

Why does absolute geometry with the elementary continuity axiom have only two models?

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Summary. Wanda Szmielew showed in 1959 that $\overline{\mathcal{A}}$, plane absolute geometry with the elementary continuity axiom schema, has precisely two models: Euclidean planes and hyperbolic planes over real-closed fields. In this note we determine the reason why this surprising result holds. We find that two axioms are responsible for it, the circle axiom and Aristotle's axiom; both hold in $\overline{\mathcal{A}}$.

1. Introduction. In [6, Theorem 3.1], Wanda Szmielew has pointed out that there are only two types of models for plane absolute geometry with the elementary continuity axiom, Euclidean planes and Klein planes over real closed fields. This is surprising, for it states that any sentence true in plane Euclidean geometry that does not hold in absolute geometry can take the place of the Euclidean parallel postulate, and similarly for any statement in plane hyperbolic geometry that does not hold in absolute geometry. In the presence of the elementary continuity axiom, the wide variety of weakenings of the Euclidean parallel postulate and of the hyperbolic parallel postulate are thus no longer weaker than the Euclidean and hyperbolic parallel postulates, but rather equivalent to one of them. The deeper reasons why that is so is somewhat of a mystery and the aim of this paper is to provide an explanation.

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2. The axiom system for plane absolute geometry with the elementary continuity axiom. For the reader's convenience, we will first present the axiom system for absolute geometry with the elementary continuity schema. Their meaning is explained at length in [5] and will be omitted here. The axiom system was provided by Alfred Tarski [7] and further simplified in [5]. One of Tarski's axioms was shown by Makarios [2] to be superfluous, if a small change is made to another one of Tarski's axioms. Its logic is first-order logic. Its language, to be denoted by $L_{B\equiv}$, has only one sort of individual variables, to be referred to as *points*, as well as two relation symbols, a ternary one B , for *betweenness*, with $B(abc)$ to be read as "point b lies between a and c " (and b may be a or c , and $a = b = c$ is also allowed), a quaternary one \equiv for *equidistance*, with $ab \equiv cd$ to be read " b is as distant from a as d is from c " (or " ab is congruent to cd ").

A1. $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,

A2. $ab \equiv cc \rightarrow a = b$,

A3. $(\forall qabc)(\exists x) B(qax) \wedge ax \equiv bc$,

A4. $B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd' \rightarrow (dc \equiv c'd \vee a = b)$,

A5. $B(aba) \rightarrow a = b$,

A6. $(\forall abcde)(\exists f) B(cea) \wedge B(dba) \rightarrow B(efd) \wedge B(bfc)$,

A7. $(\exists abc) \neg (B(abc) \vee B(bca) \vee B(cab))$,

A8. $u \neq v \wedge au \equiv av \wedge bu \equiv bv \wedge cu \equiv cv \rightarrow (B(abc) \vee B(bca) \vee B(cab))$,

A9. $(\forall vw\dots)[(\exists z)(\forall xy) \varphi \wedge \psi \rightarrow B(zxy)] \rightarrow (\exists u)(\forall xy) (\varphi \wedge \psi \rightarrow B(xuy))$,

where φ stands for any formula in which the variables x, v, w, \dots occur free, but neither y nor z nor u occur free, and similarly for ψ , with x and y interchanged.

Let $\overline{\mathcal{A}}$ denote the theory axiomatized by **A1–A9**, \mathcal{A} the theory axiomatized by **A1–A8**, and let **E** and **H** stand for the Euclidean parallel postulate and the elementary version of the hyperbolic parallel postulate put forward by Hilbert:

E. There is at most one line parallel to a given line passing through a given point.

H. From any point P not lying on a line l there are two rays r_1 and r_2 through P , not belonging to the same line, which do not intersect l , and such that every ray through P , contained in the angle formed by r_1 and r_2 , intersects l .

Then Szmielew's theorem [6, Theorem 3.1] can be rephrased as a syntactic result in the following terms:

$$(1) \quad \overline{\mathcal{A}} \vdash E \vee H,$$

3. The two axioms that are responsible for Szmielew's Theorem.

By the Compactness Theorem for first-order logic, there must be a finite subset Σ of axioms in $\overline{\mathcal{A}}$ such that

$$(2) \quad \Sigma \vdash E \vee H.$$

Thus not all the axioms that are part of the axiom schema **A9** are responsible for the validity of $E \vee H$. There are two statements, which have been historically used as axioms, Aristotle's axiom **Ar** and the circle axiom **C**, and for which, when Σ denotes $\{\mathbf{A1-A8}, \mathbf{Ar}, \mathbf{C}\}$, we have (2). This was proved in [1, Theorem 3].

