{\delta_\Omega(x) : x \in \gamma_n([a_n, b_n])\}$. In view of (6), each g_n is M -Lipschitz with $M = \sup_{n \geq 1} M_n < \infty$. Therefore, by the Arzelà–Ascoli theorem, passing to a subsequence if necessary, we may assume:

- (i) $a_n \rightarrow a \in [-\infty, 0]$ and $b_n \rightarrow b \in [0, \infty]$,
- (ii) g_n converges uniformly on every compact subset of (a, b) to a continuous map $g : (a, b) \rightarrow \overline{\Omega}^d$, and $g_n(a_n) \rightarrow p$, $g_n(b_n) \rightarrow q$.

OBSERVATION 2. $a < b$ and a and b cannot both be finite. Indeed:

- (i) $a < b$ follows from the fact that $d(g_n(a_n), g_n(b_n)) \leq M|a_n - b_n|$ for all n .
- (ii) By Remark 5.1, $g_n : [a_n, b_n] \rightarrow (\Omega, k_\Omega)$ is an isometry. Hence

$$\begin{aligned} |b - a| &= \lim_{n \rightarrow \infty} |b_n - a_n| \\ &= \lim_{n \rightarrow \infty} k_\Omega(g_n(b_n), g_n(a_n)) = \infty \quad (\text{by Observation 1}). \end{aligned}$$

In particular, a and b cannot both be finite.

CLAIM 1. $g : (a, b) \rightarrow \overline{\Omega}^d$ is a constant map.

In view of (6), we see that $\delta_\Omega(g_n(t))$ converges uniformly to 0 on compact subsets of (a, b) . Then for $s < t$ in (a, b) we have

$$(8) \quad d(g(s), g(t)) = \lim_{n \rightarrow \infty} d(g_n(s), g_n(t)) \leq \lim_{n \rightarrow \infty} M_n |s - t| = 0.$$

Thus g is constant.

Further, using the quasihyperbolic growth in Ω , we will establish a contradiction by proving the following.

CLAIM 2. $g : (a, b) \rightarrow \overline{\Omega}^d$ is not a constant map.

Since each $g_n : [a_n, b_n] \rightarrow (\Omega, k_\Omega)$ is an isometry, for $t \in [a_n, b_n]$ we have

$$\begin{aligned} |t| &= k_\Omega(g_n(0), g_n(t)) \leq k_\Omega(x_0, g_n(0)) + k_\Omega(x_0, g_n(t)) \\ &\leq \phi\left(\frac{\delta_\Omega(x_0)}{\delta_\Omega(g_n(0))}\right) + \phi\left(\frac{\delta_\Omega(x_0)}{\delta_\Omega(g_n(t))}\right) \\ &\leq 2\phi\left(\frac{\delta_\Omega(x_0)}{\delta_\Omega(g_n(t))}\right) \quad (\text{by (7)}). \end{aligned}$$

In view of Observation 2, we have only the following two cases to consider:

- Case 1: Both a and b are infinite.
- Case 2: Exactly one of a and b is finite.

We examine Case 1; the reasoning for Case 2 is analogous. Assume that $a = -\infty$ and $b = \infty$. Since $b_n \rightarrow \infty$, $a_n \rightarrow -\infty$, and ϕ is strictly increasing with $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$, there exist $n_0 \in \mathbb{N}$ and constants $A, B > 0$ such that whenever $t \in (B, b_n]$ and $n \geq n_0$ and whenever $t \in [a_n, -A)$ and $n \geq n_0$, we have

$$\frac{|t|}{2} \in \mathbf{range}(\phi), \quad \text{and hence} \quad \delta_\Omega(g_n(t)) \leq \frac{\delta_\Omega(x_0)}{\phi^{-1}\left(\frac{|t|}{2}\right)}.$$

Choose $a^* \in (-\infty, -A)$ and $b^* \in (B, \infty)$ such that

$$(9) \quad d(p, q) > \int_{b^*}^{+\infty} \frac{\delta_\Omega(x_0)}{\phi^{-1}\left(\frac{t}{2}\right)} dt + \int_{-\infty}^{a^*} \frac{\delta_\Omega(x_0)}{\phi^{-1}\left(\frac{|t|}{2}\right)} dt.$$

Therefore,

$$\begin{aligned} d(g(b^*), g(a^*)) &= \lim_{n \rightarrow \infty} d(g_n(b^*), g_n(a^*)) \\ &\geq \limsup_{n \rightarrow \infty} [d(g_n(b_n), g_n(a_n)) - d(g_n(b_n), g_n(b^*)) - d(g_n(a_n), g_n(a^*))] \\ &\geq d(p, q) - \limsup_{n \rightarrow \infty} \int_{b^*}^{b_n} |g'_n|(t) dt - \limsup_{n \rightarrow \infty} \int_{a_n}^{a^*} |g'_n|(t) dt \\ &\geq d(p, q) - \limsup_{n \rightarrow \infty} \int_{b^*}^{b_n} \delta_\Omega(g_n(t)) dt - \limsup_{n \rightarrow \infty} \int_{a_n}^{a^*} \delta_\Omega(g_n(t)) dt \\ &\geq d(p, q) - \limsup_{n \rightarrow \infty} \int_{b^*}^{b_n} \frac{\delta_\Omega(x_0)}{\phi^{-1}\left(\frac{t}{2}\right)} dt - \limsup_{n \rightarrow \infty} \int_{a_n}^{a^*} \frac{\delta_\Omega(x_0)}{\phi^{-1}\left(\frac{|t|}{2}\right)} dt \\ &\geq d(p, q) - \int_{b^*}^{+\infty} \frac{\delta_\Omega(x_0)}{\phi^{-1}\left(\frac{t}{2}\right)} dt - \int_{-\infty}^{a^*} \frac{\delta_\Omega(x_0)}{\phi^{-1}\left(\frac{|t|}{2}\right)} dt > 0 \quad (\text{by (9)}). \end{aligned}$$

Thus, g is non-constant.

So, we proved Claim 2, which contradicts Claim 1, as declared. This yields the existence of the compact set K . Therefore, (Ω, d) is quasihyperbolically visible. This concludes the proof of Theorem 2.2. ■

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