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**JOANNA RENCLAWOWICZ and
WOJCIECH M. ZAJĄCZKOWSKI**

**Long time existence for nonhomogeneous Navier–Stokes
equations with large flux**

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Joanna Renclawowicz
Institute of Mathematics and Cryptology
Cybernetics Faculty
Military University of Technology
00-908 Warszawa, Poland
E-mail: joanna.renclawowicz@wat.edu.pl

Wojciech M. Zajączkowski
Professor Emeritus
Institute of Mathematics
Polish Academy of Sciences
00-656 Warszawa, Poland
E-mail: wz@impan.pl

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Abstract

The nonhomogeneous Navier–Stokes equations are examined in a cylindrical domain in $\mathbb{R}^3$, parallel to the x_3 -axis, with large inflow and outflow at the top and bottom of the cylinder. Additionally, slip boundary conditions are applied to the lateral surface of the cylinder. This paper establishes the long time existence of regular solutions under the condition that the inflow and outflow are close to homogeneous, and that the norms of the derivatives with respect to x_3 of both the external force and the initial velocity are sufficiently small. A key point of this study is to demonstrate that the x_3 -component of the velocity remains positive.

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1. Introduction

We analyze motions of the nonhomogeneous Navier–Stokes equations in cylindrical domains with large inflow and outflow. By “nonhomogeneous,” we mean a density-dependent system. Our objective is to establish the existence of long time strong solutions without imposing smallness restrictions on the velocity and flux. In our proof, we utilize concepts and techniques from [RZ1]. We introduce Cartesian coordinates (x_1, x_2, x_3) . Let $\Omega \subset \mathbb{R}^3$ be a cylindrical domain parallel to the x_3 -axis. The boundary of Ω , denoted as S , consists of two parts: S_1 and S_2 . Here, S_1 is parallel to the x_3 -axis, while S_2 is perpendicular to it. Furthermore, S_2 can be expressed as $S_2(-a) \cup S_2(a)$, where $a \in \mathbb{R}$ is specified. The parts $S_2(a_i)$ intersect the x_3 -axis at points a_i for $i = 1, 2$, with the conditions $a_1 = -a$ and $a_2 = a$. Lastly, S_1 and $S_2(a_i)$ meet along a curve $L(a_i)$ for each $i = 1, 2$.

Let $T > 0$ be given. We consider the following initial-boundary value problem in the domain $\Omega^T = \Omega \times (0, T)$:

$$\begin{aligned}
 \varrho(v_t + v \cdot \nabla v) - \operatorname{div} \mathbb{T}(v, p) &= \varrho f && \text{in } \Omega^T, \\
 \operatorname{div} v &= 0 && \text{in } \Omega^T, \\
 \varrho_t + v \cdot \nabla \varrho &= 0 && \text{in } \Omega^T, \\
 v \cdot \bar{n} &= 0 && \text{on } S_1^T = S_1 \times (0, T), \\
 \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\
 v \cdot \bar{n} &= -d_1, \quad d_1 > 0 && \text{on } S_2^T(-a), \\
 v \cdot \bar{n} &= d_2, \quad d_2 > 0 && \text{on } S_2^T(a), \\
 \varrho &= \varrho_1 && \text{on } S_2^T(-a), \\
 \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2 && \text{on } S_2^T, \\
 v|_{t=0} &= v_0 && \text{in } \Omega, \\
 \varrho|_{t=0} &= \varrho_0 && \text{in } \Omega,
 \end{aligned} \tag{1.1}$$

where $\varrho = \varrho(x, t) \in \mathbb{R}^1$ denotes the density of the fluid, v is the velocity with $v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}^1$ denotes the pressure, $f = f(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ the external force field, $x = (x_1, x_2, x_3)$ are the Cartesian coordinates. We denote by $\nu > 0$ the constant viscosity coefficient and by $\gamma > 0$ the slip coefficient; moreover, $\bar{n}$ is the unit outward normal vector to the surface S . The vectors $\bar{\tau}_\alpha$, where $\alpha = 1, 2$, are tangent vectors to S . Additionally, I represents the unit matrix, while $\mathbb{D}(v)$ is the dilatation tensor, expressed as

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

Furthermore, the stress tensor $\mathbb{T}(v, p)$ is defined as follows:

$$\mathbb{T}(v, p) = \nu \mathbb{D}(v) - pI.$$

Utilizing Cartesian coordinates and considering a constant c_0 , we describe a sufficiently smooth closed curve in the plane through the equation $\varphi_0(x_1, x_2) = c_0$, with $x_3 = \text{const} \in (-a, a)$. This leads to the following definitions:

$$\begin{aligned} \Omega &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) \leq c_0, -a < x_3 < a\}, \\ S_1 &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) = c_0, -a < x_3 < a\}, \\ S_2(-a) &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_0, x_3 = -a\}, \\ S_2(a) &= \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_0, x_3 = a\}. \end{aligned} \tag{1.2}$$

To facilitate the description of inflow and outflow, we note the following relations:

$$d_1 = -v \cdot \bar{n}|_{S_2(-a)}, \quad d_2 = v \cdot \bar{n}|_{S_2(a)}, \tag{1.3}$$

where $d_i > 0$ for $i = 1, 2$, and $\bar{n}$ signifies the unit outward normal vector to S_2 .

In light of the consideration of incompressible motions, the following compatibility condition must be satisfied:

$$\int_{S_2(-a)} d_1 dS_2 = \int_{S_2(a)} d_2 dS_2. \tag{1.4}$$

We now present our main result regarding the long time existence of regular solutions to problem (1.1). To do this, we first prove the existence of local regular solutions, as noted in the theorem referenced below and as previously established in [RZ3]. Subsequently, we derive a long time estimate for these solutions. By standard methods, we can extend the local solutions over time, contingent upon the validity of the long time estimate.

THEOREM 1.1 (Local existence, see [RZ3]). *Assume the following:*

- (1) (Parameters) *Let s, σ, r satisfy $s \in (0, 1)$, $\frac{3}{s} < \sigma$, $\frac{5}{s} < r$, and $\sigma < r$.*
- (2) (Data functions)

- *The initial density $\varrho_0 \in W_r^1(\Omega)$ and $\varrho_1 \in W_r^{1,1}(S_2^t(-a))$.*
- *The initial velocity $v_0 \in W_\sigma^{2+s-\frac{2}{\sigma}}(\Omega)$.*
- *The inflow $d_1 \in W_\sigma^{2+s-\frac{1}{\sigma}, 1+\frac{s}{2}-\frac{1}{2\sigma}}(S_2^t(-a))$.*
- *The outflow $d_2 \in W_\sigma^{2+s-\frac{1}{\sigma}, 1+\frac{s}{2}-\frac{1}{2\sigma}}(S_2^t(a))$.*
- *The external force $f \in W_\sigma^{s, \frac{s}{2}}(\Omega^t)$.*

- (3) (Constants) *There exist positive constants $\bar{d}_0, d_0, d_\infty, b_0, b_1, \bar{b}_0$, and $\bar{b}_1$ such that*

$$\begin{aligned} \bar{d}_0 &\geq v_3(0) \geq d_0, & d_i &\geq d_\infty, \quad i = 1, 2, \\ \bar{b}_0 &\geq \varrho(0) \geq b_0, & \bar{b}_1 &\geq \varrho_1 \geq b_1. \end{aligned}$$

(4) (Finite quantities) *The following quantities are finite:*

$$\begin{aligned} c_1 &= |d_1|_{\infty, S_2^t(-a)} (|\varrho_{1,x}|_{r, S_2^t(-a)} + |\varrho_{1,t}|_{r, S_2^t(-a)}) \\ &\quad + \|\varrho_x(0)\|_{L_r(\Omega)} (1 + \|v(0)\|_{W_r^1(\Omega)}), \\ c_2 &= \|f\|_{W_{\sigma}^{s, s/2}(\Omega^t)} + \sum_{i=1}^2 \|d_i\|_{W_{\sigma}^{2+s-1/\sigma, 1+s/2-1/2\sigma}(S_2^t(a_i))} \\ &\quad + \|\varrho_1\|_{W_r^{1,1}(S_2^t(-a))} + \|v_0\|_{W_{\sigma}^{2+s-2/\sigma}(\Omega)}. \end{aligned}$$

Then there exists a local solution (v, p, ϱ) to the nonhomogeneous Navier–Stokes problem such that

$$v \in W_{\sigma}^{2+s, 1+s/2}(\Omega^t), \quad \nabla p \in W_{\sigma}^{s, s/2}(\Omega^t), \quad \varrho \in W_{r, \infty}^{1,1}(\Omega^t).$$

Moreover, the density remains bounded:

$$\varrho_* \equiv \frac{b_1 b_0}{b_1 + b_0} \leq \varrho(x, t) \leq \bar{b}_0 + \bar{b}_1 \equiv \varrho^*.$$

The velocity and pressure satisfy

$$\|v\|_{W_{\sigma}^{2+s, 1+s/2}(\Omega^t)} + \|\nabla p\|_{W_{\sigma}^{s, s/2}(\Omega^t)} \leq \phi(\text{data}),$$

where the data are described by assumptions (1)–(3), ϕ is an increasing positive function, and $t \leq T$ with T sufficiently small.

Finally, the x_3 -component of the velocity is positive, as there exists a positive constant $d_ = d_*(d_0, \bar{d}_0, \bar{c}_1, \bar{c}_2, d_{\infty}, \|f_3\|_{L_1(0, t; L_{\infty}(\Omega)))}$ such that*

$$v_3 \geq d_*.$$

ASSUMPTIONS 1. Assume that

1. $f \in W_{\sigma}^{s, s/2}(\Omega^t)$, $d \in W_{\sigma}^{2+s-1/\sigma, 1+s/2-1/2\sigma}(S_2^t)$, $v_0 \in W_{\sigma}^{2+s-2/\sigma}(\Omega)$, $d = (d_1, d_2)$,
2. $h(0) = v_{0, x_3} \in W_{5/3}^{4/5}(\Omega)$, $g \in L_4(\Omega^t)$, $h = v_{x_3}$, $q = p_{x_3}$, $g = f_{x_3}$,
3. $\varrho_0 \in L_{\infty}(\Omega)$, $\varrho_1 \in L_{\infty}(S_2^t(-a))$, $1/\varrho_0 \in L_{\infty}(\Omega)$, $1/\varrho_1 \in L_{\infty}(S_2^t(-a))$, $1/d_1 \in L_{\infty}(S_2^t(-a))$, $1/d_2 \in L_{\infty}(S_2^t(a))$.
4. The domain Ω contains edges L_1, L_2 such that $\bar{S}_1 \cap \bar{S}_2 \in \{L_1, L_2\}$. On the edges the following compatibility conditions hold:

$$\nu n_{\alpha} d_{i, x_{\alpha}} + \gamma d_i = 0 \quad \text{on } L_i, \quad i = 1, 2,$$

when $\bar{n} = (n_1, n_2)$ is the normal vector to S_1 .

5. Compatibility conditions:

$$\begin{aligned} \int_{S_2(-a)} d_1 dS_2 &= \int_{S_2(a)} d_2 dS_2, \\ \partial_x^{\alpha} v_{30}|_{S_2} &= \partial_x^{\alpha} d|_{t=0}, \quad \alpha = 0, 1, 2. \end{aligned}$$

6. The following quantities are finite:

$$\begin{aligned} c_1 &= \|d_1\|_{L_\infty(S_2^t(-a))} (\|\varrho_{1,x'}\|_{L_r(S_2^t(-a))} + \|\varrho_{1,t}\|_{L_r(S_2^t(-a))}) \\ &\quad + \|\varrho_{0,x}\|_{L_r(\Omega)}, \\ c_2 &= \|f\|_{W_\sigma^{s,s/2}(\Omega^t)} + \sum_{i=1}^2 \|d_i\|_{W_\sigma^{2+s-1/\sigma, 1+s/2-1/2\sigma}(S_2^t(a_i))} \\ &\quad + \|\varrho_1\|_{W_r^{1,1}(S_2^t(-a))} + \|v_0\|_{W_\sigma^{2+s-2/\sigma}(\Omega)}. \end{aligned}$$

7. We define finite parameters:

$$\begin{aligned} \Lambda_1 &= \|d_{x'}\|_{L_2(0,t;W_3^1(S_2))} + \|d_{x'}\|_{L_\infty(0,t;L_2(S_2))} + \|d_t\|_{L_2(0,t;H^1(S_2))} \\ &\quad + \|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|f_{x_3}\|_{L_4(\Omega^t)}, \quad x' = (x_1, x_2), \\ \Lambda_2 &\equiv \Lambda_2(r) = \|\varrho_{1,x'}\|_{L_r(S_2^t(-a))} + \|\varrho_{1,t}\|_{L_r(S_2^t(-a))} \\ &\quad + \|\varrho_{0,x}\|_{L_r(\Omega)} + \|\varrho_{0,t}\|_{L_r(\Omega)}, \\ \Lambda_3(4) &= \sum_{i,j=1}^2 \|d_{j,x_i}\|_{W_4^{7/4,7/8}(S_2^t(a_j))} + \|g\|_{L_4(\Omega^t)} + \|h(0)\|_{W_4^{3/2}(\Omega)}, \\ \Lambda &= \Lambda_1 + \Lambda_2 + \Lambda_3(4) + \|v_{0,x_3}\|_{L_2(\Omega)}, \end{aligned} \tag{1.5}$$

that need to be small.

