ACTA ARITHMETICA
Online First version

A reformulated strong multiplicity one theorem

by

HUI XUE

Abstract. We present a reformulated strong multiplicity one result for automorphic representations of GL_n. We also prove a reformulated strong multiplicity one theorem with a certain density condition for GL_2. Finally, we apply the first of these theorems to study the linear independence of central values of Rankin–Selberg convolutions on GL_2.

1. Introduction. We start this paper with a motivating example borrowed from Jacquet–Shalika [8, p. 777]. Let $N > 1$ be an integer. A theorem of Dirichlet states that there are infinitely many primes in each residue class modulo N that are coprime to N. This may be reformulated as follows. Let $\chi_1, \dots, \chi_m$ be distinct Dirichlet characters modulo N and $c_1, \dots, c_m$ be complex numbers. If

$$c_1\chi_1(p) + c_2\chi_2(p) + \cdots + c_m\chi_m(p) = 0$$

for all but finitely many primes p, then $c_1 = \cdots = c_m = 0$. Each character χ_i may be viewed as an automorphic representation (or form) of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$, and $\chi_i(p)$ may be viewed as the p-th Fourier coefficient of χ_i, due to its associated L-function:

$$L(\chi_i, s) = \sum_{n\geq 1} \chi_i(n)n^{-s}.$$

As the second example, we state a result due to Luo [10], which studies the determination of newforms by central values of Rankin–Selberg convolutions, in the level aspect.

THEOREM 1.1 ([10]). *Let $g_1 \in S_{k_1}(N_1)$ and $g_2 \in S_{k_2}(N_2)$ be two normalized newforms of levels N_1 and N_2, weights k_1 and k_2, respectively, c be a constant and l a positive integer. If there exist infinitely many primes p*

2020 *Mathematics Subject Classification*: Primary 11F55; Secondary 11F67.
Key words and phrases: strong multiplicity one, central values of Rankin–Selberg convolutions.
Received 17 January 2026; revised 19 May 2026.

DOI: 10.4064/aa260117-25-5

such that

$$L(f\otimes g_1,1/2)=cL(f\otimes g_2,1/2)$$

for all normalized newforms $f\in S_l(p)$*, then* $g_1=g_2$*.*

As the last motivating example we also recall a result due to Ganguly–Hoffstein–Sengupta [3], which studies the determination of newforms by central values of Rankin–Selberg convolutions, in the weight aspect.

THEOREM 1.2 ([3]). *Let* $g_1\in S_{k_1}(1)$ *and* $g_2\in S_{k_2}(1)$ *be two normalized newforms of level one. If*

$$L(f\otimes g_1,1/2)=L(f\otimes g_2,1/2)$$

for all normalized newforms $f\in S_l(1)$ *for infinitely many* l*, then* $g_1=g_2$*.*

The main goal of this paper is to extend the above three examples to other settings. In Section 2 we will investigate Dirichlet's result when each character χ_i is replaced with an automorphic representation π_i of GL_{r_i} for $1\le i\le m$. One result we obtain therein, Theorem 2.4, can be viewed as a reformulated strong multiplicity one theorem. The second result (Theorem 2.7) we shall prove in Section 2 is a reformulated strong multiplicity one theorem with certain density condition for unitary representations of GL_2 over number fields. Section 3 is devoted to the generalization of the last two motivating examples to the case when more than two newforms are involved. This is achieved by resorting to Proposition 3.1, a specialization of Theorem 2.4 to $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

2. A strong multiplicity one theorem in linear form. Let F be a global field and $\pi=\otimes_v\pi_v$ an irreducible, cuspidal, unitary automorphic representations of $\mathrm{GL}_r(\mathbb{A}_F)$. Let S be a finite set of places of F that contains all archimedean places and finite places over which some of the representations involved are ramified. The set S can be enlarged if necessary. The standard L-function of π outside S is given by

$$L_S(\pi,s)=\prod_{v\notin S}\frac{1}{(1-\alpha_{\pi,1}(v)q_v^{-s})\cdots(1-\alpha_{\pi,r}(v)q_v^{-s})},$$

where q_v is the norm of v, and $\alpha_{\pi,i}(v)$ for $1\le i\le r$ are Satake parameters of the local representation π_v. For each place $v\notin S$ and $n\ge 1$, we define

$$\pi(v^n)=\alpha_{\pi,1}(v)^n+\alpha_{\pi,2}(v)^n+\cdots+\alpha_{\pi,r}(v)^n; \tag{2.1}$$

see Jacquet–Shalika [8, (4.1.1)]. We will call $\pi(v)$ the *v-th Fourier coefficient of* π.

