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The structure of sequences with zero-sum subsequences of the same length on finite abelian groups of rank two

by

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Abstract. Let G be an additive finite abelian group, and let $\mathrm{disc}(G)$ denote the smallest positive integer t with the property that every sequence S over G with length $|S| \geq t$ contains two nonempty zero-sum subsequences of distinct lengths. In recent years, Gao et al. established the exact value of $\mathrm{disc}(G)$ for all finite abelian groups of rank 2 and resolved the corresponding inverse problem for the group $C_n \oplus C_n$. In this paper, we characterize the structure of sequences S over $G = C_n \oplus C_{nm}$ (where $m \geq 2$) when $|S| = \mathrm{disc}(G) - 1$ and all nonempty zero-sum subsequences of S have the same length.

1. Introduction. Throughout this paper, let G be an additive finite abelian group. We denote by C_n the cyclic group of n elements, and denote by C_n^r the direct sum of r copies of C_n.

Let p be a prime number. An old conjecture posed by Graham states that if S is a sequence of length $|S| = p$ over C_p such that all nonempty zero-sum subsequences of S have the same length, then S takes at most two distinct terms. In 1976, P. Erdős and E. Szemerédi [ES76] showed that Graham's conjecture holds for sufficiently large p. In 2010, W. Gao, Y. Hamidoune and G. Wang [GHW10] proved Graham's conjecture in full generality. Furthermore, they extended this result to all positive integers. Subsequently, D. Grynkiewicz [G11] provided an alternative proof. In 2012, B. Girard [G12] posed the problem of determining the smallest integer t, denoted by $\mathrm{disc}(G)$, such that every sequence S over G of length $|S| \geq t$ has two nonempty zero-sum subsequences of distinct lengths. Since then, $\mathrm{disc}(G)$ has been systematically studied by numerous authors, and its exact value has been determined for several classes of finite abelian groups, including the groups of rank at most 2, the groups of very large exponent compared to $|G|/\exp(G)$, the elementary 2-groups, additional special abelian p-groups and certain groups of rank 3 (see [GH$^+$20, GL$^+$16, GZZ15, LY24]).

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On the other hand, Gao et al. [GL$^+$16] considered the inverse problem associated with $\mathrm{disc}(G)$. In particular, they investigated the set, denoted by $\mathcal{L}_1(G)$, of all positive integers t, such that there is a sequence S over G with length $\mathrm{disc}(G)-1$ and all nonempty zero-sum subsequences of S have the same length t. They conjectured that $|\mathcal{L}_1(G)|=1$ for any finite abelian group. In 2020, Gao et al. [GH$^+$20] proved that $\mathcal{L}_1(G)=\{\exp(G)\}$ for the abelian groups of rank at most 2, for $C_{mp^n}\oplus H$ with m being a positive integer and H being a p-group with $\mathsf{D}(H)\le p^n$, and for the finite abelian groups of very large exponent compared to $|G|/\exp(G)$. Moreover, they disproved the above conjecture by demonstrating that $|\mathcal{L}_1(G)|\ge 2$ for certain abelian p-groups. To gain a deeper understanding of the sequence structures on these groups, they have characterized the structure of sequences of length $\mathrm{disc}(G)-1$ where all nonempty zero-sum subsequences have the same length on the cyclic group C_n and the group $C_n\oplus C_n$. Recently, X. Li and Q. Yin [LY24] have successfully extended the scope of application of the conjecture to some groups of rank 3, including $C_2\oplus C_{2m}\oplus C_{2mn}$ and $C_3\oplus C_{6m}\oplus C_{6m}$, where m and n are positive integers with $m\,|\,n$. Currently, knowledge on such inverse zero-sum problems remains insufficient, and existing results are mostly confined to specific group structures. To overcome this constraint, it is necessary to employ novel methodologies to characterize the structure of extremal sequences over more finite abelian groups in which all nonempty zero-sum subsequences have the same length.

In this paper, we consider more examples of general finite abelian groups G of rank 2, and we mainly characterize the structure of the sequence S when $|S|=\mathrm{disc}(G)-1$ and all nonempty zero-sum subsequences of S have the same length.

Our main result is as follows.

