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ON ZERO-SUM SUBSEQUENCES OVER FINITE ABELIAN GROUPS OF LENGTH NOT EXCEEDING A GIVEN NUMBER

BY

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Abstract. Let G be a finite abelian group and let k be an integer with $k \in [\exp(G), \mathsf{D}(G) - 1]$, where $\exp(G)$ and $\mathsf{D}(G)$ are the exponent and the Davenport constant of G, respectively. Denote by $\mathsf{s}_{\leq k}(G)$ the smallest positive integer $l \in \mathbb{N} \cup \{+\infty\}$ such that each sequence of length l over G has a nontrivial zero-sum subsequence of length at most k. We prove that $\mathsf{s}_{\leq \mathsf{D}(G)-2}(G) \leq \mathsf{D}(G) + 2$ for all finite non-cyclic abelian groups except for C_2^3, C_2^4 and $C_2 \oplus C_{2n}$. For some classes of p-groups G, we show that $\mathsf{s}_{\leq k}(G) \leq 2\mathsf{D}(G) - k$ for $k = c_1 p^{t+1} \in \big[\frac{\mathsf{D}(G)+1}{2}, \mathsf{D}(G)\big]$ with $c_1 \in [0, p-1]$ and $t \geq 0$. In addition, some lower bounds are obtained.

1. Introduction and the main results. Let $(G, +, 0)$ be a finite abelian group and $G^* = G \setminus \{0\}$. It is well-known that either $|G| = 1$ or $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$, where $r \in \mathbb{N}$ and $1 < n_1 \,|\, n_2 \,|\, \cdots \,|\, n_r$. Then $\mathsf{r}(G) := r$ is called the *rank* of G and $\exp(G) = n_r$ is called *exponent* of G. Let

$$S = g_1 \ldots g_l$$

be a sequence with terms from G. We say S is a *zero-sum sequence* if $g_1 + \cdots + g_l = 0$. Let $\mathscr{F}(G)$ denote the monoid of all sequences over G. Let $\mathscr{B}(G)$ denote the monoid of all zero-sum sequences over G, and denote by $\mathbb{1}$ the identity element of $\mathscr{B}(G)$, i.e., the trivial sequence over G. The *Davenport constant* $\mathsf{D}(G)$ of G is defined as the smallest positive integer t such that every sequence S over G of length t has a nontrivial zero-sum subsequence. Set

$$\mathsf{D}^*(G) := 1 + \sum_{i=1}^{r} (n_i - 1).$$

A simple construction shows that $\mathsf{D}(G) \geq \mathsf{D}^*(G)$.

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In zero-sum theory, people often consider the following two problems:

The *direct zero-sum problem* is to find the smallest integer $\ell \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq \ell$ has a zero-sum subsequence with prescribed length.

The inverse zero-sum problem asks about the structure of the sequence S over G without zero-sum subsequences of prescribed length.

Let $L \subset \mathbb{N}$ be a nonempty subset and let $\mathsf{s}_L(G)$ be the smallest integer $\ell \in \mathbb{N} \cup \{\infty\}$ such that every sequence S over G of length $|S| \geq \ell$ has a zero-sum subsequence T of length $|T| \in L$. Thus $\mathsf{D}(G) = \mathsf{s}_{\mathbb{N}}(G)$, $\eta(G) = \mathsf{s}_{[1,\exp(G)]}(G)$, $\mathsf{s}(G) = \mathsf{s}_{\{\exp(G)\}}(G)$, and $\mathsf{s}_{k\exp(G)}(G) = \mathsf{s}_{\{k\exp(G)\}}(G)$, where $\eta(G)$, the *Erdős–Ginzburg–Ziv constant* $\mathsf{s}(G)$, and the *generalized Erdős–Ginzburg–Ziv constant* $\mathsf{s}_{k\exp(G)}(G)$ are all classical zero-sum invariants. The invariant $\mathsf{s}_L(G)$ is also investigated for various other sets L (see e.g. (15, 17–19, 29, 38). The readers may want to consult one of the surveys or monographs (14, 20–22, 24).

In 2001, Delorme, Ordaz and Quiroz [6] systematically investigated the invariant $\mathsf{s}_L(G)$ for the case $L = [1, k]$. We denote

$$\mathsf{s}_{\leq k}(G) := \mathsf{s}_{[1,k]}(G) \quad \text{for all } k \in \mathbb{N}.$$

It is easy to see that $\mathsf{s}_{\leq k}(G) = \mathsf{D}(G)$ if $k \geq \mathsf{D}(G)$, $\mathsf{s}_{\leq k}(G) = \mathsf{D}(G) + 1$ if $k = \mathsf{D}(G) - 1$, $\mathsf{s}_{\leq k}(G) = \eta(G)$ if $k = \exp(G)$, and $\mathsf{s}_{\leq k}(G) = \infty$ if $1 \leq k < \exp(G)$. Thus it suffices to consider $\mathsf{s}_{\leq k}(G)$ for $k \in [\exp(G), \mathsf{D}(G) - 2]$.

The invariant $\mathsf{s}_{\leq k}(G)$ is a variant of the Erdős–Ginzburg–Ziv constant $\mathsf{s}(G)$, one of the most investigated invariants in zero-sum theory. For more, one can see [34]. The invariant $\mathsf{s}_{\leq k}(G)$ also has a close connection with other invariants. In [15, 16], Gao et al. studied $\mathsf{s}_{k\exp(G)}(G)$ and conjectured that $\mathsf{s}_{k\exp(G)}(G) = \mathsf{s}_{\leq k\exp(G)}(G) + k\exp(G) - 1$. Bhowmik and Schlage-Puchta [1] used $\mathsf{s}_{\leq k}(G)$ to study the Davenport constant $\mathsf{D}(G)$. For more results related to the Davenport constant, one can see [2, 6]. Freeze and Schmid [10] used $\mathsf{s}_{\leq k}(G)$ to study the kth Davenport constant $\mathsf{D}_k(G)$, which is the smallest $\ell \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq \ell$ has k disjoint nontrivial zero-sum subsequences. In [4], Cohen and Zémor pointed out a connection between $\mathsf{s}_{\leq k}(G)$ and coding theory.

Even though the direct and inverse zero-sum problems of $\mathsf{s}_{\leq k}(G)$ have been widely studied, those problems are still widely open. Freeze and Schmid [10] obtained the exact value $\mathsf{s}_{\leq 3}(C_2^r) = 1 + 2^{r-1}$. Lindström [30] studied $\mathsf{s}_{\leq 4}(C_2^r)$ when $r \to \infty$. In [32], Luo proved that for a finite abelian p-group G with a large exponent,

$$\mathsf{s}_{\leq \exp(G)} = \eta(G) = 2\mathsf{D}(G) - \exp(G).$$

For rank 2 groups G, Wang and Zhao [39] proved that

$$\mathsf{s}_{\leq k}(G) = 2\mathsf{D}(G) - k \quad \text{when } k \in [\exp(G), \mathsf{D}(G) - 1].$$

For finite abelian p-groups G with a large exponent, Roy and Thangadurai [33] showed that

$$\mathsf{s}_{\le k}(G) \le 2\mathsf{D}(G) - k \quad \text{under mild conditions on } k.$$

Recently, Zhang [40] continued to study this problem for abelian groups of rank 3. For the inverse problem, few results are known. Davydov and Tombakis [5] studied the inverse problem by using the methods from coding theory. Recently, the inverse problem of $\mathsf{s}_{\le k}(G)$ for rank-2 groups has also been completely solved (8, 25–27).

Most of the known results are for finite abelian p-groups. In this paper, we study the invariant $\mathsf{s}_{\le k}(G)$ for general finite abelian groups.

Theorem 1.1. *Let G be a finite abelian group with $\mathsf{r}(G) \ge 2$. If $G \ncong C_2^3$, C_2^4, $C_2 \oplus C_{2n}$, then $\mathsf{s}_{\le \mathsf{D}(G)-2}(G) \le \mathsf{D}(G)+2$. In particular, if $\mathsf{D}(G) = \mathsf{D}^*(G)$ and $\exp(G) \ge \frac{\mathsf{D}(G)-1}{2}$, then $\mathsf{s}_{\le \mathsf{D}(G)-2}(G) = \mathsf{D}(G) + 2$.*

Remark. For $G \cong C_2^3$, we have $\mathsf{D}(G) = 4$ and $\mathsf{s}_{\le 2}(G) = 8$. For $G \cong C_2^4$, we have $\mathsf{D}(G) = 5$ and $\mathsf{s}_{\le 3}(G) = 2^3 + 1 = 9$. These follow from Lemma 2.3(iii).

For abelian p-groups, we have the following results.

Theorem 1.2. *Let G be a finite abelian p-group and $k-1 = c_1 p^{t+1} \in \left[\frac{\mathsf{D}(G)+1}{2}, \mathsf{D}(G)-1\right]$ with $t \ge 0$ and $c_1 \in [1, p-1]$. Then $\mathsf{s}_{\le k-1}(G) \le 2\mathsf{D}(G) - k + 1$ provided that either*

(1) *$p > 2$ and $c_1 \le \frac{p-1}{3}$, or*
(2) *$p = 2$.*

Theorem 1.3. *Let G be a finite abelian p-group. Then $\mathsf{s}_{\le k-1}(G) \le 2\mathsf{D}(G) - k + 1$ provided that either*

(1) *$G \cong C_p^4$ and $k - 1 = 2p$ with $p \ge 5$, or*
(2) *$G \cong C_p^d$ and $k - 1 = (d-1)p \in [p, \mathsf{D}(C_p^d)]$.*

Remark. Theorems 1.2 and 1.3 improve greatly some known results given in Lemmas 2.2 and 2.3.

The paper is organized as follows. In Section 2, we recall some basic notions and collect several known results and some lower bounds of $\mathsf{s}_{\le k}(G)$. We give the proof of Theorem 1.1 in Section 3. In Section 4, we consider the problem when there exists a nontrivial zero-sum subsequence of length bounded by k for zero-sum sequences over finite abelian p-groups. By considering this problem, we obtain the upper bound $\mathsf{s}_{\le k}(G) \le 2\mathsf{D}(G) - k$ for some classes of p-groups G in Section 4, and we prove Theorems 1.2 and 1.3 in Section 5.

2. Preliminaries. Throughout the paper, let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_{\geq k} = \{n \in \mathbb{N}\colon n \geq k\}$ for any $k \in \mathbb{N}$. For real $a \leq b$, we let $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$ be the discrete interval. Let $\uplus$ denote the union of pairwise disjoint sets. We consider sequences as elements of the free abelian monoid $\mathscr{F}(G)$ over G and our notation and terminology coincides with [14, 22, 24].

Let

$$S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell = \prod_{g\in G} g^{\mathsf{v}_g(\alpha)}$$

be a sequence over G, where $\mathsf{v}_g(S) \in \mathbb{N}_0$ is called the *multiplicity* of g in S and $|S| = \ell = \sum_{g\in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ is called the *length* of S. The *sum* of elements in S is denoted by $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g\in G} \mathsf{v}_g(S) \cdot g \in G$, and the maximal multiplicity of a term in S is denoted by $\mathsf{h}(S)$.

A sequence S is called a *zero-sum sequence* if $\sum_{i=1}^{\ell} g_i = 0 \in G$. We say that T is a *subsequence* of S if $\mathsf{v}_g(S) \geq \mathsf{v}_g(T)$ for all $g \in G$, and write $T \mid S$. In particular, if T is a subsequence of S and $T \neq S$, then we say that T is a *proper subsequence* of S. If T is *trivial*, then we denote $T = \mathbb{1} \in \mathscr{B}(G)$. We say that S is *zero-sum free* if it contains no nontrivial zero-sum subsequence. If S is a zero-sum sequence and each proper subsequence is zero-sum free, then S is called a *minimal zero-sum sequence*.

We denote by $\mathrm{supp}(S)$ the subset of G consisting of all elements with $\mathsf{v}_g(S) > 0$ and by $\langle \mathrm{supp}(S)\rangle$ the subgroup generated by $\mathrm{supp}(S)$. If $T^n = \prod_{g\in G} g^{n\mathsf{v}_g(T)}$ is a subsequence of S for some $n \in \mathbb{N}$, then we denote by ST^{-n} the sequence obtained from S by deleting T^n. Moreover, we let

$$\varSigma(S) = \{\sigma(T) : T \mid S,\, T \neq \mathbb{1}\}.$$

Now we collect some results for the direct zero-sum problem of $\mathsf{s}_{\leq k}(G)$.

