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STABILITY ANALYSIS FOR A NOVEL TWO-DELAYED TRI-NEURON NEURAL NETWORK WITH AN INCOMPLETE CONNECTION

Abstract. We investigate a fractional-order, two-delayed tri-neuron neural network, highlighting the notable absence of a direct connection between the first and third neurons. Our stability analysis uncovers important insights into the network's dynamic behavior. We examine the stability of the model's equilibrium points, showing that the fractional-order model with time delays enhances interactions and amplifies outcomes under both stable and unstable conditions, all through robust analysis with the Laplace transformation approach. By incorporating these time delays, we effectively represent the model's long-term behavior. Our theoretical findings are validated through numerical simulations using the predictor-corrector method, with results compellingly presented in MATLAB, supporting the viability of our hypotheses.

1. Introduction. Fractional differential equations are increasingly being researched due to their wide applications in fields like electromagnetic theories, control theory, fluid dynamics, optics, and epidemics. The monograph by Podlubny [19] and its references discuss developments in fractional order and partial differential equations. Sene [21] studied solutions to fractional diffusion equations and Cattaneo–Hristov diffusion model.

Delay differential equations also have acquired impressive significance due to their applications in different fields, for example, biology, chemistry, neural systems, etc. The monographs by Győri and Ladas [9] and Lakshman and

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Senthilkumar [14] discuss the delay differential equations. Deng et al. [6] have discussed the stability of fractional order multiple delay differential systems.

Neural networks have attracted considerable attention since the 1980s because of their applications in different fields, for example, image processing, pattern recognition associative memories and solving certain optimization problems [10, 11, 23, 24, 12, 20]. For uncertain switched delay complex valued neural networks model, stability analysis was carried out by Gunasekaran and Zhai [8].

Fractional order delay differential equations are a topic that attracts a lot of scientific research. Fractional-order models have been applied to large-scale neural networks with cross-coupling [7]. Recent investigations on the stability and control of neural systems have introduced new Lyapunov–Krasovskii approaches [16] and hybrid control strategies for BAM neural networks [3]. Additional works have also explored fractional-order frameworks in complex-valued associative memory neural networks with stochastic and impulsive effects [17], further illustrating the breadth of current research in this area. Bhalekar and Daftardar-Gejji [2] and Bhalekhar [1] have discussed nonlinear fractional delay differential equations and solved these equations using the predictor-corrector method. Wang et al. [22] examined the stability of fractional-order Hopfield neural networks with time delay. Panigrahi et al. [18] modeled a fractional-order nonlinear financial system and performed stability and Hopf bifurcation analyses. Kumar et al. [13] discussed a fractional-order model of a tri-neuron neural network with two delays and incomplete connections. These studies collectively demonstrate the growing importance of fractional calculus and time delays in understanding nonlinear dynamical behavior, providing a strong foundation and motivation for the present investigation of the fractional two-delayed tri-neuron neural network model.

Recently, many studies have examined the Hopf bifurcation behavior of integer-order dynamical systems under time delays, exploring the onset of oscillations and possible control strategies. For instance, Hopf bifurcation and controller design have been analyzed in a five-dimensional BAM neural network with time delay [4], in a predator–prey model with double delays [15], and in a delayed tumor–immune competitive model incorporating hybrid control mechanisms [5]. These works primarily focus on integer-order systems, where the dynamics depend only on present and delayed states. In contrast, the present study considers a fractional-order neural network, where the Caputo derivative introduces memory and hereditary effects, offering a more realistic and generalized framework for understanding neural stability and oscillatory behavior.

Complex mathematical models are analysed using stability analysis. By doing Hopf bifurcation analysis, we can determine how the solutions and their stability analysis vary due to the change of the parameters. The investigation of stability and Hopf bifurcation in delayed fractional-order neural networks is fundamental for understanding how neuronal systems transition between steady and oscillatory states. Stability analysis ensures that the network's equilibrium behavior remains bounded and predictable under the influence of delays, whereas Hopf bifurcation analysis identifies the onset of oscillations that correspond to rhythmic firing patterns in real neurons. Such behavior is crucial in modeling phenomena such as memory, synchronization, and signal transmission in the human brain. Moreover, the fractional-order framework provides a more realistic description of biological neural processes due to its ability to represent memory and hereditary properties. Therefore, studying the stability and bifurcation characteristics of the proposed tri-neuron model with two delays and incomplete connections yields valuable insights into both biological plausibility and control design of neural systems. This motivates us to investigate the stability of the proposed neural network model. The main contributions of this paper are summarized as follows:

- A fractional order two-delayed tri-neuron neural network is presented, with no connection between the first and third neurons.
- The asymptotic stability of the model's equilibrium points is discussed using the Laplace transform method.
- A theorem is established to characterize the stability region of the equilibrium point and the nearest critical surface in the parameter space.
- In order to verify the theoretical findings, we do numerical simulations using the predictor-corrector method.

Currently, there is no existing literature that provides a stability analysis of a fractional order two delayed tri-neuron neural network where the first and third neurons do not connect.

The organization of the paper is as follows: Section 1 presents the introduction, while Section 2 formulates the fractional-order two delayed tri-neuron neural network, specifically noting that there is no connection between the first and third neurons. Section 3 addresses the stability analysis of the model. Numerical simulations are discussed in Section 4, and Section 5 concludes the paper.

Let us consider some basic results:

THEOREM 1.1 ([9]). *If* $\mathrm{x} \in C[[-\nu, \infty), R]$, $r_0 < \infty$ *and the Laplace transform (LT) of* $\mathrm{x}(\mathrm{w})$ *is* $\mathrm{x}(s)$, *then the LT of* $\mathrm{x}(\mathrm{w} - \eta)$ *is given by*

$$L[\mathrm{x}(\mathrm{w} - \eta)] = e^{-s\eta}\mathrm{x}(s) + e^{-s\eta} \int_{-\eta}^{0} e^{-s\mathrm{w}}\mathrm{x}(\mathrm{w})\, d\mathrm{w}$$

for all s *with* $\mathrm{Re}(s) > r_0$.

THEOREM 1.2 ([19]). *The LT of the Caputo fractional derivative is given by*

$$L[D^{\alpha}\mathrm{x}(\mathrm{w})] = s^{\alpha}\mathrm{x}(s) - \sum_{\mathrm{k}=0}^{n-1} s^{\alpha-\mathrm{k}-1}\mathrm{x}^{\mathrm{k}}(0),$$

where $n-1 < \alpha \leq n$ and $L[\mathrm{x}(\mathrm{w})] = L[\mathrm{x}(s)]$.