The statement of Aristotle's axiom will be informal, in terms of notions familiar from traditional geometry, such as "acute angle", "ray", "perpendicular", which, while not being part of Tarski's language, are easily seen to be definable inside that language, as done in [5]. It states:

Ar. If \widehat{xoy} is an acute angle and ab is any segment, then there exists a point p on the ray \overrightarrow{oy} and a point q on the ray \overrightarrow{ox} such that pq is perpendicular to ox and $pq > ab$.

The circle axiom **C**, in the version mentioned in [5, p. 15], states that the segment joining a point a lying inside and a point b lying outside a circle, intersects that circle.

That **C** holds in $\overline{\mathcal{A}}$ is well-known, given that the coordinatizing fields of models of $\overline{\mathcal{A}}$ are real-closed fields and these are Euclidean fields, i.e., every positive element has a square root. Models of $\mathcal{A} + \mathbf{C}$ are known to be precisely those for which the coordinate field is Euclidean.

What we thus need to show is that **Ar** holds in $\overline{\mathcal{A}}$. To do so, we will first show that a certain elementary axiom schema inspired by the Archimedean axiom holds in $\overline{\mathcal{A}}$. Since we have shown in [3] that **Ar** follows from that axiom schema, it turns out that both **C** and **Ar** hold in $\overline{\mathcal{A}}$.

4. The validity of Rautenberg's elementary Archimedean axiom schema in $\overline{\mathcal{A}}$. We know that Archimedeanity is not a first-order property of fields. However, W. Rautenberg [4] introduced a certain elementary Archimedean axiom schema Σ_{ar} , that captures a certain aspect of Archimedeanity that can be expressed in first-order logic. It consists, for every formula $\varphi(\cdot)$, containing none of the variables p, q, r, s, t, u, v , of the following statement:

$$\Sigma_{\text{ar}}. (\forall pqrst)(\exists uv) p \neq q \wedge \varphi(p) \wedge \varphi(q) \wedge [pq \equiv rs \rightarrow (\varphi(r) \leftrightarrow \varphi(s))] \\ \rightarrow [L(pqt) \rightarrow \varphi(u) \wedge \varphi(v) \wedge B(utv)].$$

The axiom schema states that, if \mathfrak{A} , a subset of the universe (point set) of a model \mathfrak{M} of \mathcal{A} , is a definable set of points (defined by φ , which means that $\mathbf{x} \in \mathfrak{A}$ if and only if $\varphi(\mathbf{x})$ holds in \mathfrak{M}), containing the distinct points \mathbf{p}

and \mathbf{q} , and such that, whenever $\mathbf{pq} \equiv \mathbf{rs}$, \mathbf{r} is in \mathfrak{A} if and only if \mathbf{s} is in \mathfrak{A} , then every point \mathbf{t} on the line \mathbf{pq} is between two points of \mathfrak{A} .

THEOREM 4.1. Σ_{ar} holds in $\overline{\mathfrak{A}}$.

Proof. Let \mathfrak{M} be a model of $\overline{\mathfrak{A}}$ and let \mathbf{p} and \mathbf{q} be two distinct points in the universe of \mathfrak{M} . Suppose Σ_{ar} does not hold in \mathfrak{M} for \mathbf{p} and \mathbf{q} , some formula φ , and a point \mathbf{t} on the line \mathbf{pq} . This means that

(*) for all \mathbf{m} and \mathbf{n} with $\varphi(\mathbf{m})$ and $\varphi(\mathbf{n})$, we have $B(\mathbf{mnt})$ or $B(\mathbf{nmt})$.

Let ψ_t be the formula defined by

$$(3) \quad \psi_t(y) := (\forall x) \varphi(x) \rightarrow B(xyt).$$

Applying **A9** with φ and ψ interchanged and with \mathbf{t} acting as z , we conclude that there exists \mathbf{u} such that, for all \mathbf{x} and \mathbf{y} , if $\varphi(\mathbf{x})$ and $\psi_t(\mathbf{y})$ we have $B(\mathbf{xuy})$. Since $\psi_t(\mathbf{t})$, for all \mathbf{x} with $\varphi(\mathbf{x})$, we have $B(\mathbf{xut})$, so, by (3), we have $\psi_t(\mathbf{u})$.

Notice that $\mathbf{u} \neq \mathbf{p}$, for, if $\mathbf{u} = \mathbf{p}$, then, with \mathbf{q}' denoting the reflection of \mathbf{q} in \mathbf{p} , we would have $B(\mathbf{qpt})$ and $B(\mathbf{q'pt})$, so, given that $\mathbf{p} \neq \mathbf{t}$, $B(\mathbf{pqq'}) \vee B(\mathbf{pq'q})$, contradicting $B(\mathbf{qpq'})$.