We make the following definition for bounds of the local Satake parameters. Note that the Ramanujan–Petersson conjecture claims that every π of $\mathrm{GL}_r(\mathbb{A}_F)$ satisfies hypothesis $H(0)$; see Shahidi [15] and Remark 2.5 below.

Definition 2.1. We say a unitary cuspidal automorphic representation π of $\mathrm{GL}_r(\mathbb{A}_F)$ satisfies *hypothesis* $H(\delta)$ if $|\alpha_{\pi,i}(v)| \le q_v^{\delta}$ for all $v \notin S$ and $1 \le i \le r$.

We shall consider the Rankin–Selberg convolution of two unitary cuspidal automorphic representations π and π'. Suppose $\alpha_{\pi',j}(v)$ for $1 \le j \le r'$ are Satake parameters for π'_v. Then

$$L_S(\pi \times \pi', s) = \prod_{v \notin S} \Big(\prod_{\substack{1 \le i \le r \\ 1 \le j \le r'}} (1 - \alpha_{\pi,i}(v)\alpha_{\pi',j}(v) q_v^{-s}) \Big)^{-1}.$$

Following Jacquet–Shalika [8, (4.1.2)], for $\Re(s) > 1$, let

$$\ell(\pi \times \pi', s) = \sum_{v \notin S} \sum_{n \ge 1} \frac{\pi(v^n)\pi'(v^n)}{n q_v^{ns}}. \tag{2.2}$$

Then $\ell(\pi \times \pi', \sigma)$ converges absolutely and defines a continuous function for $\Re(s) > 1$, so that

$$L_S(\pi \times \pi', s) = \exp \ell(\pi \times \pi', s);$$

see [8, (4.1.3)].

More generally, let $\pi_1, \dots, \pi_m$ be unitary, cuspidal automorphic representations and formally define, for $\Re(s) \gg 0$,

$$\ell(\pi_1 \times \cdots \times \pi_m, s) = \sum_{v \notin S} \sum_{n \ge 1} \frac{\pi_1(v^n) \cdots \pi_m(v^n)}{n q_v^{ns}} \tag{2.3}$$

so that

$$L_S(\pi_1 \times \cdots \times \pi_m, s) = \exp \ell(\pi_1 \times \cdots \times \pi_m, s).$$

We now single out the $n = 1$ term in (2.2) or (2.3) and write it as

$$a(\pi \times \pi', \sigma) := \sum_{v \notin S} \pi(v)\pi'(v) q_v^{-\sigma},$$

or

$$a(\pi_1 \times \cdots \times \pi_m, \sigma) = \sum_{v \notin S} \pi_1(v) \cdots \pi_m(v) q_v^{-\sigma}.$$

The following result shows that $a_{\pi \times \pi'}(\sigma)$ is a good approximation of $\ell(\pi \times \pi', \sigma)$.

Lemma 2.2.

(1) *Let* π *and* π' *be irreducible unitary cuspidal automorphic representations of* $\mathrm{GL}_r(\mathbb{A}_F)$ *and* $\mathrm{GL}_{r'}(\mathbb{A}_F)$, *respectively. Assume that* π *satisfies hypothesis* $H(\delta)$ *and* π' *satisfies* $H(\delta')$ *such that* $\delta + \delta' = \frac{1}{2} - \epsilon$ *for some small* $\epsilon > 0$. *Then, as the real* σ *tends to* 1^+,

$$|\ell(\pi \times \pi', \sigma) - a(\pi \times \pi', \sigma)|$$

is continuous in σ *and is uniformly bounded.*

(2) *More generally, suppose* π_i *satisfies* $H(\delta_i)$ *such that* $\delta_1 + \cdots + \delta_m = \frac{1}{2} - \epsilon$ *for* $\epsilon > 0$, *and suppose* $\ell(\pi_1 \times \cdots \times \pi_m, s)$ *converges absolutely for* $\Re(s) > 1$ *so that* $L_S(\pi_1 \times \cdots \times \pi_m, s)$ *is meromorphic at* $s = 1$. *Then, as the real* σ *tends to* 1^+,

$$|\ell(\pi_1 \times \cdots \times \pi_m, \sigma) - a(\pi_1 \times \cdots \times \pi_m, \sigma)|$$

is continuous in σ *and is uniformly bounded.*

Proof. (1) Clearly, $a(\pi \times \pi', \sigma)$ converges absolutely for $\sigma > 1$ because $\ell(\pi \times \pi', \sigma)$ does. From their definition, hypotheses $H(\delta)$ and $H(\delta')$ imply that, for $v \notin S$,

$$|\alpha_{\pi,i}(v)| \le q_v^{\delta}, \qquad |\alpha_{\pi',j}(v)| \le q_v^{\delta'}.$$

