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Weighted Stepanov-like pseudo almost automorphic solutions to the non-autonomous Oseen–Navier–Stokes equations

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Abstract. In this paper, the existence, uniqueness, and stability of (weighted) Stepanov-like pseudo almost automorphic solutions are investigated for the non-autonomous Oseen–Navier–Stokes equations (ONSE) in an exterior domain. Our method is based on the one hand on a combination of differential inequalities and interpolations functors for the case of linearized equations, and on the other hand on fixed-point arguments for semilinear equations.

1. Introduction. The study of periodic and almost periodic solutions, along with their generalizations to evolution equations, constitutes a significant research direction related to the long-term behavior of solutions to such equations. For periodic solutions, several standard approaches are commonly employed, including Massera's principle [28], Tikhonov's fixed-point theorem [39], and Lyapunov functionals [47], each of which is particularly suited to specific classes of differential equations. Among the most widely used techniques for establishing the existence of periodic solutions are demonstrating the ultimate boundedness of solutions and leveraging the compactness of the Poincaré map through appropriate compact embeddings (see [39, 40, 47] and references therein). However, in the context of partial differential equations posed on unbounded domains or those admitting unbounded solutions, such compact embeddings may no longer hold, making the existence of bounded solutions more challenging to establish. Identifying suitable initial conditions that ensure the boundedness of solutions is often non-trivial. A key method

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to address these difficulties is the so-called Massera-type theorem, which, broadly speaking, asserts that if a differential equation admits a bounded solution, then it also possesses a periodic one. This approach, in combination with interpolation spaces, has been successfully applied to fluid dynamics equations in [31, 18], where interpolation functors were used in conjunction with ergodic methods (see [31]) or topological arguments (see [18]).

For fluid flow problems, the existence of periodic solutions to the Navier–Stokes equations and related models remains an important area of research. This line of inquiry was initiated by Serrin [40], who demonstrated the existence of periodic solutions to the Navier–Stokes equations in a bounded domain via stability arguments. Subsequent works have extended these results to the entire space $\mathbb{R}^n$ and the whole timeline $\mathbb{R}$ (see [22, 24, 27, 29]). Moreover, leveraging the interpolation structure of weak-L^d spaces, Yamazaki [46] proved the existence and stability of periodic solutions in exterior domains. For further results on periodic solutions in exterior domains, we refer to [17, 43].

For the case of almost periodic solutions, several approaches have been developed by Bochner, Stepanov, Besicovitch and Weyl after the fundamental definition given by H. Bohr [6, 7] in 1925. These classes of functions play an important role in many areas of mathematics, among them differential equations, dynamical systems and harmonic analysis. For general information about the theory of almost periodic functions we refer to [11, 25] and [1, Chapter II]. Furthermore, the concept of almost automorphic functions was introduced by Bochner as a generalization of almost periodic functions [3, 4, 5] and to combine the function spaces with certain features of differential geometry. Since then, the study of this concept and its extensions to various types of solutions to abstract differential equations has been intensively conducted during recent years (see [10, 36, 35]).

Then, N'Guérékata et al. [15] and Xiao et al. [45] generalized the notion of almost automorphic functions to the so-called pseudo almost automorphic functions and established the existence and uniqueness of pseudo almost automorphic solutions to a certain class of semilinear evolution equations. Recently, Blot et al. [2] has introduced the notion of *weighted pseudo almost automorphic functions* which generalizes the concept of pseudo almost periodic functions [12] connected with the pseudo almost automorphic ones in a unified approach. Also in [2, 38, 37], some important properties of the space of weighted pseudo almost automorphic functions were proved, including their completeness and the composition theorem.

The notion of Stepanov-like almost automorphic (S^pAA) functions has originally been introduced by V. Casarino [9] as a generalization of almost automorphic functions in the spirit of Stepanov. Then weighted Stepanov-like pseudo almost automorphic functions (WS^pAA-functions) were defined by Xia and Fan [44]. Then there have been many works on the existence and uniqueness of S^pAA-solutions and WS^pAA-solutions to abstract differential equations (see [13, 14, 20, 26] and references therein).

In the present paper we will show the existence, uniqueness and stability of S^pAA and WS^pAA mild solutions to the non-autonomous Oseen–Navier–Stokes equations (ONSE) in an unbounded domain $\Omega \subset \mathbb{R}^3$ exterior to a rigid body D, i.e., $\Omega = \mathbb{R}^3 \setminus D$, with the data belonging to L^p-spaces. Concretely, we consider the flow of an incompressible, viscous fluid in the exterior of a rotating obstacle that is translating with a time-dependent velocity. Here the angular velocity of the obstacle also depends on time and the axis of rotation may change. The equations describing such a problem are the Oseen–Navier–Stokes equations (ONSE) in a time-dependent exterior domain with a prescribed velocity field at infinity. After rewriting the problem on a fixed exterior domain $\Omega \subset \mathbb{R}^3$, the system is reduced to

$$(1.1)\qquad \begin{cases} u_t + (u\cdot\nabla)u - \Delta u + \nabla p & \\ \quad = (\eta + \omega\times x)\cdot\nabla u - \omega\times u + \operatorname{div} F & \text{in } \Omega\times\mathbb{R},\\ \nabla\cdot u = 0 & \text{in } \Omega\times\mathbb{R},\\ u = \eta + \omega\times x & \text{on } \partial\Omega\times\mathbb{R},\\ \lim\limits_{|x|\to\infty} u = 0, & \end{cases}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))^T$ is the velocity of the fluid; $p = p(x,t)$ the pressure; and $\operatorname{div} F$ is the external force for a 2nd-order tensor $F = F(x,t)$. Meanwhile, $\eta = (0,0,a(t))^T$ and $\omega = (0,0,k(t))^T$ stand for the translational and angular velocities respectively of the obstacle. Here $\Omega = \mathbb{R}^3 \setminus D(0)$ with $D(0)$ being the position of $D \subset \mathbb{R}^3$ at $t = 0$.

