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ALGEBRAIC GEOMETRY

A dodecic surface with 320 cusps

by

Cédric BONNAFÉ

Presented by Jarosław WIŚNIEWSKI

Summary. We construct a degree 12 homogeneous invariant of the complex reflection group G_{29} (in Shephard–Todd's notation) whose associated surface has 320 singularities of type A_2, thereby improving previous records for dodecic surfaces.

For a type X of isolated surface singularity, and for $d \geqslant 1$, let $\mu_X(d)$ denote the maximal number of singularities of type X a surface of degree d in $\mathbf{P}^3(\mathbb{C})$ might have. Determining $\mu_X(d)$ is a classical problem. The exact value is known only in a few cases (for instance for $X = A_1$ and $d \leqslant 6$) but there have been many works trying to give upper and lower bounds for $\mu_X(d)$. For quotient singularities, and for d even or $d \geqslant 14$, the best-known upper bounds are given by Miyaoka [Miy].

Lower bounds are obtained by constructing explicit examples of surfaces of degree d with many singularities. For type A singularities and for any d, lower bounds have been obtained by several authors [Chm, Lab1, Lab3, Esc1, Esc2, Esc3, . . .]. In small degrees and in type A_1, some exceptional examples give better lower bounds (see the works of Togliatti [Tog], Barth [Bar], Endraß [End], Labs [Lab1, Lab2], Sarti [Sar1, Sar2, Sar3], . . .).

For small degrees and singularities of type A_2 (which will be called *cusps* in this paper), Labs [Lab1, Theorem 7.1] proved that $\mu_{A_2}(6) \geqslant 35$ while Borisov and Gunnels [BoGu] proved that $\mu_{A_2}(8) \geqslant 84$: these are the best-known lower bounds. Our aim in this paper is to improve the best-known lower bound for $\mu_{A_2}(12)$, obtained by Escudero [Esc2], who constructed a dodecic surface with 301 cusps:

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Theorem. *There exists a homogeneous invariant of degree* 12 *of the complex reflection group* G_{29} (*in Shephard–Todd's notation* [ShTo]) *whose zero set is a surface with exactly* 320 *cusps, which form a single* G_{29}*-orbit.*

Together with Miyaoka's upper bound, our result says that

$$320 \leqslant \mu_{A_2}(12) \leqslant 363.$$

It must be said that the results of Escudero are much more general, since he got lower bounds for all degrees divisible by 3 and A_j-singularities for any $j \geqslant 2$. For instance, he proved that

$$96k^2(4k-1)+14k-1 \leqslant \mu_{A_2}(12k) \leqslant 3k(12k-1)^2$$

(here, the upper bound is again due to Miyaoka). Using the classical trick consisting in lifting a surface of degree d in $\mathbf{P}^3(\mathbb{C})$ to a surface of degree kd through the morphism $\mathbf{P}^3(\mathbb{C}) \to \mathbf{P}^3(\mathbb{C})$, $[x:y:z:t] \mapsto [x^k : y^k : z^k : t^k]$, allows one to construct, thanks to our surface, a surface of degree $12k$ with $320k^3$ cusps; this gives the lower bound $\mu_{A_2}(12k) \geqslant 320k^3$, which is better than Escudero's only for $k=1$.

We also investigate several other singular dodecic surfaces defined by a fundamental invariant of G_{29}; some of them might be of interest for algebraic geometers. Among them, we retrieve a singular dodecic surface with 160 singularities of type D_4 which was already constructed by the author [Bon], and still gives the best known lower bound for $\mu_{D_4}(12)$.

Notation. We set $V = \mathbb{C}^4$ and we denote by (x, y, z, t) the dual basis of the canonical basis of $\mathbb{C}^4$: the algebra $\mathbb{C}[V]$ of polynomial functions on V is the polynomial algebra $\mathbb{C}[x, y, z, t]$. We identify $\mathbf{GL}_{\mathbb{C}}(V)$ with $\mathbf{GL}_4(\mathbb{C})$ and the projective space $\mathbf{P}(V)$ with $\mathbf{P}^3(\mathbb{C})$.