Thus, by (2.1) we have

$$\begin{aligned} |\pi(v^n)| &= |\alpha_{\pi,1}(v)^n + \alpha_{\pi,2}(v)^n + \cdots + \alpha_{\pi,r}(v)^n| \le r q_v^{n\delta}, \\ |\pi'(v^n)| &= |\alpha_{\pi',1}(v)^n + \alpha_{\pi',2}(v)^n + \cdots + \alpha_{\pi',r'}(v)^n| \le r' q_v^{n\delta'}. \end{aligned}$$

Combining (2.1), (2.2) and the assumption that $\delta + \delta' = \frac{1}{2} - \epsilon$, for $\sigma > 1$ we get

$$\begin{aligned} |\ell(\pi \times \pi', \sigma) - a(\pi \times \pi', \sigma)| &= \Big|\sum_{v \notin S} \sum_{n \ge 2} \pi(v^n) \pi'(v^n) n^{-1} q_v^{-n\sigma}\Big| \\ &\le rr' \sum_{v \notin S} \sum_{n \ge 2} q_v^{n(1/2 - \epsilon - \sigma)} \\ &= rr' \sum_{v \notin S} q_v^{1 - 2\epsilon - 2\sigma} \frac{1}{1 - q_v^{1/2 - \epsilon - \sigma}} = O(1), \end{aligned}$$

where the implied constant only depends on r and r'.

(2) This part follows from the same argument as in (1). ■

In the following, $\widetilde{\pi}$ denotes the contragredient representation of π. For automorphic representations π and π' define

$$\delta_{\pi,\pi'} = \begin{cases} 1 & \text{if } \widetilde{\pi} \cong \pi', \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2.3.

(1) *Retain the hypothesis of Lemma* 2.2(1). *Then*

$$\lim_{\sigma \to 1^+} \frac{a(\pi \times \pi', \sigma)}{\log(\sigma - 1)} = -\delta_{\pi,\pi'}.$$

(2) *Retain the hypothesis of Lemma* 2.2(2). *Then*

$$\lim_{\sigma \to 1^+} \frac{a(\pi_1 \times \cdots \times \pi_m, \sigma)}{\log(\sigma - 1)} = \mathrm{ord}_{s=1} L_S(\pi_1 \times \cdots \pi_m, s).$$

Proof. (1) By Jacquet–Shalika [8, Theorem 4.1], we have

$$\lim_{\sigma\to 1^+}\frac{\ell(\pi\times\pi',\sigma)}{\log(\sigma-1)}=-\delta_{\pi,\pi'}.$$

This together with Lemma 2.2 implies the desired result.

(2) By the assumption, $L_S(\pi_1\times\cdots\times\pi_m,\sigma)=(\sigma-1)^{\mathrm{ord_{s=1}}}f(\sigma)$, where $f(\sigma)$ is continuous and is nonzero at $\sigma=1$. Then taking the logarithms of both sides, dividing them by $\log(\sigma-1)$ and passing to the limit finishes the proof. In fact, the proof of [8, Theorem 4.1] uses the exactly same argument and the fact that $\mathrm{ord}_{s=1}L_S(\pi\times\pi',s)=-\delta_{\pi,\pi'}$. ■

We are now ready to state the first main result of this section. It reformulates the classical strong multiplicity one theorem in a linear form. For a not necessarily unitary automorphic representation ρ, we can write it as $\rho=\pi\otimes|\det|^t$ with π unitary and t real, and define (see [8, (4.2.2)])

$$\rho(v^n)=q_v^{-nt}\pi(v^n). \tag{2.4}$$

Readers may find that the statement and the proof of the next result closely resemble those of [8, Theorem 4.2].