Lemma 2.1 ([39, Lemma 8]). *Let G be a finite abelian group with $\mathsf{r}(G) \geq 2$. Then $\mathsf{s}_{\leq \mathsf{D}(G)-1}(G) = \mathsf{D}(G) + 1$.*

Lemma 2.2 ([1]). *We have $\mathsf{s}_{\leq \frac{3p-1}{2}}(C_p^d) \leq (6p-4)p^{d-3} + 1$ for $d \geq 3$, $\mathsf{s}_{\leq 2p}(C_p^d) \leq (6p-4)p^{d-4} + 1$ for $d \geq 4$ and $\mathsf{s}_{\leq (d-1)p}(C_p^d) \leq (d-1)p$.*

Lemma 2.3. *We have*

(i) $\mathsf{s}_{\leq 3}(C_3^3) = 17$, $\mathsf{s}_{\leq 4}(C_3^3) = 10$, $\mathsf{s}_{\leq 5}(C_3^3) = 9$, $\mathsf{s}_{\leq 6}(C_3^3) = 8$, $\mathsf{s}_{\leq 7}(C_3^3) = \mathsf{D}(C_3^3) = 7$ ([2]);

(ii) $\mathsf{s}_{\leq 5}(C_5^3) = 33$, $\mathsf{s}_{\leq 6}(C_5^3) = 24$, $\mathsf{s}_{\leq 7}(C_5^3) = 19$, $\mathsf{s}_{\leq 8}(C_5^3) = 18$, $\mathsf{s}_{\leq 9}(C_5^3) = 17$, $\mathsf{s}_{\leq 10}(C_5^3) = 15$, $\mathsf{s}_{\leq 11}(C_5^3) = 14$, $\mathsf{s}_{\leq 12}(C_5^3) = 14$, $\mathsf{s}_{\leq 13}(C_5^3) = \mathsf{D}(C_5^3) = 13$ ([37]);

(iii) $\mathsf{s}_{\leq 2}(C_2^r) = 2^r$; $\mathsf{s}_{\leq 3}(C_2^r) = 2^{r-1}+1$; $\mathsf{s}_{\leq 4}(C_2^d) \leq 2^{(d+1)/2}+1$; $\mathsf{s}_{\leq r-k}(C_2^r) = r+2$ *for all* $r-k \in \left[\lceil \frac{2r+2}{3} \rceil, r\right]$; $\mathsf{s}_{\leq 2m}(C_2^r) \leq (m+1) + (m!2^r)^{1/m}$ ([10, 30, 39]);

(iv) $\mathsf{s}_{\leq k}(G) = 2\mathsf{D}(G) - k$ *for a finite abelian group* G *with* $\mathsf{r}(G) = 2$ *and* $k \in [\exp(G), \mathsf{D}(G)]$ ([39]);

(v) $\mathsf{s}_{\leq \mathsf{D}(G)-2}(G) = \mathsf{D}(G) + 1$ *for* $G = C_p^r$ *with* $3 \leq r < p$; $\mathsf{s}_{\leq \mathsf{D}(G)-p^n}(G) = \mathsf{D}(G) + p^n$ *for* $G = C_{p^n}^3$ *with* p *odd prime* ([40]).

LEMMA 2.4 ([33, Theorem 1.1]). *Let* H *be a finite abelian* p*-group with exponent* $\exp(H) = p^m$ *for some integer* $m \geq 1$ *and for a prime number* $p > 2\mathsf{r}(H)$. *Suppose that* $\mathsf{D}(H) - 1 = kp^m + t$ *for some integers* $k \geq 1$ *and* $0 \leq t \leq p^m - 1$. *Let* $G = C_{p^n} \oplus H$ *be a finite abelian* p*-group for some integer* n *satisfying* $p^n \geq 2(\mathsf{D}(H) - 1)$. *Let* ℓ *be any integer satisfying* $\ell = ap^m + t'$ *for some integer* a *satisfying* $0 \leq a \leq k-1$ *and for some integer* t' *satisfying* $0 \leq t' \leq t$. *Then*

$$\mathsf{s}_{\leq \exp(G)+\ell}(G) \leq 2\mathsf{D}(G) - \exp(G) - \ell.$$

LEMMA 2.5 ([32, 36]). *Let* p *be an odd prime and let* G *be a finite abelian* p*-group with* $\exp(G) = n$ *and* $\mathsf{D}(G) \leq 2n - 1$. *Then*

$$2\mathsf{D}(G) - 1 \leq \eta(G) + n - 1 \leq \mathsf{s}(G) \leq \mathsf{D}(G) + 2n - 2.$$

Furthermore, if $\mathsf{D}(G) = 2n - 1$, *then* $\mathsf{s}(G) = \eta(G) + n - 1 = 4n - 3$; *if* $G' = G \oplus C_k$ *where* k *is a positive integer not divisible by* p, *then* $\eta(G') = 2\mathsf{D}(G') - \exp(G')$.

Here, we list some results for the inverse zero-sum problem for $\mathsf{s}_{\leq k}(C_n \oplus C_n)$.

LEMMA 2.6 ([8, 25, 26, 27]). *Let* $n \geq 2$, *let* $G = C_n \oplus C_n$, *let* $k \in [2, n-2]$, *and let* S *be a sequence of terms from* G *of length* $|S| = \mathsf{D}(G) + k - 1 = 2n - 2 + k$ *having no nontrivial zero-sum subsequence of length at most* $\mathsf{D}(G) - k = 2n - 1 - k$. *Then there exists a basis* (e_1, e_2) *for* G *such that*

$$S = e_1^{n-1} \boldsymbol{\cdot} e_2^{n-1} \boldsymbol{\cdot} (e_1 + e_2)^k.$$

We also collect some results on the groups G with $\mathsf{D}(G) = \mathsf{D}^*(G)$.

LEMMA 2.7. *Let* G *be a finite abelian group. Then* $\mathsf{D}(G) = \mathsf{D}^*(G)$, *provided that one of the following holds:*

(a) $\mathsf{r}(G) \leq 2$ ([20, Theorem 4.2.10] *or* [22, Theorem 5.8.3]);
(b) G *is a finite abelian* p*-group* ([20, Theorem 2.2.6] *or* [22, Theorem 5.5.9]);
(c) $G = K \oplus C_{km}$, *where* K *is a* p*-group with* $\mathsf{D}(K) \leq m$ *and* m *is a power of* p ([20, Corollary 4.2.13]);
(d) $G = C_2 \oplus C_{2m} \oplus C_{2n}$ *with* $m|n$ ([11, 13]);
(e) $G = C_3 \oplus C_{6m} \oplus C_{6n}$ *with* $m|n$ ([11, 13]);
(f) $G = C_{2p^a} \oplus C_{2p^b} \oplus C_{2p^c}$ *with* $a \leq b \leq c$ *being nonnegative integers, and* p *being a prime* ([11]);
(g) $G = C_2^3 \oplus C_{2n}$ ([11]);

(h) $G = C_2^4 \oplus C_{2n}$ *with* $n \geq 70$ *and* n *even* ([35, Theorem 5.8])*;*
(i) $G = C_2^5 \oplus C_{2n}$ *with* $n \geq 149$ *and* n *even.* ([41, Theorem 2]).

We first get some lower bounds of $\mathsf{s}_{\leq k}(G)$.

LEMMA 2.8. *Let* $G = C_n^r$ *with* $n, r \in \mathbb{N}_{\geq 2}$ *and* $k \in [0, n-1]$ *be an integer. Then* $\mathsf{s}_{\leq 2n-1-k}(G) \geq 2^{r-1}(n-1) + 1 + k$*.*

REMARK. By Lemma 2.3(iii)–(iv), one can easily see that the lower bound is sharp for $n = 2$ or $r = 2$.

Proof of Lemma 2.8. By induction on r, we can show that there exists a sequence $S = T_0^{n-1} g_0^k$ over C_n^r of length $2^{r-1}(n-1) + k$ such that each zero-sum subsequence of S has length $\geq 2n - k$. If $r = 2$, then Lemma 2.6 implies our result. Assume that we have proved the results for cases $\leq r-1$ and that $S_1' = {g'}_1^{n-1} \dots {g'}_m^{n-1} {g'}_0^k$ is the sequence over C_n^{r-1} of length $2^{r-2}(n-1) + k$ such that each zero-sum subsequence of S_1' has length $\geq 2n - k$. Set $C_n^r = C_n^{r-1} \oplus \langle e_r \rangle$. Let $S_1 = g_1^{n-1} \dots g_m^{n-1} g_0^k$ be a sequence over C_n^r obtained by attaching 0 to each element of S_1' as the rth entry. Obviously, each zero-sum subsequence of S_1 has length $\geq 2n - k$. Let

$$S = S_1(S_1 g_0^{-k} + e_r) = g_0^k \prod_{i=1}^{m} g_i^{n-1} (g_i + e_r)^{n-1}.$$

It is easy to see that each nontrivial zero-sum subsequence T in S must have the form

$$T = g_0^{k_0} \prod_{i \in I_1} g_i^{v_i} \prod_{i \in I_2} (g_i + e_r)^{u_i},$$

where $k_0 \in [0, k]$, $I_1, I_2 \subset [1, m]$, and $u_i, v_i \in [1, n-1]$. It suffices to prove that $|T| \geq 2n - k$.

Obviously, $\sum_{i \in I_2} u_i \equiv 0 \bmod n$. In addition,

$$T' := g_0^{k_0} \prod_{i \in I_1} g_i^{v_i} \prod_{i \in I_2} (g_i)^{u_i}$$

is zero-sum and has the form

$${T'}_1^n T_2' g_0^{k_0}$$

with $\mathsf{v}_g(T_2' g_0^{k_0}) < n$ for any $g \,|\, T_2'$. It is easy to see that $\sigma({T'}_1^n) = 0$ and $T_2' g_0^{k_0} \,|\, S_1$. Hence, $T_2' g_0^{k_0}$ is a zero-sum subsequence of S_1 of length $|T_2' g_0^{k_0}| \leq |T'| = |T|$. Combining this with the structure of S_1 shows that either $|T| \geq |T_2' g_0^{k_0}| \geq 2n - k$ or $T_2' g_0^{k_0}$ is empty. If the former holds, then we are done. If the latter holds, then $k_0 = 0$ and

$$T' = \prod_{i \in I_1} g_i^{v_i} \prod_{i \in I_2} g_i^{u_i} = {T'}_1^n.$$

As $u_i, v_i \in [1, n-1]$ we must have $I_1 = I_2 = I$ and $u_i + v_i = n$ for any $i \in I$. Hence,

$$T = \prod_{i \in I} g_i^{n-u_i} \prod_{i \in I} (g_i + e_r)^{u_i}$$

with $I \subset [1, m]$, $u_i \in [1, n-1]$, $\sum_{i \in I} u_i \equiv 0 \mod n$ and $|T| = |I|n$. If $|I| \geq 2$, then we are done. If $|I| = 1$, then $T = g_i^{n-u_i}(g_i + e_r)^{u_i}$ for some $i \in [1, m]$, where $u_i \in [1, n-1]$ and $u_i \equiv 0 \mod n$. Obviously, this is impossible. So Lemma 2.8 is proved. ■

Lemma 2.9. *Let G be a finite abelian group with $\mathsf{r}(G) \geq 2$. If $\exp(G) \leq \mathsf{D}^*(G) - k \leq 2\exp(G) - 1$, then $\mathsf{s}_{\leq \mathsf{D}^*(G)-k}(G) \geq \mathsf{D}^*(G) + k$.*

In particular, if G is a finite abelian p-group with $\mathsf{r}(G) \geq 2$ and $\mathsf{D}(G) \leq 2\exp(G) - 1$, then $\mathsf{s}_{\leq k}(G) \geq 2\mathsf{D}(G) - k$ for $\exp(G) \leq k \leq \mathsf{D}(G)$.

Remark. By the above result one can easily see that $\mathsf{s}_{\leq \mathsf{D}(G)-k}(G) = \mathsf{D}(G) + k$ in Lemma 2.4.

Proof of Lemma 2.9. Set

$$G = H \oplus C_{n_r} = C_{n_1} \oplus \cdots \oplus C_{n_{r-1}} \oplus C_{n_r} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_{r-1} \rangle \oplus \langle e_r \rangle,$$

where $1 < n_1 \mid \cdots \mid n_{r-1} \mid n_r$ and $\mathrm{ord}(e_i) = n_i$. Let

$$S = e_r^x \prod_{i=1}^{r-1} e_i^{n_i - 1}(e_r - e_i)^{n_i - 1}$$

be a sequence over G of length $\mathsf{D}^*(G) + k - 1$. Since $\exp(G) \leq \mathsf{D}^*(G) - k \leq 2\exp(G) - 1$, we have $k + 1 \leq \mathsf{D}^*(H) \leq \exp(G) + k$ and

$$x = |S| - \sum_{i=1}^{r-1} 2(n_i - 1) = \exp(G) + k - \mathsf{D}^*(H) \in [0, \exp(G) - 1].$$

Obviously, each nontrivial zero-sum subsequence T of S has the form

$$T = e_r^{x_1} \prod_{i \in I} e_i^{v_i}(e_r - e_i)^{v_i},$$

where $x_1 \in [0, x]$, $v_i \in [1, n_i - 1]$ and $I \subset [1, r-1]$. Since $\sigma(T) = \big(x_1 + \sum_{i \in I} v_i\big)e_r = 0$, we have $n_r \big| \big(x_1 + \sum_{i \in I} v_i\big)$, which implies that

$$\begin{aligned} |T| &= x_1 + \sum_{i \in I} 2v_i = 2\Big(x_1 + \sum_{i \in I} v_i\Big) - x_1 \geq 2n_r - x \\ &= 2n_r - (n_r + k - \mathsf{D}^*(H)) = \mathsf{D}^*(G) - k + 1. \end{aligned}$$

Hence, $\mathsf{s}_{\leq \mathsf{D}^*(G)-k}(G) \geq |S| + 1 = \mathsf{D}^*(G) + k$ and the first assertion is proved. The second assertion follows immediately from the first. ■

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1. In the proof, we use a technique from [28].

LEMMA 3.1 ([22]). *Let G be a finite abelian group and S be a minimal zero-sum sequence over G of length* $\mathsf{D}(G)$. *Then* $\Sigma(S) = G$.

LEMMA 3.2 ([22]). *Let G be a cyclic group of order n and S be a sequence over G of length n. If S is minimal zero-sum, then $S = g^n$, where g is a generator of G.*

By the above we can easily see that the following result holds.

COROLLARY 3.3. *Let G be a cyclic group of order n and S be a sequence over G of length $\geq n$. If every zero-sum subsequence of S has length $\geq n$, then $S = g^{|S|}$, where g is a generator of G.*

Let G be a finite abelian group and let A be a finite nonempty subset of G. The *stabilizer* of A is defined by $\mathrm{Stab}(A) = \{g \in G : g + A = A\}$.