THEOREM 1 (Upper and lower estimates for density). *Let $s \in (0, 1)$, $3/s < \sigma$, $5/s < r$, $\sigma < r$ and Assumptions 1 hold. Let*

$$\begin{aligned} \varrho_* &= \frac{1}{\left\| \frac{1}{\varrho_0} \right\|_{L_\infty(\Omega)} + \left\| \frac{1}{\varrho_1} \right\|_{L_\infty(S_2^t(-a))}}, \\ \varrho^* &= \|\varrho_1\|_{L_\infty(S_2^t(-a))} + \|\varrho_0\|_{L_\infty(\Omega)}. \end{aligned}$$

Then

$$\varrho_* \leq \varrho(x, t) \leq \varrho^*. \tag{1.6}$$

THEOREM 2 (Positivity of v_3). *Let $s \in (0, 1)$, $3/s < \sigma$, $5/s < r$, $\sigma < r$ and Assumptions 1 hold. Let $d_0, \bar{d}_0, d_\infty, \bar{d}$ be positive constants with $\bar{d}_0 \geq v_3(0) \geq d_0$, $d_\infty \leq d_i \leq \bar{d}$, $i = 1, 2$. Assume $f_3 \in L_1(0, t; L_\infty(\Omega))$.*

Then there exists a positive constant d_ such that*

$$v_3 \geq d_* \tag{1.7}$$

where $d_* = d_*(d_0, \bar{d}_0, c_1, c_2, d_\infty, \|f_3\|_{L_1(0,t;L_\infty(\Omega))})$.

THEOREM 3 (Energy estimate for v). *Assume that $s \in (0, 1)$, $3/s < \sigma$, $5/s < r$, $\sigma < r$, and Assumptions 1 hold. Let $\varrho_1 \in L_\infty(S_2^t)$, $d_1 \in L_6(0, t; L_3(S_2))$, $f \in L_2(0, t; L_{6/5}(\Omega))$, $v_0 \in L_2(\Omega)$. Suppose that $\tilde{d}_i$ is an extension of d_i , $i = 1, 2$, such that $\tilde{d}_i|_{S_2(a_i)} = d_i$, and*

assume $\tilde{d}_i \in L_\infty(0, t; W_{3,\infty}^1(\Omega)) \cap L_2(0, t; W_{3,\infty}^1(\Omega))$, $\tilde{d}_{i,t} \in L_2(0, t; W_{6/5}^1(\Omega))$. Then

$$\begin{aligned} & \|v\|_{L_\infty(0,t;L_2(\Omega))}^2 + \|\nabla v\|_{L_2(\Omega^t)}^2 \equiv \|v\|_{V(\Omega^t)}^2 \leq A_1^2, \\ \text{and } A_1^2 &= \phi(\|\varrho_1\|_{L_\infty(S_2^t)}, \|d_1\|_{L_6(0,t;L_3(S_2(-a)))}, \varrho^*, \varrho_*) \\ & \cdot \{\phi(\sup_t \|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*)[\|\tilde{d}\|_{L_2(0,t;W_{3,\infty}^1(\Omega))}^2 \\ & + \|\tilde{d}_t\|_{L_2(0,t;W_{6/5}^1(\Omega))}^2 + \|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2] + \varrho^* \|v_0\|_{L_2(\Omega)}^2\}, \end{aligned} \quad (1.8)$$

where $\tilde{d}$ denotes either $\tilde{d}_1$ or $\tilde{d}_2$ and ϕ is an increasing positive function.

THEOREM 4 (Long time estimate and existence). *Let $s \in (0, 1)$, $3/s < \sigma$, $5/s < r$, and $\sigma < r$. Suppose Assumptions 1 hold. Define*

$$\begin{aligned} D &= \|f\|_{W_\sigma^{s,s/2}(\Omega^t)} + \|d\|_{W_\sigma^{2+s-1/\sigma, 1+s/2-1/2\sigma}(S_2^t)} \\ &+ \|v_0\|_{W_\sigma^{2+s-2/\sigma}(\Omega)} + \|f_{x_3}\|_{L_{5/3}(\Omega^t)} + \|d_{x'}\|_{W_{5/3}^{7/5, 7/10}(S_2^t)} \\ &+ \|v_{0,x_3}\|_{W_{5/3}^{4/5}(\Omega)} + \|f_{2,x_1} - f_{1,x_2}\|_{L_2(0,t;L_{6/5}(\Omega))} < \infty. \end{aligned}$$

For sufficiently small Λ , solutions to problem (1.1) exist and satisfy the following estimate, provided that $t < \infty$ and $\bar{a} > 0$:

$$\begin{aligned} & \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s,s/2}(\Omega^t)} + \|v_{x_3}\|_{W_4^{2,1}(\Omega^t)} \\ & + \|p_{x_3}\|_{L_4(0,t;W_1^4(\Omega^t))} \leq \phi\left(\varrho^*, \frac{1}{\varrho_*}, A_1, D\right), \\ & \|\varrho_x\|_{L_\infty(0,t;L_r(\Omega))} + \|\varrho_t\|_{L_\infty(0,t;L_r(\Omega))} \leq \phi\left(\varrho^*, \frac{1}{\varrho_*}, A_1, D, \right) \cdot t^{\bar{a}} \Lambda_2. \end{aligned} \quad (1.9)$$

REMARK 1.2. We define Sobolev and other spaces in Section 2. The norms that appear in the above theorems are somewhat atypical, so we refer to Definitions 2.1 and 2.3 for the concepts of $W_r^{1,1}$, $W_{r,\infty}^{1,1}$ and $W_{p,\infty}^1$.

REMARK 1.3. In the theorems mentioned above, we do not prove global existence because we cannot approach the limit $\Lambda = 0$ as $t \rightarrow \infty$ without assigning a value of zero to some initial data. This is explained in more detail in Remark 11.3.

This paper aims to demonstrate the existence of long time regular solutions to problem (1.1) for arbitrarily large flux and fluid motion velocities. To establish such a result, we require certain smallness conditions. Specifically, in order to prove the existence of long time regular solutions to (1.1), we need the following quantities to be small in some norms, denoted by the smallness parameter Λ :

$$v_{0,x_3}, d_{i,x'}, d_{i,t}, i = 1, 2, f_3, f_{x_3}, \varrho_{0,x}, \varrho_{0,t}, \varrho_{1,x'}, \varrho_{1,t},$$

where $x' = (x_1, x_2)$, f represents the external force, v_0 is the initial velocity, and ϱ_0 is the initial density. Problem (1.1) describes the coupling between the Navier–Stokes equations and the continuity equation for density. It is important to note that the continuity equation is hyperbolic, which means the density must also be prescribed for the inflow part of the domain under consideration.

In Section 5, we derive an estimate for the derivatives of density, which relies on the norms of $\varrho_{0,x}$, $\varrho_{0,t}$, $\varrho_{1,x'}$, $\varrho_{1,t}$. This implies that ϱ_x and ϱ_t are small. However, to establish this bound, we require the velocity component along the x_3 -axis, denoted as v_3 , to be positive. The estimate for v_3 from below is provided in Section 6 and takes the form:

$$I \equiv c \exp[-(\|p_{x_3}\|_{L_1(0,t;L_\infty(\Omega))} + \|f_3\|_{L_1(0,t;L_\infty(\Omega))})]$$

To keep the quantity I positive we have to know that

$$\|p_{x_3}\|_{L_1(0,t;L_\infty(\Omega))} \quad \text{is finite for all } t.$$

We show in Section 10 that

$$\|p_{x_3}\|_{L_1(0,t;L_\infty(\Omega))} \leq \phi\Lambda, \quad (1.10)$$

where ϕ is an increasing positive function of t and Λ is a small parameter. The bound (1.10) means that for small Λ , time can be assumed large. Passing with Λ to zero we conclude that t tends to infinity.

Now, we will briefly explain how the long time estimate for regular solutions of (1.1) has been achieved. To do this, we utilize the energy estimate in the following norm:

$$\|u\|_{V(\Omega^t)}^2 = \|u\|_{L_\infty(0,t;L_2(\Omega))}^2 + \|\nabla u\|_{L_2(\Omega^t)}^2.$$

In Section 3, we derive the energy inequality for solutions to (1.1) based on the initial data. In Section 4, we establish an energy estimate for solutions to (2.6) concerning the quantities $h = v_{x_3}$ and $q = p_{x_3}$, as described in (4.19). The resulting estimate takes the form

$$\|h\|_{V(\Omega^t)} \leq \phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)})\Lambda, \quad (1.11)$$

where $\bar{a} > 0$, $s \in (0, 1)$, $\sigma > 3/s$. The inequality suggests that we have to look for long time regular solutions to problem (1.1) such that $v \in W_\sigma^{2+s,1+s/2}(\Omega^t)$.

In Section 5, we establish the estimate for the derivatives of the density:

$$\|\varrho_x\|_{L_\infty(0,t;L_r(\Omega))} + \|\varrho_t\|_{L_\infty(0,t;L_r(\Omega))} \leq \phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)})\Lambda. \quad (1.12)$$

To validate (1.12), it is essential to meet the requirement given in Section 6:

$$v_3 \geq \exp[-(\|q\|_{L_1(0,t;L_\infty(\Omega))} + \|f\|_{L_1(0,t;L_\infty(\Omega))})] > 0. \quad (1.13)$$

In order to determine a bound for $\|q\|_{L_1(0,t;L_\infty(\Omega))}$, it is necessary that $h \in W_4^{2,1}(\Omega^t)$. This estimate is derived in Section 10, specifically in (10.9):

$$\|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_2(0,t;W_4^1(\Omega))} \leq \phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)})\Lambda. \quad (1.14)$$

Furthermore, it is required that $v \in W_\sigma^{2+s,1+s/2}(\Omega^t)$. In Section 7, we address the problem described in (2.8) for $\chi = \text{rot}_3 v$. Subsequently, in (7.11), we arrive at

$$\begin{aligned} \|\chi\|_{V(\Omega^t)} &\leq \phi(D)(\|v'\|_{W_2^{1,1/2}(\Omega^t)} + \|v'\|_{L_\infty(0,t;W_2^{5/6}(\Omega))}) \\ &\quad + \phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)})\Lambda + D, \end{aligned} \quad (1.15)$$

where D represents data that may not be small.

The main idea to demonstrate the long time estimate is to verify the inequality

$$\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)} \leq \phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)})\Lambda + D \quad (1.16)$$

which for sufficiently small $t^{\bar{a}}\Lambda$ eventually implies

$$\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)} \leq \phi(D). \quad (1.17)$$

However, the first two terms on the r.h.s. of (1.15) are not of the form of the r.h.s. of (1.16), so in order to absorb them, we use the elliptic problem

$$\begin{aligned} v_{2,x_1} - v_{1,x_2} &= \chi & \text{in } \Omega', \\ v_{1,x_1} + v_{2,x_2} &= -h_3 & \text{in } \Omega', \\ v' \cdot \bar{n}' &= 0 & \text{on } S'_1, \end{aligned} \quad (1.18)$$

where x_3 is treated as a parameter, Ω' is the cross-section of Ω with the plane perpendicular to the x_3 -axis and passing through the point $x_3 \in (-a, a)$, and S'_1 is the cross-section of S_1 with the same plane. Then S'_1 is the boundary of Ω' . Derivation of estimates for solutions to (1.18) makes it possible to increase regularity and absorb norms of v' in (1.15). It is described at the beginning of Section 9.

Implementing the notation

$$X = \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)} + \|\nabla p\|_{W_{\sigma}^{s,s/2}(\Omega^t)} + \|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega^t))},$$

the following inequality is attained in Section 11 through complicated calculations:

$$X \leq \phi(X)t^{\bar{a}}\Lambda + D, \quad \bar{a} > 0,$$

which for sufficiently small $t^{\bar{a}}\Lambda$ implies the bound

$$X \leq \phi(D).$$

The estimate enables us to substantiate the long-term existence of regular solutions.

Subsequently, we shall present a thorough comparison of the results shown here with those found in our earlier studies, [RZ1]–[RZ4]. As previously mentioned, this paper demonstrates the existence of regular solutions in Besov spaces. Specifically, the velocity v , pressure p , and density ϱ are associated with special Besov spaces, namely, Sobolev–Slobodetski spaces:

$$\begin{aligned} v &\in W_{\sigma}^{2+s,1+s/2}(\Omega^T), \quad \nabla p \in W_{\sigma}^{s,s/2}(\Omega^t), \quad h \in W_4^{2,1}(\Omega^T), \\ q &\in L_4(0,t;W_4^1(\Omega^t)), \quad \varrho_x, \varrho_t \in L_{\infty}(0,t;L_r(\Omega)), \end{aligned}$$

and $s \in (0, 1)$, $\frac{3}{s} < \sigma$, $\frac{5}{s} < r$, $\sigma < r$.