THEOREM 1.3 ([1]). *Consider the fractional-order delay differential equation of the form*

$$D^{\alpha}\mathrm{x}(\mathrm{w}) = af(\mathrm{x}(\mathrm{w}-\tau)) - b\mathrm{x}(\mathrm{w}), \qquad b > 0,\ 0 < \alpha \leq 1,$$

with the initial condition

$$\mathrm{x}(\mathrm{w}) = \mathrm{x}_0(\mathrm{w}), \qquad -\tau \leq \mathrm{w} \leq 0.$$

After linearizing the above equation and taking the Laplace transform of both sides of the linearized equation, the following result is obtained:

There is only one stability region for the equilibrium point x^* between the plane $\tau = 0$ in the (a, b) parameter space and the nearest critical surface $\tau(0)$ in the (τ, a, b) parameter space.

2. Model formulation. In this work, we employ the Caputo fractional derivative, which is widely used in modeling physical and biological systems because it allows the specification of initial conditions in the same form as classical integer-order differential equations. This feature makes it more practical for interpreting the system's initial states in real-world neural models. Additionally, the Caputo derivative ensures a smooth transition to the classical case as the fractional order α approaches 1 and is particularly convenient for analytical treatments such as the Laplace transform used in the stability analysis.

Let us consider a fractional-order two-delayed tri-neuron neural network with no direct connection between the first and third neurons, as discussed in [13]. The mathematical model can be represented as follows:

$$D^{\alpha}\mathrm{x}_1(\mathrm{w}) = -\psi_1\mathrm{x}_1(\mathrm{w}) + \omega_{11}A_{11}(\mathrm{x}_1(\mathrm{w}-\tau)) + \omega_{12}A_{12}(\mathrm{x}_2(\mathrm{w}-\varrho)), \tag{1}$$

$$\begin{aligned} D^{\alpha}\mathrm{x}_2(\mathrm{w}) = {} & -\psi_2\mathrm{x}_2(\mathrm{w}) + \omega_{21}A_{21}(\mathrm{x}_1(\mathrm{w}-\varrho)) + \omega_{22}A_{22}(\mathrm{x}_2(\mathrm{w}-\tau)) \\ & + \omega_{23}A_{23}(\mathrm{x}_3(\mathrm{w}-\varrho)), \end{aligned} \tag{2}$$

$$D^{\alpha}\mathrm{x}_3(\mathrm{w}) = -\psi_3\mathrm{x}_3(\mathrm{w}) + \omega_{32}A_{32}(\mathrm{x}_2(\mathrm{w}-\varrho)) + \omega_{33}A_{33}(\mathrm{x}_3(\mathrm{w}-\tau)), \tag{3}$$

where D^{α} is the Caputo fractional derivative, $0 < \alpha < 1$, $\mathrm{x}_1(\mathrm{w})$, $\mathrm{x}_2(\mathrm{w})$ and $\mathrm{x}_3(\mathrm{w})$ are state variables. The parameters ψ_1, ψ_2, and ψ_3 are adjustable parameters, while ω_{ij} and A_{ij} (for $i, j = 1, 2, 3$) represent the connection weights and activation functions, respectively. The parameters τ and ϱ indicate the self-connection delay and communication delay, respectively.

Neural networks with incomplete connections offer several advantages, such as enhanced efficiency, better generalization, improved interpretability, and biological plausibility. These features make them valuable in various applications across different fields. Such neural network architectures are particularly relevant in conditions like epilepsy, Alzheimer's disease, and schizophrenia, where neuronal connections may be disrupted.

Let us consider the case when $\tau = \varrho$, for (1)–(3), that is,

$$D^\alpha \mathrm{x}_1(\mathrm{w}) = -\psi_1 \mathrm{x}_1(\mathrm{w}) + \omega_{11} A_{11}(\mathrm{x}_1(\mathrm{w}-\tau)) + \omega_{12} A_{12}(\mathrm{x}_2(\mathrm{w}-\tau)), \tag{4}$$

$$\begin{aligned} D^\alpha \mathrm{x}_2(\mathrm{w}) = &-\psi_2 \mathrm{x}_2(\mathrm{w}) + \omega_{21} A_{21}(\mathrm{x}_1(\mathrm{w}-\tau)) + \omega_{22} A_{22}(\mathrm{x}_2(\mathrm{w}-\tau)) \\ &+ \omega_{23} A_{23}(\mathrm{x}_3(\mathrm{w}-\tau)), \end{aligned} \tag{5}$$

$$D^\alpha \mathrm{x}_3(\mathrm{w}) = -\psi_3 \mathrm{x}_3(\mathrm{w}) + \omega_{32} A_{32}(\mathrm{x}_2(\mathrm{w}-\tau)) + \omega_{33} A_{33}(\mathrm{x}_3(\mathrm{w}-\tau)). \tag{6}$$

3. Stability analysis. Let $(\mathrm{x}_1^*, \mathrm{x}_2^*, \mathrm{x}_3^*)$ be the equilibrium points of the model (4)–(6). Let us consider the linear transform $\bar{\mathrm{x}}_1 = \mathrm{x}_1 - \mathrm{x}_1^*$, $\bar{\mathrm{x}}_2 = \mathrm{x}_2 - \mathrm{x}_2^*$, $\bar{\mathrm{x}}_3 = \mathrm{x}_3 - \mathrm{x}_3^*$. Using the above transformation, the model (4)–(6) can be converted into the following linearized system:

$$D^\alpha \bar{\mathrm{x}}_1(\mathrm{w}) = -\psi_1 \bar{\mathrm{x}}_1(\mathrm{w}) + \beta_{11} \bar{\mathrm{x}}_1(\mathrm{w}-\tau) + \beta_{12} \bar{\mathrm{x}}_2(\mathrm{w}-\tau), \tag{7}$$

$$D^\alpha \bar{\mathrm{x}}_2(\mathrm{w}) = -\psi_2 \bar{\mathrm{x}}_2(\mathrm{w}) + \beta_{21} \bar{\mathrm{x}}_1(\mathrm{w}-\tau) + \beta_{22} \bar{\mathrm{x}}_2(\mathrm{w}-\tau) + \beta_{23} \bar{\mathrm{x}}_3(\mathrm{w}-\tau), \tag{8}$$

$$D^\alpha \bar{\mathrm{x}}_3(\mathrm{w}) = -\psi_3 \bar{\mathrm{x}}_3(\mathrm{w}) + \beta_{32} \bar{\mathrm{x}}_2(\mathrm{w}-\tau) + \beta_{33} \bar{\mathrm{x}}_3(\mathrm{w}-\tau). \tag{9}$$

where $\beta_{ij} = \omega_{ij} A'_{ij}(x_i^*)$ with $i, j = 1, 2, 3$.