LEMMA 3.4 ([22]). *Let G be a finite abelian group, and let A and B be finite nonempty subsets of G. If $B \subset \mathrm{Stab}(A)$, then $\langle B \rangle$ is a subgroup of* $\mathrm{Stab}(A)$.

Furthermore, the following statements are equivalent:

(a) $B \subset \mathrm{Stab}(A)$.
(b) $A = A + B$.
(c) $A = (g_1 + \langle B \rangle) \cup \cdots \cup (g_r + \langle B \rangle)$ *for some* $r \in \mathbb{N}$ *and* $g_1, \ldots, g_r \in A$.

LEMMA 3.5. *Let G be a finite abelian group with $\mathsf{r}(G) \geq 2$ and suppose G is not isomorphic to $C_2 \oplus C_{2m}$ for any $m \geq 2$. Let S be a sequence over G of length $\mathsf{D}(G) + 1$. If every zero-sum subsequence of S has length $\geq \mathsf{D}(G) - 1$, then the rank of $\langle \mathrm{supp}(S) \rangle$ is at least 2.*

Proof. Assume to the contrary that the rank of $\langle \mathrm{supp}(S) \rangle$ is 1. Thus $\langle \mathrm{supp}(S) \rangle \cong C_n$. Since $\mathsf{r}(G) \geq 2$, $\langle \mathrm{supp}(S) \rangle$ is a proper subgroup of G, which implies that $\mathsf{D}(G) \geq n + 1$. It follows that every zero-sum subsequence of S has length $\geq \mathsf{D}(G) - 1 \geq n$. By Corollary 3.3 we have $S = g^{\mathsf{D}(G)+1}$, where g is a generator of $\langle \mathrm{supp}(S) \rangle$. Obviously, S has a zero-sum subsequence of length n. Hence, $n \geq \mathsf{D}(G) - 1 \geq n$, i.e., $\mathsf{D}(G) = n + 1$. By a simple analysis, we must have $G \cong C_2 \oplus C_n$ with $2 \mid n$, where $C_n = \langle \mathrm{supp}(S) \rangle = \langle g \rangle$, contrary to assumption. ■

LEMMA 3.6. *Let G be a finite abelian group with $\mathsf{r}(G) \geq 2$ and suppose G is not isomorphic to $C_2 \oplus C_{2m}$ for any $m \geq 2$. Let $S = Tg^2$ be a sequence over G of length $\mathsf{D}(G) + 1$, where T is a zero-sum subsequence of length $\mathsf{D}(G) - 1$. Then S has a zero-sum subsequence of length at most $\mathsf{D}(G) - 2$.*

Proof. Assume to the contrary that every zero-sum subsequence of S has length $\geq \mathsf{D}(G) - 1$. For any $g_1 \mid S$ with $g_1 \notin \{g, 2g\}$, we have $\sigma(Sg_1^{-1}) =$

$\sigma(T)+2g-g_1 = 2g-g_1 \neq 0$. Since $|Sg_1^{-1}| = |S|-1 = \mathsf{D}(G)$ and every zero-sum subsequence of Sg_1^{-1} has length $\geq \mathsf{D}(G)-1$, every zero-sum subsequence of Sg_1^{-1} must have length $\mathsf{D}(G)-1$. It follows that Sg_1^{-1} has an element $\sigma(Sg_1^{-1}) = 2g-g_1$. Hence $g_1(2g-g_1) \mid S$.

We claim that every zero-sum subsequence of $S' := S(g_1(2g-g_1))^{-1}g^2$ has length $\geq \mathsf{D}(G)-1$ for any $g_1 \mid S$ with $g_1 \notin \{g, 2g\}$. As $g_1 \notin \{g, 2g\}$, we know that g_1 and $2g-g_1$ are not equal to g. Hence,

$$S' = S(g_1(2g-g_1))^{-1}g^2 = T(g_1(2g-g_1))^{-1}g^4.$$

It follows that every zero-sum subsequence T' of S' has the form $T' = T_1'g^x$ with $T_1' \mid T(g_1(2g-g_1))^{-1}$ and $x \leq 4$. If $x \leq 2$, then $T' \mid S$, which implies that $|T'| \geq \mathsf{D}(G)-1$. If $x = 3$ or 4, then $T' = T_1'g^x = (T_1'g^2)g^{x-2}$ with $1 \leq x-2 \leq 2$ and $\sigma(T_1'g^2) = \sigma(T_1'g_1(2g-g_1))$. Obviously, $T_1'g_1(2g-g_1)g^{x-2} \mid Tg^2$ and $\sigma(T_1'g_1(2g-g_1)g^{x-2}) = \sigma(T') = 0$. Hence

$$|T_1'g_1(2g-g_1)g^{x-2}| = |(T_1'g^2)g^{x-2}| = |T'| \geq \mathsf{D}(G)-1.$$

For any $g_1 \mid S$ with $g_1 \notin \{g, 2g\}$, we may repeat the replacement procedure $S(g_1(2g-g_1))^{-1}g^2$. Finally, we can obtain a sequence $S_1 = g^a(2g)^b$ with $|S_1| = |S| = \mathsf{D}(G)+1$ such that every zero-sum subsequence of S_1 has length $\geq \mathsf{D}(G)-1$. Obviously, $\mathsf{r}(\langle S_1\rangle) = 1$, contradicting Lemma 3.5. ∎

Lemma 3.7. *Let G be a finite abelian group with $\mathsf{r}(G) \geq 3$. Let $S = Tg_1g_2g_3$ be a sequence over G of length $\mathsf{D}(G)+2$, where g_1, g_2, g_3 are pairwise distinct and T is a zero-sum subsequence of length $\mathsf{D}(G)-1$. If every zero-sum subsequence of S has length $\geq \mathsf{D}(G)-1$, then S must have each of the following forms:*

$$\begin{aligned} S &= \prod_{i=1}^{\ell} a_i(g_1+g_2-a_i)(g_1+g_2)^{\sigma}g_1g_2g_3 \\ &= \prod_{i=1}^{\ell} b_i(g_1+g_3-b_i)(g_1+g_3)^{\sigma}g_1g_2g_3 \\ &= \prod_{i=1}^{\ell} c_i(g_2+g_3-c_i)(g_2+g_3)^{\sigma}g_1g_2g_3, \end{aligned} \tag{3.1}$$

with all elements of S pairwise distinct, $\sigma = 0$ or 1, $\ell = \frac{\mathsf{D}(G)-1-\sigma}{2}$ and $g_j+g_k \notin \{g_i, 2g_i\}$ for $\{i,j,k\} = \{1,2,3\}$.

Proof. Since $\mathsf{r}(G) \geq 3$, G is not isomorphic to $C_2 \oplus C_{2m}$ for any $m \geq 2$. By the assumption , it is also easy to see that T is minimal zero-sum.

Claim 1. *Let $i, j \in \{1,2,3\}$ and $i \neq j$. For any $g \mid T$ with $g \notin \{g_i, g_j, g_i+g_j\}$ we have $g(g_i+g_j-g) \mid T$ and $g \neq g_i+g_j-g$.*

Obviously, every zero-sum subsequence of $Tg^{-1}g_ig_j$ has length $\geq \mathsf{D}(G) - 1$, $|Tg^{-1}g_ig_j| = \mathsf{D}(G)$ and $\sigma(Tg^{-1}g_ig_j) = g_i + g_j - g \neq 0$. Thus every zero-sum subsequence of $Tg^{-1}g_ig_j$ must have length $\mathsf{D}(G)-1$, which implies that $\sigma(Tg^{-1}g_ig_j) = g_i + g_j - g \mid Tg^{-1}g_ig_j$. Since $g \notin \{g_i, g_j, g_i + g_j\}$, we see that $g_i + g_j - g \notin \{g_i, g_j\}$. Hence, $g_i + g_j - g \mid Tg^{-1}$, i.e., $g(g_i + g_j - g) \mid T$.

If $g = g_i + g_j - g$, then

$$S = Tg_1g_2g_3 = T_1g(g_i + g_j - g)g_1g_2g_3 = (T_1g_ig_jg^2)g_k,$$

where $\{i, j, k\} = \{1, 2, 3\}$. Obviously, $T_1g_ig_j$ is a zero-sum subsequence of S of length $\mathsf{D}(G) - 1$ and every zero-sum subsequence of $T_1g_ig_jg^2$ has length $\geq \mathsf{D}(G) - 1$. By Lemma 3.6 this is impossible and the claim is proved. ∎

Claim 2. *$g_j + g_k \notin \{g_i, 2g_i\}$ for $\{i, j, k\} = \{1, 2, 3\}$.*

Without loss of generality, we can let $i = 1$, $j = 2$ and $k = 3$. If $2g_1 = g_2 + g_3$, then consider the sequence Tg_1^2. Note that T is minimal zero-sum. Thus any zero-sum subsequence T' of Tg_1^2 has one of the following forms: (1) T; (2) T_1g_1, where $T_1 \mid T$ with $\sigma(T_1) = -g_1$; (3) $T_2g_1^2$, where $T_2 \mid T$ with $\sigma(T_2) = -2g_1$. If (1) holds, then $|T'| = |T| = \mathsf{D}(G) - 1$. If (2) holds, then obviously $T_1g_1 \mid S$, which implies that $|T'| = |T_1g_1| \geq \mathsf{D}(G) - 1$. If (3) holds, then $2g_1 = g_2 + g_3$ implies that $\sigma(T_2g_1^2) = \sigma(T_2g_2g_3) = 0$, and $T_2g_2g_3$ is a zero-sum subsequence of S of length $\geq \mathsf{D}(G) - 1$, whence $|T'| = |T_2g_1^2| = |T_2g_2g_3| \geq \mathsf{D}(G) - 1$. Hence, every zero-sum subsequence of Tg_1^2 has length $\geq \mathsf{D}(G) - 1$. This is impossible by Lemma 3.6.

If $g_1 = g_2 + g_3$ and there exists some $g \mid T$ with $g \notin \{g_1, g_2, g_3\}$, then by Claim 1 we know that $g(g_2 + g_3 - g) \mid T$ and

$$S = Tg_1g_2g_3 = g(g_2 + g_3 - g)T'g_2g_3(g_2 + g_3)$$

with $\sigma(T') = -g_2 - g_3$. Thus $T'(g_2 + g_3)$ is a zero-sum subsequence of S of length $|T| - 1 = \mathsf{D}(G) - 2 < \mathsf{D}(G) - 1$, a contradiction.

If $g_1 = g_2 + g_3$ and $\mathrm{supp}(T) = \{g_1, g_2, g_3\}$, then set $T = g_1^{a-1}g_2^{b-1}g_3^{c-1}$ with a, b, c being positive integers, so that $S = g_1^a g_2^b g_3^c$. If $a - 1 > 0$, then as $g_1 = g_2 + g_3$ we find that $Tg_1^{-1}g_2g_3$ is a zero-sum subsequence of S of length $|T| + 1 = \mathsf{D}(G)$. Obviously, $Tg_1^{-1}g_2g_3$ is minimal zero-sum, since otherwise $Tg_1^{-1}g_2g_3$ has a nontrivial zero-sum subsequence of length $< \mathsf{D}(G) - 1$. By Lemma 3.1, $\Sigma(Tg_1^{-1}g_2g_3) = G$, whence $\mathsf{r}(\langle \mathrm{supp}(S) \rangle) = \mathsf{r}(G) \geq 3$. This is impossible since $\mathsf{r}(\langle \mathrm{supp}(S) \rangle) = \mathsf{r}(\langle g_1, g_2, g_3 \rangle) = \mathsf{r}(\langle g_2 + g_3, g_2, g_3 \rangle) \leq 2$. If $b - 1 > 0$ and $c - 1 > 0$, then $T(g_2g_3)^{-1}g_1$ is a zero-sum subsequence of S of length $|T| - 1 = \mathsf{D}(G) - 2 < \mathsf{D}(G) - 1$, a contradiction. If either $a - 1 = 0$, $b - 1 = 0$ or $a - 1 = 0$, $c - 1 = 0$, then $T = g_2^{b-1}$ or g_3^{c-1} is minimal zero-sum of length $\mathsf{D}(G) - 1$. Thus $\mathsf{D}(G) - 1 = \mathrm{ord}(g_2)$ or $\mathrm{ord}(g_3)$. It follows that G must be isomorphic to $C_2 \oplus C_{2m}$ for some positive integer m and $C_{2m} \cong \langle g_2 \rangle$ or $\langle g_3 \rangle$. This contradicts $\mathsf{r}(G) \geq 3$ and the claim is proved. ∎

Claim 3. *$g_1, g_2, g_3 \nmid T$.*

Assume to the contrary that there exists some $i \in \{1,2,3\}$, say $i = 3$, such that $g_3 \,|\, T$. By Claim 2 we have $g_1 + g_2 \neq g_3$. Since g_1, g_2, g_3 are pairwise distinct, we see that $g_3 \notin \{g_1, g_2, g_1+g_2\}$. This together with Claim 1 shows that $g_3(g_1+g_2-g_3) \,|\, T$. Thus

$$S = Tg_1g_2g_3 = T_1g_3(g_1+g_2-g_3)g_1g_2g_3 = (T_1g_1g_2g_3^2)(g_1+g_2-g_3)$$

with $\sigma(T_1) = -g_1 - g_2$. It follows that $T_1g_1g_2$ is a zero-sum subsequence of S of length $\mathsf{D}(G)-1$ and every zero-sum subsequence of $T_1g_1g_2g_3^2$ has length $\geq \mathsf{D}(G)-1$. By Lemma 3.6 this is impossible and we obtain the result. ■