For the *autonomous* case (i.e., when η and w do not depend on time t), the existence and stability of weighted Stepanov-like pseudo almost automorphic solutions are proved in Nguyen et al. [33] for a large class of linear and semilinear *autonomous* evolution equations in interpolation spaces. Moreover, Nguyen et al. [34] investigated the existence, uniqueness and polynomial stability of weighted pseudo almost automorphic solutions to *autonomous* parabolic equations and gave applications to autonomous fluid flow problems.

In the present paper we will consider Stepanov-like (resp. weighted Stepanov-like) pseudo almost automorphic solutions to the *non-autonomous* Oseen–Navier–Stokes equations (ONSE), i.e., the case that the angular and translation velocities η and w depend on time t, respectively, with the data belonging to L^p-spaces. This non-autonomous case leads to a time-dependent family of "parabolic" partial differential operators $\mathcal{L}(t)$, $t \in \mathbb{R}$, which generates an evolution family $(U(t,s))_{t\geq s}$. Our method is based on the L^p-L^q smoothing of the evolution family $(U(t,s))_{t\geq s}$ in combination with interpolation functors. For the case of linear equations, the keys of our strategy are the duality estimates, the smoothing properties and interpolation functors for the evolution family $(U(t,s))_{t\geq s}$. Then, we can pass to the semilinear equations using fixed-point arguments. Our main results on existence and stability of S^pAA (or WS^pAA) mild solutions to ONSE are contained in Theorem 2.5. Our results extend the results in [33] in which the autonomous ONSE was considered.

2. Preliminaries and statement of main results

2.1. Preliminaries. We begin by recalling fundamental concepts of almost automorphic and Stepanov-like functions, along with their weighted counterparts, known as weighted Stepanov-like almost automorphic functions. These notions will play a crucial role in our subsequent analysis. As usual, we denote by $BC(\mathbb{R},X)$ the space of bounded, continuous, X-valued functions defined on $\mathbb{R}$, equipped with the supremum norm.

Definition 2.1. A function $f \in BC(\mathbb{R},X)$ is called *almost automorphic* if for any sequence (s'_n) of real numbers, there exists a subsequence (s_n) such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f\left(t+s_n-s_m\right) = f\left(t\right) \tag{2.1}$$

for any $t\in\mathbb{R}$. The limit (2.1) is then understood in the sense that there exists a function $g(t)$ such that

$$g(t) = \lim_{n\to\infty} f(t+s_n), \qquad f(t) = \lim_{n\to\infty} g(t-s_n) \text{ for each } t\in\mathbb{R}. \tag{2.2}$$

We denote the set of all almost automorphic functions with values in a Banach space X by $AA(\mathbb{R},X)$. It is a Banach space with the sup-norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}}\|f(t)\|.$$

Denote by $L^p_{\mathrm{loc}}(\mathbb{R},X)$ the space of all functions from $\mathbb{R}$ to X that are locally Bochner p-integrable. Then a function $f\in L^p_{\mathrm{loc}}(\mathbb{R},X)$ is called *p-Stepanov bounded* (or *S^p-bounded*) if

$$\|f\|_{S^p} := \sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s)\|^p\,ds\Big)^{1/p} < \infty.$$

We denote by $L^p_s(\mathbb{R},X)$ the set of all S^p-bounded functions from $\mathbb{R}$ to X.

Definition 2.2. A function $f\in L^p_s(\mathbb{R},X)$ said to be *Stepanov-like almost automorphic* (or *S^p-almost automorphic*) if for every sequence (s'_n) of real numbers there exists a subsequence (s_n) and a function $\tilde{f}\in L^p_{\mathrm{loc}}(\mathbb{R},X)$ such that

$$\Big(\int_0^1 \|f(t+s_n+s)-\tilde f(t+s)\|_X^p\,ds\Big)^{1/p}\to 0$$

and

$$\Big(\int_0^1 \|f(t-s_n+s)-\tilde f(t+s)\|_X^p\,ds\Big)^{1/p}\to 0$$

as $n\to\infty$ pointwise on $\mathbb{R}$.

We denote the set of all Stepanov-like almost automorphic functions by $S^pAA(\mathbb{R},X)$.

REMARK 2.3. (i) For a function $f\in L_s^p(\mathbb{R},X)$ we define its Bochner transform $f^b:\mathbb{R}\to L^p(\mathbb{R},([0,1],X)))$ by $f^b(t)(s):=f(t+s)$ for all $s\in[0,1]$ and $t\in\mathbb{R}$. Then the function $f\in L_s^p(\mathbb{R},X)$ is Stepanov-like almost automorphic if

$$f^b\in AA(\mathbb{R};L^p(\mathbb{R},([0,1],X))).$$

(ii) $S^pAA(\mathbb{R},X)$ is a Banach space with the norm

$$\|f\|_{S^p}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s)\|^p\,ds\Big)^{1/p}.$$

Let $\mathcal{U}$ be the set of all real-valued functions defined on $\mathbb{R}$, which are positive and locally integrable over $\mathbb{R}$. Then, for a given $r>0$, we put $m(r,\rho):=\int_{-r}^r\rho(x)\,dx$ for each $\rho\in\mathcal{U}$, and define

$$\mathcal{U}_\infty:=\Big\{\rho\in\mathcal{U}:\lim_{r\to\infty}m(r,\rho)=\infty\Big\},$$
$$\mathcal{U}_b:=\Big\{\rho\in\mathcal{U}_\infty:\rho\text{ is bounded and }\inf_{x\in\mathbb{R}}\rho(x)>0\Big\}.$$

It is clear that $\mathcal{U}_b\subset\mathcal{U}_\infty\subset\mathcal{U}$. Now for each $\rho\in\mathcal{U}_\infty$, we define

$$PAA_0(\mathbb{R},\rho):=\Big\{f\in BC(\mathbb{R},X):\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^r\|f(s)\|_X\rho(s)\,ds=0\Big\}.$$

A function belonging to $PAA_0(\mathbb{R},\rho)$ is said to have the *weighted property.*

DEFINITION 2.4. A continuous function $f:\mathbb{R}\to X$ is called *ρ-weighted pseudo almost automorphic* (or *ρ-weighted Stepanov-like pseudo almost automorphic*) if it can be decomposed as $f=g+\phi$ where $g\in AA(\mathbb{R},X)$ (or $g\in S^pAA(\mathbb{R},X)$, respectively) and $\phi\in PAA_0(\mathbb{R},\rho)$. We denote by $WPAA(\mathbb{R},X)$ the set of all ρ-weighted pseudo almost automorphic functions, and by $WS^pAA(\mathbb{R},X)$ the ρ-weighted Stepanov-like pseudo almost automorphic functions, respectively.