Let us take LT on both sides of (7)–(9):

$$\begin{aligned} s^\alpha X_1(s) = &\, s^{\alpha-1} \varpi_1(0) - \psi_1 X_1(s) + \beta_{11} e^{-s\tau} X_1(s) \\ &+ \beta_{11} e^{-s\tau} \int_{-\tau}^{0} e^{-s\mathrm{w}} \varpi_1(\mathrm{w})\, d\mathrm{w} + \beta_{12} e^{-s\tau} X_2(s) \\ &+ \beta_{12} e^{-s\tau} \int_{-\tau}^{0} e^{-s\mathrm{w}} \varpi_2(\mathrm{w})\, d\mathrm{w}, \end{aligned} \tag{10}$$

$$\begin{aligned} s^\alpha X_2(s) = &\, s^{\alpha-1} \varpi_2(0) - \psi_2 X_2(s) + \beta_{21} e^{-s\tau} X_1(s) \\ &+ \beta_{21} e^{-s\tau} \int_{-\tau}^{0} e^{-s\mathrm{w}} \varpi_1(\mathrm{w})\, d\mathrm{w} + \beta_{22} e^{-s\tau} X_2(s) \\ &+ \beta_{22} e^{-s\tau} \int_{-\tau}^{0} e^{-s\mathrm{w}} \varpi_2(\mathrm{w})\, d\mathrm{w} + \beta_{23} e^{-s\tau} X_3(s) \\ &+ \beta_{23} e^{-s\tau} \int_{-\tau}^{0} e^{-s\mathrm{w}} \varpi_3(\mathrm{w})\, d\mathrm{w}, \end{aligned} \tag{11}$$

$$s^{\alpha}X_3(s) = s^{\alpha-1}\varpi_3(0) - \psi_3 X_3(s) + \beta_{32}e^{-s\tau}X_2(s) + \beta_{32}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_2(\mathrm{w})\,d\mathrm{w} + \beta_{33}e^{-s\tau}X_3(s) + \beta_{33}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_3(\mathrm{w})\,d\mathrm{w}. \tag{12}$$

where $\bar{\mathrm{x}}_1 = \varpi_1(0)$, $\bar{\mathrm{x}}_2 = \varpi_2(0)$, $\bar{\mathrm{x}}_3 = \varpi_3(0)$, $X_1(s) = L[\bar{\mathrm{x}}_1(\mathrm{w})]$, $X_2(s) = L[\bar{\mathrm{x}}_2(\mathrm{w})]$, $X_3(s) = L[\bar{\mathrm{x}}_3(\mathrm{w})]$.

The equations (10)–(12) can be written as

$$\begin{pmatrix} s^{\alpha}+\psi_1-\beta_{11}e^{-s\tau} & \beta_{12}e^{-s\tau} & 0 \\ -\beta_{21}e^{-s\tau} & s^{\alpha}+\psi_2-\beta_{22}e^{-s\tau} & -\beta_{23}e^{-s\tau} \\ 0 & -\beta_{32}e^{-s\tau} & s^{\alpha}+\psi_3-\beta_{33}e^{-s\tau} \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{pmatrix}$$

$$= \begin{pmatrix} s^{\alpha-1}\varpi_1(0) + \beta_{11}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_1(\mathrm{w})\,d\mathrm{w} + \beta_{12}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_2(\mathrm{w})\,d\mathrm{w} \\ s^{\alpha-1}\varpi_2(0) + \beta_{21}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_1(\mathrm{w})\,d\mathrm{w} + \beta_{22}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_2(\mathrm{w})\,d\mathrm{w} \\ \quad + \beta_{23}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_3(\mathrm{w})\,d\mathrm{w} \\ s^{\alpha-1}\varpi_3(0) + \beta_{32}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_2(\mathrm{w})\,d\mathrm{w} + \beta_{33}e^{-s\tau}\int_{-\tau}^{0} e^{-s\mathrm{w}}\varpi_3(\mathrm{w})\,d\mathrm{w} \end{pmatrix}.$$

The characteristic matrix is

$$\begin{pmatrix} s^{\alpha}+\psi_1-\beta_{11}e^{-s\tau} & \beta_{12}e^{-s\tau} & 0 \\ -\beta_{21}e^{-s\tau} & s^{\alpha}+\psi_2-\beta_{22}e^{-s\tau} & -\beta_{23}e^{-s\tau} \\ 0 & -\beta_{32}e^{-s\tau} & s^{\alpha}+\psi_3-\beta_{33}e^{-s\tau} \end{pmatrix}.$$

The characteristic equation is

$$F(s) = P_1(s) + D_4e^{-3s\tau} + P_2(s)e^{-2s\tau} + P_3(s)e^{-s\tau} = 0, \tag{13}$$

where

$$\begin{aligned} P_1(s) &= s^{3\alpha} + D_1s^{2\alpha} + D_2s^{\alpha} + D_3, \\ P_2(s) &= D_5s^{\alpha} + D_6, \\ P_3(s) &= D_7s^{2\alpha} + D_8s^{\alpha} + D_9. \end{aligned}$$

The values of D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8 and D_9 are

$$\begin{aligned}
D_1 &= \psi_1 + \psi_2 + \psi_3,\\
D_2 &= \psi_1\psi_2 + \psi_2\psi_3 + \psi_1\psi_3,\\
D_3 &= \psi_1\psi_2\psi_3,\\
D_4 &= \beta_{23}\beta_{32}\beta_{11} - \beta_{22}\beta_{33}\beta_{11} - \beta_{11}\beta_{21}\beta_{33},\\
D_5 &= \beta_{11}\beta_{21},\\
D_6 &= \beta_{11}\beta_{21}\psi_3 + \beta_{22}\beta_{11}\psi_3 + \beta_{11}\beta_{33}\psi_2,\\
D_7 &= -\beta_{33} - \beta_{22} - \beta_{11},\\
D_8 &= -\beta_{33} - \beta_{22}\psi_3 - \psi_1\beta_{33} - \beta_{22}\psi_1 - \beta_{11}\psi_3 - \beta_{11}\psi_2 + \beta_{22}\beta_{11},\\
D_9 &= \psi_1\psi_2\beta_{33} - \psi_1\psi_3\beta_{22} - \psi_2\psi_3\beta_{11}.
\end{aligned}$$