THEOREM 2.4. *Let ρ_i be inequivalent irreducible cuspidal automorphic representations of $\mathrm{GL}_{r_i}(\mathbb{A}_F)$ for $1\le i\le m$. Write $\rho_i=\pi_i\otimes|\det|^{t_i}$ with π_i unitary and t_i real. Assume that each π_i satisfies hypothesis $H(\delta_i)$ such that $\delta_i+\delta_j=\frac12-\epsilon$ for some small $\epsilon>0$ and all $1\le i,j\le m$. If $c_1,\dots,c_m$ are complex numbers such that*

$$c_1\rho_1(v)+\cdots+c_m\rho_m(v)=0 \tag{2.5}$$

for all $v\notin S$, then

$$c_1=\cdots=c_m=0.$$

Proof. We may assume that $t_1\le\cdots\le t_m$ and none of the c_i's is zero. Using (2.4) and multiplying (2.5) through by $q_v^{t_1}$ we get

$$c_1\pi_1(v)+c_2\pi_2(v)q_v^{t_1-t_2}+\cdots+c_m\pi_m(v)q_v^{t_1-t_m}=0.$$

Taking the product with $\widetilde{\pi}_1(v)q_v^{-\sigma}$ and summing over $v\notin S$ for $\sigma>1$ we obtain

$$\sum_{i=1}^m c_i\sum_{v\notin S}\pi_i(v)\widetilde{\pi}_1(v)q_v^{t_1-t_i-\sigma}=0,$$

or equivalently

$$\sum_{1\le i\le m}c_i a(\pi_i\times\widetilde{\pi}_1,\sigma+t_i-t_1)=0.$$

Note that if $t_i - t_1 > 0$, then $a(\pi_i \times \widetilde{\pi}_1, \sigma + t_i - t_1)$ is finite at $\sigma = 1$ (because $\ell(\pi_i \times \widetilde{\pi}_1, \sigma + t_i - t_1)$ is so), that is,

$$\lim_{\sigma \to 1^+} \frac{a(\pi_i \times \widetilde{\pi}_1, \sigma + t_i - t_1)}{\log(\sigma - 1)} = 0.$$

Now, suppose $t_1 = \cdots = t_l < t_{l+1}$ for some $l \geq 1$. From Proposition 2.3(1) we get

$$0 = \sum_{1 \leq i \leq l} c_i \lim_{\sigma \to 1^+} \frac{a(\pi_i \times \widetilde{\pi}_1, \sigma)}{\log(\sigma - 1)} = -\sum_{1 \leq i \leq l} c_i \delta_{\pi_i, \pi_1}.$$

For $1 \leq i \leq l$, as $\pi_i \cong \pi_1$ only when $i = 1$, we conclude that $c_1 = 0$, which is a contradiction. ■

Remark 2.5. (1) Hypothesis $H\left(\frac{7}{64}\right)$ has been proved by Blomer–Brumley [1] for automorphic representations of GL_2 over any number field. Thus, Theorem 2.4 holds true unconditionally for automorphic representations of GL_2, and furthermore for Gelbart–Jacquet lifts on GL_3.

(2) It is a little surprising that the traces $\rho(v)$ for almost all v uniquely determine ρ for $r \geq 2$, because one usually needs more equations on Satake parameters to determine them. This is why [8, Theorem 4.2] requires (2.5) to hold true for $\rho_i(v^n)$ for all $n \geq 1$.

(3) We will also call $\rho(v)$ the *v-th Fourier coefficient* of ρ. When $m = 2$, Theorem 2.4 more or less generalizes the classical multiplicity one theorem for GL_2; see Section 3.

(4) Since hypothesis $H(0)$ holds true for holomorphic, unitary, cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ over any totally real field (see [2]), Corollary 2.6(2) below holds true for such π_1.

Corollary 2.6.

(1) *If F is a function field, then the conclusion of Theorem* 2.4 *holds true unconditionally.*

(2) *Let F be a number field and π_i be inequivalent unitary cuspidal representations of $\mathrm{GL}_{r_i}(\mathbb{A}_F)$ for $1 \leq i \leq m$. Suppose further that π_1 satisfies hypothesis $H(0)$. If $c_1, \ldots, c_m$ are complex numbers such that*

$$c_1 \pi_1(v) + \cdots + c_m \pi_m(v) = 0,$$

then $c_1 = 0$.