Claim 4. *For any $g \,|\, T$, we have $\mathsf{v}_g(T) = 1$.*

Assume to the contrary that there exists some $g \,|\, T$ with $\mathsf{v}_g(T) \geq 2$. By Claim 3 we know that $g \notin \{g_1, g_2, g_3\}$. If $g \neq g_1 + g_2$, then $g \notin \{g_1, g_2, g_1+g_2\}$ and Claim 1 implies that $g(g_1+g_2-g) \,|\, T$ and $g \neq g_1+g_2-g$. Thus

$$S = Tg_1g_2g_3 = g(g_1+g_2-g)T'g_1g_2g_3 = (T'g_1g_2)g(g_1+g_2-g)g_3$$

with $g \,|\, T'$ and $\sigma(T') = -g_1 - g_2$. Obviously, $T'g_1g_2$ is a zero-sum subsequence of S of length $\mathsf{D}(G)-1$. By Lemma 3.6, g, g_1+g_2-g, g_3 must be pairwise distinct, since otherwise S has a subsequence of the form $(T'g_1g_2)g_x^2$ and $(T'g_1g_2)g_x^2$ has a zero-sum subsequence with length at most $\mathsf{D}(G)-2$. By Claim 3 we have $g \nmid T'g_1g_2$, which contradicts $g \,|\, T'$. Hence, $\mathsf{v}_g(T) = 1$ for any $g \,|\, T$ with $g \neq g_1+g_2$. Similarly, $\mathsf{v}_g(T) = 1$ for any $g \,|\, T$ with $g \notin \{g_1+g_3, g_2+g_3\}$. Since g_1, g_2, g_3 are pairwise distinct, we find that $g_1+g_2 \notin \{g_1+g_3, g_2+g_3\}$. Hence, $\mathsf{v}_{g_1+g_2}(T) = 1$ if $g_1+g_2 \,|\, T$. The claim is proved. ■

By the above claims, we can easily see that S must have each of the following forms:

$$\begin{aligned} S &= \prod_{i=1}^{\ell_1} a_i(g_1+g_2-a_i)(g_1+g_2)^{\sigma_1}g_1g_2g_3 \\ &= \prod_{i=1}^{\ell_2} b_i(g_1+g_3-b_i)(g_1+g_3)^{\sigma_2}g_1g_2g_3 \\ &= \prod_{i=1}^{\ell_3} c_i(g_2+g_3-c_i)(g_2+g_3)^{\sigma_3}g_1g_2g_3, \end{aligned}$$

with all elements of S pairwise distinct, and each σ_i being 0 or 1. Since $|S| = 2\ell_1+\sigma_1+3 = 2\ell_2+\sigma_2+3 = 2\ell_3+\sigma_3+3 = \mathsf{D}(G)+2$ and $\sigma_i = 0$ or 1, one can easily see that $\sigma := \sigma_1 = \sigma_2 = \sigma_3 = 0$ or 1 and $\ell := \ell_1 = \ell_2 = \ell_3 = \frac{\mathsf{D}(G)-1-\sigma}{2}$. Thus Lemma 3.7 is proved.

Lemma 3.8. *Let G be a finite abelian group. Then $\mathsf{D}(G) - 2 < \exp(G)$ if and only if either G is cyclic or G is isomorphic to $C_2 \oplus C_{2m}$ for some positive integer m.*

Proof. Set $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \,|\, n_2 \,|\, \cdots \,|\, n_r$. If $\mathsf{D}(G) - 2 < \exp(G)$, then since $\mathsf{D}(G) \geq \mathsf{D}^*(G)$, we have

$$n_r + 1 \geq \mathsf{D}(G) \geq \mathsf{D}^*(G) = \sum_{i=1}^{r-1}(n_i - 1) + n_r.$$

It follows that $\sum_{i=1}^{r-1}(n_i - 1) \leq 1$, i.e., either $r = 1$ or $r = 2$, $n_1 = 2$. Hence, either G is cyclic or G is isomorphic to $C_2 \oplus C_{2m}$ for some positive integer m. By Lemma 2.7(a), the converse is obvious. ∎

Proof of Theorem 1.1. If $\mathsf{r}(G) = 2$, then by Lemmas 3.8 and 2.3(iv) we are done. Hence we can suppose that $\mathsf{r}(G) \geq 3$.

Claim 1. $\mathsf{D}(G) \geq 6$ *and* $\operatorname{ord}(g) \leq \mathsf{D}(G) - 2$ *for any* $g \in G$.

Since $\mathsf{D}(G) - 2 \geq \exp(G) \geq 2$, we have $\mathsf{D}(G) \geq 4$. If $\mathsf{D}(G) = 4$, then $\exp(G) = 2$ and $G \cong C_2^3$, a contradiction. If $\mathsf{D}(G) = 5$, then by a simple analysis, we find that either $\mathsf{r}(G) \leq 2$ or $G \cong C_2^4$, a contradiction. Hence, $\mathsf{D}(G) \geq 6$. If there exists some $g \in G$ such that $\operatorname{ord}(g) \geq \mathsf{D}(G) - 1$, then $\mathsf{D}(G) - 1 \leq \exp(G) \leq \mathsf{D}(G) - 2$, a contradiction. The claim is proved. ∎

Let S be a sequence over G of length $\mathsf{D}(G) + 2$. It suffices to prove that S has a zero-sum subsequence of length $\leq \mathsf{D}(G) - 2$. Assume to the contrary that every zero-sum subsequence of S has length $\geq \mathsf{D}(G) - 1$. Since $\mathsf{s}_{\leq \mathsf{D}(G)-1}(G) = \mathsf{D}(G) + 1 < |S|$, S has a zero-sum subsequence T of length $\mathsf{D}(G) - 1$. Thus we can set

$$S = T g_1 g_2 g_3,$$

where T is a minimal zero-sum subsequence of S of length $\mathsf{D}(G) - 1$. By Lemma 3.6, g_1, g_2, g_3 are pairwise distinct. It follows from Lemma 3.7 that S must have the form (3.1) with $\sigma = 0$ or 1 and $|S| = 2\ell + \sigma + 3 = \mathsf{D}(G) + 2 \geq 8$, which implies that $\ell \geq 2$.

Claim 2. $\sigma = 0$.

Assume to the contrary that $\sigma = 1$. Since g_1, g_2, g_3 are pairwise distinct, $g_1 + g_2, g_1 + g_3, g_2 + g_3$ are pairwise distinct elements in S. By (3.1) we can suppose that $g_1 + g_3 = a_1$, $g_2 + g_3 = b_1$, $g_1 + g_2 = c_1$, which implies that T contains $g_1 + g_2 - a_1 = g_2 - g_3$, $g_1 + g_3 - b_1 = g_1 - g_2$, $g_2 + g_3 - c_1 = g_3 - g_1$. By Lemma 3.7 we have $g_j + g_k \neq 2g_i$ for $\{i, j, k\} = \{1, 2, 3\}$. Thus $g_1 - g_2, g_2 - g_3, g_3 - g_1$ are pairwise distinct. Hence, $(g_1 - g_2)(g_2 - g_3)(g_3 - g_1)$ is a zero-sum subsequence of S of length $3 \leq \mathsf{D}(G) - 2$, a contradiction. ∎

Since $\sigma = 0$, by (3.1) all elements of T are pairwise distinct, and T can be viewed as a subset of G.

Claim 3. *T has the form*

$$T = \bigcup_{x=1}^{t} \{v_x + g_1 + H, -v_x + g_1 + H\} \cup \bigcup_{y=1}^{s} (e_y + g_1 + H),$$

where $H = \langle g_1 - g_2, g_1 - g_3 \rangle$ is a subgroup of $\mathrm{Stab}(T)$, $2v_x \notin H$ and $e_y \in G$ with $\mathrm{ord}(e_y) \le 2$. In particular, if $s > 0$, then H is of even order.

Since $g_j + g_k - g \in T$ and $g \neq g_j + g_k - g$ for any $g \in T$ and $j, k \in \{1, 2, 3\}$ with $j \neq k$, we have

$$\{g_1 + g_2, g_1 + g_3, g_2 + g_3\} - T = T.$$

It follows that

$$\begin{aligned} T &= \{g_1 + g_2, g_1 + g_3, g_2 + g_3\} - T \\ &= \{g_1 + g_2, g_1 + g_3, g_2 + g_3\} - (\{g_1 + g_2, g_1 + g_3, g_2 + g_3\} - T) \\ &= \{g_i - g_j : i, j \in \{1, 2, 3\}\} + T. \end{aligned}$$

Thus

$$\mathrm{Stab}(T) \supseteq \langle g_i - g_j : i, j \in \{1, 2, 3\} \rangle = \langle g_1 - g_2, g_1 - g_3 \rangle = H.$$

By Lemma 3.4, there exist some $t' \in \mathbb{N}$ and $w_1, \ldots, w_{t'} \in T - g_1$ such that

$$T = T + H = \bigcup_{i=1}^{t'} (w_i + g_1 + H).$$

Note that $g_1 + H = g_2 + H = g_3 + H$. For any $h \in H$, $i \in [1, t']$ and $j, k \in \{1, 2, 3\}$ with $j \neq k$, we have

$$\begin{aligned} g_j + g_k - (w_i + g_1 + h) &= -w_i + g_j + (g_k - g_1 - h) \\ &\in -w_i + g_j + (g_k - g_1 + H) = -w_i + g_j + H = -w_i + g_1 + H. \end{aligned}$$

Note that $g_j + g_k - g \in T$ for any $g \in T$ and $j, k \in \{1, 2, 3\}$ with $j \neq k$. Hence, we can set

$$T = \bigcup_{x=1}^{t} \{v_x + g_1 + H, -v_x + g_1 + H\} \cup \bigcup_{y=1}^{s} (u_y + g_1 + H),$$

where $2v_x \notin H$ and $2u_y$ is in H. Since each $2u_y$ is in H, there must exist some $e_y \in G$ of order ≤ 2 such that $u_y + H = e_y + H$ and we obtain the structure of T. Since $H = \langle g_1 - g_2, g_1 - g_3 \rangle$, we find that $g_j + g_k - (e_y + g_1 + h) \in e_y + g_1 + H$ for any $h \in H$ and $j, k \in \{1, 2, 3\}$ with $j \neq k$. Note that $e_y + g_1 + h \neq g_j + g_k - (e_y + g_1 + h)$. Thus $|H|$ is even and the claim is proved. ∎

Denote by H_1 the set of elements of order ≤ 2 in H. Thus there exists $A \subset H$ such that $H = H_1 \uplus A \uplus (-A)$, where H_1, A and $-A$ are pairwise disjoint sets. Since T is minimal zero-sum of length $\mathsf{D}(G) - 1$, by Claim 3 we have

$$|T| = 2t|H| + s|H| = \mathsf{D}(G) - 1$$

and

$$\begin{aligned}T' &:= \prod_{x=1}^{t} \prod_{h \in H} ((v_x + g_1 + h) + (-v_x + g_1 - h)) \\ &\quad \cdot \prod_{y=1}^{s} \prod_{h \in A} ((e_y + g_1 + h) + (e_y + g_1 - h)) \prod_{y=1}^{s} \sum_{h \in H_1} (e_y + g_1 + h) \\ &= (2g_1)^{t|H|+s|A|} \prod_{y=1}^{s} \sum_{h \in H_1} (e_y + g_1 + h)\end{aligned}$$

is minimal zero-sum. Since $H = \langle g_1 - g_2, g_1 - g_3 \rangle$, we have $\mathsf{r}(H) \le 2$. This together with $|H|$ even (if $s > 0$) shows that $H_1 \cong C_2$ or C_2^2 if $s > 0$. Hence,

$$T' = \begin{cases} (2g_1)^{t|H|} & \text{if } s = 0, \\ (2g_1)^{t|H|+s|A|}(2g_1 + e)^s & \text{if } s > 0 \text{ and } H_1 = \langle e \rangle \cong C_2, \\ (2g_1)^{t|H|+s|A|}(4g_1)^s & \text{if } s > 0 \text{ and } H_1 \cong C_2^2, \end{cases} \tag{3.2}$$

is minimal zero-sum.

Claim 4. If $t|H| + s|A| > 0$, then $\mathrm{ord}(g_1) = \mathrm{ord}(2g_1) = \frac{\mathsf{D}(G)-1}{2}$ is odd, and for any $i \in \left[1, \frac{\mathsf{D}(G)-1}{2}\right]$, there exists $L \mid T'$ of length $\le i$ such that $\sigma(L) = i(2g_1)$.