To achieve the long time estimate, we require smallness assumptions on some data functions, mainly on derivatives with respect to x_3 . Then, to prove existence, we use a step-by-step procedure, which is based on local-in-time existence.

This work is a continuation of [RZ3], where we have shown local-in-time existence using the method of successive approximations. Since the existence time was small, we could apply it as the smallness parameter. Thus, we did not have to make any restrictions on data functions. Spaces applied in both papers are similar, except that in [RZ3] we have lower regularity, thus third derivatives with respect to x_3 are not used:

$$v \in W_{\sigma}^{2+s,1+s/2}(\Omega^t), \quad \nabla p \in W_{\sigma}^{s,s/2}(\Omega^t), \quad \varrho \in W_{r,\infty}^{1,1}(\Omega^t).$$

We now discuss [RZ2]. In that paper, we considered the Navier–Stokes system in Ω^T , the same as in this article, but without flux and homogeneous, with no density.

We showed the existence of regular solutions with v and p belonging to Sobolev spaces, $v \in W_2^{2,1}(\Omega^T)$, $h \in W_\sigma^{2,1}(\Omega^T)$, $\nabla p \in L_2(\Omega^T)$, $\nabla q \in L_\sigma(\Omega^T)$, $5/3 < \sigma < 3$, through the Leray–Schauder fixed point theorem.

The paper [RZ4], describing solutions to homogeneous Stokes system with no flux, also serves as an auxiliary one to the current paper. Actually, in Section 12, we extend results to the nonhomogeneous Stokes system, since establishing the existence to the Stokes system in Besov spaces $W_\sigma^{2+s,1+s/2}(\Omega^T)$ is crucial. In [RZ4], we showed for the homogeneous Stokes system that the velocity belongs to $W_\sigma^{2+s,1+s/2}(\Omega^T)$ and the gradient of the pressure to $W_\sigma^{s,s/2}(\Omega^T)$, where $\sigma \in (1, \infty)$, $s \in (0, 1)$.

We also refer to the book [RZ1], where we discussed the homogeneous Navier–Stokes system with large flux. We established the existence in Sobolev spaces, with $h, v \in W_2^{2,1}(\Omega^T)$, by using the Leray–Schauder fixed point theorem. Let us examine in detail the similarities and differences with the existing work. In Section 4, we discover an energy estimate for the solutions to problem (1.1), specifically in (3.37). It can be noted that even though in (1.1) we are dealing with density and the estimate relies on this variable, more precisely, on its lower and upper bounds, the estimate is similar to the one presented in [RZ1]. In both papers, we utilize sophisticated weighted estimates for the Neumann problem related to the Poisson equation. In the following step, we must establish an energy estimate for $h = v_x$, as provided in (5.1). Once more, the approach is akin to that outlined in [RZ1]; however, we require estimates for $\|\varrho_{x3}\|_{L_\infty(0,t;L_3(\Omega))}$, as demonstrated in (5.15). Additionally, to estimate the norms of ϱ_x , it is essential to establish that v_3 is distinct from zero in Ω , as described in Section 6. Furthermore, the relation in (5.1) depends on $\|v_t\|_{L_1(0,t;L_\infty(\Omega))}$. To overcome this difficulty, in Lemma 8.1, we utilize the embedding

$$\|v_t\|_{L_1(0,t;L_\infty(\Omega))} \leq \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, \quad \sigma > 3/s, \quad s \in (0, 1).$$

In view of this inequality, we can close estimates for solutions to (1.1) by demonstrating the existence of solutions to the Stokes system in the Besov spaces $W_\sigma^{2+s,1+s/2}(\Omega^t)$, as proved in [RZ4]. Therefore, the estimates for derivatives of ϱ , construction of the lower bound for v_3 and solvability of the Stokes system in Besov spaces render the proof in this paper much more complicated than that presented in [RZ1]. We emphasize that to avoid the Besov spaces it would be necessary to look for solutions to problem (1.1) with

$$v \in W_p^{4,2}(\Omega^t), \quad h \in W_p^{4,2}(\Omega^t).$$

Then the calculations in this paper would be far more complex and longer.

The idea of enhancing the regularity of weak solutions is analogous to the one in [RZ1]. By utilizing the energy estimate for h (see (4.19)) and for $\chi = (rot v)_3$ (see (7.11)) and applying the elliptic problem (9.2), we show that $v \in W_{5/3}^{2,1}(\Omega^t)$ (see (9.11)). Next, we apply the Stokes system once more to obtain an estimate for $v \in W_2^{2,1}(\Omega^t)$, as indicated in (9.22). Note that estimates in [RZ1] could be closed already for $W_2^{2,1}(\Omega^t)$. Namely, in [RZ1], we deduced the estimates

$$\begin{aligned} \|h\|_{W_\sigma^{2,1}(\Omega^t)} + |\nabla q|_{\sigma,\Omega^t} &\leq \varphi_1(\text{data}), \quad 5/3 \leq \sigma \leq 10/3, \\ \|v\|_{W_2^{2,1}(\Omega^t)} + |\nabla p|_{2,\Omega^t} &\leq \varphi_2(\text{data}), \end{aligned}$$

for sufficiently small parameter depending on norms of some data functions:

$$f_{x_3}, f_3, v_{0,x_3}, d_{i,x'}, d_{i,t}, i = 1, 2.$$

In the present paper, we prove estimates in $W_{5/2}^{2,1}(\Omega^t)$; see (9.26). Furthermore, we get an estimate for $v \in W_{5'}^{2,1}(\Omega^t)$, where $5' < 5$, but arbitrarily close to 5. Finally, employing the smallness of some parameter Λ , we attain an estimate

$$\|v\|_{W_{\sigma}^{2+s, 1+s/2}(\Omega^t)} + \|\nabla p\|_{W_{\sigma}^{s, s/2}(\Omega^t)} + \|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_4(0,t; W_4^1(\Omega^t))} \leq \phi(D).$$

As previously stated, the following quantities, represented by the smallness parameter Λ , must be small in certain norms in order to demonstrate the long time estimate for solutions to (1.1):

$$f_{x_3}, f_3, v_{0,x_3}, d_{i,x'}, d_{i,t}, i = 1, 2, \varrho_{0,x}, \varrho_{0,t}, \varrho_{1,x'}, \varrho_{1,t},$$

where $x' = (x_1, x_2)$, f represents the external force, v_0 is the initial velocity, and ϱ_0 is the initial density.

We had no restrictions regarding the magnitude of T , the finite time of existence in [RZ1] for solutions such that $v \in W_2^{2,1}(\Omega^T)$ and $h \in W_2^{2,1}(\Omega^T)$. We also demonstrated global existence using a step-by-step approach. By dividing time into intervals $(kT, (k+1)T)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with a sufficiently large T , we show the following inequalities:

$$\begin{aligned} \|v((k+1)T)\|_{H^1(\Omega)} &\leq \|v(kT)\|_{H^1(\Omega)}, \\ \|h((k+1)T)\|_{H^1(\Omega)} &\leq \|h(kT)\|_{H^1(\Omega)}, \end{aligned}$$

for the solutions where

$$h, v \in W_2^{2,1}(\Omega \times (kT, (k+1)T)),$$

with initial conditions at $t = kT$. Thus, the corresponding norms do not escalate with increasing k . This was established through the application of the energy method at the level of H^1 -functions. In the current paper, it was not feasible to prove global existence, as discussed in Remark 1.3.

We now discuss some other results related to the global or long-term existence of regular solutions to nonhomogeneous Navier–Stokes equations. In [DZ], the global existence and uniqueness of solutions to the nonhomogeneous Navier–Stokes system in the half-space $\mathbb{R}_+^n$ ($n \geq 2$) have been established. This applies when the initial density is bounded and close enough to a positive constant, the initial velocity belongs to a certain critical Besov space, and certain smallness conditions are met. Specifically, the L_∞ norm of the inhomogeneity and the critical norm of the horizontal components of the initial velocity are assumed to be very small compared to the exponential norm of the vertical component of the initial velocity.

In the paper [DM], the boundary value problem for the incompressible inhomogeneous Navier–Stokes equations in the half-space is analyzed under small data with critical regularity, specifically in dimensions $n \geq 3$. It is shown that if the initial density is close to a positive constant in $L_\infty \cap \dot{W}_n^1(\mathbb{R}_+^n)$ and the initial velocity is sufficiently small with respect to the viscosity in the homogeneous Besov space $\dot{B}_{n,1}^0(\mathbb{R}_+^n)$, then the equations have a unique global solution.

In [Z3], the author considered the equations in a bounded cylinder under boundary slip conditions. Assuming that the derivatives of the density, velocity, and external force with respect to the third coordinate are sufficiently small in certain norms, the existence of long time regular solutions in Sobolev spaces has been proven, specifically $v \in H^{s+2, s/2+1}(\Omega^t)$ for $s \in (1/2, 1)$.

In [LS], Ladyzhenskaya and Solonnikov obtained existence results for nonhomogeneous Navier–Stokes equations with $v \in W_q^{2,1}$, $\nabla p \in L_q$ (where $q > n$), and $\varrho \in C^1$ for small times with arbitrary v_0 and f , as well as for any given time interval with sufficiently small v_0 and f . This problem was analyzed in a bounded domain in $\mathbb{R}^n$ with boundary $S \in C^2$ and condition $v|_{S\tau} = 0$.

We also refer to some papers concerning nonhomogeneous magnetohydrodynamics equations; for more on Navier–Stokes equations and magnetohydrodynamics, see [Z], [CLX], and [BWY].

1.1. Sketch of proofs of Theorems 1–4

Proof of Theorem 1. The lower and upper bounds for density, as outlined in (1.6), are established in Lemma 2.4. We emphasize that this result can be independently verified, and the density in Theorem 1 is bounded by the data alone. ■

Proof of Theorem 2. Positivity of v_3 is analyzed and proved in Lemma 6.1. ■

Proof of Theorem 3. The global energy estimate presented in Section 3 is formally established in Theorem 3.4. In demonstrating this theorem, we draw upon the fact that the density is constrained by both lower and upper bounds, specifically denoted as ϱ^* and ϱ_* , as outlined in Theorem 1. ■

Proof of Theorem 4. The long time existence of problem (1.1) is proved step by step in time using local existence: by the standard argument we can extend in time local regular solutions as long as the long time estimate holds, under the assumption that $v_3 \geq d_* > 0$. As we can see, the energy estimate does not give enough regularity to accomplish such a goal; thus, we have to study derivatives of the velocity and find appropriate bounds and inequalities in order to establish a long time estimate (1.9). Therefore, we consider $h = v_{x_3}$, which is a solution to problem (2.6), and we derive Corollary 4.3 and inequality (4.19):

$$\begin{aligned} \|h\|_{V(\Omega^t)} \leq & c\phi(\varrho_*, \varrho^*, \mathcal{D}_1, V_1, A_1)[\Lambda_1 + \|\varrho_{x_3}\|_{L_\infty(0,t;L_3(\Omega))}(\|f\|_{L_2(\Omega^t)} \\ & + \|v_t\|_{L_2(\Omega^t)} + \|v\|_{L_\infty(0,t;L_\infty(\Omega))})] + \|h(0)\|_{L_2(\Omega)}, \end{aligned}$$

where $\mathcal{D}_1 \equiv D_1 = \|d_1\|_{L_2(0,t;L_3(S_2))}$, $V_1 = \|v\|_{L_2(0,t;W_3^1(\Omega))}$ are introduced in Lemma 4.2 and the norm V in Definition 2.1. A new difficulty arises: we have some norms of the velocity v to deal with but also ϱ_{x_3} . Fortunately, we can use the continuity equation

$$\varrho_t + v_3 \varrho_{x_3} + v_\alpha \varrho_{x_\alpha} = 0,$$

where $\alpha = 1, 2$ and the summation convention with respect to repeated α is assumed, to conclude

$$\varrho_{x_3} = -\frac{1}{v_3}(\varrho_t + v_\alpha \varrho_{x_\alpha})$$

as long as the condition (1.7): $v_3 \geq d_* > 0$ holds, in accordance with Theorem 2. In Section 5, specifically in Lemma 5.1, we find the following density relation with $x' = (x_1, x_2)$:

$$(\|\varrho_{x'}(t)\|_{L_r(\Omega)}^r + \|\varrho_t(t)\|_{L_r(\Omega)}^r)^{1/r} \leq \phi(v) \cdot \|d_1\|_{L_\infty(S_2^t)} \Lambda_2.$$

where $\phi(v) = \phi(\|v\|_{L_\infty(\Omega^t)}, \|v_x\|_{L_1(0,t;L_\infty(\Omega))}, \|v_t\|_{L_1(0,t;L_\infty(\Omega))})$ and Λ_2 is a small parameter defined in (1.5).

