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# THE REIDEMEISTER ZETA FUNCTIONS AND THE GAUSS CONGRUENCES FOR THE REIDEMEISTER NUMBERS

BY

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**Abstract.** We prove a dichotomy between rationality and a natural boundary for the analytic behavior of the Reidemeister zeta function for tame endomorphisms of $\mathbb{Z}_p^d$, where $\mathbb{Z}_p$ is the additive group of $p$-adic integers. We also prove the rationality of the coincidence Reidemeister zeta function for tame endomorphism pairs of finitely generated torsion-free nilpotent groups, based on a weak commutativity condition. Furthermore, we prove the Gauss congruences for Reidemeister coincidence numbers of iterations of tame endomorphism pairs of finitely generated torsion-free nilpotent groups.

**1. Introduction.** Let $G$ be a group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \beta \in G$ are said to be *$\phi$-conjugate* or *twisted conjugate* if there exists $g \in G$ such that $\beta = g\alpha\phi(g^{-1})$. We shall write $\{x\}_\phi$ for the *$\phi$-conjugacy* or *twisted conjugacy* class of the element $x \in G$. The number of $\phi$-conjugacy classes is called the *Reidemeister number* of the endomorphism $\phi$ and is denoted by $R(\phi)$. If $\phi$ is the identity map then the $\phi$-conjugacy classes are the usual conjugacy classes in the group $G$. We call the endomorphism $\phi$ *tame* if the Reidemeister numbers $R(\phi^n)$ are finite for all $n \in \mathbb{N}$. Taking a dynamical point of view, we consider the iterations of a tame endomorphism $\phi$, and following [8] we define the Reidemeister zeta function of $\phi$ as a power series:

$$R_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n\right),$$

where $z$ denotes a complex variable. The following problems were investigated [10]: for which groups and endomorphisms is the Reidemeister zeta function a rational function? Is it an algebraic function?

In [8, 10, 18, 11, 9], the rationality of the Reidemeister zeta function $R_\phi(z)$ was proven in the following cases: the group is finitely generated and

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an endomorphism is eventually commutative; the group is finite; the group is a direct sum of a finite group and a finitely generated free abelian group; the group is finitely generated, nilpotent and torsion-free. In [26] the rationality of the Reidemeister zeta function was proven for tame endomorphisms of fundamental groups of infra-nilmanifolds under some sufficient conditions. The rationality of the Reidemeister zeta function was also proven for tame endomorphisms of fundamental groups of infra-nilmanifolds [5]; for tame endomorphisms of fundamental groups of infra-solvmanifolds of type (R) [13]; for automorphisms of crystallographic groups with diagonal holonomy $\mathbb{Z}_2$ and for automorphisms of almost crystallographic groups up to dimension 3 [6]; for the right shifts of a non-finitely-generated, non-abelian torsion groups $G = \bigoplus_{i\in\mathbb{Z}} F_i, F_i \cong F$ where $F$ is a finite non-abelian group [25].

Let $G$ be a group and $\phi, \psi : G \to G$ two endomorphisms. Two elements $\alpha, \beta \in G$ are said to be $(\phi,\psi)$-*conjugate* if there exists $g \in G$ with

$$\beta = \psi(g)\alpha\phi(g^{-1}).$$

The number of $(\phi,\psi)$-conjugacy classes is called the Reidemeister coincidence number of endomorphisms $\phi$ and $\psi$, denoted by $R(\phi,\psi)$. If $\psi$ is the identity map then the $(\phi, \mathrm{id})$-conjugacy classes are the $\phi$-conjugacy classes in the group $G$ and $R(\phi, \mathrm{id}) = R(\phi)$. The Reidemeister coincidence number $R(\phi,\psi)$ has useful applications in Nielsen coincidence theory. We call the pair $(\phi,\psi)$ of endomorphisms *tame* if the Reidemeister numbers $R(\phi^n,\psi^n)$ are finite for all $n \in \mathbb{N}$. For such a tame pair of endomorphisms we define the *coincidence Reidemeister zeta function* as a power series

$$R_{\phi,\psi}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n,\psi^n)}{n} z^n\right).$$

If $\psi$ is the identity map then $R_{\phi,id}(z) = R_\phi(z)$. In the theory of dynamical systems, the coincidence Reidemeister zeta function counts the synchronization points of two maps, i.e. the points whose orbits intersect under simultaneous iterations of two endomorphisms; see [22], for instance.

In [15], in analogy to works of Bell, Miles and Ward [2] and Byszewski and Cornelissen [3, §5] about Artin–Mazur zeta function, the Pólya–Carlson dichotomy between rationality and a natural boundary for analytic behavior of the coincidence Reidemeister zeta function was proven for tame pair of commuting automorphisms of non-finitely-generated torsion-free abelian groups that are subgroups of $\mathbb{Q}^d, d \geq 1$.

In [12] the Pólya–Carlson dichotomy was proven for the coincidence Reidemeister zeta function of a tame pair of endomorphisms of non-finitely-generated torsion-free nilpotent groups of finite Prüfer rank by means of profinite completion techniques.

In this paper we prove a dichotomy between rationality and a natural boundary for the Reidemeister zeta function of tame endomorphisms of the groups $\mathbb{Z}_p^d$, $d \geq 1$, where $\mathbb{Z}_p$, for $p$ a prime number, is the additive group of $p$-adic integers.

Based on a weak commutativity condition, we also prove the rationality of the coincidence Reidemeister zeta function for a tame pair of endomorphisms of finitely generated torsion-free nilpotent groups.

Furthermore, we prove the Gauss congruences for the Reidemeister coincidence numbers of iterations of a tame pair of endomorphisms of finitely generated torsion-free nilpotent groups.

**2. Pólya–Carlson dichotomy for the Reidemeister zeta function of endomorphisms of the groups $\mathbb{Z}_p^d$.** In this section we prove a Pólya–Carlson dichotomy between rationality and a natural boundary for analytic behavior of the Reidemeister zeta function of endomorphisms of groups $\mathbb{Z}_p^d, d \geq 1$, where $\mathbb{Z}_p$, for $p$ a prime number, denotes the additive group of $p$-adic integers. The group $\mathbb{Z}_p$ is the most basic infinite pro-$p$ group, it is a totally disconnected compact abelian torsion-free group. The field of $p$-adic numbers is denoted by $\mathbb{Q}_p$ and the $p$-adic absolute value (as well as its unique extension to the algebraic closure $\overline{\mathbb{Q}}_p$) by $|\cdot|_p$.

We recall the definition of a natural boundary (see [24, Sec. 6.2]).

Definition 2.1. Suppose that an analytic function $F$ is defined in a region $D$ of the complex plane. If there is no point of the boundary $\partial D$ of $D$ over which $F$ can be analytically continued, then $\partial D$ is called the *natural boundary* for $F$.

We need the following statement:

Lemma 2.2 (cf. [2]). *Let $Z(z) = \sum_{n=1}^{\infty} R(\phi^n)z^n$. If $R_\phi(z)$ is rational then $Z(z)$ is rational. If $R_\phi(z)$ has an analytic continuation beyond its circle of convergence, then so does $Z(z)$. In particular, the existence of a natural boundary at the circle of convergence for $Z(z)$ implies the existence of a natural boundary for $R_\phi(z)$.*

*Proof.* This follows from the fact that $Z(z) = z \cdot R'_\phi(z)/R_\phi(z)$. ∎

One of the most important links between the arithmetic properties of coefficients of a complex power series and its analytic behavior is given by the Pólya–Carlson theorem [24].

Theorem 2.3 (Pólya–Carlson). *A power series with integer coefficients and radius of convergence* 1 *either is rational or has the unit circle as a natural boundary.*

Lemma 2.4. $\mathrm{End}(\mathbb{Z}_p) = \mathbb{Z}_p$ *for abelian group* $\mathbb{Z}_p$.