Theorem 1.1. *Let $G=C_n\oplus C_{nm}$ where $n,m\ge 2$ are integers. Let S be a sequence over G with length* $\mathrm{disc}(G)-1$. *Then all nonempty zero-sum subsequences of S have the same length if and only if there exists a generating set $\{g_1,g_2\}$ of G with* $\mathrm{ord}(g_2)=nm$ *such that S has one of the following forms:*

(1) $S=g_2^{2nm-1}\prod_{i=1}^{n-1}(x_ig_2+g_1)$, *where* $\mathrm{ord}(g_1)=n$ *and* $x_1,\dots,x_{n-1}\in[0,nm-1]$.
(2) $S=g_1^{n-2}g_2^{2nm-1}(-(n-1)g_1+g_2)$.
(3) $S=g_1^{2nm-1}\prod_{i=1}^{n-1}(-y_ig_1+g_2)$, *where* $\mathrm{ord}(g_1)=nm$ *and* $\sum_{i=1}^{n-1}y_i\in[0,n-1]$.
(4) $S=g_1^{rn-1}g_2^{2nm+n-1-rn}$, *where* $\mathrm{ord}(g_1)=nm$ *and* $r\in[1,2m]$.

The rest of the paper is organized as follows. Section 2 provides some basic notation and preliminaries. Section 3 gives the proof of our main result.

2. Preliminaries. Throughout this paper, our notation and terminology are consistent with [GG06, GH06]; we now briefly present some key concepts. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} : a \le x \le b\}$.

Let G be an abelian group. A family $(e_i)_{i \in I}$ of nonzero elements of G is said to be *independent* if

$$\sum_{i \in I} m_i e_i = 0 \text{ implies } m_i e_i = 0 \text{ for all } i \in I, \text{ where } m_i \in \mathbb{Z}.$$

If $I = [1, r]$ and $(e_1, \ldots, e_r)$ is independent, then we simply say that $e_1, \ldots, e_r$ are independent elements of G. The tuple $(e_i)_{i \in I}$ is called a *basis* if $(e_i)_{i \in I}$ is independent and $\langle \{e_i : i \in I\} \rangle = G$. If $1 < |G| < \infty$, then we have

$$G \cong C_{n_1} \oplus \cdots \oplus C_{n_r},$$

where C_n denotes a cyclic group with n elements, $i \in \mathbb{N}$ and $1 < n_1 \mid \cdots \mid n_r$. Then $r = \mathrm{r}(G)$ is the *rank* of G and $n_r = \exp(G)$ is the *exponent* of G.

We denote by $\mathcal{F}(G)$ the free (abelian, multiplicative) monoid with basis G. An element $S \in \mathcal{F}(G)$ is called a *sequence* over G and will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathrm{v}_g(S)},$$

where $\mathrm{v}_g(S) \ge 0$ is called the *multiplicity* of g in S, and we call

- $\mathrm{supp}(S) = \{g \in G : \mathrm{v}_g(S) > 0\}$ the *support* of S,
- $|S| = l = \sum_{g \in G} \mathrm{v}_g(S) \in \mathbb{N}_0$ the *length* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathrm{v}_g(S) g \in G$ the *sum* of S,
- $\Sigma_k(S) = \{\sum_{i \in I} g_i : I \subset [1, l] \text{ with } |I| = k\}$ the *set of k-term subsums* of S, for all $k \in \mathbb{N}$,
- $\Sigma_{\ge k}(S) = \bigcup_{j \ge k} \Sigma_j(S)$,
- $\Sigma(S) = \Sigma_{\ge 1}(S)$ the *set of all subsums* of S,
- $T = \prod_{g \in G} g^{\mathrm{v}_g(T)}$ a *subsequence* of S if $\mathrm{v}_g(T) \le \mathrm{v}_g(S)$ for all $g \in G$,
- T a *proper subsequence* of S if T is a subsequence of S and $1 \le |T| < |S|$,
- $ST^{-1} = \prod_{g \in G} g^{\mathrm{v}_g(S) - \mathrm{v}_g(T)}$ the subsequence obtained from S by deleting T,
- S a *zero-sum sequence* if $\sigma(S) = 0$,
- S a *zero-sum free sequence* if there is no nonempty zero-sum subsequence of S,
- S a *minimal zero-sum sequence* if it is zero-sum and has no proper zero-sum subsequence.