Note that $(WPAA(\mathbb{R},X),\|\cdot\|_\infty)$ and $(WS^pAA(\mathbb{R},X),\|\cdot\|_\infty)$ are Banach spaces for the supremum norm $\|\cdot\|_\infty$.

We refer the reader to [12] for examples and a more detailed discussion of almost automorphic, Stepanov-like almost automorphic functions and generalized concepts.

Let Ω be an exterior domain of class $C^{1,1}$ in $\mathbb{R}^3$. Then we consider the space

$$L^p_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^p}} \tag{2.3}$$

where $C^\infty_{0,\sigma}(\Omega) := \{v \in C^\infty_0(\Omega) : \operatorname{div} v = 0 \text{ in } \Omega\}$. We note that, as in [19, 21], the regularity of boundary is needed for the well-posedness and L^p-L^q-smoothing property of the linearized problem, and $C^{1,1}$-regularity was enough for such properties. The Lorentz space $L^{r,q}(\Omega)$ $(1 < r < \infty,\ 1 \le q \le \infty)$ was defined in [23, 42], and here $L^{r,r}(\Omega) = L^r(\Omega)$. Moreover, in case $q = \infty$, the space $L^{r,\infty}(\Omega)$ is called the *weak-L^r space*. We denote $L^r_w(\Omega) := L^{r,\infty}(\Omega)$.

Denote by $\|\cdot\|_{r,w}$ the norm in $L^r_w(\Omega)$. We recall the following inequality from [8, Lemma 2.1] which is known as the *weak Hölder inequality*: For $f \in L^p_w,\ g \in L^q_w$ we have

$$\|fg\|_{r,w} \le C\|f\|_{p,w}\|g\|_{q,w} \tag{2.4}$$

for $1 < p \le \infty, 1 < q < \infty$, $1 < r < \infty$, and $1/p + 1/q = 1/r$, where C is a positive constant depending only on p and q. Here, note that $L^\infty_w = L^\infty$. Recall the *Leray–Helmholtz decomposition* (see [8])

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus \{\nabla p \in L^r : p \in L^r_{\mathrm{loc}}(\bar{\Omega})\}$$

and denote by $\mathbb{P} = \mathbb{P}_r$ the *Helmholtz projection* on $L^r(\Omega)$ $(1 < r < \infty)$, i.e., the projection onto $L^r_\sigma(\Omega)$ relative to the above Leray–Helmholtz decomposition. Next, for each $t \in \mathbb{R}$ we define the operator $L(t)$ as follows:

(2.5)
$$\begin{aligned} D(\mathcal{L}(t)) &:= \{u \in L^r_\sigma \cap W^{1,r}_0 \cap W^{2,r} : (\omega(t) \times x) \cdot \nabla u \in L^r(\Omega)\}, \\ \mathcal{L}(t)u &:= -\mathbb{P}[\Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u] \text{ for } u \in D(\mathcal{L}(t)). \end{aligned}$$

It is known that the family $\{\mathcal{L}(t)\}_{t\in\mathbb{R}}$ of operators generates a bounded evolution family $\{U(t,s)\}_{t\ge s}$ on $L^r_\sigma(\Omega)$ for each $1 < r < \infty$ under the conditions that $\eta, \omega \in C^\theta_{loc}(\mathbb{R};\mathbb{R}^3)$ for some $\theta \in (0,1)$ (see [16, 19]). Furthermore, the *solenoidal Lorentz spaces* are defined (see [8]) by

$$L^{r,q}_\sigma(\Omega) := (L^{r_1}_\sigma, L^{r_2}_\sigma)_{\theta,q} \tag{2.6}$$

where $1 < r_0 < r < r_1 < \infty$, $1 \le q \le \infty$ and $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Then, by interpolation, the family $\{U(t,s)\}_{t\ge s}$ extends to a strongly continuous, bounded evolution family on $L^{r,q}_\sigma(\Omega)$. Denote also $L^r_{\sigma,w}(\Omega) := L^{r,\infty}_\sigma(\Omega)$.

Moreover, for $1 \le q < \infty$, we have the dual space

$$(L^{r,q}_\sigma)' = L^{r',q'}_\sigma \quad \text{where} \quad r' = \frac{r}{r-1},\ q' = \frac{q}{q-1} \text{ and } q' = \infty \text{ if } q = 1. \tag{2.7}$$

We analyze the case in which both $\eta(t)$ and $\omega(t)$ are functions such that

$$\eta, \omega \in C^\theta(\mathbb{R};\mathbb{R}^3) \cap C^1(\mathbb{R};\mathbb{R}^3) \cap L^\infty(\mathbb{R};\mathbb{R}^3) \text{ with some } \theta \in (0,1). \tag{2.8}$$

Let us introduce the following notations:

$$|(\eta,\omega)|_0 := \sup_{t\in\mathbb{R}}(|\eta(t)| + |\omega(t)|), \qquad |(\eta,\omega)|_1 := \sup_{t\in\mathbb{R}}(|\eta'(t)| + |\omega'(t)|), \tag{2.9}$$
$$|(\eta,\omega)|_\theta := \sup_{t>s} \frac{|\eta(t)-\eta(s)| + |\omega(t)-\omega(s)|}{(t-s)^\theta}.$$

We suppose that there is a positive, real constant m such that

$$|(\eta,\omega)|_0 + |(\eta,\omega)|_1 + |(\eta,\omega)|_\theta \le m. \tag{2.10}$$

2.2. Main results. We take $R_0 > 0$ satisfying

$$\mathbb{R}^3 \setminus \Omega \subset B_{R_0} := \{x \in \mathbb{R}^3 : |x| < R_0\}. \tag{2.11}$$