If $H_1 = \langle e \rangle \cong C_2$ and $2 \nmid s$, then $|A| = \frac{|H|-|H_1|}{2} = \frac{|H|-2}{2}$ and

$$\sigma(T') = (t|H| + s|A| + s)(2g_1) + e = (t + s/2)|H|(2g_1) + e = 0,$$

whence $e = (t + s/2)|H|(2g_1)$. Since $2t|H| + s|H| = \mathsf{D}(G) - 1$, we have $2e = (2t|H| + s|H|)(2g_1) = (\mathsf{D}(G) - 1)(2g_1)$, which implies that $\mathrm{ord}(2g_1) \mid (\mathsf{D}(G) - 1)$. Since $T' = (2g_1)^{t|H|+s|A|}(2g_1 + e)^s$ is minimal zero-sum, the sequence

$$(2g_1)^{t|H|+s|A|}(2g_1 + e)^{s-1} = (2g_1)^{t|H|+\frac{s|H|}{2}-s}(2g_1 + e)^{s-1}$$

is zero-sum free. Since $t|H| + s|A| > 0$ and $2t|H| + s|H| = \mathsf{D}(G) - 1$, we have

$$\begin{aligned}\Sigma(T') &\supset \{i(2g_1) : i \in [1, t|H| + s|H|/2 - 1]\} \\ &= \left\{ i(2g_1) : i \in \left[1, \frac{\mathsf{D}(G) - 3}{2}\right] \right\} \not\ni 0.\end{aligned}$$

It follows that $\mathrm{ord}(2g_1) > \frac{\mathsf{D}(G)-3}{2}$. By Claim 1 we have $\mathrm{ord}(2g_1) < \mathsf{D}(G) - 1$. Since $\mathrm{ord}(2g_1) \mid (\mathsf{D}(G) - 1)$, we must have

$$\mathrm{ord}(2g_1) = \frac{\mathsf{D}(G) - 1}{2}.$$

Since $0 \in \Sigma(T')$, it follows that

$$\Sigma(T') \supset \left\{ i(2g_1) : i \in \left[1, \frac{\mathsf{D}(G)-1}{2}\right] \right\}.$$

From the structure (3.2) of T', it is easy to see that for any $i \in \left[1, \frac{\mathsf{D}(G)-1}{2}\right]$, there exists $L \mid T'$ of length $\le i$ such that $\sigma(L) = i(2g_1)$.

If $s = 0$ or $H_1 = \langle e \rangle \cong C_2$ with $2 \mid s$ or $H_1 \cong C_2^2$, then it follows from (3.2) that the sequence

$$T'' := \begin{cases} (2g_1)^{t|H|+s|A|} & \text{if } s = 0, \\ (2g_1)^{t|H|+s|A|}(4g_1)^{\frac{s}{2}} & \text{if } s > 0 \text{ and } H_1 = \langle e \rangle \cong C_2 \text{ with } 2 \mid s, \\ (2g_1)^{t|H|+s|A|}(4g_1)^{s} & \text{if } s > 0 \text{ and } H_1 \cong C_2^2, \end{cases}$$

is minimal zero-sum. Since $t|H| + s|A| > 0$ and $2t|H| + s|H| = \mathsf{D}(G) - 1$, by a simple calculation we find that

$$\Sigma(T') \supset \Sigma(T'') = \left\{ i(2g_1) : i \in \left[1, \frac{\mathsf{D}(G)-1}{2}\right] \right\}.$$

This together with the minimality of T'' yields

$$\operatorname{ord}(2g_1) = \frac{\mathsf{D}(G)-1}{2}.$$

From the structure (3.2) of T', it is easy to see that for any $i \in \left[1, \frac{\mathsf{D}(G)-1}{2}\right]$, there exists $L \mid T'$ of length $\le i$ such that $\sigma(L) = i(2g_1)$.

Since $\operatorname{ord}(2g_1) = \frac{\mathsf{D}(G)-1}{2} \mid \operatorname{ord}(g_1)$ and $\operatorname{ord}(g_1) < \mathsf{D}(G) - 1$ (by Claim 1), we must have $\operatorname{ord}(g_1) = \operatorname{ord}(2g_1) = \frac{\mathsf{D}(G)-1}{2}$. It follows that $\frac{\mathsf{D}(G)-1}{2}$ is odd and we obtain the result. ∎

In the following, we will distinguish two cases to complete our proof.

Case 1: $t|H| + s|A| > 0$. Since $t|H| + s|A| > 0$, Claim 4 implies that $\operatorname{ord}(g_1) = \operatorname{ord}(2g_1) = \frac{\mathsf{D}(G)-1}{2}$ is odd, and there exists $L \mid T'$ of length $\le \frac{\mathsf{D}(G)-3}{4}$ such that $\sigma(L) = \frac{\mathsf{D}(G)-3}{4}(2g_1)$. Since any element of T' is the sum of at most four elements in T, there must exist a subsequence $T_1 \mid T$ of length $\le \frac{\mathsf{D}(G)-3}{4} \times 4 = \mathsf{D}(G) - 3$ such that $\sigma(T_1) = \frac{\mathsf{D}(G)-3}{4}(2g_1)$. Since $S = Tg_1g_2g_3$ and $\operatorname{ord}(g_1) = \frac{\mathsf{D}(G)-1}{2}$, we find that $\sigma(T_1g_1) = \frac{\mathsf{D}(G)-3}{4}(2g_1) + g_1 = 0$. Hence, T_1g_1 is a zero-sum subsequence of S with $|T_1g_1| \le \mathsf{D}(G) - 3 + 1 < \mathsf{D}(G) - 1$, a contradiction.

Case 2: $t|H| + s|A| = 0$. Since $t|H| + s|A| = 0$ and $2t|H| + s|H| = \mathsf{D}(G) - 1$, we must have $s > 0$ and $t = |A| = 0$, which implies that $H = H_1$ and $2t|H| + s|H| = s|H_1| = \mathsf{D}(G) - 1$. If $H_1 = \langle e \rangle \cong C_2$, then $H = H_1 = \langle g_1 - g_2, g_1 - g_3 \rangle = \langle e \rangle$. Since g_1, g_2, g_3 are pairwise distinct, it follows that $g_1 - g_2, g_1 - g_3$ are both distinct and nonzero. This is impossible. Hence $H =$

$H_1 \cong C_2^2$ and $s = \frac{\mathsf{D}(G)-1}{|H_1|} \frac{\mathsf{D}(G)-1}{4}$. It follows from (3.2) that $T' = (4g_1)^{\frac{\mathsf{D}(G)-1}{4}}$ is minimal zero-sum. Thus $\mathrm{ord}(4g_1) = \frac{\mathsf{D}(G)-1}{4} = s$, and by Claim 3, we have

$$T = \bigcup_{y=1}^{s} (e_y + g_1 + H).$$

It is easy to see that

$$\mathrm{ord}(4g_1) = \frac{\mathsf{D}(G)-1}{4} = s \text{ is odd},$$

since otherwise $\mathrm{ord}(g_1) = 2\mathrm{ord}(2g_1) = 4\mathrm{ord}(4g_1) = \mathsf{D}(G)-1$, a contradiction to Claim 4. Since $H = \langle g_1 - g_2, g_1 - g_3 \rangle \cong C_2^2$, and $g_1 - g_2, g_1 - g_3$ are both distinct and nonzero, we see that $e_1 := g_1 - g_2$ and $e_2 := g_1 - g_3$ are of order 2 and constitute a basis of H. Combining the structures of T yields

$$\begin{aligned} S &= g_1 g_2 g_3 T \\ &= g_1(g_1+e_1)(g_1+e_2) \\ &\quad \cdot \prod_{y=1}^{s} (e_y+g_1)(e_y+g_1+e_1)(e_y+g_1+e_2)(e_y+g_1+e_1+e_2) \end{aligned}$$

with $s = \frac{\mathsf{D}(G)-1}{4} \ge 2$ odd (since $\mathsf{D}(G) \ge 6$). Since $\mathrm{ord}(4g_1) = \frac{\mathsf{D}(G)-1}{4} \mid \mathrm{ord}(g_1)$ and $\mathrm{ord}(g_1) < \mathsf{D}(G) - 1$ (by Claim 1), we have

$$\mathrm{ord}(g_1) = \frac{\mathsf{D}(G)-1}{4} \text{ or } \frac{\mathsf{D}(G)-1}{2}.$$

If $\mathrm{ord}(g_1) = \frac{\mathsf{D}(G)-1}{2}$, then since $s \ge 2$ is odd, we have $0 \le \frac{s-3}{2} < s-1$ and

$$\begin{aligned} &(g_1+e_1)(g_1+e_2)(e_s+g_1)(e_s+g_1+e_1)(e_{s-1}+g_1)(e_{s-1}+g_1+e_2) \\ &\qquad \cdot \prod_{y=1}^{\frac{s-3}{2}} (e_y+g_1)(e_y+g_1+e_1)(e_y+g_1+e_2)(e_y+g_1+e_1+e_2) \end{aligned}$$

is a zero-sum subsequence of S of length $2s = \frac{\mathsf{D}(G)-1}{2} < \mathsf{D}(G) - 1$. This is a contradiction.

If $\mathrm{ord}(g_1) = \frac{\mathsf{D}(G)-1}{4}$ and $s = \frac{\mathsf{D}(G)-1}{4} \equiv 1 \bmod 4$, then $s \ge 5$ and

$$\begin{aligned} &g_1(g_1+e_1)(g_1+e_2)(e_s+g_1+e_1)(e_s+g_1+e_2) \\ &\qquad \cdot \prod_{y=1}^{\frac{s-5}{4}} (e_y+g_1)(e_y+g_1+e_1)(e_y+g_1+e_2)(e_y+g_1+e_1+e_2) \end{aligned}$$

is a zero-sum subsequence of S of length $s = \frac{\mathsf{D}(G)-1}{4} < \mathsf{D}(G) - 1$. This is a contradiction.

If $\operatorname{ord}(g_1) = \frac{\mathsf{D}(G)-1}{4}$ and $s = \frac{\mathsf{D}(G)-1}{4} \equiv 3 \bmod 4$, then $s \geq 3$ and

$$(g_1+e_1)(e_s+g_1)(e_s+g_1+e_1)$$
$$\cdot \prod_{y=1}^{\frac{s-3}{4}} (e_y+g_1)(e_y+g_1+e_1)(e_y+g_1+e_2)(e_y+g_1+e_1+e_2)$$

is a zero-sum subsequence of S of length $s = \frac{\mathsf{D}(G)-1}{4} < \mathsf{D}(G)-1$. This is a contradiction which completes the proof of the first assertion.

The second assertion immediately follows from the first assertion and Lemma 2.9.

4. The upper bound of $\mathsf{s}_{\leq k}(G)$. In this section, to obtain our upper bound, we first study the problem of when there exists a nontrivial zero-sum subsequence of length bounded by k for zero-sum sequences over finite abelian p-groups. For this problem, one can also see [9, 32].

We will use the following lemmas repeatedly.

LEMMA 4.1 (Lucas' Theorem, [31]; see also [7] and [23, p. 271]). *Let a, b be positive integers with $a = a_n p^n + \cdots + a_1 p + a_0$ and $b = b_n p^n + \cdots + b_1 p + b_0$ being the p-adic expansions, where p is a prime. Then*

$$\binom{a}{b} \equiv \binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}} \cdots \binom{a_0}{b_0} \bmod p.$$

For any $k \in \mathbb{N}_0$, $g \in G$, and a sequence S over G, we denote by

$$\mathsf{N}_g^k(S) = \left| \left\{ I \subset [1, |S|] : \sum_{i \in I} g_i = g, |I| = k \right\} \right|$$

the number of subsequences T of S having sum $\sigma(T) = g$ and length $|T| = k$ (counted with the multiplicity of their appearance in S). Denote by $\mathsf{N}_g^+(S)$ (resp. $\mathsf{N}_g^-(S)$) the number of subsequences T of S of even (resp. odd) length having sum $\sigma(T) = g$ (each counted with the multiplicity of its appearance in S). If $g = 0$, we also write $\mathsf{N}_0^k(S)$ as $\mathsf{N}^k(S)$.

LEMMA 4.2 ([22, Proposition 5.5.8]). *Let p be a prime, let G be an abelian p-group, and let $S = g_1 \cdot \ldots \cdot g_\ell \in \mathscr{F}(G)$. If $\ell \geq \mathsf{D}(G)$, then $\mathsf{N}_g^+(S) \equiv \mathsf{N}_g^-(S) \bmod p$ for all $g \in G$. In particular, $\mathsf{N}_0^+(S) \equiv \mathsf{N}_0^-(S) \mod p$.*

LEMMA 4.3. *Let x be an integer and c, k, u_1, u_2, λ be positive integers. Let*

$$A = \begin{pmatrix} 1+x & 1 & 1 & \cdots & 1 \\ \binom{c+u_1}{1} & \binom{c+u_2}{1} & \binom{c+k}{1} & \cdots & \binom{c}{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1}{\lambda} & \binom{c+u_2}{\lambda} & \binom{c+k}{\lambda} & \cdots & \binom{c}{\lambda} \end{pmatrix}_{(\lambda+1)\times(k+3)}.$$

Then by some row transformations, A can be transferred to the form

$$A' = \begin{pmatrix} \binom{u_1}{0} + (-1)^0 x \times \binom{c}{0} & \binom{u_2}{0} & \binom{k}{0} & \cdots & \binom{0}{0} \\ \binom{u_1}{1} + (-1)^1 x \times \binom{c}{1} & \binom{u_2}{1} & \binom{k}{1} & \cdots & \binom{0}{1} \\ \binom{u_1}{2} + (-1)^2 x \times \binom{c+1}{2} & \binom{u_2}{2} & \binom{k}{2} & \cdots & \binom{0}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{u_1}{\lambda} + (-1)^\lambda x \times \binom{\lambda+c-1}{\lambda} & \binom{u_2}{\lambda} & \binom{k}{\lambda} & \cdots & \binom{0}{\lambda} \end{pmatrix}.$$

Proof. We make some row transformations of A (executed from top to bottom each time by using $\binom{n}{i} - \binom{n-1}{i-1} = \binom{n-1}{i}$). It is easy to see that

$$A = A_{0,0} = \begin{pmatrix} \binom{c+u_1-1}{0} + x & \binom{c+u_2-1}{0} & \binom{c+k-1}{0} & \cdots & \binom{c-1}{0} \\ \binom{c+u_1}{1} & \binom{c+u_2}{1} & \binom{c+k}{1} & \cdots & \binom{c}{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1}{\lambda} & \binom{c+u_2}{\lambda} & \binom{c+k}{\lambda} & \cdots & \binom{c}{\lambda} \end{pmatrix}.$$