We apply some embeddings for norms of v in $\phi(v)$. Let t_0 be a given positive number and $t \geq t_0$. Then

$$\begin{aligned} \|v\|_{L_\infty(\Omega^t)} &\leq c \sup_t \|v\|_{W_\sigma^{2+s-2/\sigma}(\Omega)} \leq ct^{1/\sigma'} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, \quad \frac{5}{\sigma} < 2+s, \\ \|v_x\|_{L_1(0,t;L_\infty(\Omega))} &\leq t^{1/\sigma'} \left(\int_0^t \|v_x(t')\|_{L_\infty(\Omega)}^\sigma dt' \right)^{1/\sigma} \\ &\leq ct^{1/\sigma'} \left(\int_0^t \|v(t')\|_{W_\sigma^{2+s}(\Omega)}^\sigma dt' \right)^{1/\sigma} \leq ct^{1/\sigma'} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, \quad \frac{3}{\sigma} < 1+s, \\ \|v_t\|_{L_1(0,t;L_\infty(\Omega))} &\leq t^{1/\sigma'} \left(\int_0^t \|v_t(t')\|_{L_\infty(\Omega)}^\sigma dt' \right)^{1/\sigma} \\ &\leq ct^{1/\sigma'} \left(\int_0^t \|v(t')\|_{W_\sigma^s(\Omega)}^\sigma dt' \right)^{1/\sigma} \leq ct^{1/\sigma'} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, \quad \frac{3}{\sigma} < s, \\ \|p_{x_3}\|_{L_1(0,t;L_\infty(\Omega))} &\leq t^{1/\sigma'} \left(\int_0^t \|p_{x_3}(t')\|_{L_\infty(\Omega)}^\sigma dt' \right)^{1/\sigma} \\ &\leq ct^{1/\sigma'} \left(\int_0^t \|p_{x_3}(t')\|_{W_\sigma^s(\Omega)}^\sigma dt' \right)^{1/\sigma} \leq ct^{1/\sigma'} \|p_{x_3}\|_{W_\sigma^{s,s/2}(\Omega^t)}, \quad \frac{3}{\sigma} < s, \end{aligned} \tag{1.19}$$

where $1/\sigma + 1/\sigma' = 1$. Given these facts, we can prove that (see (5.15))

$$\begin{aligned} \|\varrho_{x_3}\|_{L_\infty(0,t;L_r(\Omega^t))} &\leq \frac{1}{d_*} (1 + \|v'\|_{L_\infty(\Omega^t)}) (\|\varrho_{x'}(t)\|_{L_r(\Omega)}^r + \|\varrho_t(t)\|_{L_r(\Omega)}^r)^{1/r} \\ &\leq \frac{1}{d_*} (1 + \|v'\|_{L_\infty(\Omega^t)}) \phi(\|v\|_{W_\sigma^{2+s,1+\frac{s}{2}}(\Omega^t)}) \|d_1\|_{L_\infty(S_2^t(-a))} \Lambda_2, \end{aligned}$$

where $v' = (v_1, v_2)$.

In order to increase the regularity of v' , we consider the third component of the vorticity, $\chi = v_{2,x_1} - v_{1,x_2}$, and the following elliptic rot-div problem relating χ and v' :

$$\begin{aligned} v_{2,x_1} - v_{1,x_2} &= \chi && \text{in } \Omega', \\ v_{1,x_1} + v_{2,x_2} &= -h_3 && \text{in } \Omega', \\ v' \cdot \vec{n}' &= 0 && \text{on } S_1', \end{aligned} \tag{1.20}$$

where x_3 is treated as a parameter, Ω' is the cross-section of Ω with the plane perpendicular to the x_3 -axis and passing through the point $x_3 \in (-a, a)$, and S_1' is the cross-section of S_1 with the same plane. Then S_1' is the boundary of Ω' . In Section 7, we analyze χ

and we come up with the following estimate:

$$\begin{aligned} \|\chi\|_{V(\Omega^t)} &\leq \phi(t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)})(1 + \mathcal{D}_3)\Lambda_2 \\ &\quad + \phi(1/\varrho_*, \varrho^*, A_1, \bar{d})(\|v'\|_{W_2^{1,1/2}(\Omega^t)} + \|v'\|_{L_{\infty}(0,t;W_2^{5/6}(\Omega))}) \\ &\quad + \|h\|_{L_{\infty}(0,t;L_3(\Omega))} + \mathcal{D}_2, \end{aligned} \quad (1.21)$$

where $\bar{d}$ is a bound for $d = (d_1, d_2)$ defined in Theorem 2, $\mathcal{D}_2 = D_2$, $\mathcal{D}_3 = D_3$ are constants defined in (8.15), depending on data: f , χ_0 , initial data for χ , $F = (\text{rot } f)_3$, the r.h.s. in χ problem, and A_1 , the estimate for $\|v\|_{V(\Omega^t)}$ defined in (1.8) with the V norm given by Definition 2.1.

Then, in Section 9, we utilize the rot-div problem (1.20) to address v' terms, concluding with the estimate (9.1) for v' in higher norms (see Definition 2.1):

$$\begin{aligned} \|v'\|_{V^1(\Omega^t)} &\leq \bar{\phi} \cdot [\|v'\|_{L_2(\Omega;H^{1/2}(0,t))} + \|h\|_{L_{\infty}(0,t;L_3(\Omega))} \\ &\quad + \mathcal{D}_2 + (1 + \mathcal{D}_3)\phi(t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)})\Lambda], \end{aligned} \quad (1.22)$$

where $\bar{\phi} \equiv \phi(\frac{1}{\varrho_*}, \varrho^*, A_1, \mathcal{D}_1, \bar{d}, \|v_0\|_{L_{\infty}(\Omega)}, \|f\|_{L_2(\Omega^t)})$.

We emphasize that on the r.h.s. of (1.22) only the terms $\|v'\|_{L_2(\Omega;H^{1/2}(0,t))}$ and $\|h\|_{L_{\infty}(0,t;L_3(\Omega))}$ are possibly multiplied by large parameters, whereas other norms of v and p are multiplied by a small parameter Λ . For the h term, we postpone the discussion (briefly, we estimate this through norms of v at the end). For the v' term, we can employ interpolation:

$$\|v'\|_{L_2(\Omega;H^{1/2}(0,t))} \leq \varepsilon\|v\|_{W_{5/3}^{2,1}(\Omega^t)} + c(1/\varepsilon)A_1$$

and deal with the $W_{5/3}^{2,1}$ norm instead.

To estimate $\|v\|_{W_{5/3}^{2,1}(\Omega^t)}$ and to enhance regularity, we will apply the regularity theory for the Stokes problem

$$\begin{aligned} \varrho v_t - \text{div } \mathbb{T}(v, p) &= -\varrho v' \cdot \nabla v - \varrho v_3 h + \varrho f && \text{in } \Omega^T, \\ \text{div } v &= 0 && \text{in } \Omega^T, \\ v \cdot \bar{n} = 0, \quad \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_{\alpha} + \gamma v \cdot \bar{\tau}_{\alpha} &= 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\ v \cdot \bar{n} = d, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_{\alpha} &= 0, \quad \alpha = 1, 2 && \text{on } S_2^T, \\ v|_{t=0} &= v_0 && \text{in } \Omega, \end{aligned} \quad (1.23)$$

where ϱ is considered as given.

The coefficients we are dealing with depend on the density ϱ and are therefore not constants. Consequently, in the case of a variable ϱ , we must consider Hölder continuity. Suitable estimates for solutions to the Stokes system can be found in Appendix A. To demonstrate the Hölder continuity of ϱ , we require that ϱ lies in $W_{r,\infty}^{1,1}(\Omega^t)$ for $r > 3$ and utilize the embedding

$$\|\varrho\|_{\dot{C}^{\alpha}(\Omega^t)} \leq \|\varrho\|_{C^{\alpha}(\Omega^t)} \leq \|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)},$$

which is valid for $\frac{3}{r} + \alpha < 1$. The estimate for ϱ in the Sobolev space norm $\|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)}$ is established in Section 5.

For a constant density $\varrho = \text{const}$, this problem was addressed in [RZ4]. In that paper, we established estimates and proved the existence of solutions in Besov spaces. It is

evident that we can also demonstrate the existence of solutions in anisotropic Sobolev spaces.

Regarding the norm of v in $W_{5/3}^{2,1}(\Omega^t)$, we show in Lemma 9.2 (see (9.1)) that

$$\begin{aligned} \|v\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5/3}(\Omega^t)} &\leq c(H + \mathcal{D}_4) \\ &\quad + \bar{\phi} \cdot \phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, \|\nabla p\|_{W_{\sigma}^{s,s/2}(\Omega^t)}) \Lambda, \end{aligned}$$

with

$$H = \|h\|_{L_{\infty}(0,t;L_3(\Omega))} + \|h\|_{L_{10/3}(\Omega^t)} + \|h(0)\|_{L_2(\Omega)}.$$

Additionally, $\mathcal{D}_4 = D_2 + D_4$, as defined by formula (9.9), is a constant that depends on the norms of f , d , and v_0 .

We proceed to increase the regularity in Lemmas 9.3–9.5 and derive similar inequalities for the following norms:

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^t)}, \|\nabla p\|_{L_2(\Omega^t)} &\quad \text{in (9.22) with } D_2, D_5, \\ \|v\|_{W_{5/2}^{2,1}(\Omega^t)}, \|\nabla p\|_{L_{5/2}(\Omega^t)} &\quad \text{in (9.26) with } D_7, \\ \|v\|_{W_5^{2,1}(\Omega^t)}, \|\nabla p\|_{L_5(\Omega^t)} &\quad \text{in (9.32) with } D_8, \end{aligned}$$

where D_5 and D_8 are constants depending on norms of f, d, v_0 , and D_7 additionally depends on norms of χ_0 , $F = (\text{rot } f)_3$ and A_1 . Finally, (9.34) in Lemma 9.6 yields

$$\begin{aligned} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)} + \|\nabla p\|_{W_{\sigma}^{s,s/2}(\Omega^t)} \\ \leq c(H, \mathcal{D}_5) + \bar{\phi}\phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, \|p\|_{W_{\sigma}^{s,s/2}(\Omega^t)}) \cdot \Lambda \end{aligned} \quad (1.24)$$

with $\mathcal{D}_5 = D_9$ depending on norms of f, d, v_0 . Since

$$H \leq c\|h\|_{W_{5/3}^{2,1}(\Omega^t)},$$

we get as well the relation with $h = v_{x_3}$, $q = p_{x_3}$ and $\mathcal{D}_6(g, d, h_0) = D_{10}$, $g = f_{x_3}$, given in (10.6) in Lemma 10.2:

$$\begin{aligned} \|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla q\|_{L_{5/3}(\Omega^t)} \\ \leq c\mathcal{D}_6 \bar{\phi}\phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|\nabla p\|_{W_{\sigma}^{s,s/2}(\Omega^t)}), \\ t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}, \|h\|_{W_{5/3}^{2,1}(\Omega^t)}, \|q\|_{L_{5/3}(\Omega^t)}) \cdot \Lambda. \end{aligned} \quad (1.25)$$

By employing the localized problem and a partition of unity, we can also infer

$$\begin{aligned} \|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega))} \\ \leq \phi(t^{\bar{a}} X_1, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}, \|f\|_{L_2(0,t;L_4(\Omega))}) \Lambda + c\Lambda_3(4), \end{aligned} \quad (1.26)$$

with a small parameter $\Lambda_3(4)$ defined in (10.10).

Let

$$\begin{aligned} X(t) &= \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)} + \|\nabla p\|_{W_{\sigma}^{s,s/2}(\Omega^t)} + \|h\|_{W_{5/3}^{2,1}(\Omega^t)} \\ &\quad + \|h\|_{W_4^{2,1}(\Omega^t)} + \|\nabla q\|_{L_{5/3}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega))}. \end{aligned}$$

Consequently, (1.24)–(1.26) imply (refer to Remark 11.1)

$$\begin{aligned} X &\leq \phi(X)t^{\bar{a}}\Lambda + \phi(1/\varrho_*, \varrho^*, \mathcal{D}_7, \bar{\phi}), \\ \mathcal{D}_7 &= \mathcal{D}_5 + \mathcal{D}_6 + \|f_{2,x_1} - f_{1,x_2}\|_{L_2(0,t;L_{6/5}(\Omega))} + A_1, \end{aligned}$$

Moreover, for $t^{\bar{a}}\Lambda$ sufficiently small, as established in Lemma 11.2, we arrive at

$$X \leq \phi(1/\varrho_*, \varrho^*, \mathcal{D}_7, \bar{\phi}).$$

Furthermore, applying (1.9), we can extend local solutions to long time solutions, for arbitrarily large, finite time. This completes the proof. ■

2. Notation and auxiliary results

First we introduce simplified notation.