Proof. (1) From the work of Lafforgue [9] we know $H(0)$ is true for all unitary, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ over any function field F.

(2) By the work of Luo–Rudnick–Sarnak [11], every unitary cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_F)$ satisfies hypothesis $H\left(\frac{1}{2} - \frac{1}{r^2+1}\right)$. This bound, coupled with the bound $H(0)$ for π_1 and Lemma 2.2, implies

the conclusion of Proposition 2.3 and hence the argument used in the proof of Theorem 2.4 still works in this situation. ■

In the spirit of Ramakrishnan [12] and Walji [16], we re-consider Theorem 2.4 for GL_2 when the linear equation (2.5) is satisfied over places of certain density. Our goal is to exemplify our method, and not to get the optimal bound on the density. Recall that the lower Dirichlet density of a set of places X of a number field F is given by (see [16, Section 1])

$$\underline{\delta}(X) := \liminf_{\sigma\to 1^+} \frac{\sum_{v\in X} q_v^{-\sigma}}{-\log(\sigma-1)}.$$

THEOREM 2.7. *For $1 \le i \le m$, let π_i be inequivalent unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ over a number field F. Let X with $X \cap S = \emptyset$ be a set of finite places of F such that $\underline{\delta}(X) < \frac{1}{16m^2}$. If $c_1, \dots, c_m$ are complex numbers such that*

$$c_1\pi_1(v) + \cdots + c_m\pi_m(v) = 0 \tag{2.6}$$

for every finite place $v \notin X \cup S$, then $c_1 = \cdots = c_m = 0$.

Proof. Again, we may suppose that no c_i is zero. For each quadruple of indices $1 \le i, j, u, w \le m$, by Lemma 2.8 below, we can see that $L_S(\pi_i \times \pi_j \times \overline{\pi}_u \times \overline{\pi}_w, s)$ converges absolutely for $\Re(s) > 1$ and is meromorphic at $s = 1$ with a pole of order at most 16. Also, according to Blomer–Brumley [1], the Satake parameters of $\pi_i, \pi_j, \overline{\pi}_u, \overline{\pi}_w$ satisfy hypothesis $H(\frac{7}{64})$. Thus Proposition 2.3(2) shows that, for real σ tending to 1^+,

$$\begin{aligned}\Big|\sum_{v\notin S} \pi_i(v)\pi_j(v)\overline{\pi}_u(v)\overline{\pi}_w(v)q_v^{-\sigma}\Big| &= |a(\pi_i \times \pi_j \times \overline{\pi}_u \times \overline{\pi}_w, \sigma)|\\ &= |\mathrm{ord}_{s=1} L_S(\pi_i \times \pi_j \times \overline{\pi}_u \times \overline{\pi}_w, s)\log(\sigma-1) + o(\log(\sigma-1))|\\ &\le -16\log(\sigma-1) + o(\log(\sigma-1)).\end{aligned} \tag{2.7}$$

By Proposition 2.3(1) we have, for real σ tending to 1^+,

$$\begin{aligned}\sum_{v\notin S} |c_1\pi_1(v) + \cdots + c_m\pi_m(v)|^2 q_v^{-\sigma} &= \sum_{v\notin S} (c_1\pi_1(v) + \cdots + c_m\pi_m(v))\overline{(c_1\pi_1(v) + \cdots + c_m\pi_m(v))}q_v^{-\sigma}\\ &= -(|c_1|^2 + \cdots + |c_m|^2)\log(\sigma-1) + o(\log(\sigma-1)).\end{aligned} \tag{2.8}$$

On the other hand, using the Cauchy–Schwarz inequality and (2.6), for $\sigma > 1$

we get

$$\begin{aligned}
(2.9)\qquad &\sum_{v\notin S} |c_1\pi_1(v)+\cdots+c_m\pi_m(v)|^2 q_v^{-\sigma}\\
&= \sum_{v\notin S} |c_1\pi_1(v)+\cdots+c_m\pi_m(v)|^2 q_v^{-\sigma/2}\mathbf{1}_X(v)q_v^{-\sigma/2}\\
&\le \sqrt{\sum_{v\notin S} |c_1\pi_1(v)+\cdots+c_m\pi_m(v)|^4 q_v^{-\sigma}}\sqrt{\sum_{v\in X} q_v^{-\sigma}},
\end{aligned}$$

where $\mathbf{1}_X$ denotes the characteristic function of X. By (2.7) and (2.9), for $\sigma \to 1^+$,