If m is a monomial in x, y, z and t, we denote by $\Sigma_4(m)$ the sum of all the monomials obtained from m by permutation of these four variables. For instance,

$$\Sigma_4(xy) = xy + xz + xt + yz + yt + zt \quad \text{and} \quad \Sigma_4(xyzt) = xyzt.$$

1. The complex reflection group G_{29}. All the computational results stated without proof in this section (structure of G_{29}, fundamental invariants) can be easily checked with Magma.

1.A. Definition. Let $i \in \mathbb{C}$ denote a square root of -1 and let

$$s_1 = \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & . & . & . \\ . & . & 1 & . \\ . & 1 & . & . \\ . & . & . & 1 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} . & -i & . & . \\ i & . & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}, \qquad s_4 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

The matrices s_1, s_2, s_3 and s_4 are reflections of order 2 and they generate a finite subgroup of $\mathbf{GL}_4(\mathbb{C})$, which can be taken as a model for the complex reflection group denoted by G_{29} in Shephard–Todd's classification [ShTo]. So we set

$$G_{29} = \langle s_1, s_2, s_3, s_4 \rangle \subset \mathbf{GL}_4(\mathbb{Q}[i]) \subset \mathbf{GL}_4(\mathbb{C}).$$

Note that G_{29} is irreducible and that

$$|G_{29}| = 7\,680 \quad \text{and} \quad \mathrm{Z}(G_{29}) = \boldsymbol{\mu}_4 \,\mathrm{Id}_V\,. \tag{1.1}$$

1.B. Invariants. Note also that G_{29} is stable under the complex conjugacy (because $\bar{s}_3 = s_1 s_3 s_1$). Since $\mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q})$ is generated by the complex conjugacy, a theorem of Marin–Michel [MaMi] implies that one can choose a family of fundamental invariants in $\mathbb{Q}[x,y,z,t]$. Moreover, the subgroup $G_{29} \cap \mathbf{GL}_4(\mathbb{Q})$ is the rational reflection group denoted by $G(2,1,4)$ in Shephard–Todd's classification (it is isomorphic to the Weyl group of type B_4). In particular, any G_{29}-invariant polynomial is a linear combination of elements of the form $\Sigma_4(m)$, where m is a monomial in x^2, y^2, z^2 and t^2. We set

$$\begin{aligned} f_1 &= \Sigma_4(x^4) - 6\,\Sigma_4(x^2y^2), \\ f_2 &= \Sigma_4(x^8) + 4\,\Sigma_4(x^6y^2) + 6\,\Sigma_4(x^4y^4) - 20\,\Sigma_4(x^4y^2z^2) + 152\,x^2y^2z^2t^2, \\ f_3 &= \Sigma_4(x^8y^2z^2) - \Sigma_4(x^6y^4z^2) + 2\,\Sigma_4(x^6y^2z^2t^2) \\ &\quad - 2\,\Sigma_4(x^4y^4z^4) + 2\,\Sigma_4(x^4y^4z^2t^2). \end{aligned}$$

Then f_1, f_2, f_3 are homogeneous G_{29}-invariant polynomials of respective degrees 4, 8 and 12 and there exists a homogeneous invariant f_4 of degree 20 such that

$$\mathbb{C}[V]^{G_{29}} = \mathbb{C}[f_1, f_2, f_3, f_4].$$

Note that f_3 is, up to a scalar, the unique fundamental invariant of degree 12 whose degree in x is $\leqslant 8$. Also, f_2 is, up to a scalar, the Hessian of f_1.