Multiplying the first row of $A_{0,0}$ by -1 and then adding it to the second row of $A_{0,0}$ yields

$$A_{0,1} = \begin{pmatrix} \binom{c+u_1-1}{0} + x & \binom{c+u_2-1}{0} & \binom{c+k-1}{0} & \cdots & \binom{c-1}{0} \\ \binom{c+u_1-1}{1} - x & \binom{c+u_2-1}{1} & \binom{c+k-1}{1} & \cdots & \binom{c-1}{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1}{\lambda} & \binom{c+u_2}{\lambda} & \binom{c+k}{\lambda} & \cdots & \binom{c}{\lambda} \end{pmatrix}.$$

Repeat this process $\lambda - 1$ times, i.e., for $1 \le i \le \lambda - 1$ multiply the $(i+1)$th row of $A_{0,i}$ by -1 and then add it to the $(i+2)$th row of $A_{0,i}$. It follows that

$$A_{0,\lambda} = \begin{pmatrix} \binom{c+u_1-1}{0} + (-1)^0 x & \binom{c+u_2-1}{0} & \binom{c+k-1}{0} & \cdots & \binom{c-1}{0} \\ \binom{c+u_1-1}{1} + (-1)^1 x & \binom{c+u_2-1}{1} & \binom{c+k-1}{1} & \cdots & \binom{c-1}{1} \\ \binom{c+u_1-1}{2} + (-1)^2 x & \binom{c+u_2-1}{2} & \binom{c+k-1}{2} & \cdots & \binom{c-1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1-1}{\lambda} + (-1)^\lambda x & \binom{c+u_2-1}{\lambda} & \binom{c+k-1}{\lambda} & \cdots & \binom{c-1}{\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} \binom{c+u_1-2}{0} + (-1)^0 x \times \binom{1}{0} & \binom{c+u_2-2}{0} & \binom{c+k-2}{0} & \cdots & \binom{c-2}{0} \\ \binom{c+u_1-1}{1} + (-1)^1 x \times \binom{1}{1} & \binom{c+u_2-1}{1} & \binom{c+k-1}{1} & \cdots & \binom{c-1}{1} \\ \binom{c+u_1-1}{2} + (-1)^2 x \times \binom{2}{2} & \binom{c+u_2-1}{2} & \binom{c+k-1}{2} & \cdots & \binom{c-1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1-1}{\lambda} + (-1)^\lambda x \times \binom{\lambda}{\lambda} & \binom{c+u_2-1}{\lambda} & \binom{c+k-1}{\lambda} & \cdots & \binom{c-1}{\lambda} \end{pmatrix}.$$

Multiplying the first row of $A_{0,\lambda}$ by -1 and then adding it to the second row of $A_{0,\lambda}$ yields

$$A_{1,1}=\begin{pmatrix} \binom{c+u_1-2}{0}+(-1)^0x\times\binom{1}{0} & \binom{c+u_2-2}{0} & \binom{c+k-2}{0} & \cdots & \binom{c-2}{0} \\ \binom{c+u_1-2}{1}+(-1)^1x\times\binom{2}{1} & \binom{c+u_2-2}{1} & \binom{c+k-2}{1} & \cdots & \binom{c-2}{1} \\ \binom{c+u_1-1}{2}+(-1)^2x\times\binom{2}{2} & \binom{c+u_2-1}{2} & \binom{c+k-1}{2} & \cdots & \binom{c-1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1-1}{\lambda}+(-1)^\lambda x\times\binom{\lambda}{\lambda} & \binom{c+u_2-1}{\lambda} & \binom{c+k-1}{\lambda} & \cdots & \binom{c-1}{\lambda} \end{pmatrix}.$$

Repeat this process $\lambda-1$ times, i.e., for $1\le i\le\lambda-1$ multiply the $(i+1)$th row of $A_{1,i}$ by -1 and then add it to the $(i+2)$th row of $A_{1,i}$. It follows that

$$A_{1,\lambda}=\begin{pmatrix} \binom{c+u_1-2}{0}+(-1)^0x\times\binom{2}{0} & \binom{c+u_2-2}{0} & \binom{c+k-2}{0} & \cdots & \binom{c-2}{0} \\ \binom{c+u_1-2}{1}+(-1)^1x\times\binom{2}{1} & \binom{c+u_2-2}{1} & \binom{c+k-2}{1} & \cdots & \binom{c-2}{1} \\ \binom{c+u_1-2}{2}+(-1)^2x\times\binom{3}{2} & \binom{c+u_2-2}{2} & \binom{c+k-2}{2} & \cdots & \binom{c-2}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1-2}{\lambda}+(-1)^\lambda x\times\binom{\lambda+1}{\lambda} & \binom{c+u_2-2}{\lambda} & \binom{c+k-2}{\lambda} & \cdots & \binom{c-2}{\lambda} \end{pmatrix}.$$

Again repeating the above row transformations $l-1$ $(1\le l\le c)$ times, one obtains

$$A_{l-1,\lambda}=\begin{pmatrix} \binom{c+u_1-l}{0}+(-1)^0x\times\binom{l}{0} & \binom{c+u_2-l}{0} & \binom{c+k-l}{0} & \cdots & \binom{c-l}{0} \\ \binom{c+u_1-l}{1}+(-1)^1x\times\binom{l}{1} & \binom{c+u_2-l}{1} & \binom{c+k-l}{1} & \cdots & \binom{c-l}{1} \\ \binom{c+u_1-l}{2}+(-1)^2x\times\binom{l+1}{2} & \binom{c+u_2-l}{2} & \binom{c+k-l}{2} & \cdots & \binom{c-l}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{c+u_1-l}{\lambda}+(-1)^\lambda x\times\binom{\lambda+l-1}{\lambda} & \binom{c+u_2-l}{\lambda} & \binom{c+k-l}{\lambda} & \cdots & \binom{c-l}{\lambda} \end{pmatrix}.$$

In particular,

$$A':=A_{c-1,\lambda}=\begin{pmatrix} \binom{u_1}{0}+(-1)^0x\times\binom{c}{0} & \binom{u_2}{0} & \binom{k}{0} & \cdots & \binom{0}{0} \\ \binom{u_1}{1}+(-1)^1x\times\binom{c}{1} & \binom{u_2}{1} & \binom{k}{1} & \cdots & \binom{0}{1} \\ \binom{u_1}{2}+(-1)^2x\times\binom{c+1}{2} & \binom{u_2}{2} & \binom{k}{2} & \cdots & \binom{0}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{u_1}{\lambda}+(-1)^\lambda x\times\binom{\lambda+c-1}{\lambda} & \binom{u_2}{\lambda} & \binom{k}{\lambda} & \cdots & \binom{0}{\lambda} \end{pmatrix}. \blacksquare$$

We now give some conditions for a zero-sum sequence T over a p-group G of length $|T|\ge 2k$ to have a zero-sum subsequence of length $\le k-1$.

LEMMA 4.4. *Let p be a prime and k be a positive integer. Let G be a finite abelian p-group and T be a zero-sum sequence over G of length $|T|\ge$*

$2k \ge \mathsf{D}(G)+2$. If there exists some $i \in [1, 2k-\mathsf{D}(G)]$ such that

$$a_i := \binom{|T|-k}{k-i} + (-1)^i \binom{|T|-k+i-1}{k-1} \not\equiv 0 \bmod p,$$

then T has a nontrivial zero-sum subsequence of length $\le k-1$.

Proof. Note that $a_1 = 0$. Assume to the contrary that each zero-sum subsequence of T has length $\ge k$. Since $\sigma(T) = 0$ and $|T| \ge 2k \ge \mathsf{D}(G)+2$, T is not minimal zero-sum and each proper zero-sum subsequence of T has length $\le |T|-k$. Hence,

$$\mathsf{N}^i(T) = 0 \quad \text{for } i \in [1, k-1] \cup [|T|-k+1, |T|-1].$$

It follows from Lemma 4.2 that

$$\begin{aligned} (4.1)\qquad & 1 + (-1)^k \mathsf{N}^k(T) + \cdots + (-1)^{|T|-k}\mathsf{N}^{|T|-k}(T) + (-1)^{|T|}\mathsf{N}^{|T|}(T) \\ &= (1+(-1)^{|T|}) + (-1)^k \mathsf{N}^k(T) + \cdots + (-1)^{|T|-k}\mathsf{N}^{|T|-k}(T) \equiv 0 \bmod p, \end{aligned}$$

and for any $T_1 \mid T$ with $|T_1| \in [\mathsf{D}(G), |T|-1]$,

$$1 + (-1)^k \mathsf{N}^k(T_1) + \cdots + (-1)^{|T|-k}\mathsf{N}^{|T|-k}(T_1) \equiv 0 \bmod p.$$

Thus for $1 \le t \le |T| - \mathsf{D}(G)$,

$$\sum_{T_1 \mid T,\, |T_1| = |T|-t} \left(1 + (-1)^k \mathsf{N}^k(T_1) + \cdots + (-1)^{|T|-k}\mathsf{N}^{|T|-k}(T_1)\right) \equiv 0 \bmod p.$$

Analyzing the number of times each subsequence is counted, we obtain

$$\begin{aligned} (4.2)\qquad & \binom{|T|}{|T_1|} + (-1)^k \binom{|T|-k}{|T_1|-k}\mathsf{N}^k(T) \\ &\qquad + \cdots + (-1)^{|T|-k}\binom{|T|-(|T|-k)}{|T_1|-(|T|-k)}\mathsf{N}^{|T|-k}(T) \\ &= \binom{|T|}{t} + (-1)^k \binom{|T|-k}{t}\mathsf{N}^k(T) \\ &\qquad + \cdots + (-1)^{|T|-k}\binom{k}{t}\mathsf{N}^{|T|-k}(T) \equiv 0 \bmod p. \end{aligned}$$

Set $X := (1, (-1)^k\mathsf{N}^k(T), \ldots, (-1)^{|T|-k}\mathsf{N}^{|T|-k}(T))^T$ and

$$A := \begin{pmatrix} \binom{|T|}{0} + (-1)^{|T|} & \binom{|T|-k}{0} & \cdots & \binom{k}{0} \\ \binom{|T|}{1} & \binom{|T|-k}{1} & \cdots & \binom{k}{1} \\ \binom{|T|}{2} & \binom{|T|-k}{2} & \cdots & \binom{k}{2} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{|T|}{|T|-\mathsf{D}(G)} & \binom{|T|-k}{|T|-\mathsf{D}(G)} & \cdots & \binom{k}{|T|-\mathsf{D}(G)} \end{pmatrix}.$$

On the one hand, it can be deduced from (4.1) and (4.2) that

$$AX \equiv 0 \bmod p.$$

Since $|T| \geq 2k \geq \mathsf{D}(G)+2$, Lemma 4.3 implies that by some row transformations, A can be transformed into

$$A' = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} =$$

$$\left(\begin{array}{c|ccc}
\binom{|T|-k}{0} + (-1)^0(-1)^{|T|}\binom{k}{0} & \binom{|T|-2k}{0} & \cdots & \binom{0}{0} \\
\binom{|T|-k}{1} + (-1)^1(-1)^{|T|}\binom{k}{1} & \binom{|T|-2k}{1} & \cdots & \binom{0}{1} \\
\binom{|T|-k}{2} + (-1)^2(-1)^{|T|}\binom{k+1}{2} & \binom{|T|-2k}{2} & \cdots & \binom{0}{2} \\
\vdots & \vdots & \vdots & \vdots \\
\binom{|T|-k}{|T|-2k} + (-1)^{|T|-2k}(-1)^{|T|}\binom{|T|-k-1}{|T|-2k} & \binom{|T|-2k}{|T|-2k} & \cdots & \binom{0}{|T|-2k} \\
\hdashline
\binom{|T|-k}{|T|-2k+1} + (-1)^{|T|-2k+1}(-1)^{|T|}\binom{|T|-k}{|T|-2k+1} & \binom{|T|-2k}{|T|-2k+1} & \cdots & \binom{0}{|T|-2k+1} \\
\binom{|T|-k}{|T|-2k+2} + (-1)^{|T|-2k+2}(-1)^{|T|}\binom{|T|-k+1}{|T|-2k+2} & \binom{|T|-2k}{|T|-2k+2} & \cdots & \binom{0}{|T|-2k+2} \\
\vdots & \vdots & \vdots & \vdots \\
\binom{|T|-k}{|T|-\mathsf{D}(G)} + (-1)^{|T|-\mathsf{D}(G)}(-1)^{|T|}\binom{|T|-\mathsf{D}(G)+k-1}{|T|-\mathsf{D}(G)} & \binom{|T|-2k}{|T|-\mathsf{D}(G)} & \cdots & \binom{0}{|T|-\mathsf{D}(G)}
\end{array}\right).$$

Since $\binom{j}{|T|-2k+i} = 0$ for $j < |T|-2k+i$, each element in A_{22} is 0. Obviously,

$$A'X \equiv 0 \bmod p.$$

For the ith component of A_{21}, we have

$$\begin{aligned}\binom{|T|-k}{|T|-2k+i} &+ (-1)^{|T|-2k+i}(-1)^{|T|}\binom{|T|-k+i-1}{|T|-2k+i} \\ &= \binom{|T|-k}{k-i} + (-1)^i\binom{|T|-k+i-1}{k-1} = a_i.\end{aligned}$$

Thus

$$\begin{pmatrix} A_{21} & A_{22} \end{pmatrix} X = A_{21} = \begin{pmatrix} a_1 \\ \vdots \\ a_{2k-\mathsf{D}(G)} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \bmod p.$$

This contradicts the existence of some $i \in [1, 2k-\mathsf{D}(G)]$ such that $a_i \not\equiv 0 \bmod p$, completing the proof. ∎

In the following results, we obtain some values of i such that $a_i \not\equiv 0 \bmod p$.