*Proof.* Let $\phi \in \mathrm{End}(\mathbb{Z}_p)$. We have $p^n\phi(x) = \phi(p^n x)$, where $x \in \mathbb{Z}_p$, $n \in \mathbb{N}$. Then $\phi(p^n\mathbb{Z}_p) \subset p^n\mathbb{Z}_p$, so $|\phi(x)|_p \leq |x|_p$ for all $x \in \mathbb{Z}_p$ and for $x, y \in \mathbb{Z}_p$ we have $|\phi(x)-\phi(y)|_p = |\phi(x-y)|_p \leq |x-y|_p$. Thus $\phi$ is continuous. For every $x \in \mathbb{Z}_p$ there exists a sequence of integers $x_n$ converging to $x$. Then

$$\phi(x) = \lim \phi(x_n) = \lim x_n\phi(1) = \phi(1)x,$$

so $\phi$ is multiplication by $\phi(1)$. ∎

Let $\phi \in \mathrm{End}(\mathbb{Z}_p)$; then $\phi(x) = ax$, where $a \in \mathbb{Z}_p$. We have $\phi^n(x) = a^n x$. By definition,

$$y \sim_\phi x \Leftrightarrow \exists b \in \mathbb{Z}_p : y = b + x - ab = x + b(1-a) \Leftrightarrow y \equiv x(\mathrm{mod}(1-a)).$$

This implies that $R(\phi) = |\mathbb{Z}_p/(1-a)\mathbb{Z}_p|$. On the other hand,

$$(1-a)\mathbb{Z}_p = p^{v_p(1-a)}\mathbb{Z}_p = |1-a|_p^{-1}\mathbb{Z}_p,$$

so we can write $R(\phi) = |1-a|_p^{-1} = |a-1|_p^{-1}$ and, more generally,

$$R(\phi^n) = |1-a^n|_p^{-1} = |a^n-1|_p^{-1} \tag{2.1}$$

for all $n \in \mathbb{N}$.

Now consider a group $\mathbb{Z}_p^d$, $d \geq 2$. It follows easily from Lemma 2.4 that $\mathrm{End}(\mathbb{Z}_p^d) = M_d(\mathbb{Z}_p)$. Let $A \in M_d(\mathbb{Z}_p)$. Then, by the Smith normal form algorithm, there exist unimodular matrices $E, F \in M_d(\mathbb{Z}_p)$ and a diagonal matrix $D \in M_d(\mathbb{Z}_p)$ such that $D = EAF$.

Lemma 2.5. *For $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ we have*

$$R(\phi_p) = |\mathrm{Coker}(1-\phi_p)| = |\det(\Phi_p - \mathrm{Id})|_p^{-1},$$

*where $\Phi_p$ is the matrix of $\phi_p$.*

*Proof.* Let $D, E, F \in M_d(\mathbb{Z}_p)$ be such that $D = E(\mathrm{Id} - \Phi_p)F$, where $D = (a_i)$ is a diagonal matrix, $a_i \in \mathbb{Z}_p, 1 \leq i \leq d$, and the matrices $E, F$ are unimodular. Then

$$\begin{aligned} R(\phi_p) &= |\mathrm{Coker}(1-\phi_p)| = |\mathbb{Z}_p^d/(\mathrm{Id}-\Phi_p)\mathbb{Z}_p^d| = |\mathbb{Z}_p^d/D\mathbb{Z}_p^d| \\ &= |\mathbb{Z}_p/a_1\mathbb{Z}_p| \cdot |\mathbb{Z}_p/a_2\mathbb{Z}_p| \cdots |\mathbb{Z}_p/a_d\mathbb{Z}_p||a_1|_p^{-1} \cdots |a_d|_p^{-1} \\ &= |\det(D)|_p^{-1} = |\det(\mathrm{Id}-\Phi_p)|_p^{-1} = |\det(\Phi_p - \mathrm{Id})|_p^{-1}. \ ∎ \end{aligned}$$

Now we need to handle the sequence $R(\phi^n) = |a^n - 1|_p^{-1}$, $n \in \mathbb{N}$. The following technical lemma is a useful tool for that, since it provides a way to evaluate expressions of the form $|a^n - 1|_p$ when $|a|_p = 1$.

Lemma 2.6 (cf. [21, Lemma 4.9], [2, Lemma 2]). *Let $K_v$ be a non-archimedean local field and suppose $x \in K_v$ has $|x|_v = 1$ and infinite multiplicative order. Let $p > 0$ be the characteristic of the residue field $F_v$ and $\gamma \in \mathbb{N}$ the multiplicative order of the image of $x$ in $F_v$. Then $|x^n - 1|_v = 1$ whenever the greatest common divisor $(\gamma, n)$ is 1 and $\gamma \neq 1$. Furthermore, if*

$\mathrm{char}(K_v) = 0$, *there are constants* $0 < C < 1$ *and* $r_0 \geq 0$ *such that whenever* $n = k\gamma p^r$ *with* $(p, k) = 1$ *and* $r > r_0$, *then* $|x^n - 1|_v = C|p|_v^r$.

Now we prove the Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behavior of the Reidemeister zeta function for endomorphisms of the groups $\mathbb{Z}_p^d$, $d \geq 1$.

THEOREM 2.7. *Let* $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ *be a tame endomorphism and* $\lambda_1, \dots, \lambda_d \in \overline{\mathbb{Q}}_p$ *be the eigenvalues of* $\Phi_p$, *listed with multiplicities. Then the Reidemeister zeta function* $R_{\phi_p}(z)$ *either is a rational function or has the unit circle as a natural boundary. Furthermore, the latter occurs if and only if* $|\lambda_i|_p = 1$ *for some* $i \in \{1, \dots, d\}$.

*Proof.* First, let us consider the case of the group $\mathbb{Z}_p$ separately. This case illustrates some ideas needed to prove the dichotomy in the general case when $d \geq 1$. So let $d = 1$. Lemma 2.4 yields $\phi_p(x) = ax$, where $a \in \mathbb{Z}_p$. Hence $|a|_p \leq 1$. Then according to (2.1) we have $R(\phi_p^n) = |a^n - 1|_p^{-1}$ for all $n \in \mathbb{N}$. If $|a|_p < 1$, then $R(\phi_p^n) = |a^n - 1|_p^{-1} = 1$ for all $n \in \mathbb{N}$. Hence the radius of convergence of $R_{\phi_p}(z)$ equals 1 and $R_{\phi_p}(z) = \frac{1}{1-z}$ is a rational function.

From now on, we shall write $a(n) \ll b(n)$ if there is a constant $c$ independent of $n$ for which $a(n) < c \cdot b(n)$. When $|a|_p = 1$, we will show that the radius of convergence of $R_{\phi_p}(z)$ equals 1 by deriving the bound

$$\frac{1}{n} \ll |a^n - 1|_p \leq 1. \tag{2.2}$$

The upper bound in (2.2) follows from the definition of the $p$-adic norm. The case when $|a^n - 1|_p = 1$ is trivial, so we may suppose that $|a^n - 1|_p < 1$. Let $F$ denote the smallest field which contains $\mathbb{Q}_p$ and is both algebraically closed and complete with respect to $|\cdot|_p$. The $p$-adic logarithm $\log_p$ is defined by

$$\log_p(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n,$$

and converges for all $z \in F$ such that $|z|_p < 1$. Setting $z = a^n - 1$ we

$$|\log_p(a^n)|_p = \left| \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (a^n - 1)^m \right|_p$$

(by the strong triangle inequality for the $p$-adic norm)

$$= \left| \sum_{m=1}^{N} \frac{(-1)^{m+1}}{m} (a^n - 1)^m \right|_p \leq \max_{1 \leq m \leq N} \left( \left| \frac{(a^n - 1)^m}{m} \right|_p \right)$$

for some $N$. So, there is a constant $C > 0$ independent of $n$ for which

$$C \cdot |\log_p(a^n)|_p \leq |a^n - 1|_p.$$

We always have

$$\frac{1}{n} \ll |n \log_p(a)|_p = |\log_p(a^n)|_p,$$

so this establishes (2.2). Alternatively, we can use Lemma 2.6 rather than the $p$-adic logarithm to prove the lower bound in (2.2).

By the Cauchy–Hadamard formula, the bound (2.2) implies that the radius of convergence of $R_{\phi_p}(z)$ equals 1. Now it remains to show that the Reidemeister zeta function $R_{\phi_p}(z)$ is irrational if $|a|_p = 1$. Then $R_{\phi_p}(z)$ has the unit circle as a natural boundary by Lemma 2.2 and by the Pólya–Carlson theorem. For a contradiction, assume that $R_{\phi_p}(z)$ is rational. Then Lemma 2.2 implies that $Z_p(z) = \sum_{n=1}^{\infty} R(\phi_p^n) z^n$ is rational as well. Hence the sequence $R(\phi_p^n)$ satisfies a linear recurrence relation. Define $n = \gamma p^r$, where the integer constants $r \geqslant 0$ and $\gamma$ are as in Lemma 2.6. Applying Lemma 2.6, we see that

$$R(\phi_p^{kn}) = R(\phi_p^n)$$

whenever $k$ is coprime to $n$. Hence the sequence $R(\phi_p^n)$ takes an infinite number of values infinitely many times, and so it cannot satisfy a linear recurrence by a result of Myerson and van der Poorten [23, Proposition 2], giving a contradiction.