LEMMA 1.2. *Any fundamental invariant of degree* 12 *of* G_{29} *is irreducible.*

Proof. Let F be a fundamental invariant of degree 12 of G_{29} and let f be an irreducible divisor of F. Note that f is necessarily homogeneous. We set

$$G = \{g \in G_{29} \mid g(f) \in \mathbb{C}^\times f\}.$$

Write

$$f_\sharp = \prod_{g \in [G_{29}/G]} g(f),$$

where $[G_{29}/G]$ is a set of representatives of the cosets in G_{29}/G. Then $g(f_\sharp) \in \mathbb{C}^\times f_\sharp$ for all $g \in G_{29}$. Note that $f_\sharp$ divides F because F is G_{29}-invariant. Note also that f is G'-invariant (where G' denotes the derived subgroup of G) because the map $G \to \mathbb{C}^\times$, $g \mapsto g(f)/f$, is a linear character of G.

Now, let $\theta : G_{29} \to \mathbb{C}^\times$, $g \mapsto g(f_\sharp)/f_\sharp$: it is a linear character of G_{29}. There are only two linear characters of G_{29}, the trivial one and the restriction of the determinant. If θ is the restriction of the determinant, then it follows for instance from [LeTa, Theorem 9.19] that $f_\sharp$ has degree larger than the number of reflecting hyperplanes of G_{29}, which is equal to 40. This is impossible because $f_\sharp$ divides F.

This shows that $f_\sharp$ is G_{29}-invariant. Three cases may occur:

- If $\deg(f_\sharp) = 4$, then f_1 divides F, so F/f_1 is homogeneous of degree 8 and G_{29}-invariant, so it is of the form $\alpha f_2 + \beta f_1^2$ for some α, $\beta \in \mathbb{C}$. This is impossible since this would give an algebraic relation between F, f_1 and f_2.
- If $\deg(f_\sharp) = 8$, then $F/f_\sharp$ is a homogeneous G_{29}-invariant divisor of F of degree 4. We conclude that this is impossible as in the previous case.
- If $\deg(f_\sharp) = 12$, then

  $$F = \kappa \prod_{g \in [G_{29}/G]} g(f)$$

  for some $\kappa \in \mathbb{C}^\times$. Write $d = \deg(f)$ and $r = |G_{29}/G|$. Then $dr = 12$ and f is G'-invariant. But one can easily check with MAGMA that, for any subgroup G of G_{29} of index $r \in \{2, 3, 4, 6, 12\}$, the derived subgroup of G has no non-zero homogeneous invariant of degree $12/r$. This shows that $d = 12$ and $r = 1$, i.e. $F = \kappa f$, as expected.

The proof of the lemma is complete. ■

2. The family of invariant dodecics. If $f \in \mathbb{C}[V]$ is homogeneous, we denote by $\mathscr{Z}(f)$ the scheme $\mathrm{Proj}(\mathbb{C}[V]/\langle f \rangle)$. Any reduced, irreducible surface defined by a homogeneous invariant of degree 12 of G_{29} is of the form

$$\mathscr{S}_{12}^{\lambda,\mu} = \mathscr{Z}(f_3 + \lambda f_2 f_1 + \mu f_1^3) \quad \text{for some } (\lambda, \mu) \in \mathbf{A}^2(\mathbb{C}).$$

Lemma 1.2 says that the converse holds, that is,

$$\mathscr{S}_{12}^{\lambda,\mu} \textit{ is reduced and irreducible} \tag{2.1}$$

for any $(\lambda, \mu) \in \mathbf{A}^2(\mathbb{C})$.