LEMMA 4.5. *Let p be a prime and $k = c_1p^{t+1}+d$ be a positive integer with $t \geq 0$, $d \in [0, p-1]$ and $c_1 \in [1, p-1]$. Let G be a finite abelian p-group and T be a zero-sum sequence over G of length $|T| \geq 2k$. Suppose* *that* $|T|-k = u_1p^{t+1}+v_1$ *with* $v_1 \in [0,p^{t+1}-1]$, $u_1 \in [1,p-1]$ *and* $u_1+c_1+1 \not\equiv 0 \bmod p$.

(1) *If* $2k-\mathsf{D}(G) \geq p+d-v$, *then* $a_{p+d-v} \not\equiv 0 \bmod p$.
(2) *If* $d=1$ *and* $2k-\mathsf{D}(G) \geq 2$, *then* $a_2 \not\equiv 0 \bmod p$.

Proof. Set $v_1 = u_2p+v$ with $v \in [0,p-1]$. Since $|T| \geq 2k$, we have

$$|T|-k = u_1p^{t+1}+u_2p+v \geq k = c_1p^{t+1}+d,$$

which implies that

$$p-1 \geq u_1 \geq c_1 \geq 1. \tag{4.3}$$

CLAIM. $\binom{u_1}{c_1-1}+(-1)^l\binom{u_1+1}{c_1} \not\equiv 0 \bmod p$ *for any integer* l.

If l is odd, then

$$\binom{u_1}{c_1-1}+(-1)^l\binom{u_1+1}{c_1} = \binom{u_1}{c_1-1}-\binom{u_1+1}{c_1} = -\binom{u_1}{c_1} \not\equiv 0 \bmod p,$$

where $\binom{u_1}{c_1} \not\equiv 0 \bmod p$ since $p-1 \geq u_1 \geq c_1 \geq 1$. If l is even, then

$$\begin{aligned}\binom{u_1}{c_1-1}+(-1)^l\binom{u_1+1}{c_1} &= \binom{u_1}{c_1-1}+\binom{u_1}{c_1-1}\cdot\frac{u_1+1}{c_1}\\ &= \binom{u_1}{c_1-1}\cdot\frac{u_1+c_1+1}{c_1} \not\equiv 0 \bmod p,\end{aligned}$$

where $\binom{u_1}{c_1-1} \not\equiv 0 \bmod p$ because $p-1 \geq u_1 \geq c_1 \geq 1$, and $\frac{u_1+c_1+1}{c_1} \not\equiv 0 \bmod p$ because $u_1+c_1+1 \not\equiv 0 \mod p$ and $p-1 \geq c_1 \geq 1$. ■

(1) Since $2k-\mathsf{D}(G) \geq p+d-v$, by the definition of a_i in Lemma 4.4 we have

$$\begin{aligned}a_{p+d-v} &= \binom{|T|-k}{k-(p+d-v)}+(-1)^{p+d-v}\binom{|T|-k+(p+d-v)-1}{k-1}\\ &= \binom{u_1p^{t+1}+u_2p+v}{c_1p^{t+1}-p+v}+(-1)^{p+d-v}\binom{u_1p^{t+1}+u_2p+p+d-1}{c_1p^{t+1}+d-1}\\ &\equiv \binom{u_1p^t+u_2}{c_1p^t-1}+(-1)^{p+d-v}\binom{u_1p^t+u_2+1}{c_1p^t} \mod p.\end{aligned}$$

The last congruence follows from Lemma 4.1 If $t=0$, then $u_2=0$ and by the Claim we have

$$a_{p+d-v} \equiv \binom{u_1}{c_1-1}+(-1)^{p+d-v}\binom{u_1+1}{c_1} \not\equiv 0 \bmod p.$$

Now suppose that $t \geq 1$. If $u_2 \in [0, p^t - 2]$, then

$$\begin{aligned} a_{p+d-v} &\equiv \binom{u_1p^t + u_2}{c_1p^t - 1} + (-1)^{p+d-v}\binom{u_1p^t + u_2 + 1}{c_1p^t} \\ &\equiv \binom{u_1}{c_1 - 1}\binom{u_2}{p^t - 1} + (-1)^{p+d-v}\binom{u_1}{c_1}\binom{u_2 + 1}{0} \\ &= (-1)^{p+d-v}\binom{u_1}{c_1} \not\equiv 0 \bmod p \quad \text{(by (4.3))}. \end{aligned}$$

If $u_2 = p^t - 1$, then

$$\begin{aligned} a_{p+d-v} &\equiv \binom{u_1p^t + u_2}{c_1p^t - 1} + (-1)^{p+d-v}\binom{u_1p^t + u_2 + 1}{c_1p^t} \\ &= \binom{(u_1 + 1)p^t - 1}{c_1p^t - 1} + (-1)^{p+d-v}\binom{(u_1 + 1)p^t}{c_1p^t} \\ &\equiv \binom{u_1}{c_1 - 1}\binom{p^t - 1}{p^t - 1} + (-1)^{p+d-v}\binom{u_1 + 1}{c_1}\binom{0}{0} \\ &= \binom{u_1}{c_1 - 1} + (-1)^{p+d-v}\binom{u_1 + 1}{c_1} \not\equiv 0 \bmod p \quad \text{(by the Claim)}. \end{aligned}$$

(2) Since $2k - \mathsf{D}(G) \geq 2$, by the definition of a_i in Lemma 4.4 we have

$$\begin{aligned} a_2 &= \binom{|T| - k}{k - 2} + (-1)^2\binom{|T| - k + 1}{k - 1} \\ &= \binom{u_1p^{t+1} + v_1}{c_1p^{t+1} - 1} + \binom{u_1p^{t+1} + v_1 + 1}{c_1p^{t+1}}. \end{aligned}$$

If $v_1 \in [0, p^{t+1} - 2]$, then $v_1 + 1 \in [1, p^{t+1} - 1]$, and by Lemma 4.1,

$$\begin{aligned} a_2 &= \binom{u_1p^{t+1} + v_1}{c_1p^{t+1} - 1} + \binom{u_1p^{t+1} + v_1 + 1}{c_1p^{t+1}} \\ &\equiv \binom{u_1}{c_1 - 1}\binom{v_1}{p^t - 1} + \binom{u_1}{c_1}\binom{v_1 + 1}{0} \\ &= \binom{u_1}{c_1} \not\equiv 0 \bmod p, \end{aligned}$$

where $\binom{u_1}{c_1} \not\equiv 0 \bmod p$ because $p - 1 \geq u_1 \geq c_1 \geq 1$. If $v_1 = p^{t+1} - 1$, then by Lemma 4.1 we have

$$\begin{aligned} a_2 &\equiv \binom{u_1p^{t+1} + v_1}{c_1p^{t+1} - 1} + \binom{u_1p^{t+1} + v_1 + 1}{c_1p^{t+1}} \\ &= \binom{(u_1 + 1)p^{t+1} - 1}{c_1p^t - 1} + \binom{(u_1 + 1)p^{t+1}}{c_1p^{t+1}} \end{aligned}$$

$$\equiv \binom{u_1}{c_1-1}\binom{p^{t+1}-1}{p^{t+1}-1} + \binom{u_1+1}{c_1}\binom{0}{0}$$
$$= \binom{u_1}{c_1-1} + \binom{u_1+1}{c_1} \not\equiv 0 \bmod p \quad \text{(by the Claim).} \blacksquare$$

In the following, we will prove the upper bound $\mathsf{s}_{\leq k}(G) \leq 2\mathsf{D}(G) - k$ for some classes of p-groups G by using Lemmas 4.4 and 4.5. In our proofs, the following lemma is important.

LEMMA 4.6. *Let G be a finite abelian p-group and $k \in [\exp(G)+1, \mathsf{D}(G)]$ be a positive integer. Let S be a sequence over G of length $2\mathsf{D}(G) - k + 1$. Suppose that $\binom{\mathsf{D}(G)}{k-1} \not\equiv 0 \bmod p$. If $\mathsf{N}^i(S) = 0$ for $i \in [\mathsf{D}(G)+1, |S|]$, then S has a nontrivial zero-sum subsequence T with $|T| \leq k-1$.*

Proof. Assume to the contrary that each zero-sum subsequence of S has length $\geq k$. Since $\mathsf{N}^i(S) = 0$ for $i \in [\mathsf{D}(G)+1, |S|]$, by Lemma 4.2 the following congruence holds for any $T \mid S$ with $|T| \in [\mathsf{D}(G), |S|]$:

$$1 + (-1)^k \mathsf{N}^k(T) + \cdots + (-1)^{\mathsf{D}(G)} \mathsf{N}^{\mathsf{D}(G)}(T) \equiv 0 \bmod p.$$

Thus for $0 \leq t \leq |S| - \mathsf{D}(G)$,

$$\sum_{T|S,\ |T|=|S|-t} \left(1 + (-1)^k \mathsf{N}^k(T) + \cdots + (-1)^{\mathsf{D}(G)} \mathsf{N}^{\mathsf{D}(G)}(T)\right) \equiv 0 \bmod p.$$

Analyzing the number of times each subsequence is counted, one obtains

$$\begin{aligned} (4.4) \qquad & \binom{|S|}{|T|} + (-1)^k \binom{|S|-k}{|T|-k} \mathsf{N}^k(S) \\ & \qquad + \cdots + (-1)^{\mathsf{D}(G)} \binom{|S|-\mathsf{D}(G)}{|T|-\mathsf{D}(G)} \mathsf{N}^{\mathsf{D}(G)}(S) \\ & = \binom{|S|}{t} + (-1)^k \binom{|S|-k}{t} \mathsf{N}^k(S) \\ & \qquad + \cdots + (-1)^{\mathsf{D}(G)} \binom{|S|-\mathsf{D}(G)}{t} \mathsf{N}^{\mathsf{D}(G)}(S) \equiv 0 \bmod p. \end{aligned}$$

Set $X := (1, (-1)^k \mathsf{N}^k(S), \ldots, (-1)^{\mathsf{D}(G)} \mathsf{N}^{\mathsf{D}(G)}(S))^T$ and

$$A := \begin{pmatrix} \binom{|S|}{0} & \binom{|S|-k}{0} & \cdots & \binom{|S|-\mathsf{D}(G)}{0} \\ \binom{|S|}{1} & \binom{|S|-k}{1} & \cdots & \binom{|S|-\mathsf{D}(G)}{1} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{|S|}{|S|-\mathsf{D}(G)} & \binom{|S|-k}{|S|-\mathsf{D}(G)} & \cdots & \binom{|S|-\mathsf{D}(G)}{|S|-\mathsf{D}(G)} \end{pmatrix}_{(|S|-\mathsf{D}(G)+1)(\mathsf{D}(G)-k+2)}.$$

On the one hand, it can be deduced from (4.4) that

$$AX \equiv 0 \bmod p.$$

It follows from Lemma 4.3 that by some row transformations, A can be transformed into

$$A' = \begin{pmatrix} \binom{\mathsf{D}(G)}{0} & \binom{\mathsf{D}(G)-k}{0} & \cdots & \binom{0}{0} \\ \binom{\mathsf{D}(G)}{1} & \binom{\mathsf{D}(G)-k}{1} & \cdots & \binom{0}{1} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{\mathsf{D}(G)}{|S|-\mathsf{D}(G)} & \binom{\mathsf{D}(G)-k}{|S|-\mathsf{D}(G)} & \cdots & \binom{0}{|S|-\mathsf{D}(G)} \end{pmatrix}_{(|S|-\mathsf{D}(G)+1)(\mathsf{D}(G)-k+2)}$$

and

$$AX \equiv A'X \equiv 0 \bmod p.$$

Since $|S| = 2\mathsf{D}(G) - k + 1$, we have $|S| - \mathsf{D}(G) + 1 = \mathsf{D}(G) - k + 2$. It follows that A' is a square matrix and $|S| - \mathsf{D}(G) > \mathsf{D}(G) - k$. Note that $\binom{a}{b} = 0$ if $0 \le a < b$. We have

$$\binom{\mathsf{D}(G) - k}{|S| - \mathsf{D}(G)} = \cdots = \binom{0}{|S| - \mathsf{D}(G)} = 0.$$

Thus

$$\begin{aligned} &\left(\binom{\mathsf{D}(G)}{|S|-\mathsf{D}(G)} \quad \binom{\mathsf{D}(G)-k}{|S|-\mathsf{D}(G)} \quad \cdots \quad \binom{0}{|S|-\mathsf{D}(G)}\right)X \\ &\qquad\qquad = \binom{\mathsf{D}(G)}{|S| - \mathsf{D}(G)} = \binom{\mathsf{D}(G)}{k-1} \equiv 0 \bmod p. \end{aligned}$$

This contradicts $\binom{\mathsf{D}(G)}{k-1} \not\equiv 0 \bmod p$. ■

By Lemma 4.6, we can obtain the following result.

Theorem 4.7. *Let G be a finite abelian p-group and $k = c_1p^{t+1} + d \in [\exp(G) + 1, \mathsf{D}(G)]$ with $d \in [0, p-1]$, $t \ge 0$ and $c_1 \in [1, p-1]$. Let S be a sequence over G of length $2\mathsf{D}(G) - k + 1$. Suppose that $\binom{\mathsf{D}(G)}{k-1} \not\equiv 0 \bmod p$ and $2k - \mathsf{D}(G) \ge p + d - v$ with $v \in [0, p-1]$.*

If either

(1) $p > 2$ *and* $2\mathsf{D}(G) - k + 1 < (p-1)p^{t+1}$, *or*
(2) $p = 2$ *and* $2\mathsf{D}(G) - 2k + 1 < p^{t+2}$,

then S has a nontrivial zero-sum subsequence T with $|T| \le k - 1$.