Then we fix a cut-off function $\phi \in C_0^\infty(B_{3R_0})$ such that $\phi = 1$ on B_{2R_0} and set

$$b(x,t) = \tfrac{1}{2}\mathrm{rot}\{\phi(x)(\eta(t) \times x - |x|^2\omega(t))\},$$

which fulfills

$$\operatorname{div} b = 0, \quad b|_{\partial\Omega} = \eta + \omega \times x, \quad b(t) \in C_0^\infty(B_{3R_0}).$$

Moreover, an elementary calculation shows that

$$\omega \times b = \operatorname{div}\begin{pmatrix} -(a(t))^2|x|^2\phi(x)/2 & 0 & a(t)k(t)x_2\phi(x) \\ 0 & -(a(t))^2|x|^2\phi(x)/2 & -a(t)k(t)x_1\phi(x) \\ 0 & 0 & 0 \end{pmatrix} \tag{2.12}$$
$$= \operatorname{div}(-F_1),$$

$$b_t = \operatorname{div}\begin{pmatrix} 0 & -a'(t)|x|^2\phi(x)/2 & -k'(t)x_1\phi(x)/2 \\ a'(t)|x|^2\phi(x)/2 & 0 & -k'(t)x_2\phi(x)/2 \\ k'(t)x_1\phi(x) & k'(t)x_2\phi(x) & 0 \end{pmatrix} \tag{2.13}$$
$$= \operatorname{div}(-F_2).$$

If we set $z(x,t) = u(x,t) - b(x,t)$ then we find that u fulfills (1.1) if and only if z satisfies

$$\begin{cases} z_t - \Delta z - (\eta + \omega \times x)\cdot\nabla z + \omega \times z + \nabla p & \\ = \operatorname{div}\mathbb{G} - (z\cdot\nabla)z - (b\cdot\nabla)z - (z\cdot\nabla)b - (b\cdot\nabla)b & \text{in } \Omega\times\mathbb{R}, \\ \nabla\cdot z = 0 & \text{in } \Omega\times\mathbb{R}, \\ z = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ \lim\limits_{|x|\to\infty} u = 0, & \end{cases} \tag{2.14}$$

where

$$\mathbb{G} := F + F_1 + F_2 + \nabla b + (\eta + \omega \times x) \otimes \nabla b. \tag{2.15}$$

Applying the Helmholtz operator $\mathbb{P}$ to (2.14) we may rewrite the equation as a non-autonomous evolution equation on the whole line $\mathbb{R}$,

$$z_t + \mathcal{L}(t)z = \mathbb{P}\operatorname{div}(\mathcal{F} - z \otimes z - b \otimes z - z \otimes b), \quad t \in \mathbb{R}, \tag{2.16}$$

where $\mathcal{L}(t)$ is defined as in (2.5) and

$$\mathcal{F} := F + F_1 + F_2 + \nabla b + (\eta + \omega \times x) \otimes \nabla b - b \otimes b. \tag{2.17}$$

We now put $G(z) = -z \otimes z - b \otimes z - z \otimes b$. Then (2.16) becomes

$$z_t + \mathcal{L}(t)z = \mathbb{P}\operatorname{div}(G(z) + \mathcal{F}(t)), \quad t \in \mathbb{R}. \tag{2.18}$$

As proved in [19], the family of operators $(\mathcal{L}(t))_{t\in\mathbb{R}}$ generates an evolution family $(U(t,s))_{t\geq s}$ in the sense, roughly speaking, that $z(t) = U(t,s)x$ is the solution to the Cauchy problem $z_t + \mathcal{L}(t)z = 0$, $z(s) = x$. Furthermore, similarly to Kozono and Nakao [24], we can define a *mild solution* of (2.18) as the function $z(t)$ fulfilling the following integral equation in which the integral is understood in the weak sense as in [46, Remark 1.2]:

$$z(t) = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(G(z) + \mathcal{F}(\tau))\, d\tau \quad \text{for } t \in \mathbb{R}. \tag{2.19}$$

We write $\|\cdot\|_{s,w}$ for the norm in $L^s_{\sigma,w}$. The following space is also needed in our strategy:

$$C_{w^*,b}(\mathbb{R}, L^s_{\sigma,w}) := \Big\{v : \mathbb{R} \to L^s_{\sigma,w} : v \text{ is weak}^* \text{ continuous and } \sup_{t\in\mathbb{R}} \|v(t)\|_{s,w} < \infty\Big\} \tag{2.20}$$

endowed with the norm

$$\|v\|_{\infty,s,w} := \sup_{t\in\mathbb{R}} \|v(t)\|_{s,w}.$$

For $R > 0$ we set

$$B_{S^pAA}(0,R) := \big\{v \in S^pAA(\mathbb{R}, L^3_{\sigma,w}) : \|v\|_{\infty,3,w} \leq R\big\},$$
$$B_{WS^pAA}(0,R) := \big\{v \in WS^pAA(\mathbb{R}, L^3_{\sigma,w}) : \|v\|_{\infty,3,w} \leq R\big\}.$$

Our main results are stated in the following theorem which proves the existence, uniqueness, and stability of mild solutions to (2.18) in a small ball $B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$) under certain smallness conditions.

THEOREM 2.5. *Consider an exterior domain Ω in $\mathbb{R}^3$ with $C^{1,1}$-boundary. Suppose that F is an S^pAA-function* (*resp. a WS^pAA-function*) *and belongs to $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. Let η and ω be S^pAA-functions* (*resp. WS^pAA-functions*) *fulfilling* (2.8) *and* (2.10). *Then the following assertions hold:*

(1) *If $\|F\|_{\infty,3/2,w}$ and m are sufficiently small, then problem* (2.18) *has a unique mild solution $\hat{z} \in S^pAA(\mathbb{R}, L^3_{\sigma,w}) \cap C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$ (resp. $WS^pAA(\mathbb{R}, L^3_{\sigma,w}) \cap C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$) in $B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$) provided R is small enough.*

(2) *The solution $\hat{z}$ to* (2.18) *obtained in* (1) *is asymptotically stable in the sense that for any other mild solution $z \in C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$ to* (2.18) *such that $\|z(0)-\hat{z}(0)\|_{3,w}$ is sufficiently small, we have*

$$\|z(t)-\hat{z}(t)\|_{r,w} \le Ct^{-\gamma} \quad \text{for all } t > 0, \tag{2.21}$$

where r is any fixed number satisfying $r > 3$ and $\gamma = \frac{1}{2} - \frac{3}{2r}$.