2.A. Singular invariant dodecics. The subset $\mathscr{C}$ of $\mathbf{A}^2(\mathbb{C})$ formed by the elements (λ,μ) such that $\mathscr{S}_{12}^{\lambda,\mu}$ is singular is a closed subset of $\mathbf{A}^2(\mathbb{C})$. The computation of $\mathscr{C}$ was too long for our computer: adapting slightly the algorithm of [Bon, §3], we computed with Magma the subset $\mathscr{C}_0$ of $\mathscr{C}$ defined as the set of elements $(\lambda,\mu) \in \mathbf{A}^2(\mathbb{C})$ such that $\mathscr{S}_{12}^{\lambda,\mu}$ has a singular point with last coordinate 0. Then $\mathscr{C}_0$ is of pure dimension 1 and is the union of eight irreducible curves:

- six lines $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, $\mathscr{L}_4$, $\mathscr{L}_5^+$ and $\mathscr{L}_5^-$ defined by the equations

$$\begin{aligned}
&(\mathscr{L}_1) && \mu = 0,\\
&(\mathscr{L}_2) && \lambda = 0,\\
&(\mathscr{L}_3) && \lambda+\mu = 0,\\
&(\mathscr{L}_4) && \lambda - 15\,\mu - \frac{1}{45} = 0,\\
&(\mathscr{L}_5^+) && \lambda - (4+2i)\,\mu - \frac{3+i}{320} = 0,\\
&(\mathscr{L}_5^-) && \lambda - (4-2i)\,\mu - \frac{3-i}{320} = 0,
\end{aligned}$$

- a cubic curve $\mathscr{A}$ defined by the equation

$$20\,480\,\lambda^3 - 256\,\lambda^2 + \lambda + \mu = 0,$$

- a sextic curve $\mathscr{B}$ defined by

$$\begin{aligned}
1\,342\,177\,280\,\lambda^6 &- 100\,663\,296\,\lambda^5 + 3\,014\,656\,\lambda^4 - 3\,538\,944\,\lambda^3\mu\\
&- 45\,056\,\lambda^3 + 73\,728\,\lambda^2\mu + 336\,\lambda^2 - 288\,\lambda\mu - \lambda - 432\,\mu^2 - \mu = 0.
\end{aligned}$$

We do not know whether $\mathscr{C}_0$ and $\mathscr{C}$ are equal.

2.B. 320 cusps. The first seven irreducible components are isomorphic to $\mathbf{A}^1(\mathbb{C})$, and $\mathscr{B}$ has two singular points $(\lambda^\pm,\mu^\pm)$, where

$$\lambda^\pm = \frac{3\pm\sqrt{3}}{384} \quad\text{and}\quad \mu^\pm = \frac{-5\pm 3\sqrt{3}}{6\,912}.$$

We set

$$\mathscr{S}_{12}^{\pm} = \mathscr{Z}(f_3 + \lambda^\pm f_2 f_1 + \mu^\pm f_1^3).$$

The main result of this paper is the following; the proof is obtained through a straightforward Magma [Mag] computation, which takes only a few minutes on a standard laptop.

Theorem 2.2. *The surfaces $\mathscr{S}_{12}^{+}$ and $\mathscr{S}_{12}^{-}$ are reduced, irreducible, of degree 12 and admit 320 cusps and no other singular point. These cusps form a single G_{29}-orbit.*

Unfortunately, even though the surfaces $\mathscr{S}_{12}^{\pm}$ are defined over $\mathbb{Q}(\sqrt{3})$, none of their singular points is real, so drawing it with SURFER [Sur] does not lead to a beautiful picture. We do not know whether this surface can be defined over $\mathbb{Q}$.

2.C. Some other singular dodecics. The following can also be checked by a computer calculation with MAGMA:

(1) For generic (λ, μ) in $\mathscr{L}_1$, the surface $\mathscr{S}_{12}^{\lambda,\mu}$ has 20 singularities of type $T_{4,4,4}$ (1).
(2) For generic (λ, μ) in $\mathscr{L}_2$, the surface $\mathscr{S}_{12}^{\lambda,\mu}$ has 120 singularities of type A_1.
(3) For generic (λ, μ) in $\mathscr{L}_3$, the surface $\mathscr{S}_{12}^{\lambda,\mu}$ has 40 singularities of type A_1.
(4) For generic (λ, μ) in $\mathscr{L}_4$, the surface $\mathscr{S}_{12}^{\lambda,\mu}$ has 160 singularities of type A_1.
(5) For generic (λ, μ) in $\mathscr{L}_5^{\pm}$, the surface $\mathscr{S}_{12}^{\lambda,\mu}$ has 80 singularities of type A_1.
(6) For generic (λ, μ) in $\mathscr{A}$, the surface $\mathscr{S}_{12}^{\lambda,\mu}$ has 480 singularities of type A_1.
(7) For generic (λ, μ) in $\mathscr{B}$, the surface $\mathscr{S}_{12}^{\lambda,\mu}$ has 320 singularities of type A_1.

Theorem 2.2 shows for instance that, for (λ, μ) running in $\mathscr{B}$, the 320 singularities of type A_1 degenerate to A_2-singularities when (λ, μ) reaches a singular point of $\mathscr{B}$. Note also that the two singular points $(\lambda^{\pm}, \mu^{\pm})$ of $\mathscr{B}$ do not belong to any other irreducible component of $\mathscr{C}_0$.

Since $\mathscr{B}$ is the unique singular irreducible component of $\mathscr{C}_0$, the singular locus of $\mathscr{C}_0$ consists of $(\lambda^{\pm}, \mu^{\pm})$ and the points lying on at least two irreducible components of $\mathscr{C}_0$. There are 39 such intersection points. We have computed with MAGMA the singularities of the 39 associated dodecic surfaces and we have checked the following facts:

(a) If (λ_0, μ_0) belongs to only two irreducible components $\mathscr{I}$ and $\mathscr{J}$ of $\mathscr{C}_0$ and if $\mathscr{I}$ and $\mathscr{J}$ intersect transversely at (λ_0, μ_0), then the dodecic surface $\mathscr{S}_{12}^{\lambda_0,\mu_0}$ cumulates the singularities "coming from $\mathscr{I}$" and those "coming from $\mathscr{J}$". There are 33 singular points of $\mathscr{C}_0$ satisfying this property. For instance:
    - The point $p_{1,4} = (1/45, 0)$ belongs only to $\mathscr{L}_1$ and $\mathscr{L}_4$, so $\mathscr{S}_{12}^{p_{1,4}}$ has 20 singularities of type $T_{4,4,4}$ and 160 singularities of type A_1. These are its only singular points; they form two G_{29}-orbits.
    - The point $p_5^{\pm} = (-(-1 \pm i)/320, -(3 \pm i)/1600)$ belongs only to $\mathscr{L}_5^{\pm}$ and $\mathscr{A}$, and $\mathscr{L}_5^{\pm}$ and $\mathscr{A}$ intersect transversely at this point, so $\mathscr{S}_{12}^{p_5^{\pm}}$ has 560 singularities of type A_1, and no other singular point; they form two G_{29}-orbits, of respective cardinality 80 and 480.

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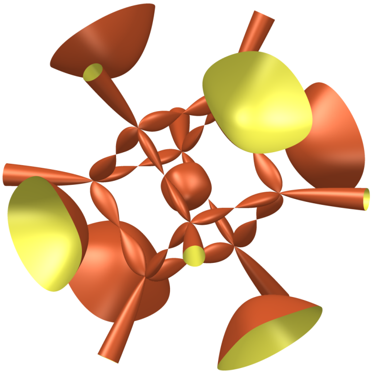
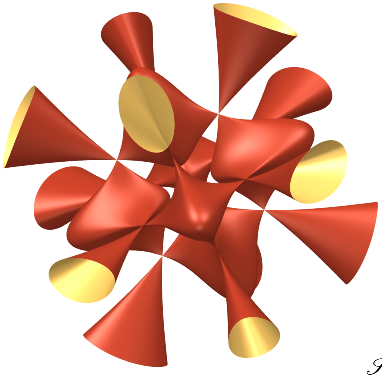
(1) A singularity is said to be of type $T_{4,4,4}$ if it is equivalent to the singularity at the point 0 of the affine surface $\{(a, b, c) \in \mathbf{A}^3(\mathbb{C}) \mid abc + a^4 + b^4 + c^4 = 0\}$. Such a singularity has multiplicity 3, Milnor number 11 and Tjurina number 10.