$$\begin{aligned}
(2.10)\qquad &\sqrt{\sum_{v\notin S} |c_1\pi_1(v)+\cdots+c_m\pi_m(v)|^4 q_v^{-\sigma}}\\
&= \sqrt{\sum_{v\notin S} (c_1\pi_1(v)+\cdots+c_m\pi_m(v))^2\,\overline{(c_1\pi_1(v)+\cdots+c_m\pi_m(v))}^2 q_v^{-\sigma}}\\
&= \sqrt{\sum_{1\le i,j,u,w\le m} c_ic_j\bar{c}_u\bar{c}_w \sum_{v\notin S} \pi_i(v)\pi_j(v)\overline{\pi}_u(v)\overline{\pi}_w(v)q_v^{-\sigma}}\\
&\le \sqrt{\sum_{1\le i,j,u,w\le m} |c_ic_jc_uc_w|\cdot\Big|\sum_{v\notin S} \pi_i(v)\pi_j(v)\overline{\pi}_u(v)\overline{\pi}_w(v)q_v^{-\sigma}\Big|}\\
&\le \sqrt{-16(|c_1|+\cdots+|c_m|)^4\log(\sigma-1)} + o(\sqrt{-\log(\sigma-1)})\\
&= 4(|c_1|+\cdots+|c_m|)^2\sqrt{-\log(\sigma-1)} + o(\sqrt{-\log(\sigma-1)}).
\end{aligned}$$

Combining the following inequality with (2.8)–(2.10):

$$(|c_1|+\cdots+|c_m|)^2 \le m(|c_1|^2+\cdots+|c_m|^2),$$

dividing out both sides by $-\log(\sigma-1)$ and taking the lim inf we obtain

$$1 \le 4m\sqrt{\underline{\delta}(X)},$$

which implies that $\underline{\delta}(X) \ge \frac{1}{16m^2}$, contradicting our assumption. ■

LEMMA 2.8. *Let π_i for $1\le i\le 4$ be unitary cuspidal irreducible automorphic representations of* $\mathrm{GL}_2(\mathbb{A}_F)$. *Then $L_S(\pi_1\times\pi_2\times\pi_3\times\pi_4, s)$ converges absolutely for $\Re(s)>1$ and is meromorphic at $s=1$ with a pole of order at most* 16.

Proof. By the work of Ramakrishnan [13, §3, Theorem M] on the automorphicity of Rankin–Selberg convolutions on $\mathrm{GL}_2\times\mathrm{GL}_2$, we know that

$$L_S(\pi_1\times\pi_2\times\pi_3\times\pi_4, s) = L_S(\pi_1\boxtimes\pi_2\times\pi_3\boxtimes\pi_4, s),$$

where $\pi_1 \boxtimes \pi_2$ and $\pi_3 \boxtimes \pi_4$ are unitary isobaric representations of $\mathrm{GL}_4(\mathbb{A}_F)$ such that

$$L_S(\pi_1 \boxtimes \pi_2, s) = L_S(\pi_1 \times \pi_2, s) \quad \text{and} \quad L_S(\pi_3 \boxtimes \pi_4, s) = L_S(\pi_3 \times \pi_4, s).$$

Note that

$$\pi_1 \boxtimes \pi_2 = \boxplus_{i=1}^{e} \eta_i \quad \text{and} \quad \pi_3 \boxtimes \pi_4 = \boxplus_{j=1}^{f} \eta'_j$$

as isobaric representations, where e and f are integers between 1 and 4, and each η_i (resp. η'_j) is a cuspidal representation on GL_{n_i} (resp. $\mathrm{GL}_{n'_j}$), such that $n_1 + \cdots + n_e = 4$ (resp. $n'_1 + \cdots + n'_f = 4$). By [7, Theorem 9.5] (the bi-additivity of Rankin–Selberg convolutions),

$$\begin{aligned} L_S(\pi_1 \times \pi_2 \times \pi_3 \times \pi_4, s) &= L_S(\pi_1 \boxtimes \pi_2 \times \pi_3 \boxtimes \pi_4, s) \\ &= \prod_{1\le j\le f} \prod_{1\le i\le e} L_S(\eta_i \times \eta'_j, s). \end{aligned}$$