3. Proofs of the main results. We recall the following $L^{r,q}$-$L^{p,q}$ estimates taken from [21, Theorems 2.1 and 2.2].

Proposition 3.1. *Suppose that η and ω fulfill* (2.8) *and* (2.10) *for $m \in (0,\infty)$. Denote by $\|\cdot\|_{r,q}$ the norm in $L^{r,q}$ (here $1 < r < \infty$, $1 \le q \le \infty$). Then the following assertions hold:*

(i) *For $1 < p \le r < \infty$ we have*

$$\|U(t,s)x\|_{r,q}, \|U(t,s)^*x\|_{r,q} \le M(t-s)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\|x\|_{p,q} \quad \text{for all } t > s. \tag{3.1}$$

(ii) *When $1 < p \le r < 3$ we have*

$$\|\nabla U(t,s)x\|_{r,q}, \|\nabla U(t,s)^*x\|_{r,q} \le M(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\|x\|_{p,q} \quad \text{for all } t > s. \tag{3.2}$$

(iii) *For $1 < p \le r \le 3$ and $1 \le q < \infty$ we have*

$$\|\nabla U(t,s)^*x\|_{r,q} \le M(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\|x\|_{p,q} \quad \text{for all } t > s. \tag{3.3}$$

If in particular $\frac{1}{p}-\frac{1}{r}=\frac{1}{3}$ as well as $1 < p \le r \le 3$, then

$$\int_0^t \|\nabla U(t,s)^*x\|_{r,1}\, ds \le M\|x\|_{p,1} \quad \text{for all } t \ge 0. \tag{3.4}$$

Proof. Using the interpolation theorem and L^p-L^q decay estimates obtain by Hishida [21, Theorem 2.1] we obtain the estimates (3.1) and (3.2). The assertions of (iii) have been proved in [21, Theorem 2.2]. ∎

Remark 3.2. We would like to note that (3.4) also holds when the integral is considered on $(-\infty, t)$. The proof can be done in the same way as in [21, Theorem 2.2] without major changes (see details in [41, Theorem 3.1.15]).

We consider the linear equation

$$z_t + \mathcal{L}(t)z = \mathbb{P}\,\mathrm{div}(\mathcal{F}(t)) \quad \text{for all } t \in \mathbb{R}. \tag{3.5}$$

Similarly to Kozono and Nakao [24], a mild solution of (3.5) is defined to be a function z satisfying

$$z(t) = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

REMARK 3.3. (i) Let η and ω be S^pAA-functions satisfying (2.8) and (2.10). Let the external force F be an S^pAA-function and belong to $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. Then $\mathcal{F}$ is also an S^pAA-function belonging to $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. Moreover,

$$\|\mathcal{F}\|_{\infty,3/2,w} \le \|F\|_{\infty,3/2,w} + Cm. \tag{3.7}$$

(ii) Let η and ω belong to $PAA_0(\mathbb{R},\rho)$ satisfying (2.8) and (2.10). Let the external force F belong to $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3}) \cap PAA_0(\mathbb{R},\rho)$. Then $\mathcal{F}$ belongs to $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3}) \cap PAA_0(\mathbb{R},\rho)$, and moreover

$$\|\mathcal{F}\|_{\infty,3/2,w} \le \|F\|_{\infty,3/2,w} + Cm. \tag{3.8}$$

We have the following lemma on the boundedness of mild solutions to (3.5):

LEMMA 3.4. *Consider an exterior domain Ω in $\mathbb{R}^3$ with $C^{1,1}$-boundary. Suppose that $F \in C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. Then* (3.5) *has a unique mild solution $z \in C_{w^*,b}(\mathbb{R}, L^{3}_{\sigma,w}(\Omega))$ expressed by* (3.6). *Also,*

$$\|z\|_{\infty,3,w} \le \tilde{M}\|\mathcal{F}\|_{\infty,3/2,w}, \tag{3.9}$$

where $\tilde{M}$ is a positive constant independent of z and F.

Proof. The assertion has been proved in [30, Lemma 2.6]. ■

Before stating the theorem containing our main results of this section we give the following lemma.

LEMMA 3.5. *Consider an exterior domain Ω in $\mathbb{R}^3$ with $C^{1,1}$-boundary. Then the following assertions hold:*

(1) *Suppose that the second-order tensor field $F \in C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ is an S^pAA-function. Let η and ω be S^pAA-functions fulfilling* (2.8) *and* (2.10). *Then the mild solution $\hat{z} = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau$ to* (3.5) *belongs to $S^pAA(\mathbb{R}, L^3_{\sigma,w}(\Omega)) \cap C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ with $p \ge 1$.*
(2) *Let η and ω belong to $PAA_0(\mathbb{R},\rho)$ satisfying* (2.8) *and* (2.10). *Let the external force F belongs to $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3}) \cap PAA_0(\mathbb{R},\rho)$. Then the mild solution $\hat{z} = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau$ to* (3.5) *belongs to $PAA_0(\mathbb{R},\rho)$.*

Proof. (1) Since $\mathcal{F} \in C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3}) \cap S^pAA(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$, by Remark 3.3 we know that for each real sequence (s'_n), there exists a subsequence (s_n) and $\tilde{\mathcal{F}} \in L^p_{\mathrm{loc}}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ such that

$$\Big(\int_0^1 \|\mathcal{F}(t+s_n+s) - \tilde{\mathcal{F}}(t+s)\|^p_{3/2,w}\,ds\Big)^{1/p} \to 0$$

and

$$\Big(\int_0^1 \|\mathcal{F}(t-s_n+s) - \tilde{\mathcal{F}}(t+s)\|^p_{3/2,w}\,ds\Big)^{1/p} \to 0$$

as $n \to \infty$, pointwise on $\mathbb{R}$.