Now we consider the general case of a tame endomorphism $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$, $d \geq 1$. According to Lemma 2.5 we have

$$R(\phi_p^n) = |\mathrm{Coker}(1 - \phi_p^n)| = |\det(\Phi_p^n - \mathrm{Id})|_p^{-1} = \prod_{i=1}^{d} |\lambda_i^n - 1|_p^{-1},$$

where $\Phi_p$ is a matrix of $\phi_p$ and $\lambda_1, \ldots, \lambda_d \in \overline{\mathbb{Q}}_p$ are the eigenvalues of $\Phi_p$, including multiplicities. The polynomial $\prod_{i=1}^{d}(X - \lambda_i)$ has coefficients in $\mathbb{Z}_p$; moreover, $|\lambda_i|_p \leq 1$ for every $i \in \{1, \ldots, d\}$ (see [12]). If $|\lambda_i|_p < 1$ for every $i \in \{1, \ldots, d\}$, then $R(\phi_p^n) = \prod_{i=1}^{d} |\lambda_i^n - 1|_p^{-1} = 1$ for all $n \in \mathbb{N}$. Hence the radius of convergence of $R_{\phi_p}(z)$ equals 1 and $R_{\phi_p}(z) = \frac{1}{1-z}$ is a rational function. If $|\lambda_i|_p = 1$, for some $i \in \{1, \ldots, d\}$, then (2.2) implies

$$\frac{1}{n^d} \ll \prod_{i=1}^{d} |\lambda_i^n - 1|_p \leq 1. \tag{2.3}$$

Hence the radius of convergence of $R_{\phi_p}(z)$ equals 1 by the Cauchy–Hadamard formula and (2.3). Now for the proof of the theorem it remains to show that $R_{\phi_p}(z)$ is irrational if $|\lambda_i|_p = 1$ for some $i \in \{1, \ldots, d\}$. Then $R_{\phi_p}(z)$ has the unit circle as a natural boundary by Lemma 2.2 and by the Pólya–Carlson theorem. For a contradiction, assume that $R_{\phi_p}(z)$ is rational. Then Lemma 2.2 implies that so is $Z_p(z) = \sum_{n=1}^{\infty} R(\phi_p^n) z^n$. Hence the sequence $R(\phi_p^n)$ satisfies a linear recurrence relation. Define $n = \gamma p^r$, where the integer constants $r \geqslant 0$ and $\gamma$ are as in Lemma 2.6. Applying Lemma 2.6, we see that $R(\phi_p^{kn}) = R(\phi_p^n)$ whenever $k$ is coprime to $n$. Hence the sequence $R(\phi_p^n)$ takes an infinite number of values infinitely many times, and so it cannot satisfy a linear recurrence [23, Proposition 2], giving a contradiction. ∎

## 3. The rationality of the coincidence Reidemeister zeta function for endomorphisms of finitely generated torsion-free nilpotent groups

EXAMPLE 3.1 ([12, Example 1.3]). Let $G = \mathbb{Z}$ be the infinite cyclic group, written additively, and let

$$\phi\colon \mathbb{Z} \to \mathbb{Z},\ x \mapsto d_\varphi x, \quad \text{and} \quad \psi\colon \mathbb{Z} \to \mathbb{Z},\ x \mapsto d_\psi x,$$

for $d_\phi, d_\psi \in \mathbb{Z}$. The coincidence Reidemeister number $R(\phi, \psi)$ of endomorphisms $\phi, \psi$ of an abelian group $G$ coincides with the cardinality of the quotient group $\mathrm{Coker}(\phi - \psi) = G/\mathrm{Im}(\phi - \psi)$ (or $\mathrm{Coker}(\psi - \phi) = G/\mathrm{Im}(\psi - \phi)$). Hence

$$R(\phi^n, \psi^n) = \begin{cases} |d_\psi^n - d_\phi^n| & \text{if } d_\phi^n \neq d_\psi^n, \\ \infty & \text{otherwise.} \end{cases}$$

Consequently, $(\phi, \psi)$ is tame precisely when $|d_\phi| \neq |d_\psi|$ and, in this case,

$$R_{\phi,\psi}(z) = \frac{1 - d_2 z}{1 - d_1 z}, \quad \text{where} \quad d_1 = \max\{|d_\phi|, |d_\psi|\} \text{ and } d_2 = \frac{d_\phi d_\psi}{d_1}.$$

This simple example (or at least special cases of it) is known. The aim of the current section is to generalize this example to finitely generated torsion-free nilpotent groups.

LEMMA 3.2. *Let $\phi, \psi : G \to G$ be two automorphisms of a finitely generated group $G$. Elements $x, y \in G$ are $\psi^{-1}\phi$-conjugate if and only if $\psi(x)$ and $\psi(y)$ are $(\phi, \psi)$-conjugate. Therefore, $R(\psi^{-1}\phi)$ is equal to $R(\phi, \psi)$. For a tame pair of commuting automorphisms $\phi, \psi : G \to G$ the coincidence Reidemeister zeta function $R_{\phi,\psi}(z)$ is equal to $R_{\psi^{-1}\phi}(z)$.*

*Proof.* If $x$ and $y$ are $\psi^{-1}\phi$-conjugate, then there exists $g \in G$ such that $x = gy\psi^{-1}\phi(g^{-1})$. This implies $\psi(x) = \psi(g)\psi(y)\phi(g^{-1})$, hence $\psi(x)$ and $\psi(y)$ are $(\phi, \psi)$-conjugate. The opposite statement follows if we reverse the direction of the previous implications. The equality of the Reidemeister numbers is then straightforward. Additionally, if $\phi$ and $\psi$ are commuting, then

$$R((\psi^{-1}\phi)^n) = R(\psi^{-n}\phi^n) = R(\phi^n, \psi^n)$$

and it follows that $R_{\psi^{-1}\phi}(z) = R_{\phi,\psi}(z)$. ∎

We assume $X$ is a connected, compact polyhedron and $f : X \to X$ is a continuous map. The *Lefschetz zeta function* of the discrete dynamical system $f^n$ is defined by $L_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right)$, where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr}[f_{*k}^n : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})]$$

is the *Lefschetz number* of the iteration $f^n$ of $f$. The Lefschetz zeta function is a rational function of $z$ and is given by

$$L_f(z) = \prod_{k=0}^{\dim X} \det(I - f_{*k}.z)^{(-1)^{k+1}}.$$

In this section we consider a finitely generated torsion-free nilpotent group $\Gamma$. It is well known [19] that such a $\Gamma$ is a uniform discrete subgroup of a simply connected nilpotent Lie group $G$ (uniform means that the coset space $G/\Gamma$ is compact). The coset space $M = G/\Gamma$ is called a *nilmanifold*. Since $\Gamma = \pi_1(M)$ and $M$ is $K(\Gamma, 1)$, every endomorphism $\phi : \Gamma \to \Gamma$ can be realized by a self-map $f : M \to M$ such that $f_* = \phi$ and thus $R(f) = R(\phi)$. Any endomorphism $\phi : \Gamma \to \Gamma$ can be uniquely extended to an endomorphism $F : G \to G$. Let $\tilde{F} : \tilde{G} \to \tilde{G}$ be the corresponding Lie algebra endomorphism induced from $F$ and let $\operatorname{Spectr}(\tilde{F})$ be the set of eigenvalues of $\tilde{F}$.

LEMMA 3.3 (cf. [9, Theorem 23] and [11, Theorem 5]). *Let $\phi : \Gamma \to \Gamma$ be a tame endomorphism of a finitely generated torsion-free nilpotent group and $f$ as above. Then $R_\phi(z) = R_f(z)$ is a rational function and*

$$R_\phi(z) = R_f(z) = L_f((-1)^s z)^{(-1)^r}, \tag{3.1}$$

*where $s$ is the number of $\mu \in \operatorname{Spectr}(\tilde{F})$ such that $\mu < -1$ and $r$ is the number of real eigenvalues of $\tilde{F}$ whose absolute value is $> 1$.*

Every pair of automorphisms $\phi, \psi : \Gamma \to \Gamma$ of a finitely generated torsion-free nilpotent group $\Gamma$ can be realized by a pair of homeomorphisms $f, g : M \to M$ such that $f_* = \phi$, $g_* = \psi$ and thus $R(g^{-1}f) = R(\psi^{-1}\phi) = R(\phi, \psi)$. An automorphism $\psi^{-1}\phi : \Gamma \to \Gamma$ can be uniquely extended to an automorphism $L : G \to G$. Let $\tilde{L} : \tilde{G} \to \tilde{G}$ be the corresponding Lie algebra automorphism.

Lemmas 3.2 and 3.3 imply the following theorem.