For a finite abelian group G, let $\mathsf{D}(G)$ denote the Davenport constant of G, which is defined as the smallest positive integer d such that every sequence over G of length at least d has a nonempty zero-sum subsequence.

We next give several lemmas which will be used in what follows.

Lemma 2.1 ([GH06, Theorem 5.8.3]). *Let $G = C_n \oplus C_{nm}$ where n, m are integers. Then* $\mathsf{D}(G) = n + nm - 1$.

Lemma 2.2 ([GZZ15, Theorem 1.2]). *Let G be a finite abelian group with* $\mathrm{r}(G) \le 2$. *Then* $\mathrm{disc}(G) = \mathsf{D}(G) + \exp(G)$.

Lemma 2.3 ([GH$^+$20, Theorem 1.4]). *Let G be a finite abelian group with* $\mathrm{r}(G) \le 2$. *Then* $\mathcal{L}_1(G) = \{\exp(G)\}$.

Lemma 2.4 ([GH06, Proposition 5.1.4]). *Let G be a finite abelian group and let S be a zero-sum free sequence over G with $|S| = \mathsf{D}(G) - 1$. Then* $|\Sigma(S)| = |G| - 1$.

Lemma 2.5 ([S10, Theorem 3.2]). *Let $G = C_{n_1} \oplus C_{n_2}$ be a finite abelian group with $1 < n_1 \mid n_2$ and S be a sequence over G. Then S is a minimal zero-sum sequence of length $|S| = n_1 + n_2 - 1$ if and only if S has form either*

$$S = e_j^{\mathrm{ord}(e_j)-1} \prod_{\nu=1}^{\mathrm{ord}(e_k)} (x_\nu e_j + e_k),$$

where $\{e_1, e_2\}$ is a basis of G with $\mathrm{ord}(e_i) = n_i$ for $i \in [1,2]$, $\{j,k\} = \{1,2\}$, $x_1, \ldots, x_{\mathrm{ord}(e_k)} \in [0, \mathrm{ord}(e_j) - 1]$ and $x_1 + \cdots + x_{\mathrm{ord}(e_k)} \equiv 1 \pmod{\mathrm{ord}(e_j)}$, or

$$S = g_1^{sn_1-1} \prod_{\nu=1}^{n_2+(1-s)n_1} (-x_\nu g_1 + g_2),$$

where $\{g_1, g_2\}$ is a generating set of G with $\mathrm{ord}(g_2) = n_2$, $x_1, \ldots, x_{n_2+(1-s)n_1} \in [0, n_1 - 1]$ and $x_1 + \cdots + x_{n_2+(1-s)n_1} = n_1 - 1$, $s \in [1, n_2/n_1]$ and either $s = 1$ or $n_1 g_1 = n_1 g_2$.

3. Proof of Theorem 1.1. In this section, we present the proof of our main result. To begin, we establish a crucial lemma.

Lemma 3.1. *Let S be a sequence over a finite abelian group G of length $|S| = \mathrm{disc}(G) - 1$, where all nonempty zero-sum subsequences of S have the same length. Suppose T is a nonempty zero-sum subsequence of S. Then*

$$\mathrm{supp}(T) \cap \Sigma_{\ge 2}(ST^{-1}) = \emptyset.$$

Proof. Assume to the contrary that there exists a subsequence $T' \mid ST^{-1}$ with $|T'| \ge 2$ such that $\sigma(T') = g$, where $g \mid T$. Then $T'Tg^{-1}$ is a zero-sum subsequence of S of length $|T'Tg^{-1}| > |T|$, which is a contradiction. Hence, $\mathrm{supp}(T) \cap \Sigma_{\ge 2}(ST^{-1}) = \emptyset$. ∎

Proof of Theorem 1.1. Sufficiency is straightforward. We prove necessity.

By Lemmas 2.1 and 2.2, we have $|S| = \mathrm{disc}(G) - 1 = \mathsf{D}(G) + \exp(G) - 1 = n + 2nm - 2$. And it follows from Lemma 2.3 that all nonempty zero-sum subsequences of S have the same length nm.