Proof. If $\mathsf{N}^i(S) = 0$ for $i \in [\mathsf{D}(G) + 1, |S|]$, then by Lemma 4.6 we are done.

Now suppose that S has a zero-sum subsequence T of length $|T| \in [\mathsf{D}(G) + 1, |S|]$. If $|T| < 2k$, then since T is a zero-sum subsequence of S of length $|T| \ge \mathsf{D}(G) + 1$, by the definition of $\mathsf{D}(G)$ we find that T has proper zero-sum subsequences T_1, TT_1^{-1}. Obviously, $\min\{|T_1|, |TT_1^{-1}|\} \le k - 1$, and we are done.

Suppose that $|T| \geq 2k$.

(1) Since $p > 2$ and $|T| \leq |S| = 2\mathsf{D}(G) - k + 1 < (p-1)p^{t+1}$ with $k = c_1 p^{t+1} + d$, $d \in [0, p-1]$, $t \geq 0$ and $c_1 \in [1, p-1]$, we have

$$|T| - k \geq k = c_1 p^{t+1} + d \geq p^{t+1}$$

and

$$|T| - k \leq |S| - k < (p-1)p^{t+1} - (c_1 p^{t+1} + d) = (p-1-c_1)p^{t+1} - d < p^{t+2}.$$

Hence, we can set $|T| - k = u_1 p^{t+1} + v_1$ with $u_1 \in [1, p-1]$ and $v_1 \in [0, p^{t+1} - 1]$. Since $2k - \mathsf{D}(G) \geq p + d - v$, by Lemmas 4.4 and 4.5(1), it suffices to prove that

$$u_1 + c_1 + 1 \not\equiv 0 \bmod p.$$

Since

$$\begin{aligned} |T| &= |T| - k + k = u_1 p^{t+1} + v_1 + c_1 p^{t+1} + d = (u_1 + c_1)p^{t+1} + v_1 + d \\ &\leq |S| < (p-1)p^{t+1}, \end{aligned}$$

we have $u_1 + c_1 < p - 1$, which implies that

$$u_1 + c_1 + 1 \not\equiv 0 \bmod p$$

as required.

(2) Since $p = 2$ and $|T| \leq |S| = 2\mathsf{D}(G) - k + 1$, $2\mathsf{D}(G) - 2k + 1 < p^{t+2}$ and $k = c_1 p^{t+1} + d$ with $d \in [0, p-1]$, $t \geq 0$ and $c_1 \in [1, p-1]$, we have $c_1 = 1$,

$$|T| - k \geq k = 2^{t+1} + d \geq 2^{t+1}$$

and

$$|T| - k \leq |S| - k = 2\mathsf{D}(G) - 2k + 1 < 2^{t+2}.$$

Hence, we can set $|T| - k = u_1 2^{t+1} + v_1$ with $u_1 = 1$ and $v_1 \in [0, 2^{t+1} - 1]$. Since $2k - \mathsf{D}(G) \geq p + d - v$ and $u_1 + c_1 + 1 = 3 \not\equiv 0 \bmod 2$, by Lemmas 4.4 and 4.5(1), we get (2). ■

Similarly, we can also obtain the following result.

Theorem 4.8. *Let G be a finite abelian p-group and $k = c_1 p^{t+1} + 1 \in [\exp(G) + 1, \mathsf{D}(G)]$ with $t \geq 0$ and $c_1 \in [1, p-1]$. Let S be a sequence over G of length $2\mathsf{D}(G) - k + 1$. Suppose that $\mathsf{D}(G) < p^{t+2}$ and $2k - \mathsf{D}(G) \geq 2$.*

If either

(1) *$p > 2$ and $2\mathsf{D}(G) - k + 1 < (p-1)p^{t+1}$, or*
(2) *$p = 2$ and $2\mathsf{D}(G) - 2k + 1 < p^{t+2}$,*

then S has a nontrivial zero-sum subsequence T with $|T| \leq k - 1$.

Proof. Since $k = c_1 p^{t+1} + 1 \leq \mathsf{D}(G) < p^{t+2}$ with $c_1 \geq 1$, we can set $\mathsf{D}(G) = ap^{t+1} + b$ with $p - 1 \geq a \geq c_1 \geq 1$ and $b \in [0, p^{t+1} - 1]$. Thus

$$\binom{\mathsf{D}(G)}{k-1} = \binom{ap^{t+1} + b}{c_1 p^{t+1}} \equiv \binom{a}{c_1}\binom{b}{0} = \binom{a}{c_1} \not\equiv 0 \bmod p,$$

where $\binom{a}{c_1} \not\equiv 0 \bmod p$ because $p-1 \geq a \geq c_1 \geq 1$. This together with Lemma 4.6 shows that if $\mathsf{N}^i(S) = 0$ for $i \in [\mathsf{D}(G)+1, |S|]$, then we are done.

Now suppose that S has a zero-sum subsequence T of length $|T| \in [\mathsf{D}(G)+1, |S|]$. Then by imitating the proof of Theorem 4.7 we get the assertion. ∎

5. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Let S be a sequence over G of length $2\mathsf{D}(G)-k+1$. It suffices to prove that S has a nontrivial zero-sum subsequence T with $|T| \leq k-1$.

(1) Since $c_1 \leq \frac{p-1}{3}$ and $k-1 = c_1 p^{t+1} \in [\frac{\mathsf{D}(G)+1}{2}, \mathsf{D}(G)-1]$ with $t \geq 0$ and $c_1 \in [1, p-1]$, we have

$$\begin{aligned}
\mathsf{D}(G) &\leq 2(k-1)-1 = 2c_1p^{t+1}-1 \leq 2\frac{p-1}{3}p^{t+1}-1 \\
&= \frac{2}{3}p^{t+2} - \frac{2}{3}p^{t+1} - 1 < p^{t+2}, \\
2k - \mathsf{D}(G) &\geq 2k-(2k-3) = 3 \geq 2, \\
2\mathsf{D}(G)-k+1 &\leq 2(2k-3)-k+1 = 3c_1p^{t+1}-2 \leq 3\frac{p-1}{3}p^{t+1}-2 \\
&< (p-1)p^{t+1}.
\end{aligned}$$

By Theorem 4.8(1), we get (1).

(2) Since $p=2$, we have $c_1 = 1$. Since $k-1 = 2^{t+1} \geq \frac{\mathsf{D}(G)+1}{2}$ with $t \geq 0$, we have

$$\begin{aligned}
&2k-\mathsf{D}(G) \geq 2(\frac{\mathsf{D}(G)+1}{2}+1) - \mathsf{D}(G) = 3 > 2, \\
&\mathsf{D}(G) \leq 2(k-1)-1 = 2^{t+2}-1 < 2^{t+2}, \\
&2\mathsf{D}(G)-2k+1 \leq 2(2k-3)-2k+1 = 2k-5 = 2^{t+2}-3 < 2^{t+2}.
\end{aligned}$$

By Theorem 4.8(2), we deduce (2). ∎

Proof of Theorem 1.3. Let S be a sequence over G of length $|S| = 2\mathsf{D}(G)-k+1$. In the following, we will prove that S has a nontrivial zero-sum subsequence of length $\leq k-1$. If there exists some $i \in [\mathsf{D}(G)+1, 2k-1]$ such that $\mathsf{N}^i(S) \neq 0$, then S has a zero-sum subsequence T of length $|T| \in [\mathsf{D}(G)+1, 2k-1]$. By the definition of $\mathsf{D}(G)$, T has proper zero-sum subsequences T_1, TT_1^{-1}. Obviously, $\min\{|T_1|, |TT_1^{-1}|\} \leq k-1$ and we are done. Hence we can suppose that

$$\mathsf{N}^i(S) = 0 \quad \text{for } i \in [\mathsf{D}(G)+1, 2k-1]. \tag{5.1}$$

(1) Since $\mathsf{D}(C_p^4) = 4p-3$ and $k-1=2p$, by Lemma 4.1 we have

$$\binom{\mathsf{D}(C_p^4)}{k-1} = \binom{4p-3}{2p} \equiv \binom{3}{2}\binom{p-3}{0} = 3 \not\equiv 0 \bmod p,$$

because $p \geq 5$. If $\mathsf{N}^i(S) = 0$ for $i \in [\mathsf{D}(C_p^4)+1, |S|] = [4p-3, 6p-6]$, then by Lemma 4.6 we are done. If there exists some $i \in [\mathsf{D}(C_p^4)+1, |S|] = [4p-3, 6p-6]$ such that $\mathsf{N}^i(S) \neq 0$, then since $2k = 4p+2 > \mathsf{D}(C_p^4)+1$, (5.1) implies that S has a zero-sum subsequence T of length $|T| \in [4p+2, 6p-6]$. Thus $|T| \geq 4p+2 = 2k$ and $|T|-k \in [2p+1, 4p-7]$.

If $3p-1 \notin [2p+1, 4p-7]$, i.e., $p=5$, then $|T|-k \in [2p+1, 3p-2]$. Set $|T|-k = 2p+v_1$ with $v_1 \in [1, p-2]$. Thus by Lemma 4.1 we have

$$\begin{aligned} a_2 &= \binom{|T|-k}{k-2} + \binom{|T|-k+1}{k-1} = \binom{2p+v_1}{2p-1} + \binom{2p+v_1+1}{2p} \\ &\equiv \binom{2}{1}\binom{v_1}{p-1} + \binom{2}{2}\binom{v_1+1}{0} = 1 \not\equiv 0 \bmod p. \end{aligned}$$

If $3p-1 \in [2p+1, 4p-7]$, i.e., $p \geq 6$, then set $|T|-k = u_1p+v_1$ with $u_1 = 2$ or 3 and $v_1 \in [0, p-1]$. If $|T|-k \neq 3p-1$, then $v_1 \in [0, p-2]$. This together with Lemma 4.1 yields

$$\begin{aligned} a_2 &= \binom{|T|-k}{k-2} + \binom{|T|-k+1}{k-1} = \binom{u_1p+v_1}{2p-1} + \binom{u_1p+v_1+1}{2p} \\ &\equiv \binom{u_1}{1}\binom{v_1}{p-1} + \binom{u_1}{2}\binom{v_1+1}{0} = \binom{u_1}{2} \not\equiv 0 \bmod p. \end{aligned}$$

If $|T|-k = 3p-1$, then

$$\begin{aligned} a_2 &= \binom{|T|-k}{k-2} + \binom{|T|-k+1}{k-1} = \binom{3p-1}{2p-1} + \binom{3p}{2p} \\ &\equiv \binom{2}{1}\binom{p-1}{p-1} + \binom{3}{2}\binom{0}{0} = 5 \not\equiv 0 \bmod p. \end{aligned}$$

because $p \geq 6$. Since $2k - \mathsf{D}(C_p^4) = 5 \geq 2$, by Lemma 4.4, S has a zero-sum subsequence of length $\leq k-1$, proving (1).

(2) Since $\mathsf{D}(C_p^d) = (p-1)d+1$ and $k-1 = (d-1)p \in [p, \mathsf{D}(C_p^d)]$, we have $(p-1)d+1 \geq (d-1)p \geq p$, i.e., $2 \leq d \leq p+1$. It follows that $p-d+1 \in [0, p-1]$. This together with Lemma 4.1 yields

$$\begin{aligned} \binom{\mathsf{D}(C_p^d)}{k-1} &= \binom{(d-1)p+p-d+1}{(d-1)p} \equiv \binom{d-1}{d-1}\binom{p-d+1}{0} \\ &= 1 \not\equiv 0 \bmod p. \end{aligned}$$

If $\mathsf{N}^i(S) = 0$ for $i \in [\mathsf{D}(C_p^d)+1, |S|] = [(p-1)d+2, (d+1)p-2d+2]$, then by Lemma 4.6, we are done. If there exists some $i \in [\mathsf{D}(C_p^d)+1, |S|] =$

$[(p-1)d+2, (d+1)p-2d+2]$ such that $\mathsf{N}^i(S) \neq 0$, then since

$$2k - (\mathsf{D}(C_p^d)+1) = 2(d-1)p+2-(p-1)d-2 = (d-2)p+d > 0$$

(as $d \geq 2$), (5.1) implies that S has a zero-sum subsequence T with $|T| \in [2(d-1)p+2, (d+1)p-2d+2]$. It follows that $2(d-1)p+2 \leq (d+1)p-2d+2$. It is also easy to see that

$$2(d-1)p+2-((d+1)p-2d+2) = (d-3)p+2d > 0$$

if $d \geq 3$. Hence $d = 2$. Note that $\mathsf{s}_{\leq p}(C_p^2) = 3p-2 = 2\mathsf{D}(C_p^2) - p$ and the proof is complete. ■

6. Concluding remarks and a conjecture. We close the paper by making the following conjecture for finite abelian groups G. Obviously, this conjecture covers most of the results in Theorems 1.1–1.3 and Lemmas 2.1–2.5.

CONJECTURE 6.1. *Let G be a finite abelian group with $\mathsf{r}(G) \geq 2$ and $\mathsf{D}(G) = \mathsf{D}^*(G)$. Let k be a positive integer such that $k \in [\exp(G), \mathsf{D}(G)]$. If $k \geq \frac{\mathsf{D}(G)+1}{2}$, then*

$$\mathsf{s}_{\leq k}(G) \leq 2\mathsf{D}(G) - k.$$

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