THEOREM 3.4. *Let $\phi, \psi : \Gamma \to \Gamma$ be a tame pair of commuting automorphisms of a finitely generated torsion-free nilpotent group $\Gamma$ and $f, g$ as above. Then $R_{\phi,\psi}(z)$ is a rational function and*

$$R_{\phi,\psi}(z) = R_{\psi^{-1}\phi}(z) = R_{g^{-1}f}(z) = L_{g^{-1}f}((-1)^s z)^{(-1)^r}, \tag{3.2}$$

*where $s$ is the number of $\mu \in \operatorname{Spectr}(\tilde{L})$ such that $\mu < -1$, and $r$ is the number of real eigenvalues of $\tilde{L}$ whose absolute value is $> 1$.*

Following [4, p. 3], for every group $G$ we can define the $k$th commutator group $\gamma_k(G)$ inductively by $\gamma_1(G) := G$ and $\gamma_{k+1}(G) := [G, \gamma_k(G)]$. We use the convention that the commutator $[a, b]$ is $a^{-1}b^{-1}ab$ for any $a, b \in G$. Let $G$ be a group. For a subgroup $H \leqslant G$, we define the *isolator* $\sqrt[G]{H}$ of $H$ in $G$ by

$$\sqrt[G]{H} = \{g \in G \mid g^n \in H \text{ for some } n \in \mathbb{N}\}.$$

Note that the isolator of a subgroup $H \leqslant G$ does not have to be a subgroup in general. For example, the isolator of the trivial subgroup is the set of torsion elements of $G$.

LEMMA 3.5 (cf. [4, Lemmas 1.1.2 and 1.1.4)]). *Let $G$ be a group. Then*

(i) *for all $k \in \mathbb{N}$, $\sqrt[G]{\gamma_k(G)}$ is a fully characteristic subgroup of $G$,*
(ii) *for all $k \in \mathbb{N}$, the factor $G/\sqrt[G]{\gamma_k(G)}$ is torsion-free,*
(iii) *for all $k, l \in \mathbb{N}$, $[\sqrt[G]{\gamma_k(G)}, \sqrt[G]{\gamma_l(G)}] \leqslant \sqrt[G]{\gamma_{k+l}(G)}$,*
(iv) *for all $k, l \in \mathbb{N}$ such that $l \geq k$ if $M := \sqrt[G]{\gamma_l(G)}$, then*

$$\sqrt[G/M]{\gamma_k(G/M)} = \sqrt[G]{\gamma_k(G)}/M.$$

We define the *adapted lower central series* of a group $G$ to be

$$G = \sqrt[G]{\gamma_1(G)} \geqslant \cdots \geqslant \sqrt[G]{\gamma_k(G)} \geqslant \cdots ,$$

where $\gamma_k(G)$ is the $k$th commutator of $G$. The adapted lower central series will terminate if and only if $G$ is a torsion-free nilpotent group. Moreover, all factors $\sqrt[G]{\gamma_k(G)}/\sqrt[G]{\gamma_{k+1}(G)}$ are torsion-free.

We are particularly interested in the case where $G$ is a finitely generated torsion-free nilpotent group. In this case the factors of the adapted lower central series are finitely generated torsion-free abelian groups, i.e. for all $k \in \mathbb{N}$ we have $\sqrt[G]{\gamma_k(G)}/\sqrt[G]{\gamma_{k+1}(G)} \cong \mathbb{Z}^{d_k}$ for some $d_k \in \mathbb{N}$.

Let $N$ be a normal subgroup of $G$ and $\phi, \psi \in \mathrm{End}(G)$ with $\phi(N) \subseteq N, \psi(N) \subseteq N$. We denote the restriction of $\phi$ to $N$ by $\phi|_N$, $\psi$ to $N$ by $\psi|_N$ and the induced endomorphisms on the quotient $G/N$ by $\phi', \psi'$ respectively. Then we get the following commutative diagrams with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i"] \arrow[d, "{\phi|_N,\psi|_N}"] & G \arrow[r, "p"] \arrow[d, "{\phi,\psi}"] & G/N \arrow[r] \arrow[d, "{\phi',\psi'}"] & 1 \\
1 \arrow[r] & N \arrow[r, "i"] & G \arrow[r, "p"] & G/N \arrow[r] & 1
\end{tikzcd} \tag{3.3}$$

where $i$ denotes the injection and $p$ the projection, as usual. Note that both $i$ and $p$ induce functions $\hat{i}, \hat{p}$ on the set of Reidemeister classes so that the sequence

$$\mathfrak{R}[\phi|_N, \psi|_N] \xrightarrow{\hat{i}} \mathfrak{R}[\phi, \psi] \xrightarrow{\hat{p}} \mathfrak{R}[\phi', \psi'] \longrightarrow 0$$

is exact, i.e. $\hat{p}$ is surjective and $\hat{p}^{-1}[1] = \mathrm{Im}(\hat{i})$, where 1 is the identity element of $G/N$ (see also [16]). Let $Z(G)$ denote the center of a group $G$.

LEMMA 3.6. *If $R(\phi|_N, \psi|_N) < \infty, R(\phi', \psi') < \infty$ and $N \subseteq Z(G)$, then*

$$R(\phi, \psi) \leq R(\phi|_N, \psi|_N) R(\phi', \psi').$$

*Proof.* Let

$$\mathfrak{R}(\phi|_N, \psi|_N) = \{[n_1]_{\phi|_N, \psi|_N}, \ldots, [n_{R(\phi|_N, \psi|_N)}]_{\phi|_N, \psi|_N}\}$$

and

$$\mathfrak{R}(\phi', \psi') = \{[g_1 N]_{\phi', \psi'}, \ldots, [g_{R(\phi', \psi')} N]_{\phi', \psi'}\}$$

be the $(\phi|_N, \psi|_N)$-Reidemeister classes and $(\phi', \psi')$-Reidemeister classes respectively. Let $g \in G$. Then $gN \in [g_i N]_{\phi', \psi'}$ for some $i$, so there exists $hN \in G/N$ such that

$$gN = \psi'(hN) g_i N \phi'(hN)^{-1} = \psi(h) g_i \phi(h)^{-1} N.$$

It follows that there exists $n \in N$ such that $g = \psi(h) g_i \phi(h)^{-1} n$. In turn $n \in [n_j]_{\phi|_N, \psi|_N}$ for some $j$, so there is $m \in N$ such that $n = \psi|_N(m) n_j \phi|_N(m)^{-1}$. Since $n, m, n_j, \psi|_N(m), \phi|_N(m) \in N \subseteq Z(G)$, it follows that

$$g = \psi(hm)(g_i n_j)\phi(hm)^{-1},$$

i.e. $g \in [g_i n_j]_{\phi, \psi}$. Since this is true for arbitrary $g \in G$ we obtain

$$R(\phi, \psi) \leq R(\phi|_N, \psi|_N) R(\phi', \psi'). \blacksquare$$

THEOREM 3.7. *Let $N$ be a finitely generated, torsion-free, nilpotent group and*

$$N = \sqrt[N]{\gamma_1(N)} \geqslant \cdots \geqslant \sqrt[N]{\gamma_{c+1}(N)} = 1$$

*be the adapted lower central series of $N$. Suppose that $R(\phi, \psi) < \infty$ and $R(\phi_k, \psi_k) < \infty$ for a pair $\phi, \psi$ of endomorphisms of $N$ and for every pair $\phi_k, \psi_k$ of induced endomorphisms on the finitely generated torsion-free abelian factors*

$$\sqrt[N]{\gamma_k(N)} / \sqrt[N]{\gamma_{k+1}(N)} \cong \mathbb{Z}^{d_k}, \qquad d_k \in \mathbb{N},\ 1 \leq k \leq c,$$

*then*

$$R(\phi, \psi) = \prod_{k=1}^{c} R(\phi_k, \psi_k).$$

*Proof.* We will prove the product formula for the coincidence Reidemeister numbers by induction on the length of an adapted lower central series. Let us denote $\sqrt[N]{\gamma_k(N)}$ by $N_k$. If $c = 1$, the result follows trivially. Let $c > 1$ and assume the product formula holds for the adapted lower central series of length $c - 1$. Let $\phi, \psi \in \mathrm{End}(N)$, then $\phi(N_c) \subseteq N_c, \psi(N_c) \subseteq N_c$ by Lemma 3.5(i) and hence we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \phi_c,\psi_c} & & \Big\downarrow{\scriptstyle \phi,\psi} & & \Big\downarrow{\scriptstyle \phi',\psi'} & & \\
1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1
\end{array}$$

where $\phi_c, \psi_c$ are induced endomorphisms on $N_c$. The quotient $N/N_c$ is a finitely generated, nilpotent group with the adapted lower central series

$$N/N_c = N_1/N_c \geqslant N_2/N_c \geqslant \cdots \geqslant N_{c-1}/N_c \geqslant N_c/N_c = 1$$

of length $c-1$.