Since $|S| = n + 2nm - 2 > \mathsf{D}(G) = n + nm - 1$, there exists a zero-sum subsequence T of S of length nm and $0 \nmid S$. Then $|ST^{-1}| = n + nm - 2 = \mathsf{D}(G) - 1$ and ST^{-1} is zero-sum free. It follows from Lemma 2.4 that

$$\Sigma(ST^{-1}) = G \setminus \{0\}.$$

It is easy to see that $ST^{-1}(-\sigma(ST^{-1}))$ is a minimal zero-sum sequence of length $nm + n - 1 = \mathsf{D}(G)$. And by Lemma 2.5, $ST^{-1}(-\sigma(ST^{-1}))$ has the form either

$$ST^{-1}(-\sigma(ST^{-1})) = e_u^{\mathrm{ord}(e_u)-1} \prod_{i=1}^{\mathrm{ord}(e_v)} (x_i e_u + e_v), \tag{3.1}$$

where $\{e_1, e_2\}$ is a basis of G with $\mathrm{ord}(e_1)=n$ and $\mathrm{ord}(e_2)=nm$, $\{u,v\}=\{1,2\}$, $x_1, \dots, x_{\mathrm{ord}(e_v)} \in [0, \mathrm{ord}(e_u)-1]$ and $x_1 + \dots + x_{\mathrm{ord}(e_v)} \equiv 1 \pmod{\mathrm{ord}(e_u)}$, or

$$ST^{-1}(-\sigma(ST^{-1})) = g_1^{sn-1} \prod_{i=1}^{nm+(1-s)n} (-y_i g_1 + g_2), \tag{3.2}$$

where $\{g_1, g_2\}$ is a generating set of G with $\mathrm{ord}(g_2) = nm$, $y_1, \dots, y_{nm+(1-s)n} \in [0, n-1]$ and $y_1 + \dots + y_{nm+(1-s)n} = n - 1$, $s \in [1, m]$ and either $s = 1$ or $ng_1 = ng_2$.

We now divide the remaining proof into the following four cases.

Case 1: *$ST^{-1}(-\sigma(ST^{-1}))$ is of the form* (3.1) *with $u = 1$ and $v = 2$.* It follows that

$$ST^{-1}(-\sigma(ST^{-1})) = e_1^{n-1} \prod_{i=1}^{nm} (x_i e_1 + e_2),$$

where $x_1, \dots, x_{nm} \in [0, n-1]$ and $x_1 + \dots + x_{nm} \equiv 1 \pmod{n}$.

Subcase 1.1: $ST^{-1} = e_1^{n-1} \prod_{i=1}^{nm-1} (x_i e_1 + e_2)$. If $x_i \neq x_j$ for some $i \neq j \in [1, nm-1]$, then

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_1\}.$$

By Lemma 3.1 and $0 \nmid S$, we find that T is of the form e_1^{nm}. Therefore, e_1^n is a zero-sum subsequence of S of length $n < nm$, which is a contradiction.

Next we assume that $x_1 = \dots = x_{nm-1}$, i.e. $ST^{-1} = e_1^{n-1}(x_1e_1+e_2)^{nm-1}$. Replacing $x_1 e_1 + e_2$ with e_2, we have $ST^{-1} = e_1^{n-1} e_2^{nm-1}$ and

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_1, e_2\}.$$

If $e_1 \,|\, T$, then e_1^n is a zero-sum subsequence of S of length $n < nm$, a contradiction. By Lemma 3.1 and $0 \nmid S$, we know that T is of the form e_2^{nm}. Therefore,

$$S = e_1^{n-1} e_2^{2nm-1}.$$

Replacing e_1 with g_1 and e_2 with g_2, we see that $\{g_1, g_2\}$ is a generating set of G,where $\mathrm{ord}(g_1) = n$ and $\mathrm{ord}(g_2) = nm$. Thus S is of the form (1).