We write $\langle\cdot,\cdot\rangle$ to indicate the duality pairing between $L^3_{\sigma,w}$ and $L^{3/2,1}_{\sigma}$. Then, for $\psi \in L^{3/2,1}_{\sigma}$ we estimate

$$\begin{aligned}
&|\langle \hat{z}(t+s_n+s) - \tilde{\hat{z}}(t+s), \psi\rangle| \\
&\quad = \Big|\Big\langle \int_{-\infty}^{t+s_n+s} U(t+s_n+s,z)\mathbb{P}\operatorname{div}(\mathcal{F}(z))\,dz \\
&\qquad - \int_{-\infty}^{t+s} U(t+s,z)\mathbb{P}\operatorname{div}(\tilde{\mathcal{F}}(z))\,dz, \psi\Big\rangle\Big| \\
&\quad = \Big|\Big\langle \int_0^{\infty} U(t,t-z)\mathbb{P}\operatorname{div}(\mathcal{F}(t+s_n+s-z)\,dz \\
&\qquad - \int_0^{\infty} U(t,t-z)\mathbb{P}\operatorname{div}(\tilde{\mathcal{F}}(t+s-z)\,dz, \psi\Big\rangle\Big| \\
&\quad \le \int_0^{\infty} |\langle U(t,t-z)\mathbb{P}\operatorname{div}(\mathcal{F}(t+s_n+s-z) - \tilde{\mathcal{F}}(t+s-z)), \psi\rangle|\,dz \\
&\quad = \int_0^{\infty} |\langle \mathcal{F}(t+s_n+s-z) - \tilde{\mathcal{F}}(t+s-z), \nabla U(t,t-z)^*\psi\rangle|\,dz \\
&\quad \le \int_0^{\infty} \|\mathcal{F}(t+s_n+s-z) - \tilde{\mathcal{F}}(t+s-z)\|_{3/2,w}\|\nabla U(t,t-z)^*\psi\|_{3,1}\,dz \\
&\quad \le \sup_{z\in\mathbb{R}} \|\mathcal{F}(t+s_n+s-z) - \tilde{\mathcal{F}}(t+s-z)\|_{3/2,w} \int_0^{\infty} \|\nabla U(t,t-z)^*\psi\|_{3,1}\,dz \\
&\quad \le \tilde{L} \sup_{z\in\mathbb{R}} \|\mathcal{F}(t+s_n+s-z) - \tilde{\mathcal{F}}(t+s-z)\|_{3/2,w}\|\psi\|_{3/2,1}.
\end{aligned}$$

Hence

$$\|\hat{z}(t+s_n+s) - \tilde{\hat{z}}(t+s)\|_{3,w} \le \tilde{L} \sup_{z\in\mathbb{R}} \|\mathcal{F}(t+s_n+s-z) - \tilde{\mathcal{F}}(t+s-z)\|_{3/2,w}.$$

Since

$$\Big(\int_0^1 \|\mathcal{F}(t+s_n+s-z)-\tilde{\mathcal{F}}(t+s-z)\|_{3/2,w}^p\,ds\Big)^{1/p} \to 0$$

as $n\to\infty$, we see that $\|\mathcal{F}(t+s_n+s-z)-\tilde{\mathcal{F}}(t+s-z)\|_{3/2,w}$ is bounded and converges to 0 as $n\to\infty$ for almost all $s\in[0,1]$.

Therefore $\hat{z}(t+s_n+s)\to\tilde{\hat{z}}(t+s)$ as $n\to\infty$ for almost all $s\in[0,1]$. By Lemma 3.4, $\|\hat{z}(t+s_n+s)\|_{3,w}$ is bounded by $\tilde{L}\|\mathcal{F}\|_{\infty,3/2,w}$. From this fact and by dominated convergence in L^p-spaces, it follows that $\hat{z}(t+s_n+s)\to\tilde{\hat{z}}(t+s)$ as $n\to\infty$ in $L^p([0,1],L^3_{\sigma,w}(\Omega))$. This yields

$$\Big(\int_0^1 \|\hat{z}(t+s_n+s)-\tilde{\hat{z}}(t+s)\|_{3,w}^p\,ds\Big)^{1/p}\to 0 \qquad \text{as } n\to\infty \text{ pointwise on } \mathbb{R}.$$

Similarly,

$$\Big(\int_0^1 \|\hat{z}(t-s_n+s)-\tilde{\hat{z}}(t+s)\|_{3,w}^p\,ds\Big)^{1/p}\to 0 \qquad \text{as } n\to\infty \text{ pointwise on } \mathbb{R}.$$

Therefore, $\hat{z}\in S^pAA(\mathbb{R},L^3_{\sigma,w}(\Omega))$.

(2) The conclusion in (2) is equivalent to

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\Big\|\int_{-\infty}^{s}U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau\Big\|_{3,w}\rho(s)\,ds=0.$$

Next, for each $\psi\in L_\sigma^{3/2,1}$ we have

$$\begin{aligned}(3.10)\qquad &\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{s}|\langle U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau)),\psi\rangle|\,d\tau\ \rho(s)\,ds\\ &\quad=\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{-r}|\langle U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau)),\psi\rangle|\,d\tau\ \rho(s)\,ds\\ &\qquad+\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-r}^{s}|\langle U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau)),\psi\rangle|\,d\tau\ \rho(s)\,ds.\end{aligned}$$

The $L^{r,q}$-$L^{p,q}$ smoothing properties and interpolation (similarly to inequality (3.4)) yield

$$\int_{-\infty}^{s}\|\nabla U(s,\tau)^*\psi\|_{3,1}\,d\tau\le\tilde{L}\|\psi\|_{3/2,1},$$

hence, for all $\epsilon>0$, there exists $r_0\in\mathbb{R}$ such that for all $r>r_0$, we have

$$\int_{-\infty}^{-r}\|\nabla U(s,\tau)^*\psi(\tau)\|_{3,1}\,d\tau<\epsilon\|\psi\|_{3/2,1}.$$