Every factor of this series is of the form $(N_k/N_c)/(N_{k+1}/N_c) \cong N_k/N_{k+1}$ by the third isomorphism theorem, hence it is also torsion-free. Moreover, because of this natural isomorphism we know that for every induced pair of endomorphisms $(\phi'_k, \psi'_k)$ on $(N_k/N_c)/(N_{k+1}/N_c)$ it is true that $R(\phi'_k, \psi'_k) = R(\phi_k, \psi_k)$. The assumptions of the theorem imply that $R(\phi', \psi') < \infty$ and $R(\phi_c, \psi_c) < \infty$.

Moreover, let $[g_1N_c]_{\phi',\psi'}, \dots, [g_nN_c]_{\phi',\psi'}$ be the $(\phi', \psi')$-Reidemeister classes and $[c_1]_{\phi_c,\psi_c}, \dots, [c_m]_{\phi_c,\psi_c}$ the $(\phi_c, \psi_c)$-Reidemeister classes. Since $N_c \subseteq Z(N)$, by Lemma 3.6 we find that $R(\phi, \psi) \leqslant R(\phi_c, \psi_c)R(\phi', \psi')$.

To prove the opposite inequality it suffices to prove that every class $[c_ig_j]_{\phi,\psi}$ represents a different $(\phi, \psi)$-Reidemeister class. Then we obtain

$$R(\phi, \psi) = R(\phi_c, \psi_c)R(\phi', \psi')$$

and the theorem follows from the induction hypothesis.

Suppose that there exists some $h \in N$ such that $c_ig_j = \psi(h)c_ag_b\phi(h)^{-1}$. Then by taking the projection to $N/N_c$ we find that

$$g_jN_c = p(c_ig_j) = p(\psi(h)c_ag_b\phi(h)^{-1}) = \psi'(hN_c)(g_bN_c)\phi'(hN_c)^{-1},$$

hence $[g_jN_c]_{\phi',\psi'} = [g_bN_c]_{\phi',\psi'}$. Now assume that $c_ig_j = \psi(h)c_ag_j\phi(h)^{-1}$. If $h \in N_c \subseteq Z(N)$, then $c_ig_j = \psi(h)c_a\phi(h)^{-1}g_j$ and consequently

$$[c_i]_{\phi_c,\psi_c} = [c_a]_{\phi_c,\psi_c}.$$

So let us assume that $h \notin N_c$ and that $N_k$ is the smallest group in the central series which contains $h$. Then

$$\begin{aligned}
c_ig_j &= \psi(h)c_ag_j\phi(h)^{-1} \iff \\
g_jc_i &= \psi(h)c_ag_j\phi(h)^{-1} \iff \\
c_i &= g_j^{-1}\psi(h)c_ag_j\phi(h)^{-1} \iff \\
c_i &= g_j^{-1}\psi(h)g_j\phi(h)^{-1}c_a \iff \\
c_ic_a^{-1} &= g_j^{-1}\psi(h)g_j\phi(h)^{-1}
\end{aligned}$$

and therefore

$$\begin{aligned} c_i c_a^{-1} N_{k+1} &= g_j^{-1}\psi(h) g_j \phi(h)^{-1} N_{k+1} \\ &= [g_j, \psi(h)^{-1}](\psi(h)\phi(h)^{-1}) N_{k+1}. \end{aligned}$$

As $c_i c_a^{-1} \in N_c \subseteq N_{k+1}$ and $[g_j, \psi(h)^{-1}] \in N_{k+1}$ , we find that

$$(\phi_k)(hN_{k+1}) = (\psi_k)(hN_{k+1}).$$

That means that the set of coincidence points satisfies

$$\begin{aligned} \mathrm{Coin}(\phi'_k, \psi'_k) := \big\{ g(N_{k+1}/N_c) \in (N_k/N_c)/(N_{k+1}/N_c) \mid \phi'_k(g(N_{k+1}/N_c)) \\ = \psi'_k(g(N_{k+1}/N_c)) \big\} \neq \{1(N_{k+1}/N_c)\}, \end{aligned}$$

i.e. it is not a singleton. This in turn implies that $R(\phi', \psi') = \infty$, which is a contradiction. ∎

THEOREM 3.8. *Let $\phi, \psi\colon N \to N$ be a tame pair of endomorphisms of a finitely generated torsion-free nilpotent group $N$. Let $c$ denote the nilpotency class of $N$ and, for $1 \le k \le c$, let $\phi_k, \psi_k\colon G_k \to G_k$, $1 \le k \le c$, denote the tame pairs of induced endomorphisms of the finitely generated torsion-free abelian factor groups $G_k = N_k/N_{k+1} = \sqrt[N]{\gamma_k(N)}/\sqrt[N]{\gamma_{k+1}(N)} \cong \mathbb{Z}^{d_k}$, for some $d_k \in \mathbb{N}$, that arise from an adapted lower central series of $N$. Then the following statements hold:*

(1) *For each $n \in \mathbb{N}$,*

$$R(\phi^n, \psi^n) = \prod_{k=1}^{c} R(\phi_k^n, \psi_k^n).$$

(2) *For $1 \le k \le c$, let*

$$\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}\colon G_{k,\mathbb{Q}} \to G_{k,\mathbb{Q}}$$

*denote the extensions of $\phi_k, \psi_k$ to the divisible hull $G_{k,\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} G_k \cong \mathbb{Q}^{d_k}$ of $G_k$. Suppose that each pair of endomorphisms $\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}$ is simultaneously triangularizable. Let $\xi_{k,1}, \dots, \xi_{k,d_k}$ and $\eta_{k,1}, \dots, \eta_{k,d_k}$ be the eigenvalues of $\phi_{k,\mathbb{Q}}$ and $\psi_{k,\mathbb{Q}}$ in the field $\mathbb{C}$, including multiplicities, ordered in such a way that, for $n \in \mathbb{N}$, the eigenvalues of $\phi_{k,\mathbb{Q}}^n - \psi_{k,\mathbb{Q}}^n$ are $\xi_{k,1}^n - \eta_{k,1}^n, \dots, \xi_{k,d_k}^n - \eta_{k,d_k}^n$. Then for each $n \in \mathbb{N}$,*

$$R(\phi_k^n, \psi_k^n) = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|. \tag{3.4}$$

(3) *Moreover, suppose that $|\xi_{k,i}| \neq |\eta_{k,i}|$ for $1 \le k \le c$, $1 \le i \le d_k$. If $\phi, \psi$ is a tame pair of endomorphisms of $N$ and $\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}$, $1 \le k \le c$, are simultaneously triangularizable pairs of endomorphisms of $G_{k,\mathbb{Q}}$, then $R_{\phi,\psi}(z)$ is a rational function.*

*Proof.* The coincidence Reidemeister number $R(\phi,\psi)$ of automorphisms $\phi,\psi$ of an abelian group $G$ coincides with the cardinality of the quotient group $\mathrm{Coker}(\phi-\psi) = G/\mathrm{Im}(\phi-\psi)$ (or $\mathrm{Coker}(\psi-\phi) = G/\mathrm{Im}(\psi-\phi)$).

For $1 \leq k \leq c$, let tame pairs $\phi_k, \psi_k \colon G_k \to G_k$ of induced endomorphisms of the finitely generated torsion-free abelian factor groups $G_k = N_k/N_{k+1} \cong \mathbb{Z}^{d_k}$ be represented by $A_k, B_k \in M_{d_k}(\mathbb{Z})$ respectively. There is a diagonal integer matrix

$$C_k = \mathrm{diag}(c_1, \dots, c_{d_k})$$

such that $C_k = M_k(A_k - B_k)N_k$ where $M_k$ and $N_k$ are unimodular matrices.

Now we have

$$|\det C_k| = |\det(A_k - B_k)|$$

and the order of cokernel of $\phi_k - \psi_k$ is the order of the group $\bigoplus_{i=1}^{d_k} \mathbb{Z}/c_i\mathbb{Z}$. Thus, the order of cokernel of $\phi_k - \psi_k$ is

$$|\mathrm{Coker}(\phi_k - \psi_k)| = |c_1 \cdots c_{d_k}| = |\det C_k| = |\det(\phi_k - \psi_k)|.$$

Then for each $n \in \mathbb{N}$ and $1 \leq k \leq c$,

$$\begin{aligned} R(\phi_k^n, \psi_k^n) &= |\mathrm{Coker}(\phi_k^n - \psi_k^n)| = |\det(\phi_k^n - \psi_k^n)| \\ &= |\det(\phi_{k,\mathbb{Q}}^n - \psi_{k,\mathbb{Q}}^n)| = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|. \end{aligned}$$

Now we will prove the rationality of $R_{\phi,\psi}(z)$. We investigate the absolute values in the product $R(\phi_k^n, \psi_k^n) = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|$, $1 \leq k \leq c$. Complex eigenvalues $\xi_{k,i}$ with non-zero imaginary part in the spectrum of $\phi_{k,\mathbb{Q}}$, respectively $\eta_{k,i}$ in the spectrum of $\psi_{k,\mathbb{Q}}$, appear in pairs with their complex conjugate $\overline{\xi_{k,i}}$, respectively $\overline{\eta_{k,i}}$.