Subcase 1.2: $ST^{-1} = e_1^{n-2}\prod_{i=1}^{nm}(x_ie_1+e_2)$ *with* $\sum_{i=1}^{nm} x_i \equiv 1 \pmod n$. It follows that $x_i \neq x_j$ for some $i \neq j \in [1, nm]$. Without loss of generality, we may assume that $0 \le x_1 \le \cdots \le x_{nm} \le n-1$. If x_i, x_j, x_k are pairwise distinct for some $i, j, k \in [1, nm]$ or $x_i - x_j \in [2, n-2]$ for some $i, j \in [1, nm]$, then

$$\Sigma_{\ge 2}(ST^{-1}) = G \setminus \{0\},$$

a contradiction with Lemma 3.1 and $0 \nmid S$.

Next we assume that $x_i - x_j \in \{-1, 0, 1\}$ for all $i, j \in [1, nm]$. We may also assume $x_1 = \cdots = x_l = x_{l+1} - 1 = \cdots = x_{nm} - 1$, $l \in [1, nm-1]$. Thus $\sum_{i=1}^{nm} x_i = nmx_1 + (nm - l) \equiv 1 \pmod n$. We deduce that $l = nm - tn - 1$, $t \in [0, m-1]$. So $ST^{-1} = e_1^{n-2}(x_1e_1+e_2)^{nm-tn-1}((x_1+1)e_1+e_2)^{tn+1}$. Replacing $x_1e_1 + e_2$ with e_2, we have $ST^{-1} = e_1^{n-2}e_2^{nm-tn-1}(e_1+e_2)^{tn+1}$. Then

$$\Sigma_{\ge 2}(ST^{-1}) = G \setminus \{0, e_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we find that T is of the form e_2^{nm}. If $t \ge 1$ and $n \ge 3$, then $e_1^{n-2}(e_1+e_2)^2e_2^{nm-2}$ is a zero-sum subsequence of S of length $nm + n - 2 > nm$, which is a contradiction. Therefore $t = 0$ or $n = 2$. If $t = 0$, then

$$S = e_1^{n-2}e_2^{2nm-1}(e_1+e_2).$$

Replacing e_1 with g_1 and e_2 with g_2, we deduce that $\{g_1, g_2\}$ is a generating set of G, where $\mathrm{ord}(g_1) = n$ and $\mathrm{ord}(g_2) = nm$. Thus S is of the form (1).

If $n = 2$, then

$$S = e_2^{4m-2t-1}(e_1+e_2)^{2t+1}.$$

Replacing $e_1 + e_2$ with g_1, e_2 with g_2 and t with $r - 1$, we see that $\{g_1, g_2\}$ is a generating set of G, where $\mathrm{ord}(g_1) = \mathrm{ord}(g_2) = nm$. Thus S is of the form (4).

Case 2: $ST^{-1}(-\sigma(ST^{-1}))$ *is of the form* (3.1) *with* $u = 2$ *and* $v = 1$. It follows that

$$ST^{-1}(-\sigma(ST^{-1})) = e_2^{nm-1}\prod_{i=1}^{n}(x_ie_2+e_1),$$

where $x_1, \ldots, x_n \in [0, nm-1]$ and $x_1 + \cdots + x_n \equiv 1 \pmod{nm}$.

Subcase 2.1: $ST^{-1} = e_2^{nm-1}\prod_{i=1}^{n-1}(x_ie_2+e_1)$. If $x_i \neq x_j$ for some $i \neq j \in [1, n-1]$, then

$$\Sigma_{\ge 2}(ST^{-1}) = G \setminus \{0, e_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form e_2^{nm}. Therefore

$$S = e_2^{2nm-1}\prod_{i=1}^{n-1}(x_ie_2+e_1).$$

Replacing e_1 with g_1 and e_2 with g_2, we find that $\{g_1, g_2\}$ is a generating set of G, where $\mathrm{ord}(g_1) = n$ and $\mathrm{ord}(g_2) = nm$. Thus S is of the form (1).

Next we assume that $x_1 = \cdots = x_{n-1}$, i.e. $ST^{-1} = e_2^{nm-1}(x_1e_2+e_1)^{n-1}$.