The claims now follow from the fact [8] that each factor $L_S(\eta_i \times \eta'_j, s)$ converges absolutely for $\Re(s) > 1$, and has a meromorphic continuation with a pole of order at most 1 at $s = 1$. ∎

3. Linear independence of central values of Rankin–Selberg convolutions. We now apply Theorem 2.4 to extend Theorems 1.1 and 1.2 to the case of several newforms. First, we recall the following well known facts; see for example Gelbart [4, Chapter 5]. Let $f(z) \in S_k(N, \chi)$ be a normalized newform of weight k, level N and character χ such that $a_f(1) = 1$. We write its Fourier expansion as

$$f(z) = \sum_{n\ge 1} a_f(n) q^n = \sum_{n\ge 1} \lambda_f(n) n^{\frac{k-1}{2}} q^n.$$

Suppose π_f is the cuspidal unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ that f generates. Then for every finite prime $p \nmid N$ one has

$$\pi_f(p) = \lambda_f(p) \quad \text{and} \quad (\pi_f \otimes |\det|^{-\frac{k-1}{2}})(p) = a_f(p).$$

Thus, we can specialize Theorem 2.4 to the following two cases.

PROPOSITION 3.1. *Suppose $f_i \in S_{k_i}(N_i, \chi_i)$ are m distinct normalized newforms. If $c_1, \ldots, c_m$ are complex numbers such that either*

$$c_1 \lambda_{f_1}(p) + \cdots + c_m \lambda_{f_m}(p) = 0$$

for all but finitely many primes p, or

$$c_1 a_{f_1}(p) + \cdots + c_m a_{f_m}(p) = 0$$

for all but finitely many primes p, then $c_1 = \cdots = c_m = 0$.

We first generalize Theorem 1.1 to the case involving more than two newforms.

Theorem 3.2. *Suppose $g_i \in S_{k_i}(N_i)$ are pairwise different newforms for $1 \le i \le m$. Let $c_1, \ldots, c_m$ be complex numbers such that*

$$c_1 L(f \otimes g_1, 1/2) + \cdots + c_m L(f \otimes g_m, 1/2) = 0 \tag{3.1}$$

for all normalized newforms $f \in S_l(p)$ for infinitely many p. Then $c_1 = \cdots = c_m = 0$.

Proof. We quickly recall the following critical approximation for ℓ prime [10, p. 598]:

$$\sum_{f \in H_l(p)} L(f \otimes g, 1/2)\lambda_f(\ell)\frac{\Gamma(l-1)}{(4\pi)^{l-1}\langle f, f\rangle_{\Gamma_0(p)}} = \lambda_g(\ell)\ell^{-1/2}\prod_{q|N}(1-q^{-1})\log p + O(1), \tag{3.2}$$

where $H_l(p)$ denotes the set of normalized newforms in $S_l(p)$ and $g \in S_k(N)$ is a normalized newform. Now, applying the asymptotic (3.2) to (3.1) we get

$$c_1\lambda_{g_1}(\ell)\ell^{-\frac{1}{2}}\prod_{q|N_1}(1-q^{-1})\log p + \cdots + c_m\lambda_{g_m}(\ell)\ell^{-\frac{1}{2}}\prod_{q|N_m}(1-q^{-1})\log p = O(1).$$

Dividing out by large $\log p$ we deduce that, for any prime ℓ not dividing $N_1 \cdots N_m$,

$$c_1'\lambda_{g_1}(\ell) + \cdots + c_m'\lambda_{g_m}(\ell) = 0,$$

where for each i we have $c_i' = 0$ if and only if $c_i = 0$. The proof is complete by resorting to Proposition 3.1. ∎

We next extend Theorem 1.2 to the case of more than two newforms.

Theorem 3.3. *Suppose $g_i \in S_{l_i}(1)$ are pairwise distinct Hecke eigenforms for $i = 1, \ldots, m$. If $c_1, \ldots, c_m$ are complex numbers such that*

$$c_1 L(f \otimes g_1, 1/2) + \cdots + c_m L(f \otimes g_m, 1/2) = 0$$

for all normalized newforms $f \in S_k(1)$ for infinitely many k, then $c_1 = \cdots = c_m = 0$.