Moreover, such pairs can be lined up with one another in a simultaneous triangularization as follows. Write $\phi_{k,\mathbb{C}}, \psi_{k,\mathbb{C}}$ for the induced endomorphisms of the $\mathbb{C}$-vector space $V = \mathbb{C} \otimes_{\mathbb{Q}} G \cong \mathbb{C}^{d_k}$. If $v \in V$ is at the same time an eigenvector of $\phi_{k,\mathbb{C}}$ with complex eigenvalue $\xi_{k,d_k}$ and an eigenvector of $\psi_{k,\mathbb{C}}$ with eigenvalue $\eta_{k,d_k}$, then there is $w \in V$ such that $w$ is, at the same time, an eigenvector of $\phi_{k,\mathbb{C}}$ with eigenvalue $\overline{\xi_{k,d_k}} \neq \xi_{k,d_k}$ and an eigenvector of $\psi_{k,\mathbb{C}}$ with eigenvalue $\overline{\eta_{k,d_k}}$, possibly equal to $\eta_{k,d_k}$. Thus we can start our complete flag of $\{\phi,\psi\}$-invariant subspaces of $V$ with $\{0\} \subset \langle v \rangle \subset \langle v, w \rangle$, and proceed with $V/\langle v, w\rangle$ by induction to produce the rest of the flag in the same way, treating complex eigenvalues of $\psi_{k,\mathbb{C}}$ in the same way as they appear. If at least one of $\xi_{k,i}, \eta_{k,i}$ is complex so that these eigenvalues of $\phi_{k,\mathbb{Q}}$ and $\psi_{k,\mathbb{Q}}$ are paired with eigenvalues $\xi_{k,j} = \overline{\xi_{k,i}}, \eta_{k,j} = \overline{\eta_{k,i}}$, for suitable $j \neq i$, as discussed above, we see that

$$\left|\xi_{k,i}^n - \eta_{k,i}^n\right| \left|\xi_{k,j}^n - \eta_{k,j}^n\right| = \left|\xi_{k,i}^n - \eta_{k,i}^n\right|^2 = \left(\xi_{k,i}^n - \eta_{k,i}^n\right) \cdot \left(\overline{\xi_{k,i}}^n - \overline{\eta_{k,i}}^n\right).$$

If $\xi_{k,i}$ and $\eta_{k,i}$ are both real eigenvalues of $\phi_{k,\mathbb{Q}}$ and $\psi_{k,\mathbb{Q}}$, not paired up with another pair of eigenvalues, then exactly as in Example 3.1 above we have $|\xi_{k,i}^n - \eta_{k,i}^n| = \delta_{1,k,i}^n - \delta_{2,k,i}^n$, where $\delta_{1,k,i} = \max\{|\xi_{k,i}|, |\eta_{k,i}|\}$ and $\delta_{2,k,i} = \xi_{k,i} \cdot \eta_{k,i}/\delta_{1,k,i}$. Hence we can expand each product $R(\phi_k^n, \psi_k^n)$, $1 \leq k \leq c$, using an appropriate symmetric polynomial to obtain for the Reidemeister numbers $R(\phi^n, \psi^n)$ an expression of the form

$$R(\phi^n, \psi^n) = \prod_{k=1}^{c} R(\phi_k^n, \psi_k^n) = \prod_{k=1}^{c} \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n| = \sum_{j \in J} c_j w_j^n, \tag{3.5}$$

where $J$ is a finite index set, $c_j \in \{-1, 1\}$ and $\{w_j \mid j \in J\} \subseteq \mathbb{C} \setminus \{0\}$. Consequently, the coincidence Reidemeister zeta function can be written as

$$R_{\phi,\psi}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n, \psi^n)}{n} z^n\right) = \exp\left(\sum_{j \in J} c_j \sum_{n=1}^{\infty} \frac{(w_j z)^n}{n}\right).$$

and it follows immediately that $R_{\phi,\psi}(z) = \prod_{j \in J}(1 - w_j z)^{-c_j}$ is a rational function. ∎

**4. Gauss congruences for the Reidemeister coincidence numbers.** In this section we use Theorem 3.8 on the rationality of the Reidemeister coincidence zeta function for a tame pair of endomorphisms of a finitely generated torsion-free nilpotent group to prove the Gauss congruences (GR) for the Reidemeister coincidence numbers of iterations of these endomorphisms.

Let $\mu(n)$, $n \in \mathbb{N}$, be the *Möbius function*, i.e.

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes}, \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$

In number theory, the following Gauss congruence for integers is shown:

$$\sum_{d|n} \mu(n/d) \cdot a^d \equiv 0 \pmod{n}$$

for any integer $a$ and any natural number $n$. In the case of a prime power $n = p^r$, the Gauss congruence turns into the Euler congruence. Indeed, for $n = p^r$ the Möbius function $\mu(n/d) = \mu(p^r/d)$ is non-zero only in two cases: when $d = p^r$ and when $d = p^{r-1}$. Therefore, from the Gauss congruence we obtain the Euler congruence

$$a^{p^r} \equiv a^{p^{r-1}} \pmod{p}^r$$

When $(a, n) = 1, n = p^r$, these congruences are equivalent to the following classical Euler theorem:

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

These congruences have been generalized from integers to some other mathematical invariants such as the traces of powers of all integer matrices $A$ and the Lefschetz numbers of iterations of a map (see [20, 27]):

$$\sum_{d|n} \mu(n/d) \cdot \operatorname{tr}(A^d) \equiv 0 \pmod{n}, \tag{4.1}$$

$$\operatorname{tr}(A^{p^r}) \equiv \operatorname{tr}(A^{p^{r-1}}) \pmod{p}^r. \tag{4.2}$$

A. Dold [7] (see also [17, Theorem 3.1.4]) proved by a geometric argument the following congruences (DL) for the Lefschetz numbers of iterations of a map $f$ on a compact ANR $X$ and any natural number $n$:

$$\sum_{d|n} \mu(n/d) \cdot L(f^d) \equiv 0 \pmod{n}. \tag{DL}$$

These congruences are now called the *Dold congruences*. It is also shown in [20] (see also [27, Theorem 9]) that the above congruences (4.1), (4.2) and (DL) are equivalent.

The following Gauss congruences for the Reidemeister numbers of iterations of a tame automorphism $\phi$ of a polycyclic-by-finite group are proven in [14]:

$$\sum_{d|n} \mu(n/d) \cdot R(\phi^d) \equiv 0 \pmod{n}.$$

We divide the proof of the Gauss congruences for a tame pair of endomorphisms of a finitely generated torsion-free nilpotent group into several intermediate results. In some of them we follow [1, Theorem 2.1] and [17, Theorem 3.1.23].

The rationality of the coincidence Reidemeister zeta function $R_{\phi,\psi}(z)$ for a tame pair of endomorphisms of a finitely generated torsion-free nilpotent group implies that

$$R_{\phi,\psi}(z) = \frac{u_1(z)}{v_1(z)} = \frac{\prod_i (1-\beta_i z)}{\prod_j (1-\gamma_j z)}, \tag{4.3}$$

where the numerator and denominator are expanded into linear factors in terms of the roots of the conjugate polynomials $u_1(1/z), v_1(1/z)$, because the factor $z$ does not appear, as follows from $R_{\phi,\psi}(0) = 1$. We can write formula (4.3) in a different way, where $\lambda_i$ is either $\beta_i$ or $\gamma_j$:

$$R_{\phi,\psi}(z) = \prod_{i=1}^{r} (1-\lambda_i z)^{-\rho_i}$$

with all $\lambda_i$ distinct non-zero numbers and $\rho_i$ non-zero integers. Taking the logarithm on both sides of the above equality, we obtain

$$\sum_{k=1}^{\infty} \frac{R(\phi^k, \psi^k)}{k} z^k = \sum_{i=1}^{r} -\rho_i \log(1-\lambda_i z) = \sum_{i=1}^{r} \rho_i \left( \sum_{k=1}^{\infty} \frac{(\lambda_i z)^k}{k} \right)$$
$$= \sum_{k=1}^{\infty} \frac{\sum_{i=1}^{r} \rho_i \lambda_i^k}{k} z^k.$$

This implies

$$R(\phi^k, \psi^k) = \sum_{i=1}^{r} \rho_i \lambda_i^k. \tag{R1}$$

Note that $r$ is the number of zeros and poles of $R_{\phi,\psi}(z)$. This argument tells us that whenever we have a rational expression of $R_{\phi,\psi}(z)$, we can write down every $R(\phi^k, \psi^k)$ directly from the expression.