If $x_1 \pmod m = 0$, it is easy to see that $\mathrm{ord}(x_1e_2+e_1) = n$. By replacing $x_1e_2 + e_1$ with e_1, we have $ST^{-1} = e_2^{nm-1}e_1^{n-1}$ and this reduces to Case 1. Next we suppose $x_1 \pmod m \in [1, m-1]$. Then

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, e_2, x_1e_2 + e_1\}.$$

If $x_1 \pmod m \in [2, m-1]$ and $(x_1e_2 + e_1) \mid T$, then we deduce that $(x_1e_2+e_1)^n e_2^{nm-(nx_1 \,(\mathrm{mod}\, nm))}$ is a zero-sum subsequence of S of length $nm+n-(nx_1 \pmod{nm}) < nm$, which is a contradiction. By Lemma 3.1 and $0 \nmid S$, we find that T is of the form e_2^{nm}. Therefore

$$S = e_2^{2nm-1}(x_1e_2 + e_1)^{n-1}.$$

Replacing e_1 with g_1 and e_2 with g_2, we see that $\{g_1, g_2\}$ is a generating set of G, where $\mathrm{ord}(g_1) = n$ and $\mathrm{ord}(g_2) = nm$. Thus S is of the form (1).

Next we suppose that $x_1 \pmod m = 1$. Replacing $x_1e_1+e_2$ with e_1+e_2, we have $ST^{-1} = e_2^{nm-1}(e_1 + e_2)^{n-1}$. By Lemma 3.1 and $0 \nmid S$, we find that T is of the form $e_2^{nm-tn}(e_1 + e_2)^{tn}$ for $t \in [0, m]$. Therefore

$$S = e_2^{2nm-tn-1}(e_1 + e_2)^{n+tn-1}.$$

Replacing e_1+e_2 with g_1, e_2 with g_2 and t with $r-1$, we deduce that $\{g_1, g_2\}$ forms a generating set of G, where $\mathrm{ord}(g_1) = \mathrm{ord}(g_2) = nm$. Thus S is of the form (4).

Subcase 2.2: *$ST^{-1} = e_2^{nm-2}\prod_{i=1}^{n}(x_ie_2+e_1)$ with $\sum_{i=1}^{n} x_i \equiv 1 \pmod{nm}$.* It follows that $x_i \neq x_j$ for some $i \neq j \in [1, n]$. Without loss of generality, we may assume that $0 \leq x_1 \leq \cdots \leq x_n \leq nm-1$. If x_i, x_j, x_k are pairwise distinct for some $i, j, k \in [1, n]$, or if $x_i - x_j \in [2, nm-2]$ for some $i, j \in [1, n]$, we infer that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\},$$

a contradiction with Lemma 3.1 and $0 \nmid S$.

Next we assume that $x_i - x_j \in \{-1, 0, 1\}$ for all $i, j \in [1, n]$. We may assume that $x_1 = \cdots = x_l = x_{l+1} - 1 = \cdots = x_n - 1$, $l \in [1, n-1]$. Thus $\sum_{i=1}^{n} x_i = nx_1 + n - l \equiv 1 \pmod{nm}$. We deduce that $l = n-1$ and $m \mid x_1$. So $ST^{-1} = e_2^{nm-2}(x_1e_2+e_1)^{n-1}((x_1+1)e_2+e_1)$. By replacing $x_1e_2+e_1$ with e_1, we have $ST^{-1} = e_2^{nm-2}e_1^{n-1}(e_2 + e_1)$. Therefore, this reduces to Case 1 and we are done.

Case 3: *$ST^{-1}(-\sum(ST^{-1}))$ is of the form (3.2) with $ng_1 \neq ng_2$.* By (3.2), we have $s = 1$. So we can write

$$ST^{-1}(-\sigma(ST^{-1})) = g_1^{n-1}\prod_{i=1}^{nm}(-y_ig_1 + g_2).$$

Note that $y_1, \ldots, y_{nm} \in [0, n-1]$ and $y_1 + \cdots + y_{nm} = n-1$.

Let $\varphi : G \to G/\langle g_2\rangle$ denote the canonical epimorphism. Since $\mathrm{ord}(g_2) = nm$ and $\{g_1, g_2\}$ is a generating set of G, we have $n = \mathrm{ord}(\varphi(g_1)) \mid \mathrm{ord}(g_1)$. Set $\mathrm{ord}(g_1) = dn$ with $d \mid m$ and $d > 1$. Suppose $g_1 = xe_1 + t_0g_2$ for some integers $t_0 \in [0, nm-1]$ and $x \in [0, n-1]$, where $\{e_1, g_2\}$ is a basis of G. Then $ng_1 = t_0ng_2$. Since $\mathrm{ord}(g_2) = nm$, we infer that there exists $t' \in [0, m-1]$ such that

$$ng_1 = t_0ng_2 = t'ng_2.$$

Since $ng_1 \neq ng_2$, we obtain $t' \in [2, m-1]$, and it follows that $m \geq 3$.