- We denote by ϕ and ϕ_k , where $k \in \mathbb{N}$, increasing positive functions depending on quantities and norms of data that are not assumed to be small, and ϕ contains some constants denoted by c .
- D_k , $k \in \mathbb{N}$, depends linearly on norms of data that are not assumed to be small.
- The parameters Λ_1 , Λ_2 , and Λ are small and depend on norms of data that are assumed to be small.
- By dot $\cdot$ we represent multiplication of functions.
- The exponent $\bar{a} > 0$ may vary from one formula to another.
- The functions Φ'_k and ϕ'_k are used solely in proofs and serve a temporary, auxiliary purpose.

DEFINITION 2.1. Let Q be either Ω , $S \subset \partial\Omega$, or $\mathbb{R}^3$. For Lebesgue and Sobolev spaces (for general theory see [A]), we set the notation

$$\begin{aligned} \|u\|_{L_p(Q)} &= |u|_{p,Q}, \quad \|u\|_{L_p(Q^t)} = |u|_{p,Q^t}, \\ \|u\|_{L_q(0,t;L_p(Q))} &= |u|_{p,q,Q^t}, \end{aligned}$$

where $p, q \in [1, \infty]$, $Q^t = Q \times (0, t)$.

Let $W_p^s(\Omega)$, where $s \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^3$, denote the Sobolev space characterized by the following finite norm:

$$\|u\|_{W_p^s(\Omega)} = \left(\sum_{|\alpha| \leq s} \int_{\Omega} |D_x^\alpha u|^p dx \right)^{1/p},$$

where D_x^α is defined as $\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha|$ represents the sum $\alpha_1 + \alpha_2 + \alpha_3$, and α_i belongs to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for $i = 1, 2, 3$. The parameter p belongs to $[1, \infty]$.

Furthermore, we identify $H^s(\Omega)$ with $W_2^s(\Omega)$. Consequently, we denote the norms as follows:

$$\|u\|_{H^s(\Omega)} = \|u\|_{s,\Omega}, \quad \|u\|_{W_p^s(\Omega)} = \|u\|_{s,p,\Omega}.$$

In order to define $W_p^s(S)$, it is necessary to have a suitable partition of unity.

We have

$$H^0(\Omega) = L_2(\Omega), \quad W_p^0(\Omega) = L_p(\Omega).$$

We also apply the notation

$$\|u\|_{L_q(0,t;W_p^k(Q))} = \|u\|_{k,p,q,Q^t},$$

$$\|u\|_{L_q(0,t;H^k(Q))} = \|u\|_{k,q,Q^t},$$

and utilize these norms for density:

$$\|u\|_{W_r^{1,1}(Q^t)} = \|u\|_{L_r(Q^t)} + \|u_x\|_{L_r(Q^t)} + \|u_t\|_{L_r(Q^t)},$$

where $r \in [1, \infty]$ and Q is either Ω or S . Additionally, we define

$$\|u\|_{W_{r,s}^{1,1}(Q^t)} = \|u\|_{L_s(0,t;L_r(Q))} + \|u_x\|_{L_s(0,t;L_r(Q))} + \|u_t\|_{L_s(0,t;L_r(Q))},$$

where $r, s \in [1, \infty]$.

Finally, we introduce spaces that are suitable for energy-type estimates for solutions to the Navier–Stokes equations:

$$\|u\|_{V^k(\Omega^t)} = \|u\|_{L_\infty(0,t;H^k(\Omega))} + \|\nabla u\|_{L_2(0,t;H^k(\Omega))},$$

where $V^0(\Omega^t) = V(\Omega^t)$ and

$$\|u\|_{V(\Omega^t)} = \|u\|_{L_\infty(0,t;L_2(\Omega))} + \|\nabla u\|_{L_2(\Omega^t)}.$$

DEFINITION 2.2. The anisotropic Sobolev–Slobodetskii spaces $W_p^{k,l}(\Omega^T)$, $k, l \in \mathbb{R}_+$, $p \in [1, \infty]$, are defined by setting

$$\|u\|_{W_p^{k,l}(\Omega^T)} = \|u\|_{W_p^{k,0}(\Omega^T)} + \|u\|_{W_p^{0,l}(\Omega^T)}$$

and

$$\begin{aligned} \|u\|_{W_p^{k,0}(\Omega^T)} &= \left(\int_0^T \|u(t)\|_{W_p^k(\Omega)} dt \right)^{1/p}, \\ \|u\|_{W_p^{0,l}(\Omega^T)} &= \left(\int_\Omega \|u(x)\|_{W_p^l(0,T)} dx \right)^{1/p}. \end{aligned}$$

Next, we have (the theory of Besov spaces can be found in [BIN, G, N, Tr1])

$$\begin{aligned} \|u\|_{W_p^k(\Omega)} &= \sum_{|\alpha| \leq [k]} \|D_x^\alpha u\|_{L_p(\Omega)} \\ &+ \left(\sum_{|\alpha|=[k]} \iint_{\Omega \times \Omega} \frac{|D_x^\alpha u(x,t) - D_{x'}^\alpha u(x',t)|^p}{|x - x'|^{s+p(k-[k])}} dx dx' \right)^{1/p} \end{aligned}$$

and

$$\begin{aligned} \|u\|_{W_p^l(0,T)} &= \sum_{i \leq [l]} \|\partial_t^i u\|_{L_p(0,T)} \\ &+ \left(\int_0^T \int_0^T \frac{|\partial_t^{[l]} u(x,t) - \partial_{t'}^{[l]} u(x,t')|^p}{|t - t'|^{1+p(l-[l])}} dt dt' \right)^{1/p}, \end{aligned}$$

where $s = \dim \Omega$, $[m]$ is the integer part of m , $D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_s}^{\alpha_s}$, $\alpha = (\alpha_1, \dots, \alpha_s)$ is a multiindex. The following definition yields homogeneous spaces:

$$\|u\|_{\dot{W}_{r,s}^{1,1}(\Omega^t)} = |u_x|_{r,s,\Omega^t} + |u_t|_{r,s,\Omega^t}.$$

By $C^\alpha(\Omega^T)$, $\alpha \in (0, 1)$, we denote the Hölder space with the norm

$$\begin{aligned} \|u\|_{C^\alpha(\Omega^T)} = & \sup_{x, x', t} \frac{|u(x, t) - u(x', t)|}{|x - x'|^\alpha} \\ & + \sup_{x, t, t'} \frac{|u(x, t) - u(x, t')|}{|t - t'|^\alpha} + \|u\|_{\infty, \Omega^T}, \end{aligned}$$

and we consider anisotropic homogeneous Hölder spaces with

$$\|u\|_{\dot{C}^\alpha(\Omega^t)} = \sup_{t, x', x''} \frac{|u(x', t) - u(x'', t)|}{|x' - x''|^\alpha} + \sup_{x, t', t''} \frac{|u(x, t') - u(x, t'')|}{|t' - t''|^\alpha}.$$

Next, we introduce weighted spaces.

Let $\varrho = \min_{i \in \{1, 2\}} \text{dist}\{x, S_2(a_i)\}$. Then the weighted spaces $L_{p, \mu}(\Omega)$ and $V_{p, \mu}^2(\Omega)$ are defined by

$$\|u\|_{L_{p, \mu}(\Omega)} = \left(\int_{\Omega} |u|^p \varrho^{p\mu} dx \right)^{1/p}, \quad p \in (1, \infty), \mu \in \mathbb{R},$$

and

$$\|u\|_{V_{p, \mu}^2(\Omega)} = \left(\int_{\Omega} (|\nabla^2 u|^p + |\nabla u|^p \varrho^{-p} + |u|^p \varrho^{-2p}) \varrho^{p\mu} dx \right)^{1/p}.$$

DEFINITION 2.3. The anisotropic Lebesgue spaces $L_{p, \infty}(\Omega)$ and anisotropic Sobolev spaces $W_{p, \infty}^1(\Omega)$, where $p \in (1, \infty]$, are defined by the following finite norms:

$$\|u\|_{L_{p, \infty}(\Omega)} = \left(\sup_{x_3 \in (-a, a)} \int_{S_2(x_3)} |u(x', x_3)|^p dx' \right)^{1/p}.$$

Here, $S_2(x_3)$ denotes the cross-section of the domain Ω by the plane $x_3 = \text{const}$ for x_3 in $(-a, a)$, and $x' = (x_1, x_2)$ represents the coordinates in the cross-section plane.

Moreover,

$$\|u\|_{W_{p, \infty}^1(\Omega)} = \sup_{x_3 \in (-a, a)} \left(\int_{S_2(x_3)} (|u(x', x_3)|^p + |\nabla u(x', x_3)|^p) dx' \right)^{1/p}.$$

We investigate the problem

$$\begin{aligned} \varrho_t + v \cdot \nabla \varrho &= 0 \quad \text{in } \Omega^T, \\ \varrho|_{t=0} &= \varrho_0, \\ \varrho|_{S_2(-a)} &= \varrho_1 \quad \text{on } S_2, \\ \text{div } v &= 0 \quad \text{in } \Omega^T, \\ v \cdot \bar{n} &= -d_1, \quad d_1 > 0 \quad \text{on } S_2(-a). \end{aligned} \tag{2.1}$$

LEMMA 2.4. Assume that $\varrho_0 \in L_\infty(\Omega)$ and $\varrho_1 \in L_\infty(S_2^t(-a))$. Let ϱ represent a solution to (2.1). Then

$$|\varrho(t)|_{\infty, \Omega} \leq |\varrho_1|_{\infty, S_2^t(-a)} + |\varrho_0|_{\infty, \Omega} \equiv \varrho^*. \tag{2.2}$$

Furthermore, suppose that $1/\varrho_0 \in L_\infty(\Omega)$ and $1/\varrho_1 \in L_\infty(S_2^t(-a))$. Then

$$\varrho_* \equiv \frac{1}{|1/\varrho_0|_{\infty, \Omega} + |1/\varrho_1|_{\infty, S_2^t(-a)}} = \frac{\inf \varrho_1 \cdot \inf \varrho_0}{\inf \varrho_1 + \inf \varrho_0} \leq \varrho. \tag{2.3}$$

Proof. We multiply equation (2.1)₁ by $\varrho|\varrho|^{p-2}$, where $p \in \mathbb{R}_+$, and subsequently integrate over the domain Ω . This yields

$$\frac{d}{dt}|\varrho|_{p,\Omega}^p + \int_{\Omega} v \cdot \nabla |\varrho|^p dx = 0.$$

Considering the properties of the velocity $\mathbf{v}$ as well as the boundary conditions applicable to ϱ , it follows that

$$\frac{d}{dt}|\varrho|_{p,\Omega}^p \leq \int_{S_2(-a)} d_1 |\varrho_1|^p dS_2.$$

Upon integrating this inequality with respect to time, we obtain

$$|\varrho(t)|_{p,\Omega}^p \leq \int_{S_2^t(-a)} d_1 |\varrho_1|^p dS_2 dt' + |\varrho_0|_{p,\Omega}^p.$$

Therefore,

$$\begin{aligned} |\varrho(t)|_{p,\Omega} &\leq \left(\int_{S_2^t(-a)} d_1 |\varrho_1|^p dS_2 dt' \right)^{1/p} + |\varrho_0|_{p,\Omega} \\ &\leq |d_1|_{\infty, S_2^t(-a)}^{1/p} |\varrho_1|_{p, S_2^t(-a)} + |\varrho_0|_{p,\Omega}. \end{aligned}$$

Taking the limit as $p \rightarrow \infty$ leads to the conclusion in (2.2).