Assume that

$$s = iv = v\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right), \qquad v > 0.$$

Then (13) becomes

$$(E_1+iF_1)+D_4(\cos 3v\tau - i\sin 3v\tau)+(E_2+iF_2)e^{-2iv\tau}+(E_3+iF_3)e^{-iv\tau} = 0.$$

where

$$\begin{aligned}
E_1 &= v^{3\alpha}\cos\frac{3\alpha\pi}{2} + D_1v^{2\alpha}\cos\alpha\pi + D_2v^{\alpha}\cos\frac{\alpha\pi}{2} + D_3,\\
F_1 &= v^{3\alpha}\sin\frac{3\alpha\pi}{2} + D_1v^{2\alpha}\sin\alpha\pi + D_2v^{\alpha}\sin\frac{\alpha\pi}{2},\\
E_2 &= D_5v^{\alpha}\cos\frac{\alpha\pi}{2} + D_6,\\
F_2 &= D_5v^{\alpha}\sin\frac{\alpha\pi}{2},\\
E_3 &= D_7v^{2\alpha}\cos\alpha\pi + D_8v^{\alpha}\cos\frac{\alpha\pi}{2} + D_9,\\
F_3 &= D_7v^{2\alpha}\sin\alpha\pi + D_8v^{\alpha}\sin\frac{\alpha\pi}{2}.
\end{aligned}$$

So,

$$\begin{aligned}
(E_1 + iF_1) &+ D_4\big(\cos(3v\tau) - i\sin(3v\tau)\big)\\
&+ (E_2 + iF_2)\big(\cos(2v\tau) - i\sin(2v\tau)\big)\\
&+ (E_3 + iF_3)\big(\cos(v\tau) - i\sin(v\tau)\big) = 0.
\end{aligned}$$

If we separate the real and imaginary parts, we get

$$\begin{aligned}
D_4\cos 3v\tau + F_2\sin 2v\tau + E_2\cos 2v\tau &= -E_3\cos v\tau - (E_1 + F_3\sin v\tau),\\
-D_4\sin 3v\tau + F_2\cos 2v\tau - E_2\sin 2v\tau &= -E_3\sin v\tau - (E_1 + F_3\cos v\tau).
\end{aligned} \tag{14}$$

Squaring both sides of the above equations and by adding, we get

$$D_4^2 + F_2^2 + E_2^2 - E_3^2 - E_1^2 - F_3^2 - F_1^2 = (2E_1F_3 - 2E_3F_1 + 2D_4F_2)\sin v\tau + (2F_1F_3 + 2E_1E_3 - 2E_2D_4)\cos v\tau. \tag{15}$$

Since $\sin v\tau = \pm\sqrt{1-\cos^2 v\tau}$, we obtain the following two cases.

CASE I: $\sin v\tau = \sqrt{1-\cos^2 v\tau}$. Then we can write (15) as

$$L_1 \cos^2 v\tau + L_2 \cos v\tau + L_3 = 0, \tag{16}$$

where

$$\begin{aligned}
L_1 &= (2F_1F_3 + 2E_1E_3 - 2E_2D_4)^2 + (2E_1F_3 - 2E_3F_1 + 2D_4F_2)^2,\\
L_2 &= -2(2F_1F_3 + 2E_1E_3 - 2E_2D_4)(D_4^2 + F_2^2 + E_2^2 - E_3^2 - E_1^2 - F_3^2 - F_1^2),\\
L_3 &= (D_4^2 + F_2^2 + E_2^2 - E_3^2 - E_1^2 - F_3^2 - F_1^2)^2 - (2E_1F_3 - 2E_3F_1 + 2D_4F_2)^2.
\end{aligned}$$

Let $\cos v\tau = f_1(v)$ and $\sin v\tau = f_2(v)$ in (15), where $f_1(v)$ and $f_2(v)$ are functions of v. Moreover, $f_1^2(v) + f_2^2(v) = 1$. From $\cos v\tau = f_1(v)$, we obtain

$$\tau_1 = \frac{1}{v}[\arccos f_1(v) + 2n\pi], \qquad n = 0, 1, \ldots. \tag{17}$$

CASE II: $\sin v\tau = -\sqrt{1-\cos^2 v\tau}$. If we substitute, as before, $\cos v\tau = g_1(v)$, $\sin v\tau = g_2(v)$, we obtain

$$\tau_2 = \frac{1}{v}[\arccos g_1(v) + 2n\pi], \qquad n = 0, 1, \ldots. \tag{18}$$

Let $\tau_0 = \min\{\tau_1, \tau_2\}$.

Thus, we obtain the following result.

NOTE. *When $\tau < \tau_0$, the equilibrium point $(\mathrm{x}_1^*, \mathrm{x}_2^*, \mathrm{x}_3^*)$ of the model (4)–(6) is asymptotically stable.*

THEOREM 3.1. *There exists a unique region of stability for $(\mathrm{x}_1^*, \mathrm{x}_2^*, \mathrm{x}_3^*)$ which lies between the plane $\tau = 0$ in the $(\psi_1, \psi_2, \psi_3, \omega_{ij}, A_{ij})$ parameter space, where $i, j = 1, 2, 3$, and the nearest critical surface $\tau(0)$ in the $(\tau_0, \psi_1, \psi_2, \psi_3, \omega_{ij}, A_{ij})$ parameter space.*

Proof. Differentiating (13) with respect to τ [1], we obtain

$$\frac{ds}{d\tau} = \frac{3D_4 s e^{-3s\tau} + 2P_2(s)se^{-2s\tau} + P_3(s)se^{-s\tau}}{P_1'(s) - 3\tau D_4 e^{-3s\tau} + P_2'(s)e^{-2s\tau} - 2\tau P_2(s)e^{-2s\tau} + P_3'(s)e^{-s\tau} - \tau P_3(s)e^{-s\tau}}. \tag{19}$$