SUBCASE 3.1: $ST^{-1} = g_1^{n-2} \prod_{i=1}^{nm}(-y_ig_1 + g_2)$. Without loss of generality, we assume that $n-1 \geq y_1 \geq \cdots \geq y_j > y_{j+1} = \cdots = y_{nm} = 0$. Since $y_1 + \cdots + y_{nm} = n-1$, we have $j \in [1, n-1]$.

If $j \in [2, n-2]$, we obtain

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\},$$

a contradiction with Lemma 3.1 and $0 \nmid S$.

If $j = 1$, i.e. $ST^{-1} = g_1^{n-2}g_2^{nm-1}(-(n-1)g_1 + g_2)$, then we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we see that T is of the form g_2^{nm}. Therefore

$$S = g_1^{n-2}g_2^{2nm-1}(-(n-1)g_1 + g_2).$$

Thus S is of the form (2).

If $j = n-1$, i.e. $ST^{-1} = g_1^{n-2}g_2^{nm-n+1}(-g_1 + g_2)^{n-1}$, then we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, -g_1 + g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form $(-g_1 + g_2)^{nm}$. Furthermore, $g_2^{(t'-1)n}(-g_1 + g_2)^n$ is a zero-sum subsequence of S of length $t'n < nm$, which is a contradiction.

SUBCASE 3.2: $ST^{-1} = g_1^{n-1} \prod_{i=1}^{nm-1}(-y_ig_1 + g_2)$. Without loss of generality, we assume that $n-1 \geq y_1 \geq \cdots \geq y_j > y_{j+1} = \cdots = y_{nm-1} = 0$. Since $y_1 + \cdots + y_{nm} = n-1$, we have $j \in [0, n-1]$.

If $j \in [1, n-1]$, we may first assume that $\sum_{i=1}^{j} y_i = n-1$, so we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\},$$

a contradiction with Lemma 3.1 and $0 \nmid S$.

Next we consider $\sum_{i=1}^{j} y_i < n-1$. Here

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1\}.$$

Again by Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form g_1^{nm}. Furthermore, $g_1^{nm-n+\sum_{i=1}^{j} y_i} \prod_{i=1}^{t'n}(-y_ig_1+g_2)$ is a zero-sum subsequence of S of length $nm + t'n - n + \sum_{i=1}^{j} y_i > nm$, which is a contradiction.

If $j = 0$, i.e. $ST^{-1} = g_1^{n-1} g_2^{nm-1}$, it follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1, g_2\}.$$

If $g_1 \mid T$, then $g_1^n g_2^{nm-t'n}$ is a zero-sum subsequence of S of length $nm - t'n + n < nm$, which is a contradiction. By Lemma 3.1 and $0 \nmid S$, we infer that T is of the form g_2^{nm}. Therefore

$$S = g_1^{n-1} g_2^{2nm-1}.$$

Thus S is of the form (1).

Case 4: *$ST^{-1}(-\sigma(ST^{-1}))$ is of the form (3.2) with $ng_1 = ng_2$.* It follows that

$$ST^{-1}(-\sigma(ST^{-1})) = g_1^{sn-1} \prod_{i=1}^{nm+(1-s)n} (-y_i g_1 + g_2),$$

where $y_1, \ldots, y_{nm+(1-s)n} \in [0, n-1]$ and $y_1 + \cdots + y_{nm+(1-s)n} = n - 1$.

Similar to the proof of Case 3, we have $n \mid \mathrm{ord}(g_1)$. Since $ng_1 = ng_2$ and $\mathrm{ord}(g_2) = nm$, we have $\mathrm{ord}(g_1)/n = \mathrm{ord}(ng_1) = \mathrm{ord}(ng_2) = m$, and so we deduce that $\mathrm{ord}(g_1) = nm$. It is easy to see that $\mathrm{ord}(-g_1 + g_2) = n$.