We consider another generating function associated to the sequence of the Reidemeister numbers $R(\phi^k, \psi^k)$:

$$S_{\phi,\psi}(z) = \frac{d}{dz} \log R_{\phi,\psi}(z).$$

Then it is easy to see that

$$S_{\phi,\psi}(z) = \sum_{k=1}^{\infty} R(\phi^k, \psi^k) z^{k-1} = \sum_{k=1}^{\infty} \sum_{i=1}^{r} \rho_i \lambda_i^k z^{k-1} = \sum_{i=1}^{r} \frac{\rho_i \lambda_i}{1-\lambda_i z}, \tag{4.4}$$

which is a rational function with simple poles and integral residues, and 0 at infinity.

We will show that the numbers $\lambda_i$ are algebraic integers. From the rationality of $S_{\phi,\psi}(z)$ it follows that $S_{\phi,\psi}(z) = u(z)/v(z)$ where $u(z), v(z) \in \mathbb{Q}[z]$. Indeed, the coefficients of $S_{\phi,\psi}(z)$ are integers, hence the coefficients of $u(z)$ and $v(z)$, given as the solutions of a sequence of linear equations, belong to the field of quotients of $\mathbb{Z}$, i.e. to $\mathbb{Q}$. Now we will use the following Fatou lemma (see, for example, [17, Lemma 3.1.31]).

Lemma 4.1. *Let $\sum_{i=0}^{\infty} c_i z^i = u(z)/v(z)$ be a rational function where $c_i$ are integers. Then $u(z), v(z)$ are of the form*

$$u(z) = c_0 + \sum_{i=1}^{p} a_i z^i, \quad v(z) = 1 + \sum_{j=1}^{t} b_j z^j,$$

*where the coefficients $\{a_i\}_{i=1}^{p}, \{b_j\}_{j=1}^{t}$ are integers.*

By the Fatou lemma, the rational function $S_{\phi,\psi}(z)$ can be written as $S_{\phi,\psi}(z) = u(z)/v(z)$, where $u(z)$ and $v(z)$ are of the form

$$u(z) = R(\phi, \psi) + \sum_{i=1}^{p} a_i z^i, \quad v(z) = 1 + \sum_{j=1}^{t} b_j z^j$$

with $a_i$ and $b_j$ integers. Let $\tilde{v}(z)$ be the conjugate polynomial of $v(z)$, i.e. $\tilde{v}(z) = z^t v(1/z)$. Note that

$$\tilde{v}(z) = z^t v\left(\frac{1}{z}\right) = z^t\left(1 + \sum_{i=1}^{t} b_j \frac{1}{z^j}\right) = z^t + b_1 z^{t-1} + \cdots + b_t,$$

and

$$S_{\phi,\psi}(z) = \sum_{i=1}^{r} \frac{\rho_i \lambda_i}{1 - \lambda_i z} = \frac{u(z)}{v(z)},$$

where $1/\lambda_i$ are poles of $S_{\phi,\psi}(z)$. Moreover, since there are $r$ poles (counting with multiplicities) and $\deg(v) = t$, then $t = r$. Consequently, the numbers $\lambda_i$ are algebraic integers, being the roots of the polynomial $\tilde{v}(z)$.

The following can be found in the proof of [1, Theorem 2.1 (3)⇒(5)], or in the proof of [17, Theorem 3.1.23]. We rewrite the proof for completeness.

LEMMA 4.2. *If $\lambda_i$ and $\lambda_j$ are roots of the rational polynomial $\tilde{v}(z)$ which are algebraically conjugate (i.e., $\lambda_i$ and $\lambda_j$ are roots of the same irreducible polynomial), then $\rho_i = \rho_j$.*

*Proof.* Let $\Sigma = \mathbb{Q}(\lambda_1, \ldots, \lambda_r) \subset \mathbb{C}$ be the splitting field of $\tilde{v}(z)$ and let $\sigma$ be an automorphism of $\Sigma$ over $\mathbb{Q}$, i.e., $\sigma$ is the identity on $\mathbb{Q}$. The group of all such automorphisms is called the *Galois group* of $\Sigma$. Since the $\sigma(\lambda_i)$ are again the roots of $\tilde{v}(z)$, we have $\sigma(\lambda_i) = \lambda_{\sigma(i)}$. That is, $\sigma$ induces a permutation $\sigma$ of $\{1, \ldots, r\}$. Applying $\sigma$ to the sequence $R(\phi^k, \psi^k)$, we obtain

$$\sigma(R(\phi^k, \psi^k)) = \sigma\Big(\sum_{i=1}^{r} \rho_i \lambda_i^k\Big) = \sum_{i=1}^{r} \rho_i \sigma(\lambda_i)^k = \sum_{i=1}^{r} \rho_i \lambda_{\sigma(i)}^k = \sum_{i=1}^{r} \rho_{\sigma^{-1}(i)} \lambda_i^k.$$

Since the $R(\phi^k, \psi^k)$ are integers, $\sigma(R(\phi^k, \psi^k)) = R(\phi^k, \psi^k)$ and consequently

$$\sum_{i=1}^{r} \rho_{\sigma^{-1}(i)} \lambda_i^k = \sum_{i=1}^{r} \rho_i \lambda_i^k \quad \text{for } k = 1, 2, \ldots.$$

Writing the first $r$ of the above equations in matrix form gives

$$\begin{bmatrix} \lambda_1 & \cdots & \lambda_r \\ \vdots & & \vdots \\ \lambda_1^r & \cdots & \lambda_r^r \end{bmatrix} \begin{bmatrix} \rho_{\sigma^{-1}(1)} \\ \vdots \\ \rho_{\sigma^{-1}(r)} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & \lambda_r \\ \vdots & & \vdots \\ \lambda_1^r & \cdots & \lambda_r^r \end{bmatrix} \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_r \end{bmatrix}.$$

Since the $\lambda_i$'s are distinct, the matrices in this equation are non-singular (this follows from the properties of the Vandermonde determinant). Thus $\rho_i = \rho_{\sigma^{-1}(i)}$ for all $i = 1, \ldots, r$. On the other hand, it is known that the Galois group acts transitively on the set of algebraically conjugate roots. Since $\lambda_i$ and $\lambda_j$ are conjugate roots of $\tilde{v}(z)$, we can choose $\sigma$ in the Galois group so that $\sigma(\lambda_i) = \lambda_j$. Hence $\sigma(i) = j$ and so $\rho_i = \rho_j$. ∎

The following lemma shows that it is equivalent to study the behavior of the sequences $\operatorname{tr} A^m$, where $A \in M_n(\mathbb{Z})$, and the sequences $L(f^m)$ of Lefschetz numbers, where $f : X \to X$ is a map of a finite complex.

LEMMA 4.3 ([1, Theorem 2.1], [17, Proposition 3.1.12]). *Let $X$ be the bouquet of $n_2$ circles and $n_1$ 2-spheres. Then for each pair of integral matrices $A_e \in M_{n_1}(\mathbb{Z}), A_o \in M_{n_2}(\mathbb{Z})$ there exists a self-map $f : X \to X$ such that for every $k \in \mathbb{N}$ the Lefschetz numbers satisfy*

$$L(f^k) = \operatorname{tr} A_e^k - \operatorname{tr} A_o^k.$$

Finally we prove the main result of this section.