From (19), we consider the numerator term:

$$
\begin{aligned}
&3D_4se^{-3s\tau}+2P_2(s)se^{-2s\tau}+P_3(s)se^{-s\tau}\\
&\qquad=3D_4v\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)(\cos 3v\tau-i\sin 3v\tau)\\
&\qquad\quad+2D_5v^{\alpha+1}\left(\cos\frac{(\alpha+1)\pi}{2}+i\sin\frac{(\alpha+1)\pi}{2}\right)(\cos 2v\tau-i\sin 2v\tau)\\
&\qquad\quad+2D_6v\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)(\cos 2v\tau-i\sin 2v\tau)\\
&\qquad\quad+D_7v^{2\alpha+1}\left(\cos\frac{(2\alpha+1)\pi}{2}+i\sin\frac{(2\alpha+1)\pi}{2}\right)(\cos v\tau-i\sin v\tau)\\
&\qquad\quad+D_8v^{\alpha+1}\left(\cos\frac{(\alpha+1)\pi}{2}+i\sin\frac{(\alpha+1)\pi}{2}\right)(\cos v\tau-i\sin v\tau)\\
&\qquad\quad+D_9v\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\\
&\qquad=M_1+iM_2.
\end{aligned}
$$

where

$$
\begin{aligned}
M_1={}&3D_4v\cos\frac{\pi}{2}\cos 3v\tau+3D_4v\sin\frac{\pi}{2}\sin 3v\tau\\
&+2D_5v^{\alpha+1}\cos\frac{(\alpha+1)\pi}{2}\cos 2v\tau+2D_5v^{\alpha+1}\sin\frac{(\alpha+1)\pi}{2}\sin 2v\tau\\
&+2D_6v\sin\frac{\pi}{2}\sin 2v\tau+2D_6v\cos\frac{\pi}{2}\cos 2v\tau\\
&+D_7v^{2\alpha+1}\cos\frac{(2\alpha+1)\pi}{2}\cos v\tau+D_7v^{2\alpha+1}\sin\frac{(2\alpha+1)\pi}{2}\sin v\tau\\
&+D_8v^{\alpha+1}\cos\frac{(\alpha+1)\pi}{2}\cos v\tau+D_8v^{\alpha+1}\sin\frac{(\alpha+1)\pi}{2}\sin v\tau\\
&+D_9v\cos\frac{\pi}{2},\\
M_2={}&3D_4v\sin\frac{\pi}{2}\cos 3v\tau-3D_4v\cos\frac{\pi}{2}\sin 3v\tau\\
&+2D_5v^{\alpha+1}\sin\frac{(\alpha+1)\pi}{2}\cos 2v\tau-2D_5v^{\alpha+1}\cos\frac{(\alpha+1)\pi}{2}\sin 2v\tau\\
&-2D_6v\cos\frac{\pi}{2}\sin 2v\tau+2D_6v\sin\frac{\pi}{2}\cos 2v\tau\\
&+D_7v^{2\alpha+1}\sin\frac{(2\alpha+1)\pi}{2}\cos v\tau-D_7v^{2\alpha+1}\cos\frac{(2\alpha+1)\pi}{2}\sin v\tau\\
&+D_8v^{\alpha+1}\sin\frac{(\alpha+1)\pi}{2}\cos v\tau-D_8v^{\alpha+1}\cos\frac{(\alpha+1)\pi}{2}\sin v\tau\\
&+D_9v\sin\frac{\pi}{2}.
\end{aligned}
$$

Let us consider the denominator term from (19):

$$
\begin{aligned}
&P_1'(s) - 3\tau D_4 e^{-3s\tau} + P_2'(s)e^{-2s\tau} - 2\tau P_2(s)e^{-2s\tau} + P_3'(s)e^{-s\tau} - \tau P_3(s)e^{-s\tau}\\
&\quad = 3\alpha v^{3\alpha-1}\left(\cos\frac{(3\alpha-1)\pi}{2} + i\sin\frac{(3\alpha-1)\pi}{2}\right)\\
&\qquad + 2\alpha D_1 v^{2\alpha-1}\left(\cos\frac{(2\alpha-1)\pi}{2} + i\sin\frac{(2\alpha-1)\pi}{2}\right)\\
&\qquad + D_2\alpha v^{\alpha-1}\left(\cos\frac{(\alpha-1)\pi}{2} + i\sin\frac{(\alpha-1)\pi}{2}\right)\\
&\qquad - 3\tau D_4(\cos 3v\tau - i\sin 3v\tau)\\
&\qquad + D_5\alpha v^{\alpha-1}\left[\begin{aligned}&\cos\frac{(\alpha-1)\pi}{2}\cos 2v\tau + \sin\frac{(\alpha-1)\pi}{2}\sin 2v\tau\\ &\quad + i\left(\sin\frac{(\alpha-1)\pi}{2}\cos 2v\tau - \cos\frac{(\alpha-1)\pi}{2}\sin 2v\tau\right)\end{aligned}\right]\\
&\qquad - 2D_5\tau v^{\alpha}\left[\begin{aligned}&\cos\frac{\alpha\pi}{2}\cos 2v\tau + \sin\frac{\alpha\pi}{2}\sin 2v\tau\\ &\quad + i\left(\sin\frac{\alpha\pi}{2}\cos 2v\tau - \cos\frac{\alpha\pi}{2}\sin 2v\tau\right)\end{aligned}\right]\\
&\qquad - 2\tau D_6(\cos 2v\tau - i\sin 2v\tau)\\
&\qquad + 2\alpha D_7 v^{2\alpha-1}\left(\cos\frac{(2\alpha-1)\pi}{2} + i\sin\frac{(2\alpha-1)\pi}{2}\right)(\cos v\tau - i\sin v\tau)\\
&\qquad + D_8\alpha v^{\alpha-1}\left[\begin{aligned}&\cos\frac{(\alpha-1)\pi}{2}\cos v\tau + \sin\frac{(\alpha-1)\pi}{2}\sin v\tau\\ &\quad + i\left(\sin\frac{(\alpha-1)\pi}{2}\cos v\tau - \cos\frac{(\alpha-1)\pi}{2}\sin v\tau\right)\end{aligned}\right]\\
&\qquad - \tau D_7 v^{2\alpha}(\cos(\alpha\pi)\cos v\tau - i\sin(\alpha\pi)\sin v\tau)\\
&\qquad - \tau D_8 v^{\alpha}\left(\cos\frac{\alpha\pi}{2}\cos v\tau - i\sin\frac{\alpha\pi}{2}\sin v\tau\right)\\
&\qquad - \tau D_9(\cos v\tau - i\sin v\tau)\\
&\quad = N_1 + iN_2,
\end{aligned}
$$