Subcase 4.1: $ST^{-1} = g_1^{sn-2} \prod_{i=1}^{nm+(1-s)n} (-y_i g_1 + g_2)$. Without loss of generality, we assume that $n - 1 \geq y_1 \geq \cdots \geq y_j > y_{j+1} = \cdots = y_{nm+(1-s)n} = 0$. Since $y_1 + \cdots + y_{nm+(1-s)n} = n - 1$, we have $j \in [1, n-1]$.

If $j \in [2, n-2]$, we obtain

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\},$$

a contradiction with Lemma 3.1 and $0 \nmid S$.

Suppose $j = 1$, i.e. $ST^{-1} = g_1^{sn-2} g_2^{nm+n(1-s)-1}(-(n-1)g_1 + g_2)$. If $s \geq 2$, we have

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\},$$

a contradiction with Lemma 3.1 and $0 \nmid S$.

If $s = 1$, i.e. $ST^{-1} = g_1^{n-2} g_2^{nm-1}(-(n-1)g_1 + g_2)$, it follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we see that T is of the form g_2^{nm}. Therefore

$$S = g_1^{n-2} g_2^{2nm-1}(-(n-1)g_1 + g_2).$$

Thus S is of the form (2).

If $j = n - 1$, i.e. $ST^{-1} = g_1^{sn-2} g_2^{nm-sn+1}(-g_1 + g_2)^{n-1}$, then by replacing g_1 with e_2 and $-g_1 + g_2$ with e_1, we see that $\{e_1, e_2\}$ is a basis of G. It follows that $ST^{-1} = e_1^{n-1} e_2^{nm-sn+1}(e_1 + e_2)^{sn-2}$, and this reduces to Case 1.

Subcase 4.2: $ST^{-1} = g_1^{sn-1} \prod_{i=1}^{nm+(1-s)n-1} (-y_i g_1 + g_2)$. Without loss of generality, we assume that $n - 1 \geq y_1 \geq \cdots \geq y_j > y_{j+1} = \cdots = y_{nm+(1-s)n-1} = 0$. Since $y_1 + \cdots + y_{nm+(1-s)n} = n - 1$, we have $j \in [0, n-1]$.

If $j \in [1, n-2]$, we first assume that $\sum_{i=1}^{j} y_i = n-1$ and $s \leq m-1$; it then follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0\},$$

A contradiction with Lemma 3.1 and $0 \nmid S$.

Next, considering the cases where $\sum_{i=1}^{j} y_i < n-1$ or $s = m$, we infer that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form g_1^{nm}. If $s < m$, then $g_1^{nm-n+\sum_{i=1}^{j} y_i} \prod_{i=1}^{n}(-y_i g_1 + g_2)$ is a zero-sum subsequence of S with length $nm + \sum_{i=1}^{j} y_i > nm$, which is a contradiction. Therefore $s = m$ and

$$S = g_1^{2nm-1} \prod_{i=1}^{n-1}(-y_i g_1 + g_2).$$

Thus S is of the form (3).

If $j = 0$, i.e. $ST^{-1} = g_1^{sn-1} g_2^{nm+n(1-s)-1}$, it follows that

$$\Sigma_{\geq 2}(ST^{-1}) = G \setminus \{0, g_1, g_2\}.$$

By Lemma 3.1 and $0 \nmid S$, we conclude that T is of the form $g_1^{tn} g_2^{nm-tn}$ for $t \in [0, m]$. Therefore

$$S = g_1^{sn+tn-1} g_2^{2nm+n(1-s)-tn-1}.$$

Letting $r = s + t$, we see that S is of the form (4).

Suppose $j = n-1$, i.e. $ST^{-1} = g_1^{sn-1}(-g_1 + g_2)^{n-1} g_2^{nm-sn}$. Replacing g_1 with e_2 and $-g_1 + g_2$ with e_1, we see that $\{e_1, e_2\}$ is a basis of G. It follows that $ST^{-1} = e_1^{n-1} e_2^{nm-sn}(e_1 + e_2)^{sn-1}$, which reduces to Case 1. ∎

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