Therefore, for $\|\mathcal{F}\|_{\infty,3/2,w} < M$, we have

$$(3.11)\qquad \begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{-r}|\langle U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau)),\psi\rangle|\,d\tau\,\rho(s)\,ds\\
&\quad=\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{-r}|\langle\mathcal{F}(\tau),\nabla U(s,\tau)^*\psi\rangle|\,d\tau\,\rho(s)\,ds\\
&\quad\le\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{-r}\|\mathcal{F}(\tau)\|_{3/2,w}\|\nabla U(s,\tau)^*\psi\|_{3,1}\,d\tau\ \rho(s)\,ds\\
&\quad\le\frac{\tilde{L}M\|\psi\|_{3/2,1}}{\int_{-r}^{r}\rho(s)\,ds}\int_{-r}^{r}\epsilon\rho(s)\,ds=\tilde{L}M\epsilon\|\psi\|_{3/2,1}.
\end{aligned}$$

On the other hand, using (3.4) we obtain the following estimate:

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-r}^{s}|\langle U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau)),\psi\rangle|\,d\tau\ \rho(s)\,ds\\
&\quad\le\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-r}^{s}|\langle\mathcal{F}(\tau),\nabla U(s,\tau)^*\psi\rangle|\,d\tau\ \rho(s)\,ds\\
&\quad\le\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-r}^{s}\|\mathcal{F}(\tau)\|_{3/2,w}\|\nabla U(s,\tau)^*\psi\|_{3,1}\,d\tau\ \rho(s)\,ds\\
&\quad\le\frac{\tilde{L}\|\psi\|_{3/2,1}}{m(r,\rho)}\int_{-r}^{r}\|\mathcal{F}(s)\|_{3/2,w}\rho(s)\,ds.
\end{aligned}$$

From the fact that $\mathcal{F}\in PAA_0(\mathbb{R},\rho)$, we get

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\|\mathcal{F}(s)\|_{3/2,w}\rho(s)\,ds=0.$$

Hence, there exists $r_1\in\mathbb{R}$ such that for all $r>r_1$,

$$(3.12)\qquad \frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-r}^{s}|\langle U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau)),\psi\rangle|\,d\tau\ \rho(s)\,ds\le\tilde{L}\epsilon\|\psi\|_{3/2,1}.$$

By combining (3.10)–(3.12), we get

$$\frac{1}{m(r,\rho)}\int_{-r}^{r}\Big|\int_{-\infty}^{s}\langle U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau)),\psi\rangle\,d\tau\Big|\rho(s)ds\le\tilde{L}\epsilon(M+1)\|\psi\|_{3/2,1},$$

for all $r\ge\max\{r_0,r_1\}$ and all $\psi\in L_\sigma^{3/2,1}$. Therefore,

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\Big\|\int_{-\infty}^{s}U(s,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau\Big\|_{3,w}\rho(s)\,ds=0.\ \blacksquare$$

We state and prove the following results for linear equations.

THEOREM 3.6. *Let $p \geq 1$ and consider an exterior domain Ω in $\mathbb{R}^3$ with $C^{1,1}$-boundary. Then the following assertions hold:*

(a) *Suppose that the second-order tensor field $F \in C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ is an S^pAA-function. Let η and ω be S^pAA-functions fulfilling (2.8) and (2.10). Then the mild solution $\hat{z} = \int_{-\infty}^t U(t,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau$ to (3.5) belongs to $S^pAA(\mathbb{R}, L^3_{\sigma,w}(\Omega))$.*

(b) *Let $\eta, \omega \in PAA_0(\mathbb{R}, \rho)$, satisfying (2.8) and (2.10), be S^pAA-functions. If $F \in WS^pAA(\mathbb{R}, X) \cap C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$, then the mild solution $\hat{z} = \int_{-\infty}^t U(t,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau$ to (3.5) belongs to $WS^pAA(\mathbb{R}, L^3_{\sigma,w}(\Omega))$.*

Proof. By Lemma 3.5 we can see that the mild solution

$$\hat{z} = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(\mathcal{F}(\tau))\,d\tau$$

is an S^pAA-function and is weak* continuous. It follows from [26, Lemma 3.1] that $\hat{z}$ is an S^pAA-function, and hence (a) follows. To prove (b), we write $\mathcal{F} = g + \phi$ where $g \in S^pAA(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ and $\phi \in PAA_0(\mathbb{R}, \rho)$. Putting

$$\hat{z}_1(t) = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(g(\tau))\,d\tau, \qquad \hat{z}_2(t) = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(\phi(\tau))\,d\tau,$$

by (a) we see that $\hat{z}_1$ belongs to $S^pAA(\mathbb{R}, L^3_{\sigma,w}(\Omega))$, and by Lemma 3.5, $\hat{z}_2$ belongs to $PAA_0(\mathbb{R}, \rho)$. Therefore $\hat{z}$ belongs to $WS^pAA(\mathbb{R}, L^3_{\sigma,w}(\Omega))$ and (b) is proved. ■

For the non-linear operator $G(z)$, by similar arguments to those used by Nguyen et al. [32, (4.14)] (note that $G(0) = 0$) and using the weak Hölder inequality we have

$$\begin{aligned} \|G(v_1) - G(v_2)\|_{\infty,3/2,w} &\leq (\|b\|_{\infty,3,w} + \|v_1\|_{\infty,3,w} + \|v_2\|_{\infty,3,w})\|v_1 - v_2\|_{\infty,3,w} \\ &\leq (m + \|v_1\|_{\infty,3,w} + \|v_2\|_{\infty,3,w})\|v_1 - v_2\|_{\infty,3,w} \end{aligned} \tag{3.13}$$

for $v_1, v_2 \in C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$. Therefore, we also get

$$\|G(v_1)\|_{\infty,3/2,w} \leq (m + \|v_1\|_{\infty,3,w})\|v_1\|_{\infty,3,w} \tag{3.14}$$

for $v_1 \in C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$, $v_2 = 0$ in (3.13). Moreover, for $r > 3$ we have

$$\|G(v_1) - G(v_2)\|_{\infty,\frac{3r}{3+r},w} \leq (m + \|v_1\|_{\infty,3,w} + \|v_2\|_{\infty,3,w})\|v_1 - v_2\|_{\infty,r,w} \tag{3.15}$$

for $v_1, v_2 \in C_{w^*,b}(\mathbb{R}, L^r_{\sigma,w}) \cap C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$.