THEOREM 4.4. *Let $\phi, \psi$ be a tame pair of endomorphisms of a finitely generated torsion-free nilpotent group. Then under the assumptions of Theorem 3.8(3) for every $n \in \mathbb{N}$ we have*

$$\text{(GR)} \qquad \sum_{k|n} \mu(n/k) \cdot R(\phi^k, \psi^k) \equiv 0 \pmod{n}.$$

*Proof.* Under the assumptions of Theorem 3.8(3) the coincidence Reidemeister zeta function is a rational function. Let $\tilde{v}(z)$ be the conjugate polynomial of $v(z)$, i.e. $\tilde{v}(z) = z^r v(1/z)$. Let $\tilde{v}(z) = \prod_{\alpha=1}^{s} \tilde{v}_\alpha(z)$ be the decomposition of the monic integral polynomial $\tilde{v}(z)$ into irreducible polynomials $\tilde{v}_\alpha(z)$ of degree $r_\alpha$. Since the degree of $\tilde{v}(z)$ is $r$, it follows that $r = \sum_{\alpha=1}^{s} r_\alpha$ and

$$\begin{aligned}
\tilde{v}(z) &= z^r + b_1 z^{r-1} + \cdots + b_{r-1} z + b_r \\
&= \prod_{\alpha=1}^{s} (z^{r_\alpha} + b_1^\alpha z^{r_\alpha - 1} + \cdots + b_{r_\alpha - 1}^\alpha z + b_{r_\alpha}^\alpha) \\
&= \prod_{\alpha=1}^{s} \tilde{v}_\alpha(z).
\end{aligned}$$

If the $\lambda_i^{(\alpha)}$ are the roots of $\tilde{v}_\alpha(z)$, then, by Lemma 4.2, $\rho_i^{(\alpha)} = \rho_\alpha$ for every $i = 1, \ldots, r_\alpha$ in the formula (R1). Consequently, we can rewrite (R1) as

$$\begin{aligned}
R(\phi^k, \psi^k) &= \sum_{\alpha=1}^{s} \rho_\alpha \Big( \sum_{i=1}^{r_\alpha} (\lambda_i^{(\alpha)})^k \Big) \\
&= \sum_{\rho_\alpha > 0} \rho_\alpha \Big( \sum_{i=1}^{r_\alpha} (\lambda_i^{(\alpha)})^k \Big) - \sum_{\rho_\alpha < 0} |\rho_\alpha| \Big( \sum_{i=1}^{r_\alpha} (\lambda_i^{(\alpha)})^k \Big).
\end{aligned}$$

Consider the $r_\alpha \times r_\alpha$ integral companion matrices

$$M_\alpha = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b^\alpha_{r_\alpha} \\ 1 & 0 & \cdots & 0 & -b^\alpha_{r_\alpha - 1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b^\alpha_1 \end{bmatrix}.$$

The characteristic polynomial of $M_\alpha$ is $\det(zI - M_\alpha) = \tilde{v}_\alpha(z)$ and therefore $\lambda_i^{(\alpha)}$ are the eigenvalues of $M_\alpha$. This implies that

$$R(\phi^k, \psi^k) = \sum_{\alpha=1}^{s} \rho_\alpha \cdot \operatorname{tr} M_\alpha^k.$$

Next, we split the set $A = A^+ \cup A^-$ of indices $\alpha$ into two subsets according to the sign of $\rho_\alpha$.

Denote

$$M_+ := \bigoplus_{\alpha \in A^+} \bigoplus_{i=1}^{\rho_\alpha} M_\alpha, \quad M_- := \bigoplus_{\alpha \in A^-} \bigoplus_{i=1}^{|\rho_\alpha|} M_\alpha,$$

where a direct sum " $\bigoplus_{i=1}^{\rho_\alpha} M_\alpha$" means a block diagonal matrix consisting of $\rho_\alpha$ copies of $M_\alpha$ along the main diagonal, with zeros everywhere else. Observe that $M_+$ and $M_-$ are integral square matrices. Then

$$\text{(R2)} \qquad R(\phi^k, \psi^k) = \sum_{\alpha \in A^+} \rho_\alpha \operatorname{tr} M_\alpha^k - \sum_{\alpha \in A^-} |\rho_\alpha| \operatorname{tr} M_\alpha^k = \operatorname{tr} M_+^k - \operatorname{tr} M_-^k.$$

By Lemma 4.3 and the trace formula (R2) there exists a self-map $f : X \to X$ of the bouquet $X$ of circles and 2-spheres such that for every $k \in \mathbb{N}$ we have $R(\phi^k, \psi^k) = \operatorname{tr} M_+^k - \operatorname{tr} M_-^k = L(f^k)$. Hence the Dold congruences (DL) for the Lefschetz numbers $L(f^k)$ imply the Gauss congruences for $R(\phi^k, \psi^k)$. ∎

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## *REFERENCES*

[1] I. K. Babenko and S. A. Bogatyĭ, *The behavior of the index of periodic points under iterations of a mapping*, Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), 3–31 (in Russian); English transl.: Math. USSR-Izv. 38 (1992), 1–26.

[2] J. Bell, R. Miles and T. Ward, *Towards a Pólya–Carlson dichotomy for algebraic dynamics*, Indag. Math. (N.S.) 25 (2014), 652–668.

[3] J. Byszewski and G. Cornelissen, *Dynamics on abelian varieties in positive characteristic* (with an appendix by R. Royals and T. Ward), Algebra Number Theory 12 (2018), 2185–2235.

[4] K. Dekimpe, *Almost-Bieberbach Groups: Affine and Polynomial Structures*, Lecture Notes in Math. 1639, Springer, Berlin, 1996.

[5] K. Dekimpe and G.-J. Dugardein, *Nielsen zeta functions for maps on infra-nilmanifolds are rational*, J. Fixed Point Theory Appl. 17 (2015), 355–370.

[6] K. Dekimpe, S. Tertooy, and I. Van den Bussche, *Reidemeister zeta functions of low-dimensional almost-crystallographic groups are rational*, Comm. Algebra 46 (2018), 4090–4103.

[7] A. Dold, *Fixed point indices of iterated maps*, Invent. Math. 74 (1983), 419–435.

[8] A. L. Fel'shtyn, *The Reidemeister zeta function and the computation of the Nielsen zeta function*, Colloq. Math. 62 (1991), 153–166.

[9] A. Fel'shtyn, *Dynamical zeta functions, Nielsen theory and Reidemeister torsion*, Mem. Amer. Math. Soc. 147 (2020), no. 699, xii+146 pp.

[10] A. L. Fel'shtyn and R. Hill, *The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion*, K-theory 8 (1994), 367–393.

[11] A. L. Fel'shtyn, R. Hill and P. Wong, *Reidemeister numbers of equivariant maps*, Topology Appl. 67 (1995), 119–131.

[12] A. Fel'shtyn and B. Klopsch, *Pólya–Carlson dichotomy for coincidence Reidemeister zeta functions via profinite completions*, Indag. Math. (N.S.) 33 (2022), 753–767.

[13] A. Fel'shtyn and J. B. Lee, *The Nielsen and Reidemeister numbers of maps on infra-solvmanifolds of type (R)*, Topology Appl. 181 (2015), 62–103.

[14] A. Fel'shtyn and E. Troitsky, *Twisted Burnside–Frobenius theory for discrete groups*, J. Reine Angew. Math. 613 (2007), 193–210.

[15] A. Fel'shtyn and M. Zietek, *Dynamical zeta functions of Reidemeister type*, Topol. Methods Nonlinear Anal. 56 (2020), 433–455.

[16] D. L. Gonçalves and P. N.-S. Wong, *Homogeneous spaces in coincidence theory*, Forum Math. 17 (2005), 297–313.

[17] J. Jezierski and W. Marzantowicz, *Homotopy Methods in Topological Fixed and Periodic Points Theory*, Topol. Fixed Point Theory Appl. 3, Springer, Dordrecht, 2006.

[18] L. Li, *On the rationality of the Nielsen zeta function*, Adv. in Math. (China) 23 (1994), 251–256.

[19] A. Mal'cev, *On a class of homogeneous spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 13 (1949), 9–32 (in Russian).

[20] W. Marzantowicz and P. M. Przygodzki, *Finding periodic points of a map by use of a k-adic expansion*, Discrete Contin. Dynam. Systems 5 (1999), 495–514.

[21] R. Miles, *Zeta functions for elements of entropy rank-one actions*, Ergodic Theory Dynam. Systems 27 (2007), 567–582.

[22] R. Miles, *Synchronization points and associated dynamical invariants*, Trans. Amer. Math. Soc. 365 (2013), 5503–5524.

[23] G. Myerson and A. J. van der Poorten, *Some problems concerning recurrence sequences*, Amer. Math. Monthly 102 (1995), 698–705.

[24] S. L. Segal, *Nine Introductions in Complex Analysis*, revised edition, North-Holland 208 Elsevier, Amsterdam, 2008.

[25] E. Troitsky, *Two examples related to the twisted Burnside–Frobenius theory for infinitely generated groups*, Fundam. Appl. Math. 21 (2016), 231–239.

[26] P. Wong, *Reidemeister zeta function for group extensions*, J. Korean Math. Soc. 38 (2001), 1107–1116.

[27] A. V. Zarelua, *On congruences for the traces of powers of some matrices*, Tr. Mat. Inst. Steklova 263 (2008), 85–105 (in Russian); English transl.: Proc. Steklov Inst. Math. 263 (2008), 78–98.

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