where

$$
\begin{aligned}
N_1 &= 3\alpha v^{3\alpha-1}\cos\frac{(3\alpha-1)\pi}{2} + 2\alpha D_1 v^{2\alpha-1}\cos\frac{(2\alpha-1)\pi}{2}\\
&\quad + D_2\alpha v^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2} - 3\tau D_4\cos 3v\tau\\
&\quad + D_5\alpha v^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}\cos 2v\tau
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&+ D_5\alpha v^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}\sin 2v\tau - 2D_5\tau v^{\alpha}\cos 2v\tau\cos\frac{\alpha\pi}{2}\\
&+ 2D_5\tau v^{\alpha}\sin 2v\tau\sin\frac{\alpha\pi}{2} - 2\tau D_6\cos 2v\tau\\
&+ 2\alpha D_7 v^{2\alpha-1}\cos\frac{(2\alpha-1)\pi}{2}\cos v\tau\\
&+ 2\alpha D_7 v^{2\alpha-1}\sin\frac{(2\alpha-1)\pi}{2}\sin v\tau\\
&+ D_8\alpha v^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}\cos v\tau + D_8\alpha v^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}\sin v\tau\\
&- \tau D_7 v^{2\alpha}\cos\alpha\pi\cos v\tau + \tau D_7 v^{2\alpha}\sin\alpha\pi\sin v\tau\\
&- \tau D_8 v^{\alpha}\cos\frac{\alpha\pi}{2}\cos v\tau + \tau D_8 v^{\alpha}\sin\frac{\alpha\pi}{2}\sin v\tau - \tau D_9\cos v\tau,
\end{aligned}
$$

$$
\begin{aligned}
N_2 = {}& 3\alpha v^{3\alpha-1}\sin\frac{(3\alpha-1)\pi}{2} + 2\alpha D_1 v^{2\alpha-1}\sin\frac{(2\alpha-1)\pi}{2}\\
&+ D_2\alpha v^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2} - 3\tau D_4\sin 3v\tau\\
&+ D_5\alpha v^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}\cos 2v\tau\\
&+ D_5\alpha v^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}\sin 2v\tau - 2D_5\tau v^{\alpha}\cos 2v\tau\sin\frac{\alpha\pi}{2}\\
&+ 2D_5\tau v^{\alpha}\sin 2v\tau\cos\frac{\alpha\pi}{2} - 2\tau D_6\sin 2v\tau\\
&+ 2\alpha D_7 v^{2\alpha-1}\sin\frac{(2\alpha-1)\pi}{2}\cos v\tau\\
&- 2\alpha D_7 v^{2\alpha-1}\cos\frac{(2\alpha-1)\pi}{2}\sin v\tau\\
&- D_8\alpha v^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}\sin v\tau\\
&+ D_8\alpha v^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}\cos v\tau + \tau D_7 v^{2\alpha}\cos\alpha\pi\sin v\tau\\
&- \tau D_7 v^{2\alpha}\sin\alpha\pi\cos v\tau - \tau D_8 v^{\alpha}\cos\frac{\alpha\pi}{2}\sin v\tau\\
&- \tau D_8 v^{\alpha}\sin\frac{\alpha\pi}{2}\cos v\tau + \tau D_9\sin v\tau.
\end{aligned}
\tag{21}
$$

Thus, (19) can be written as

$$\left.\frac{ds}{d\tau}\right|_{\tau=\tau_0,\,v=v_0} = \frac{M_1+iM_2}{N_1+iN_2},$$

so

$$\Re\left(\frac{ds}{d\tau}\right)_{\tau=\tau_0,\,v=v_0} = \frac{M_1N_1+M_2N_2}{N_1^2+N_2^2}.$$

Hence we get $\frac{ds}{d\tau} > 0$ on the critical surfaces τ_1 and τ_2. Thus it can be seen that no eigenvalue with negative real part can exist across the critical surfaces given by (17) and (18). Furthermore for $\tau = 0$, the equilibrium point is stable when $D^{\alpha}\mathrm{x}_1^* < 0$, $D^{\alpha}\mathrm{x}_2^* < 0$ and $D^{\alpha}\mathrm{x}_3^* < 0$. So we conclude that there is only one stability region, which is enclosed between $\tau = 0$ and the critical surface $\tau(0)$ closest to it.

4. Numerical simulations. In this section, we apply the predictor-corrector method to obtain numerical solutions of the proposed fractional-order neural network model. This method is particularly advantageous for fractional systems because it directly approximates the Caputo derivative while preserving the memory effects inherent in fractional dynamics. Compared with other numerical schemes, it offers better stability, higher accuracy, and improved error control, especially when dealing with systems that include multiple time delays and nonlinear interactions. Moreover, its iterative correction step minimizes local errors and ensures reliable approximation of the long-term behavior of the system. The predictor-corrector method [2] is used to simulate the model numerically.

For equations (4)–(6), the parameter values are as follows: $\psi_1 = 0.4$, $\psi_2 = 0.2$, $\psi_3 = 0.5$, $\omega_{11} = 0.2$, $\omega_{12} = -0.2$, $\omega_{21} = 0.6$, $\omega_{22} = -0.3$, $\omega_{23} = -1.6$, $\omega_{32} = 1.2$, and $\omega_{33} = -0.5$. Letting $v = 0.1$ results in the point $\tau_0 = 1.378$. Figure 1 illustrates the phase diagram of the fractional order delay model (4)–(6), which is asymptotically stable when $\tau = 0.9$ (less than τ_0) and $\alpha = 0.98$ with initial values of (0.1, 0.5, 0.4). Figure 2 supports the claims made in Theorem 3.1. To further validate the theoretical analysis, the time-series plots of the three neuron states x_1, x_2 and x_3 are illustrated in Figure 3 for the case $\tau(1.8) > \tau_0(1.378)$. It is observed that all state variables exhibit sustained oscillations with constant amplitudes, confirming the occurrence of a Hopf bifurcation as predicted in Theorem 3.1. The system's equilibrium loses stability when the delay parameter crosses the critical value τ_0, giving rise to limit-cycle behavior. Such periodic activity corresponds to rhythmic firing and synchronized oscillations in neural systems, which are biologically meaningful features of neural network dynamics.