Proof. Recall the following asymptotic [3, eq. (13)] for any prime p:

$$\sum_{f \in H_k(1)} L(f \otimes g, 1/2)\lambda_f(p)\frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle_{\Gamma_0(1)}} = 2\frac{\lambda_g(p)}{\sqrt{p}}M_p(k, l) + 2E_{g,p}(k),$$

where $g \in S_l(1)$ is a normalized newform. Moreover, from [3, p. 856],

$$M_p(k,l) = \log k + O(1)$$

and $|E_{g,p}(k)| \ll 1$ as $k \to \infty$. Thus

$$\sum_{1\le i\le m} c_i \frac{\lambda_{g_i}(p)}{\sqrt{p}} M_p(k,l) = -\sum_{1\le i\le m} c_i E_{g_i,p}(k).$$

Dividing both sides by $\frac{\log k}{\sqrt{p}}$ and letting $k \to \infty$, we get

$$c_1\lambda_{g_1}(p) + \cdots + c_m\lambda_{g_m}(p) = 0$$

for all primes p. Applying Proposition 3.1 again gives the result. ■

At a last note, we want to point out that results similar to Theorems 3.2 and 3.3 can be obtained in other scenarios, such as for Hilbert modular forms in [6, 5], and for Maass forms in [14], to name a few.

Acknowledgements. We would like to express our gratitude to Dihua Jiang for his helpful comments on an earlier version of this paper. We also want to thank the referee for helpful suggestions.

Funding. Research of the author is supported by Simons Foundation Grant MPS-TSM-00007911.

References

[1] V. Blomer and F. Brumley, *On the Ramanujan conjecture over number fields*, Ann. of Math. (2) 174 (2011), 581–605.

[2] J.-L. Brylinski et J.-P. Labesse, *Cohomologie d'intersection et fonctions L de certaines variétés de Shimura*, Ann. Sci. École Norm. Sup. (4) 17 (1984), 361–412.

[3] S. Ganguly, J. Hoffstein, and J. Sengupta, *Determining modular forms on* $\mathrm{SL}_2(\mathbb{Z})$ *by central values of convolution L-functions*, Math. Ann. 345 (2009), 843–857.

[4] S. S. Gelbart, *Automorphic Forms on Adele Groups*, Ann. of Math. Stud. 83, Princeton Univ. Press, Princeton, NJ, and Univ. of Tokyo Press, Tokyo, 1975.

[5] A. Hamieh and N. Tanabe, *Determining Hilbert modular forms by central values of Rankin–Selberg convolutions: the level aspect*, Trans. Amer. Math. Soc. 369 (2017), 8781–8797.

[6] A. Hamieh and N. Tanabe, *Determining Hilbert modular forms by central values of Rankin–Selberg convolutions: the weight aspect*, Ramanujan J. 45 (2018), 615–637.

[7] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin–Selberg convolutions*, Amer. J. Math. 105 (1983), 367–464.

[8] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms II*, Amer. J. Math. 103 (1981), 777–815.

[9] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. 147 (2002), 1–241.

[10] W. Luo, *Special L-values of Rankin–Selberg convolutions*, Math. Ann. 314 (1999), 591–600.

[11] W. Luo, Z. Rudnick, and P. Sarnak, *On the generalized Ramanujan conjecture for* GL(n), in: Automorphic Forms, Automorphic Representations, and Arithmetic (Fort Worth, TX, 1996), Proc. Sympos. Pure Math. 66, Part 2, Amer. Math. Soc., Providence, RI, 1999, 301–310.

[12] D. Ramakrishnan, *A refinement of the strong multiplicity one theorem for* GL(2)*. Appendix to: "l-adic representations associated to modular forms over imaginary quadratic fields. II" [Invent. Math. 116 (1994), 619–643] by R. Taylor*, Invent. Math. 116 (1994), 645–649.

[13] D. Ramakrishnan, *Modularity of the Rankin–Selberg L-series, and multiplicity one for* SL(2), Ann. of Math. (2) 152 (2000), 45–111.

[14] B. Saha and J. Sengupta, *Determination of* **GL**(2) *Maass forms from twists in the level aspect*, Acta Arith. 189 (2019), 165–178.

[15] F. Shahidi, *On the Ramanujan conjecture for quasisplit groups*, Asian J. Math. 8 (2004), 813–835.

[16] N. Walji, *Further refinement of strong multiplicity one for* GL(2), Trans. Amer. Math. Soc. 366 (2014), 4987–5007.

Hui Xue
School of Mathematical and Statistical Sciences
Clemson University
Clemson, SC 29634, USA
E-mail: huixue@clemson.edu