By **A3**, there is \mathbf{v} on the ray $\overrightarrow{\mathbf{up}}$ such that $\mathbf{uv} \equiv \mathbf{pq}$. We will show that we cannot have $B(\mathbf{upv})$, for that would imply $B(\mathbf{q'up})$. Since $\varphi(\mathbf{p})$ and $\varphi(\mathbf{q}')$, we also have $B(\mathbf{put})$ and $B(\mathbf{q'ut})$, which, together with $B(\mathbf{q'up})$, would imply $\mathbf{u} = \mathbf{t}$, which is not possible, as it would entail $B(\mathbf{q'tp})$, contradicting (*). Thus $B(\mathbf{uvp})$ holds.

We now want to prove $\psi_t(\mathbf{v})$. Since $B(\mathbf{put})$ (as $\varphi(\mathbf{p})$ and $\psi_t(\mathbf{u})$) and $B(\mathbf{uvp})$, we have $B(\mathbf{pvt})$. Let \mathbf{x} be such that $\varphi(\mathbf{x})$. We want to show that neither (i) $B(\mathbf{xtv})$ nor (ii) $B(\mathbf{vxt})$ is possible.

First, notice that $\mathbf{t} \neq \mathbf{v}$, for if $\mathbf{t} = \mathbf{v}$, then $B(\mathbf{utp})$, which cannot hold, for we have $B(\mathbf{put})$. Now, suppose (i) were to hold. By the properties of betweenness, $B(\mathbf{xtv}) \wedge B(\mathbf{pvt}) \wedge \mathbf{v} \neq \mathbf{t} \rightarrow B(\mathbf{ptx})$, contradicting (*), so (i) cannot hold.

Suppose (ii) holds. Then, by the properties of betweenness, $B(\mathbf{pvt}) \wedge B(\mathbf{vxt}) \rightarrow B(\mathbf{pvx})$, and also $B(\mathbf{pvx})$, $B(\mathbf{xut})$, $B(\mathbf{pvu})$, and $B(\mathbf{put})$ imply $B(\mathbf{vxu})$. By **A3**, there is \mathbf{y} such that $B(\mathbf{pxy})$ and $\mathbf{xy} \equiv \mathbf{vu}$. Given that $B(\mathbf{pvu})$, $B(\mathbf{vxu})$, $B(\mathbf{pxy})$, and $\mathbf{xy} \equiv \mathbf{vu}$, we have $B(\mathbf{puy})$. However, since $\varphi(\mathbf{p})$ and $\varphi(\mathbf{y})$, we also have $B(\mathbf{put})$ and $B(\mathbf{yut})$, contradicting $B(\mathbf{puy})$. Thus (ii) cannot hold.

This means that, for all \mathbf{x} with $\varphi(\mathbf{x})$, we have $B(\mathbf{tvx})$, so, by (3), we have $\psi_t(\mathbf{v})$. By the definition of \mathbf{u} we should have $B(\mathbf{xuv})$ for all \mathbf{x} with $\varphi(\mathbf{x})$. Since \mathbf{p} is such that $\varphi(\mathbf{p})$, we should thus have $B(\mathbf{puv})$, which contradicts our construction of \mathbf{v} as a point for which $B(\mathbf{pvu})$. ■

Now, as shown in [3], $\mathcal{A} + \Sigma_{\text{ar}} \vdash \mathbf{Ar}$, so Theorem 4.1 implies the following.

COROLLARY. Both **C** and **Ar** hold in $\overline{\mathcal{A}}$.

Since both **C** and **Ar** hold in $\overline{\mathcal{A}}$, the validity of (2) with $\Sigma = \{\mathbf{A1}\text{--}\mathbf{A8}, \mathbf{Ar}, \mathbf{C}\}$ (established in [1, Theorem 3]) proves that (1) holds as well.

References

- [1] M. J. Greenberg, *Aristotle's axiom in the foundations of geometry*. J. Geom. 33 (1988), 53–57.
- [2] T. J. M. Makarios, *A further simplification of Tarski's axioms of geometry*, Note Mat. 33 (2013), 123–132.
- [3] V. Pambuccian, *The elementary Archimedean axiom in absolute geometry*, J. Geom. 110 (2019), art. 52, 9 pp.
- [4] W. Rautenberg, *The elementary Archimedean schema in geometry and some of its applications*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 837–843.
- [5] W. Schwabhäuser, W. Szmielew und A. Tarski, *Metamathematische Methoden in der Geometrie*, Springer, Berlin, 1983.
- [6] W. Szmielew, *Some metamathematical problems concerning elementary hyperbolic geometry*, in: The Axiomatic Method: With Special Reference to Geometry and Physics (Berkeley, CA, 1957–1958), North-Holland, Amsterdam, 1959, 30–52.
- [7] A. Tarski, *What is elementary geometry?* in: The Axiomatic Method: With Special Reference to Geometry and Physics (Berkeley, CA, 1957–1958), North-Holland, Amsterdam, 1959, 16–29.

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