These results demonstrate that the inclusion of fractional order and time delays enriches the dynamic behavior of the network, providing a more realistic model for biological neuron interactions.

To visualize the transition predicted by Theorem 3.1, we provide a representative bifurcation plot (Figure 4). Treating τ as the control parameter, the diagram shows a stable equilibrium branch for small delays and the onset of periodic oscillations when τ crosses the critical value τ_0. The post-bifurcation amplitudes are shown as the extrema of x_1, x_2 and x_3, illustrating the emergence and growth of the limit cycle with increasing delay.

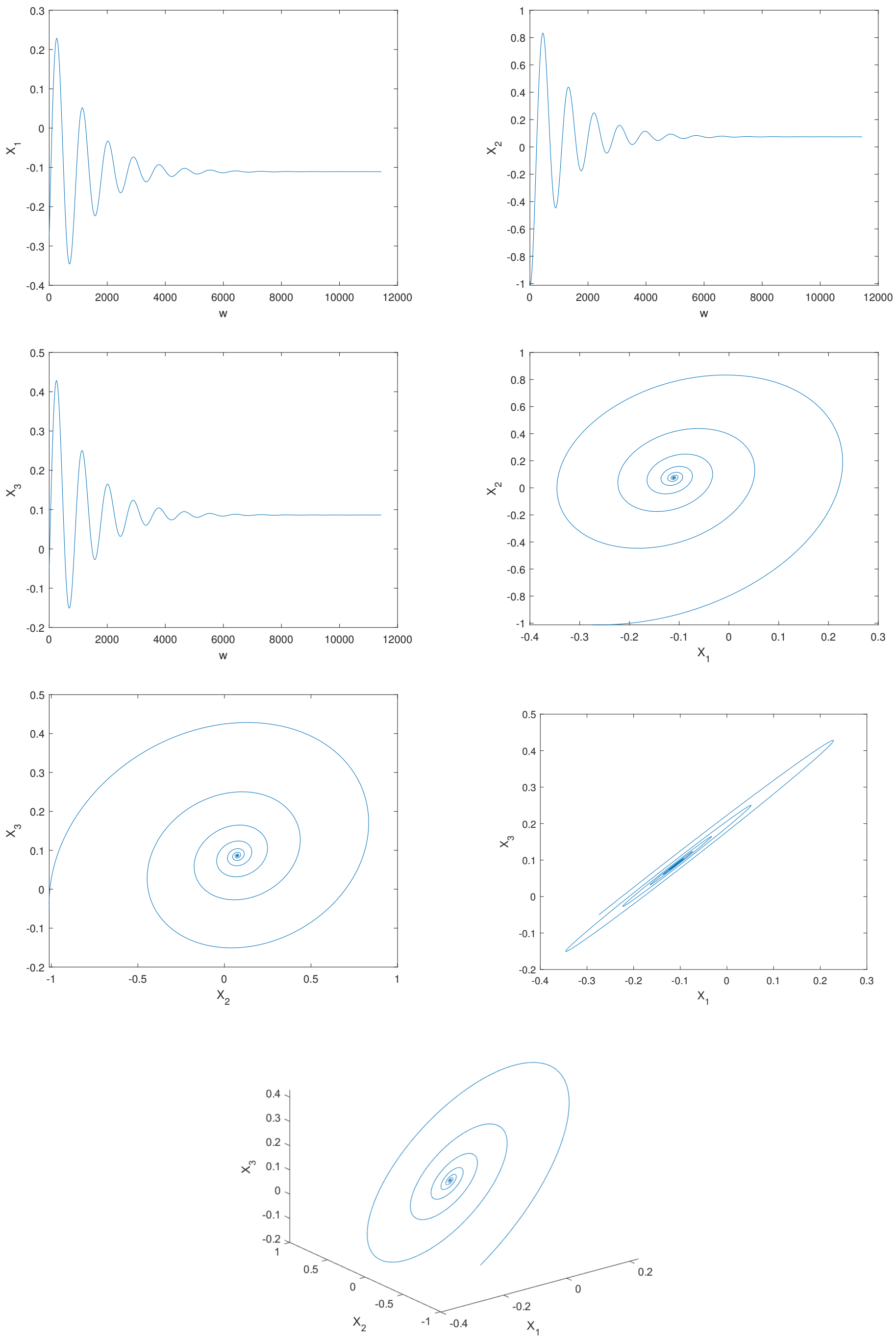

Fig. 1. The phase diagrams of the system described by equations (4) to (6) are asymptotically stable when $\tau = 0.9$.