Next, we multiply (2.1)₁ by $\varrho|\varrho|^{-p-2}$, where $p \in \mathbb{R}_+$, and integrate over Ω . This results in

$$\frac{d}{dt} \left| \frac{1}{\varrho} \right|_{p,\Omega}^p \leq \int_{S_2(-a)} d_1 \left| \frac{1}{\varrho_1} \right|^p dS_2.$$

Integrating with respect to time provides

$$\left| \frac{1}{\varrho} \right|_{p,\Omega}^p \leq \int_{S_2^t(-a)} d_1 \left| \frac{1}{\varrho_1} \right|^p dS_2 dt' + \left| \frac{1}{\varrho_0} \right|_{p,\Omega}^p.$$

Taking the $1/p$ power yields

$$\begin{aligned} \left| \frac{1}{\varrho} \right|_{p,\Omega} &\leq \left(\int_{S_2^t(-a)} d_1 \left| \frac{1}{\varrho_1} \right|^p dS_2 dt' \right)^{1/p} + \left| \frac{1}{\varrho_0} \right|_{p,\Omega} \\ &\leq |d_1|_{\infty, S_2^t(-a)}^{1/p} \left| \frac{1}{\varrho_1} \right|_{p, S_2^t(-a)} + \left| \frac{1}{\varrho_0} \right|_{p,\Omega}. \end{aligned}$$

Taking the limit as $p \rightarrow \infty$ leads to

$$\left| \frac{1}{\varrho} \right|_{\infty,\Omega} \leq \left| \frac{1}{\varrho_1} \right|_{\infty, S_2^t(-a)} + \left| \frac{1}{\varrho_0} \right|_{\infty,\Omega}.$$

This gives (2.3) and completes the proof. ■

LEMMA 2.5 (The Korn inequality (see [SS])). *Assume that*

$$E_{\Omega}(w) = |\mathbb{D}(w)|_{2,\Omega}^2 < \infty, \quad \operatorname{div} w = 0, \quad w \cdot \bar{n}|_S = 0.$$

Furthermore, assume that the domain Ω is not axially symmetric. Then there exists a constant c , independent of w , such that

$$\|w\|_{H^1(\Omega)}^2 \leq cE_\Omega(w). \quad (2.4)$$

In order to demonstrate the existence of solutions to problem (1.1) with large data, we adopt the methods established in [RZ1], [RZ2], [Z1], and [Z2]. To elucidate these techniques, we introduce the quantities

$$h = v_{x_3}, \quad q = p_{x_3}, \quad g = f_{x_3}, \quad \chi = (\operatorname{rot} v)_3, \quad F = (\operatorname{rot} f)_3. \quad (2.5)$$

LEMMA 2.6. *Let (ϱ, v, p) represent a solution to problem (1.1). Then (ϱ, h, q) is a solution to the problem*

$$\begin{aligned} \varrho h_t - \operatorname{div} \mathbb{T}(h, q) &= -\varrho(v \cdot \nabla h + h \cdot \nabla v - g) - \varrho_{x_3}(v_t + v \cdot \nabla v - f) && \text{in } \Omega^T, \\ \operatorname{div} h &= 0 && \text{in } \Omega^T, \\ h \cdot \bar{n} = 0, \quad \nu \bar{n} \cdot \mathbb{T}(h, q) \cdot \bar{\tau}_\alpha + \gamma h \cdot \bar{\tau}_\alpha &= 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\ h_i &= -d_{1,x_i}, \quad i = 1, 2, \quad h_{3,x_3} = \Delta' d_1 && \text{on } S_2^T(-a), \\ h_i &= -d_{2,x_i}, \quad i = 1, 2, \quad h_{3,x_3} = \Delta' d_2 && \text{on } S_2^T(a), \\ h|_{t=0} &= h(0) && \text{in } \Omega, \end{aligned} \quad (2.6)$$

where $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$, and ϱ is a solution to (2.1).

Proof. (2.6)_{1,2,3,6} directly follows from (1.1)_{1,2,3,4,8} by differentiation with respect to x_3 . To show (2.6)_{4,5}, we recall that

$$\begin{aligned} v_3|_{S_2(-a)} &= d_1, \quad (v_{i,x_3} + v_{3,x_i})|_{S_2(-a)} = 0, \quad i = 1, 2, \\ v_3|_{S_2(a)} &= d_2, \quad (v_{i,x_3} + v_{3,x_i})|_{S_2(a)} = 0, \quad i = 1, 2. \end{aligned}$$

Hence $v_{i,x_3}|_{S_2(a_j)} = -d_{j,x_i}$, $i, j = 1, 2$. Then

$$v_{3,x_3 x_3}|_{S_2(a_j)} = -(v_{1,x_1 x_3} + v_{2,x_2 x_3})|_{S_2(a_j)} = d_{j,x_1 x_1} + d_{j,x_2 x_2} = \Delta' d_j, \quad j = 1, 2.$$

Therefore, (2.6)_{4,5} holds. This ends the proof. ■

To create a problem for χ , we introduce

$$\begin{aligned} \bar{n}|_{S_1} &= \frac{\nabla \varphi_0}{|\nabla \varphi_0|}, \quad \bar{\tau}_1|_{S_1} = \frac{\nabla^\perp \varphi_0}{|\nabla^\perp \varphi_0|}, \quad \bar{\tau}_2|_{S_1} = (0, 0, 1) \equiv \bar{e}_3, \\ \bar{n}|_{S_2(a_j)} &= (-1)^j \bar{e}_3, \quad j = 1, 2, \quad a_1 = -a, \quad a_2 = a, \\ \bar{\tau}_1|_{S_2(a_j)} &= (1, 0, 0) \equiv \bar{e}_1, \quad \bar{\tau}_2|_{S_2(a_j)} = (0, 1, 0) \equiv \bar{e}_2. \end{aligned} \quad (2.7)$$

LEMMA 2.7. *Let ϱ , v and h be specified parameters. Then χ represents the solution to*

the problem

$$\begin{aligned}
& \varrho(\chi_t + v \cdot \nabla \chi) - \nu \Delta \chi = \varrho(F + \chi h_3 - v_{3,x_1} h_2 + v_{3,x_2} h_1) \\
& \quad + \varrho_{x_1}(v_{2,t} + v \cdot \nabla v_2 + f_2) - \varrho_{x_2}(v_{1,t} + v \cdot \nabla v_1 + f_1) \quad \text{in } \Omega^T, \\
& \chi = v_i(n_{i,x_j} \tau_{1j} + \tau_{1i,x_j} n_j) + v \cdot \bar{\tau}_1(\tau_{12,x_1} - \tau_{11,x_2}) \\
& \quad + \frac{\gamma}{\nu} v_j \tau_{1j} \equiv \chi_* \quad \text{on } S_1^T, \\
& \chi_{,x_3} = 0 \quad \text{on } S_2^T, \\
& \chi|_{t=0} = \chi(0) \quad \text{in } \Omega.
\end{aligned} \tag{2.8}$$

In the following, it is understood that the summation convention applies to the indices that are repeated.

Proof. By applying the two-dimensional rotation to the first two equations of (1.1)₁, we derive (2.8)₁. The boundary condition (2.8)₂ is demonstrated in [Z1]. To establish (2.8)₃, we calculate

$$(v_{2,x_1} - v_{1,x_2})_{,x_3} = v_{1,x_1 x_3} - v_{1,x_2 x_3} = d_{,x_2 x_1} - d_{,x_1 x_2} = 0,$$

where (1.1)₉ and $v_3|_{S_2(a_i)} = d|_{S_2(a_i)} = d_i$, $i = 1, 2$. This concludes the proof. ■

Consider the problem

$$\Delta \varphi = f \quad \text{in } \Omega, \quad \bar{n} \cdot \nabla \varphi|_S = 0, \quad \int_{\Omega} \varphi \, dx = 0. \tag{2.9}$$

LEMMA 2.8 (see [RZ1]). Assume that $f \in L_{p,\mu}(\Omega)$ for $p \in (1, \infty)$ and $\mu \in \mathbb{R}_+$. Then $\varphi \in V_{p,\mu}^2(\Omega)$. Furthermore, $\|\nabla^2 \varphi\|_{L_{p,\mu}(\Omega)} \leq c \|f\|_{L_{p,\mu}(\Omega)}$, where c is a constant.

3. Energy estimate for solutions (v, p, ϱ) to problem (1.1)

In this section, we define and analyze weak solutions to derive an energy-type estimate for the solutions to problem (1.1). To achieve this, we need to perform integration by parts, which necessitates that the function v satisfies homogeneous Dirichlet boundary conditions. Consequently, to make (1.1) homogeneous, we introduce the Hopf function

$$\eta(\sigma; \varepsilon, \varkappa) = \begin{cases} 1, & 0 \leq \sigma \leq \varkappa e^{-1/\varepsilon} \equiv r, \\ -\varepsilon \ln \frac{\sigma}{\varkappa}, & r < \sigma \leq \varkappa, \\ 0, & \varkappa < \sigma < \infty. \end{cases} \tag{3.1}$$

We find the derivative

$$\frac{d\eta}{d\sigma} = \eta'(\sigma; \varepsilon, \varkappa) = \begin{cases} 0, & 0 < \sigma \leq r, \\ -\frac{\varepsilon}{\sigma}, & r < \sigma \leq \varkappa, \\ 0, & \varkappa < \sigma < \infty, \end{cases}$$

so that $|\eta'(\sigma; \varepsilon, \varkappa)| \leq \frac{\varepsilon}{\sigma}$. We define local functions η_i in an internal neighborhood of S_2 according to the expression

$$\eta_i = \eta(\sigma_i; \varepsilon, \varkappa), \quad i = 1, 2,$$

where σ_i represents a local coordinate specified in a small neighborhood of the relevant region:

$$\begin{aligned} S_2(a_1, \varkappa) &= \{x \in \Omega: x_3 \in (-a, -a + \varkappa)\}, \\ S_2(a_2, \varkappa) &= \{x \in \Omega: x_3 \in (a - \varkappa, a)\}, \end{aligned} \quad (3.2)$$

$\sigma_1 = -x_3$, $x_3 \in (-a, -a + \varkappa)$, and $\sigma_2 = x_3$, $x_3 \in (a - \varkappa, a)$. Hence, σ_i , $i = 1, 2$, are positive. We extend the functions d_1 and d_2 with the condition that

$$\tilde{d}_i|_{S_2(a_i)} = d_i, \quad i = 1, 2, \quad a_1 = -a, \quad a_2 = a. \quad (3.3)$$

Subsequently, we establish that

$$\alpha = \sum_{i=1}^2 \tilde{d}_i \tilde{\eta}_i, \quad b = \alpha \bar{e}_3, \quad \bar{e}_3 = (0, 0, 1). \quad (3.4)$$

Next, we set

$$u = v - b. \quad (3.5)$$

Then

$$\operatorname{div} u = -\operatorname{div} b = -\alpha_{x_3} \quad \text{in } \Omega, \quad u \cdot \bar{n}|_S = 0.$$

Therefore, the boundary conditions for u are homogeneous, but u is not divergence-free. To address this issue, we define φ as a solution to the Neumann problem for the Poisson equation

$$\begin{aligned} \Delta \varphi &= -\operatorname{div} b \quad \text{in } \Omega, \\ \bar{n} \cdot \nabla \varphi &= 0 \quad \text{on } S, \\ \int_{\Omega} \varphi \, dx &= 0. \end{aligned} \quad (3.6)$$

Additionally, we set

$$w = u - \nabla \varphi = v - (b + \nabla \varphi) \equiv v - \delta. \quad (3.7)$$

Thus, w is divergence-free and $w|_S = 0$.

As a result, for a given density ϱ , a pair (w, p) is a solution to the problem

$$\begin{aligned} &\varrho(w_t + w \cdot \nabla w + w \cdot \nabla \delta + \delta \cdot \nabla w) - \operatorname{div} \mathbb{T}(w, p) \\ &= \varrho f - \varrho(\delta_t + \delta \cdot \nabla \delta) + \nu \operatorname{div} \mathbb{D}(\delta) \equiv \bar{F}(f, \varrho, \delta, t) \quad \text{in } \Omega^T, \\ &\operatorname{div} w = 0 \quad \text{in } \Omega^T, \\ &w \cdot \bar{n} = 0 \quad \text{on } S^T, \\ &\nu \bar{n} \cdot \mathbb{D}(w) \cdot \bar{\tau}_\alpha + \gamma w \cdot \bar{\tau}_\alpha = -\nu \bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha - \gamma \delta \cdot \bar{\tau}_\alpha \\ &\equiv B_{1\alpha}(\delta), \quad \alpha = 1, 2, \quad \text{on } S_1^T, \\ &\bar{n} \cdot \mathbb{D}(w) \cdot \bar{\tau}_\alpha = -\bar{n} \cdot \mathbb{D}(\delta) \cdot \bar{\tau}_\alpha \equiv B_{2\alpha}(\delta), \quad \alpha = 1, 2, \quad \text{on } S_2^T, \\ &w|_{t=0} = v(0) - \delta(0) \equiv w(0) \equiv w_0 \quad \text{in } \Omega, \end{aligned} \quad (3.8)$$

We have used the fact that $\operatorname{div} \delta = 0$. Furthermore, we have

$$\begin{aligned} \bar{n}|_{S_1} &= \frac{(\varphi_{0,x_1}, \varphi_{0,x_2}, 0)}{\sqrt{\varphi_{0,x_1}^2 + \varphi_{0,x_2}^2}}, \quad \bar{\tau}_1|_{S_1} = \frac{(-\varphi_{0,x_2}, \varphi_{0,x_1}, 0)}{\sqrt{\varphi_{0,x_1}^2 + \varphi_{0,x_2}^2}}, \\ \bar{\tau}_2|_{S_1} &= (0, 0, 1) = \bar{e}_3, \quad \bar{n}|_{S_2(-a)} = -\bar{e}_3, \\ \bar{n}|_{S_2(a)} &= \bar{e}_3, \quad \bar{\tau}_1|_{S_2(a_j)} = \bar{e}_1, \quad \bar{\tau}_2|_{S_2(a_j)} = \bar{e}_2, \quad j = 1, 2, \end{aligned} \quad (3.9)$$

where $a_1 = -a$, $a_2 = a$, $\bar{e}_1 = (1, 0, 0)$ and $\bar{e}_2 = (0, 1, 0)$.

Since Dirichlet boundary conditions for w are homogeneous and w is divergence-free, we can define weak solutions to problem (3.8).