$\mathscr{S}_{12}^{p_{1,4}}$



$\mathscr{S}_{12}^{\diamondsuit}$

$\mathscr{S}_{12}^{\mathbf{0}}$

(b) The point $\clubsuit = (1/40, 1/5400)$ belongs only to $\mathscr{L}_4$ and $\mathscr{B}$, and $\mathscr{L}_4$ and $\mathscr{B}$ intersect in $\clubsuit$ with multiplicity 2. The surface $\mathscr{S}_{12}^{\clubsuit}$ has 160 singularities of type A_3 and no other singular point; they form a single G_{29}-orbit.

(c) The point $\diamondsuit = (1/240, -13/10\,800)$ belongs only to $\mathscr{L}_4$ and $\mathscr{A}$, and $\mathscr{L}_4$ and $\mathscr{A}$ intersect in $\diamondsuit$ with multiplicity 3. The surface $\mathscr{S}_{12}^{\diamondsuit}$ has 160 singularities of type D_4 and no other singular point; they form a single G_{29}-orbit. This is the best-known lower bound for $\mu_{D_4}(12)$; this surface was already discovered by the author [Bon, Example 5.5(3)].

(d) The point $\heartsuit = (1/64, 0)$ belongs only to $\mathscr{L}_4$ and $\mathscr{B}$, and $\mathscr{L}_4$ and $\mathscr{B}$ intersect in $\heartsuit$ with multiplicity 4. The surface $\mathscr{S}_{12}^{\heartsuit}$ has 20 singular points, which form a single G_{29}-orbit; they have multiplicity 4, Milnor number 27 and Tjurina number 26 and their projective tangent cone is smooth. Using the software SINGULAR, we checked that this singularity is not in Arnold's list; let us denote this type of singularity by $X_{\heartsuit}$.

(e) The point $\spadesuit^{\pm} = ((3 \pm i)/640, (-7 \pm i)/6\,400)$ belongs to $\mathscr{L}_5^{\pm}$, $\mathscr{A}$ and $\mathscr{B}$ and to no other irreducible component of $\mathscr{C}_0$. The surface $\mathscr{S}_{12}^{\spadesuit^{\pm}}$ has 80 singularities of type $T_{4,4,4}$ and no other singular points. They form a single G_{29}-orbit.

(f) The point $\mathbf{0} = (0,0)$ belongs to $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, $\mathscr{A}$ and $\mathscr{B}$ and does not belong to $\mathscr{L}_4$ or $\mathscr{L}_5^{\pm}$. The singular locus of the surface $\mathscr{S}_{12}^{\mathbf{0}}$ is the union of 30 lines (in particular, it is of dimension 1); these lines form a single G_{29}-orbit, namely the orbit of the line defined by $z = t = 0$. This surface was also mentioned in [Bon, Example 5.5(4)].

Surface	Singularities
$\mathscr{S}_{12}^{\pm}$	$320\ A_2$
$\mathscr{S}_{12}^{\clubsuit}$	$160\ A_3$
$\mathscr{S}_{12}^{\diamondsuit}$	$160\ D_4$
$\mathscr{S}_{12}^{\heartsuit}$	$20\ X_{\heartsuit}$
$\mathscr{S}_{12}^{\spadesuit\pm}$	$80\ T_{4,4,4}$

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Cédric Bonnafé
IMAG, Université de Montpellier, CNRS
Montpellier, France
E-mail: cedric.bonnafe@umontpellier.fr