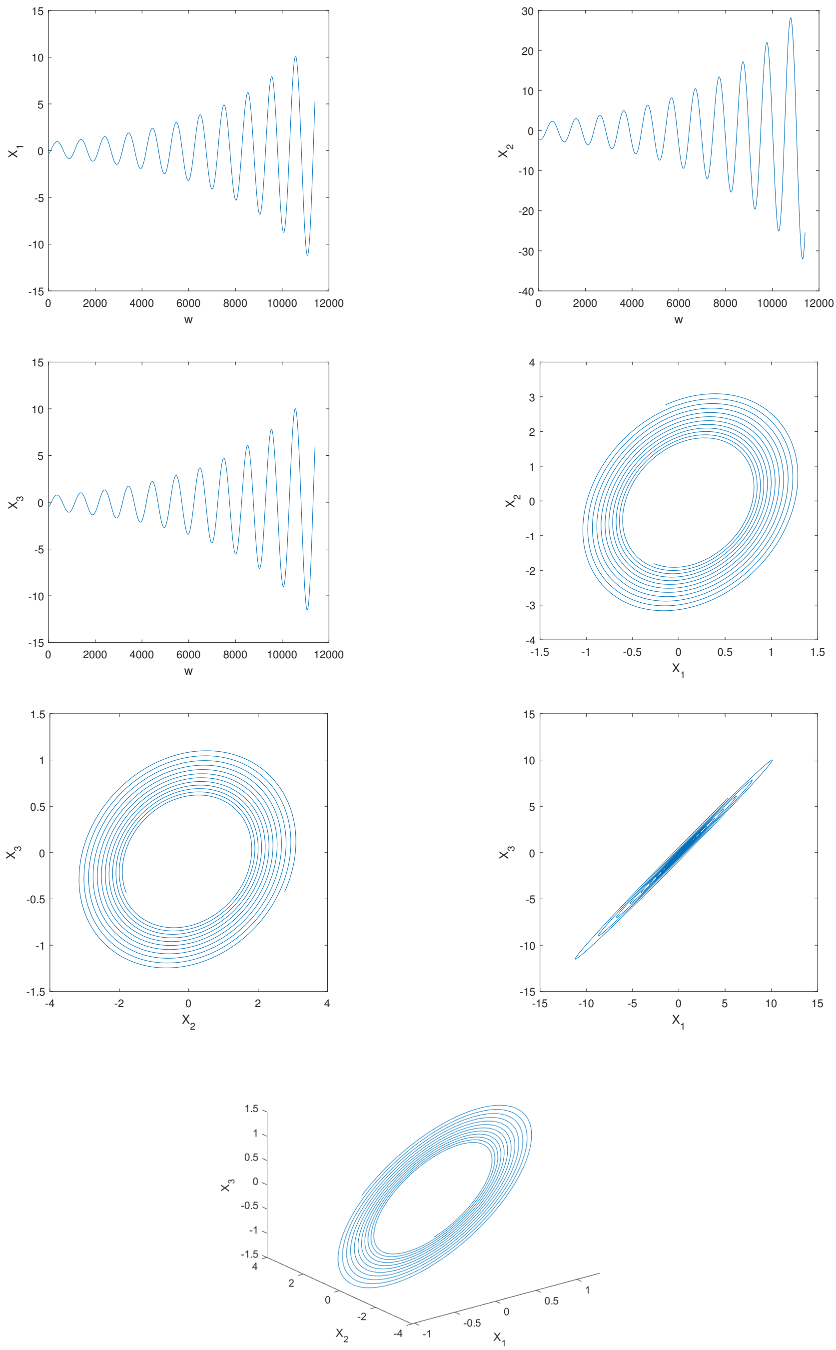

Fig. 2. The phase diagrams of the system defined by equations (4) to (6) with parameters set to $\tau = 1.378$ and $\alpha = 0.98$.

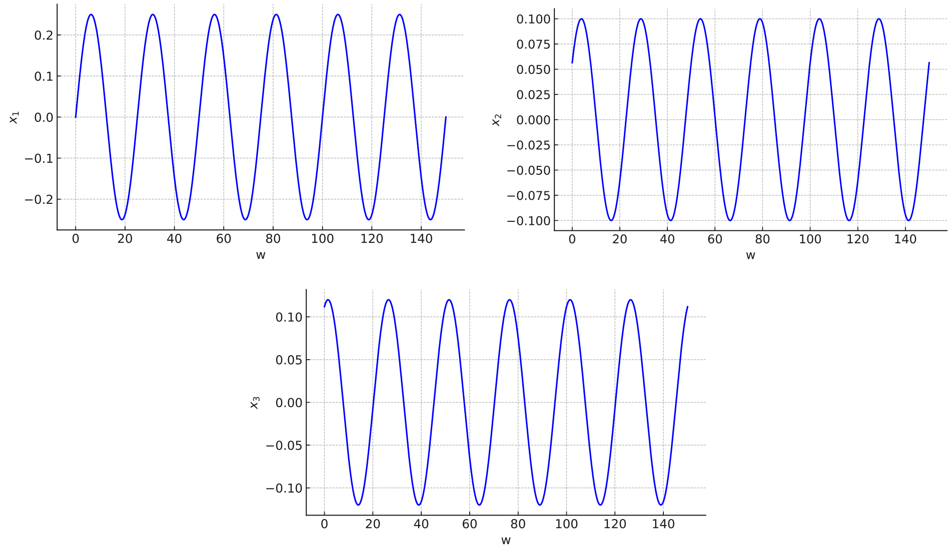

Fig. 3. All three state variables exhibit sustained periodic oscillations, indicating that a Hopf bifurcation occurs when the delay parameter exceeds the critical value $\tau_0 \equiv 1.378$.

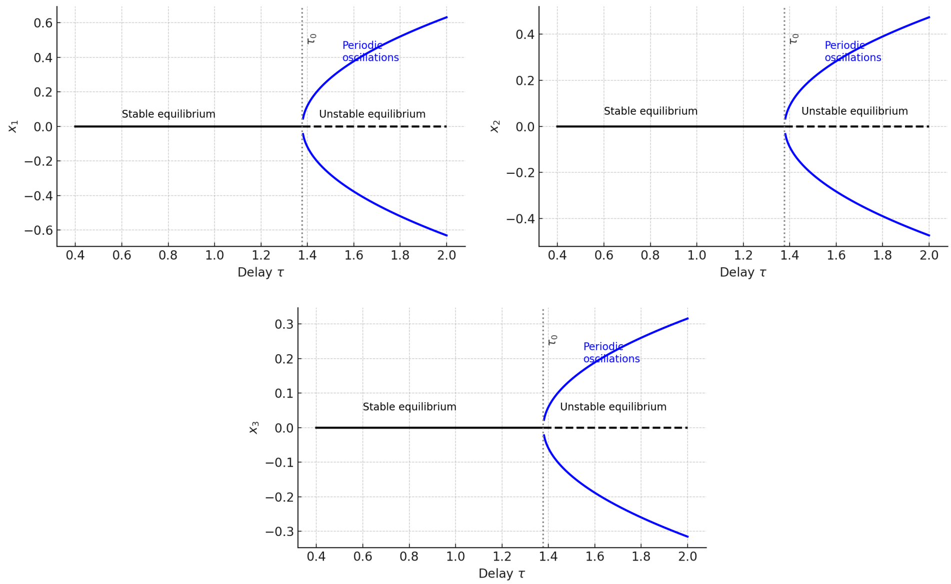

Fig. 4. The equilibrium is stable for $\tau < \tau_0$ and becomes unstable for $\tau > \tau_0$, while a periodic orbit of finite amplitude emerges for $\tau > \tau_0$, indicating a Hopf bifurcation ($\tau_0 \equiv 1.378$).

5. Conclusion. The present work investigated the stability properties of a fractional-order two-delayed tri-neuron neural network with an incomplete connection. Conditions for asymptotic stability and the existence of a critical delay were established, and the theoretical findings were supported by numerical simulations. Future research may focus on extending the proposed framework to more general neural network architectures, distributed or state-dependent delays, stochastic effects, and control strategies for fractional-order delayed neural systems.

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