DEFINITION 3.1. We call w a *weak solution* to problem (3.8) if for any sufficiently smooth function ψ such that

$$\operatorname{div} \psi|_{\Omega} = 0, \quad \psi \cdot \bar{n}|_S = 0,$$

the following integral identity holds:

$$\begin{aligned} \int_{\Omega^T} \varrho(w_t + w \cdot \nabla w + w \cdot \nabla \delta + \delta \cdot \nabla w) \cdot \psi \, dx \, dt + \nu \int_{\Omega^T} \mathbb{D}(v) \cdot \mathbb{D}(\psi) \, dx \, dt \\ + \gamma \sum_{\alpha=1}^2 \int_{S_1^T} w \cdot \bar{\tau}_\alpha \psi \cdot \bar{\tau}_\alpha \, dS_1 \, dt - \sum_{\alpha, \sigma=1}^2 \int_{S_\sigma^T} B_{\sigma\alpha} \psi \cdot \bar{\tau}_\alpha \, dS_\sigma \, dt = \int_{\Omega^T} \bar{F} \cdot \psi \, dx \, dt. \end{aligned}$$

Exploiting ideas from [RZ1, Ch. 3], we get

LEMMA 3.2. Assume (w, ϱ) is a solution to (3.8), (2.1) and there are constants ϱ_* , ϱ^* , $0 < \varrho_* < \varrho^*$ such that $\varrho_* \leq \varrho \leq \varrho^*$. Let $\varrho_1 \in L_\infty(S_1^t)$, $d_1 \in L_6(0, t; L_3(S_2))$, $\tilde{d} = (\tilde{d}_1, \tilde{d}_2) \in L_\infty(0, t; W_{3,\infty}^1(\Omega)) \cap L_2(0, t; W_{3,\infty}^1(\Omega))$, $\tilde{d}_t \in L_2(0, t; W_{6/5}^1(\Omega))$, $f \in L_2(0, t; L_{6/5}(\Omega))$, $w(0) \in L_2(\Omega)$, $t \leq T$.

Then, for some increasing positive functions ϕ_1 , ϕ and $t < T$, the following a priori inequality holds:

$$\begin{aligned} \varrho_* |w(t)|_{2,\Omega}^2 + \frac{\nu}{4} \|w(t)\|_{1,2,\Omega^t}^2 + \gamma |w \cdot \bar{\tau}_\alpha|_{2,S_1^t}^2 \\ \leq \phi_1(|\varrho_1|_{\infty,S_2^t}, |d_1|_{3,6,S_2^t}, \varrho_*) \{ \phi(\|\tilde{d}\|_{L_\infty(0,t;W_{3,\infty}^1(\Omega))}, \varrho^*) \\ \cdot [\|\tilde{d}\|_{L_2(0,t;W_{3,\infty}^1(\Omega))}^2 + \|\tilde{d}_t\|_{L_2(0,t;W_{6/5}^1(\Omega))}^2 + \|f\|_{L_2(0,t;L_{6/5}(\Omega))}^2] \\ + \varrho^* |w(0)|_{2,\Omega}^2 \}. \end{aligned} \quad (3.10)$$

Proof. We multiply (3.8)₁ by w and integrate over Ω . Then

$$\begin{aligned} \int_{\Omega} \varrho(w_t \cdot w + w \cdot \nabla w \cdot w + w \cdot \nabla \delta \cdot w + \delta \cdot \nabla w \cdot w) \, dx \\ - \int_{\Omega} \operatorname{div} \mathbb{T}(w + \delta, p) \cdot w \, dx = \int_{\Omega} (f - \varrho(\delta_t + \delta \cdot \nabla \delta)) \cdot w \, dx. \end{aligned} \quad (3.11)$$

Taking into account problem (2.1) we obtain

$$\begin{aligned}
\int_{\Omega} \varrho(w_t + w \cdot \nabla w + \delta \cdot \nabla w) \cdot w \, dx &= \int_{\Omega} \varrho(w_t + v \cdot \nabla w) \cdot w \, dx \\
&= \frac{1}{2} \int_{\Omega} (\varrho \partial_t w^2 + \varrho v \cdot \nabla w^2) \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho w^2 \, dx \\
&\quad + \frac{1}{2} \int_{S_2(-a)} \varrho_1 v \cdot \bar{n} w^2 \, dS_2 + \frac{1}{2} \int_{S_2(a)} \varrho v \cdot \bar{n} w^2 \, dS_2 \\
&= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho w^2 \, dx - \frac{1}{2} \int_{S_2(-a)} \varrho_1 d_1 w^2 \, dS_2 + \frac{1}{2} \int_{S_2(a)} \varrho d_2 w^2 \, dS_2. \quad (3.12)
\end{aligned}$$

We have to examine the following term from the first term on the l.h.s. of (3.11):

$$\begin{aligned}
I_1 &= \int_{\Omega} \varrho w \cdot \nabla \delta \cdot w \, dx \\
&= \int_{\Omega} \varrho w \cdot \nabla b \cdot w \, dx + \int_{\Omega} \varrho w \cdot \nabla^2 \varphi \cdot w \, dx \equiv I_3 + I_4.
\end{aligned}$$

In order to estimate I_3 and I_4 , we introduce the sets

$$\begin{aligned}
\tilde{S}_2(a_1, r, \varkappa) &= \{x \in \Omega: x_3 \in (-a + r, -a + \varkappa)\}, \\
\tilde{S}_2(a_2, r, \varkappa) &= \{x \in \Omega: x_3 \in (a - \varkappa, a - r)\}.
\end{aligned}$$

In view of (3.4) and the Hardy inequality, we have

$$\begin{aligned}
|I_3| &= \left| \int_{\Omega} \varrho w \cdot \nabla \sum_{i=1}^2 (\tilde{d}_i \eta_i) w_3 \, dx \right| \leq \varrho^* |w|_{6,\Omega} \left| \nabla \sum_{i=1}^2 \tilde{d}_i \eta_i \right|_{L_{3,\mu}(\Omega)} |w_3|_{L_{2,-\mu}(\Omega)} \\
&\leq c \varrho^* |w|_{6,\Omega} \|w_{3,x_3}\|_{L_{2,1-\mu}(\Omega)} \left[\varepsilon \left(\sum_{i=1}^2 \int_{S_2(a_i, r, \varkappa)} |\tilde{d}_i|^3 \frac{\sigma_i^{3\mu}}{\sigma_i^3} \, dx \right)^{1/3} \right. \\
&\quad \left. + \left(\sum_{i=1}^2 \int_{S_2(a_i, \varkappa)} |\tilde{d}_{i,x_3}|^3 |\sigma_i|^{3\mu} \, dx \right)^{1/3} \right] \equiv I'_3.
\end{aligned}$$

Since Ω is bounded, we have

$$\begin{aligned}
I'_3 &\leq c \varrho^* \|w\|_{1,\Omega}^2 \left[\varepsilon \sum_{i=1}^2 \left(\sup_{x_3} \int_{S_2(a_i, \varkappa)} |\tilde{d}_i|^3 \, dx' \int_r^{\varkappa} \frac{\sigma_i^{3\mu}}{\sigma_i^3} \, d\sigma_i \right)^{1/3} \right. \\
&\quad \left. + \sum_{i=1}^2 \left(\sup_{x_3} \int_{S_2(a_i, \varkappa)} |\tilde{d}_{i,x_3}|^3 \, dx' \int_0^{\varkappa} \sigma_i^{3\mu} \, d\sigma_i \right)^{1/3} \right] \\
&\leq c \varepsilon^* \|w\|_{1,\Omega}^2 \left[\varepsilon \varkappa^{\mu-2/3} \sum_{i=1}^2 \sup_{x_3} |\tilde{d}_i|_{3,S_2(a_i)} + c \varkappa^{\mu+1/3} \sum_{i=1}^2 \sup_{x_3} |\tilde{d}_{i,x_3}|_{3,S_2(a_i)} \right] \\
&\equiv c \varrho^* \|w\|_{1,\Omega}^2 E^2(\varepsilon, \varkappa), \quad (3.13)
\end{aligned}$$

where $\sigma_i = \text{dist} \{S_2(a_i), x\}$, $s \in S_2(a_i, \varkappa)$. Consider I_4 ,

$$|I_4| = \left| \int_{\Omega} \varrho w \cdot \nabla^2 \varphi \cdot w \, dx \right| \leq \varrho^* |w|_{6,\Omega} \|w\|_{L_{2,-\mu}(\Omega)} \|\nabla^2 \varphi\|_{L_{3,\mu}(\Omega)} \equiv I'_4,$$

where φ is a solution to problem (3.6). In view of Lemma 2.5 and the Hardy inequality,

$$I'_4 \leq c \varrho^* \|w\|_{H^1(\Omega)}^2 |\text{div } b|_{3,\mu,\Omega} \equiv I_4^2. \quad (3.14)$$

Employing the definition of b and the properties of η , we have

$$\begin{aligned} \|\text{div } b\|_{L_{3,\mu}(\Omega)} &\leq c\varepsilon \left(\sum_{i=1}^2 \int_{S_2(a_i, r, \varkappa)} |\tilde{d}_i|^3 \frac{\sigma_i^{3\mu}}{\sigma_i^3} \, dx \right)^{1/3} \\ &\quad + \left(\sum_{i=1}^2 \int_{S_2(a_i, \varkappa)} |\tilde{d}_{i,x_3}|^3 |\sigma_i(x)|^{3\mu} \, dx \right)^{1/3} \equiv I_5. \end{aligned}$$

Repeating the considerations in (3.13) implies

$$I_5 \leq cE^2(\varepsilon, \varkappa). \quad (3.15)$$

In view of boundary conditions (3.8)_{4,5}, the second integral on the l.h.s. of (3.11) can be rewritten as follows:

$$\begin{aligned} - \int_{\Omega} \text{div } \mathbb{T}(w + \delta, p) \cdot w \, dx &= - \int_{\Omega} \text{div} [\nu \mathbb{D}(w + \delta) - p \mathbb{I}] \cdot w \, dx \\ &= - \int_{\Omega} \text{div} [\nu \mathbb{D}(w + \delta)] \cdot w \, dx + \int_{\Omega} \nabla p \cdot w \, dx \\ &= \nu \int_{\Omega} D_{ij}(w + \delta) w_{j,x_i} \, dx - \nu \int_{\Omega} \partial_{x_j} [D_{ij}(w + \delta) w_i] \, dx + \int_{\Omega} \text{div}(pw) \, dx \\ &\equiv I. \end{aligned}$$

The first term in I equals

$$\frac{\nu}{2} |D_{ij}(w)|_{2,\Omega}^2 + \nu \int_{\Omega} D_{ij}(\delta) w_{j,x_i} \, dx, \quad (3.16)$$

where the summation over repeated indices is assumed. By the Green theorem, the second term in I takes the form

$$\begin{aligned} - \nu \int_{S_1} n_j D_{ij}(w + \delta) w_i \, dS_1 - \nu \int_{S_2} n_j D_{ij}(w + \delta) w_i \, dS_2 \\ = - \nu \int_{S_1} n_j D_{ij}(w + \delta) (w_{\tau_\alpha} \tau_{\alpha i} + w_n n_i) \, dS_1 \\ - \nu \int_{S_2} n_j D_{ij}(w + \delta) (w_{\tau_\alpha} \tau_{\alpha i} + w_n n_i) \, dS_2 \\ = \gamma \int_{S_1} (|w_{\tau_\alpha}|^2 + w_{\tau_\alpha} \delta \tau_\alpha) \, dS_1, \end{aligned} \quad (3.17)$$

where $w_{\tau_\alpha} = w \cdot \bar{\tau}_\alpha$, $\alpha = 1, 2$, $w_n = w \cdot \bar{n}$, and conditions (3.8)_{4,5} were used.