We now give the proof of our main results stated in Theorem 2.5.

Proof of Theorem 2.5. (1) Consider the closed sets

$$B_{S^pAA}(0,R) := \{v \in S^pAA(\mathbb{R}, L^3_{\sigma,w}) : \|v\|_{\infty,3,w} \le R\},$$

$$B_{WS^pAA}(0,R) := \{v \in WS^pAA(\mathbb{R}, L^3_{\sigma,w}) : \|v\|_{\infty,3,w} \le R\}.$$

By Remark 3.3, $\mathcal{F}$ is an S^pAA-function (resp. a WS^pAA-function) and belongs to $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. For each $v \in B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$), $G(v)$ belongs to the space $C_{w^*,b}(\mathbb{R}, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ and is an S^pAA-function (resp. a WS^pAA-function). On $B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$), we define the map $\mathcal{T}$ by

$$(\mathcal{T}v)(t) = \int_{-\infty}^{t} U(t,\tau)\mathbb{P}\operatorname{div}(G(v) + \mathcal{F}(\tau))\,d\tau.$$

Applying Theorem 3.6 to the right-hand side $G(v) + \mathcal{F}$ instead of $\mathcal{F}$ we find that $z := \mathcal{T}v$ is an S^pAA-function (resp. a WS^pAA-function) and is the unique mild solution in $C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w}(\Omega))$ to the equation

$$z_t + \mathcal{L}(t)z = \mathbb{P}\operatorname{div}(G(v) + \mathcal{F}). \tag{3.16}$$

Moreover, by (3.14), (3.9) and using the fact that $v \in B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$) we have

$$\|z\|_{\infty,3,w} \le \tilde{M}(\|G(v)\|_{\infty,3/2,w} + \|\mathcal{F}\|_{\infty,3/2,w}) \le \tilde{M}(mR + R^2 + \|\mathcal{F}\|_{\infty,3/2,w}).$$

Therefore, if $\|F\|_{\infty,3/2,w}$, R, and m are sufficiently small then $\mathcal{T}$ acts from $B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$) into itself.

For $v_1, v_2 \in B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$) putting $\mathcal{T}v_1 = z_1$ and $\mathcal{T}v_2 = z_2$, similarly by (3.13), (3.9) we obtain

$$\|z_1 - z_2\|_{\infty,3,w} \le \tilde{M}\|G(v_1) - G(v_2)\|_{\infty,3/2,w} \le \tilde{M}(m + 2R)\|v_1 - v_2\|_{\infty,3,w}.$$

Thus, if $\|F\|_{\infty,3/2,w}$, R and m are small enough then the mapping $\mathcal{T}$ acts from $B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$) into itself and is a contraction. Therefore, $\mathcal{T}$ has a unique fixed point $\hat{z}$, and by the definition of $\mathcal{T}$, (2.18) has a unique mild solution $\hat{z} \in S^pAA\left(\mathbb{R}, L^3_{\sigma,w}\right) \cap C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$ (resp. $WS^pAA(\mathbb{R}, L^3_{\sigma,w}) \cap C_{w^*,b}(\mathbb{R}, L^3_{\sigma,w})$) in $B_{S^pAA}(0,R)$ (resp. $B_{WS^pAA}(0,R)$).

(2) We can see that $z(t)$ and $\hat{z}(t)$ satisfy the following integral equations:

$$z(t) = U(t,0)u_0 + \int_0^t U(t,\tau)\mathbb{P}\operatorname{div}(G(z) + \mathcal{F}(t))\,d\tau,$$

$$\hat{z}(t) = U(t,0)\hat{z}_0 + \int_0^t U(t,\tau)\mathbb{P}\operatorname{div}(G(\hat{z}) + \mathcal{F}(t))\,d\tau.$$

By putting $v = z - \hat{z}$ we deduce that v satisfies the integral equation

$$(3.17)\qquad v(t) = U(t,0)(z_0 - \hat{z}_0) + \int_0^t U(t,\tau)\mathbb{P}\operatorname{div}(G(z)(\tau) - G(\hat{z})(\tau))\,d\tau.$$

We set $\mathbb{M} := \{v \in C_{w^*,b}(\mathbb{R}_+, L^3_{\sigma,w}) : \sup_{t\in\mathbb{R}_+} t^{\frac{1}{2}-\frac{3}{2r}}\|v(t)\|_{r,w} < \infty\}$ endowed with the norm

$$\|v\|_{\mathbb{M}} = \|v\|_{\infty,3,w} + \sup_{t\in\mathbb{R}_+} t^{\frac{1}{2}-\frac{3}{2r}}\|v(t)\|_{r,w}.$$

Now, we prove that if $\|\hat{z}\|_{\infty,3,w}$ and $\|z_0 - \hat{z}_0\|_{3,w}$ and m are small enough, then (3.17) has a unique solution on a small ball of $\mathbb{M}$. Indeed, for $v \in \mathbb{M}$, we consider the mappings

$$\Phi(v)(t) := U(t,0)(z_0 - \hat{z}_0) + \int_0^t U(t,\tau)\mathbb{P}\operatorname{div}\big(G(v+\hat{z})(\tau) - G(\hat{z})(\tau)\big)\,d\tau.$$

Setting $B_\rho := \{v \in \mathbb{M} : \|v\|_{\mathbb{M}} \le \rho\}$, by an argument similar to that in the proof of [30, Theorem 2.8], we find that for sufficiently small $\|z(0)-\hat{z}(0)\|_{3,w}$, $\|\hat{z}\|_{\infty,3,w}$, m, and ρ, the mapping Φ is from B_ρ into B_ρ, and it is a contraction. So, Φ has a unique fixed point. Therefore, the function $v = z - \hat{z}$, being the fixed point of this mapping, belongs to $\mathbb{M}$. Thus, we obtain (2.21), and hence the stability of $\hat{z}$ follows. ∎

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