Applying estimates (3.12) through (3.17) to (3.11), together with the Korn inequality and the condition that

$$\sum_{i=1}^2 \sup_{x_3} |\tilde{d}_i|_{3,S_2(a_i)} + \sum_{i=1}^2 \sup_{x_3} |\tilde{d}_{i,x_3}|_{3,S_2(a_i)} < \infty,$$

for sufficiently small values of ε and $\varkappa$, the following inequality is established:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho w^2 dx + \nu \|w\|_{1,\Omega}^2 + \gamma \sum_{\alpha=1}^2 |w \cdot \bar{\tau}_{\alpha}|_{2,S_1}^2 \\ & \leq \frac{1}{2} \int_{S_2(-a)} \varrho_1 d_1 w^2 dS_2 - \frac{1}{2} \int_{S_2(a)} \varrho d_2 w^2 dS_2 \\ & \quad + c \varrho^* \|w\|_{1,\Omega}^2 \left[\varepsilon \varkappa^{\mu-2/3} \sum_{i=1}^2 \sup_{x_3} |\tilde{d}_i|_{3,S_2(a_i)} \right. \\ & \quad + \varkappa^{\mu+1/3} \sum_{i=1}^2 \sup_{x_3} |\tilde{d}_{i,x_3}|_{3,S_2(a_i)} + c \sum_{\alpha=1}^2 |\delta \cdot \bar{\tau}_{\alpha}|_{2,S_1}^2 + c |\mathbb{D}(\delta)|_{2,\Omega}^2 \\ & \quad \left. + \left| \int_{\Omega} (f - \varrho(\delta_t + \delta \cdot \nabla \delta)) \cdot w dx \right| \right]. \end{aligned} \quad (3.18)$$

According to [RZ1, (3.17)], we have

$$\sum_{\alpha=1}^2 |\delta \cdot \bar{\tau}_{\alpha}|_{2,S_1}^2 \leq c \|\tilde{d}\|_{1,3/2,\Omega}^2 + c \frac{\varepsilon^2}{\varkappa^{2/3}} \exp\left(\frac{2}{3\varepsilon}\right) \sup_{x_3} |\tilde{d}|_{3/2,S_2}^2. \quad (3.19)$$

The following estimate is presented in [RZ1, (3.18)]:

$$|\mathbb{D}(\delta)|_{2,\Omega}^2 \leq c \sum_{i=1}^2 \left[\|\tilde{d}_i\|_{1,2,\Omega}^2 + \frac{\varepsilon^2}{\varkappa} e^{1/\varepsilon} \sup_{x_3} |\tilde{d}_i|_{2,S_2}^2 \right]. \quad (3.20)$$

Estimating the last integral on the right-hand side of (3.18) leads to

$$\begin{aligned} & \int_{\Omega} \varrho(f - \delta_t - \delta \cdot \nabla \delta) \cdot w dx \leq \varepsilon_1 |w|_{6,\Omega}^2 \\ & \quad + c(1/\varepsilon_1)(\varrho^*)^2 [|f|_{6/5,\Omega}^2 + |\delta_t|_{6/5,\Omega}^2] + \left| \int_{\Omega} \varrho \delta \cdot \nabla \delta \cdot w dx \right|. \end{aligned} \quad (3.21)$$

Based on [RZ1, (3.19)], we can derive

$$|\delta_t|_{6/5,\Omega} \leq \|\tilde{d}_t\|_{1,6/5,\Omega} + c \frac{\varepsilon}{\varkappa^{1/6}} e^{1/6\varepsilon} \sup_{x_3} |\tilde{d}_t|_{6/5,S_2}. \quad (3.22)$$

Finally, we examine (see [RZ1, (3.20)])

$$\left| \int_{\Omega} \varrho \delta \cdot \nabla \delta \cdot w dx \right| \leq \varepsilon_2 |w|_{6,\Omega}^2 + c(1/\varepsilon_2)(\varrho^*)^2 \left(\|\tilde{d}\|_{1,2,\Omega}^4 + \frac{\varepsilon^4}{\varkappa^2} e^{2/\varepsilon} \sup_{x_3} |\tilde{d}|_{2,S_2}^4 \right). \quad (3.23)$$

Finally, we review the findings of [RZ1, (3.20)]. By applying (3.19)–(3.23) in (3.18), we

obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \varrho w^2 dx + \nu \|w\|_{1,\Omega}^2 + \gamma \sum_{\alpha=1}^2 |w \cdot \bar{\tau}_{\alpha}|_{2,S_1}^2 \leq \int_{S_2(-a)} \varrho_1 d_1 w^2 dS_2 \\
& - \int_{S_2(a)} \varrho d_2 w^2 dS_2 + c \varrho^* \|w\|_{1,\Omega}^2 (\varepsilon \varkappa^{\mu-2/3} + \varkappa^{\mu+1/3}) \sum_{i=1}^2 \|\tilde{d}_i\|_{W_{3,\infty}^1(\Omega)}^2 \\
& + c \sum_{i=1}^2 \left(\|\tilde{d}_i\|_{1,2,\Omega}^2 + \frac{\varepsilon^2}{\varkappa} e^{1/\varepsilon} \|\tilde{d}_i\|_{2,\infty,\Omega}^2 \right) \\
& + c (\varrho^*)^2 \left[|f|_{6/5,\Omega}^2 + \left(1 + \frac{\varepsilon}{\varkappa^{1/6}} e^{1/6\varepsilon} \right) \sum_{i=1}^2 \|\tilde{d}_{i,t}\|_{1,6/5,\Omega}^2 \right] \\
& + c (\varrho^*)^2 \left(1 + \frac{\varepsilon^4}{\varkappa^2} e^{2/\varepsilon} \right) \sum_{i=1}^2 \|\tilde{d}_i\|_{1,2,\Omega}^4. \tag{3.24}
\end{aligned}$$

In light of the conditions that $\mu > 2/3$ and $\varkappa < 1$, we can derive

$$\varepsilon \varkappa^{\mu-2/3} + \varkappa^{\mu+1/3} \leq \varepsilon + \varkappa.$$

Furthermore, assuming that both ε and $\varkappa$ are sufficiently small, we can assert that

$$c \varrho^* (\varepsilon + \varkappa) \|\tilde{d}\|_{W_{3,\infty}^1(\Omega)} \leq \frac{\nu}{2},$$

where $\tilde{d}$ is defined as $(\tilde{d}_1, \tilde{d}_2)$. This leads to

$$\begin{aligned}
\varepsilon &= \frac{\nu}{4c \varrho^* \|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}}, \\
\varkappa &= \frac{\nu}{4c \varrho^* \|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}}, \tag{3.25}
\end{aligned}$$

We arrive at

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \varrho w^2 dx + \nu \|w\|_{1,\Omega}^2 + \gamma \sum_{\alpha=1}^2 |w \cdot \bar{\tau}_{\alpha}|_{2,S_1}^2 \\
& \leq \int_{S_2(-a)} \varrho_1 d_1 w^2 dS_2 - \int_{S_2(a)} \varrho d_2 w^2 dS_2 \\
& + \phi(\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \cdot (\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}^2 + \|d_t\|_{1,6/5,\Omega}^2 + |f|_{6/5,\Omega}^2), \tag{3.26}
\end{aligned}$$

where ϕ is an increasing, positive function.

To estimate the first term on the right-hand side of (3.26), we have

$$\int_{S_2(-a)} \varrho_1 d_1 w^2 dS_2 \leq |\varrho_1|_{\infty, S_2(-a)} |d_1|_{3, S_2(-a)} |w|_{3, S_2(-a)}^2 \equiv I.$$

By the interpolation

$$|w|_{3, S_2(-a)}^2 \leq \varepsilon^{1/3} |\nabla w|_{2,\Omega}^2 + c \varepsilon^{-5/3} |w|_{2,\Omega}^2$$

we find

$$I \leq \varepsilon |\nabla w|_{2,\Omega}^2 + c(1/\varepsilon) |\varrho_1|_{\infty, S_2(-a)}^6 |d_1|_{3, S_2(-a)}^6 |w|_{2,\Omega}^2.$$

Applying (3.26) and assuming that ε is sufficiently small, we derive

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varrho w^2 dx + \frac{\nu}{2} \|w\|_{1,\Omega}^2 + \gamma \sum_{\alpha=1}^2 |w \cdot \bar{\tau}_{\alpha}|_{2,S_1}^2 \\ \leq c |\varrho_1|_{\infty, S_2(-a)}^6 |d_1|_{3, S_2(-a)}^6 |w|_{2,\Omega}^2 \\ + \phi(\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \cdot (\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)} + \|\tilde{d}_t\|_{1,6/5,\Omega}^2 + |f|_{6/5,\Omega}^2). \end{aligned} \quad (3.27)$$

Fix $T > 0$. Our objective is to derive an energy-type estimate in the time interval $(0, T)$. To achieve this, we note that

$$\int_{\Omega} |w|^2 dx \leq \int_{\Omega} \frac{\varrho}{\varrho_*} |w|^2 dx \leq \frac{1}{\varrho_*} \int_{\Omega} \varrho |w|^2 dx.$$

Applying this to (3.27) results in

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varrho w^2 dx + \frac{\nu}{2} \|w\|_{1,\Omega}^2 + \gamma \sum_{\alpha=1}^2 |w \cdot \bar{\tau}_{\alpha}|_{2,S_1}^2 \\ \leq c |\varrho_1|_{\infty, S_2(-a)}^6 |d_1|_{3, S_2(-a)}^6 \int_{\Omega} \varrho |w|^2 dx \\ + \phi(\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \cdot (\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)} + \|\tilde{d}_t\|_{1,6/5,\Omega}^2 + |f|_{6/5,\Omega}^2). \end{aligned} \quad (3.28)$$

Next, we analyze (3.28) over $(0, T)$. We have

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} \varrho w^2 dx \exp \left[\frac{\nu}{4} t - \frac{c}{\varrho_*} \int_0^t |\varrho_1|_{\infty, S_2(-a)}^6 |d_1|_{3, S_2(-a)}^6 dt' \right] \right\} \\ \leq \phi(\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \cdot (\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)} + \|\tilde{d}_t\|_{1,6/5,\Omega}^2 + |f|_{6/5,\Omega}^2) \\ \cdot \exp \left[\frac{\nu}{4} t - \frac{c}{\varrho_*} \int_0^t |\varrho_1|_{\infty, S_2(-a)}^6 |d_1|_{3, S_2(-a)}^6 dt' \right]. \end{aligned} \quad (3.29)$$

Integrating (3.29) with respect to time gives

$$\begin{aligned} \int_{\Omega} \varrho(t) w(t)^2 dx \leq \exp \left[\int_0^t |\varrho_1|_{\infty, S_2(-a)}^6 |d_1|_{3, S_2(-a)}^6 dt' \right] \\ \cdot \int_0^t \phi(\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \cdot (\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)} + \|\tilde{d}_t\|_{1,6/5,\Omega}^2 + |f|_{6/5,\Omega}^2) dt' \\ + \exp \left[-\frac{\nu}{4} t + \frac{c}{\varrho_*} \int_0^t |\varrho_1|_{\infty, S_2(-a)}^6 |d_1|_{3, S_2(-a)}^6 dt' \right] \int_{\Omega} \varrho(0) w(0)^2 dx \end{aligned} \quad (3.30)$$

for $t \leq T$. Finally, by integrating (3.28) with respect to time from 0 to $t \leq T$ and applying

(3.30), we conclude that

$$\begin{aligned}
& \int_{\Omega} \varrho(t) w^2(t) dx + \frac{\nu}{4} \|w\|_{1,2,\Omega^t}^2 + \gamma \sum_{\alpha=1}^2 |w \cdot \bar{\tau}_{\alpha}|_{2,S_1^t}^2 \\
& \leq \frac{c}{\varrho_*} \int_0^t |\varrho_1|_{\infty,S_2}^6 |d_1|_{3,S_2}^6 dt' \left\{ \sup_{t' \leq t} \exp \left[\frac{c}{\varrho_*} \int_0^{t'} |\varrho_1|_{\infty,S_2}^6 |d_1|_{3,S_2}^6 d\tau \right] \right. \\
& \quad \cdot \int_0^{t'} \phi(\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \cdot (\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}^2 + \|\tilde{d}_t\|_{1,6/5,\Omega}^2 + |f|_{6/5,\Omega}^2) d\tau \\
& \quad + \exp \left[-\frac{\nu}{4} + \frac{c}{\varrho_*} \int_0^{t'} |\varrho_1|_{\infty,S_2(-a)}^6 |d_1|_{3,S_2(-a)}^6 d\tau \right] \int_{\Omega} \varrho(0) w(0)^2 dx \Big\} \\
& \quad + \int_0^t \phi(\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \cdot (\|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}^2 + \|\tilde{d}_t\|_{1,6/5,\Omega}^2 + |f|_{6/5,\Omega}^2) dt' \\
& \quad + \int_{\Omega} \varrho(0) w(0)^2 dx. \tag{3.31}
\end{aligned}$$

By simplifying (3.31), we arrive at

$$\begin{aligned}
& \varrho^* |w(t)|_{2,\Omega}^2 + \frac{\nu}{4} \|w\|_{1,2,\Omega^t}^2 + \gamma \sum_{\alpha=1}^2 |w \cdot \bar{\tau}_{\alpha}|_{2,S_1^t}^2 \\
& \leq \phi_1(|\varrho_1|_{\infty,S_2^t}, |d_1|_{3,6,S_2^t}, \varrho_*) \{ \phi(\sup_t \|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \\
& \quad \cdot [\|\tilde{d}\|_{L_2(0,t;W_{3,\infty}^1(\Omega))}^2 + \|\tilde{d}_t\|_{1,6/5,2,\Omega^t}^2 + |f|_{6/5,2,\Omega^t}^2] + \varrho^* |w(0)|_{2,\Omega}^2 \}, \quad t \leq T. \tag{3.32}
\end{aligned}$$

The above inequality implies (3.10) and completes the proof of Lemma 3.2. ■

In conclusion, we provide an energy estimate for the triplet (v, p, ϱ) .

LEMMA 3.3. *Under the assumptions of Lemma 3.2, any solution (v, p, ϱ) corresponding to the equations in (1.1) satisfies*

$$\begin{aligned}
\|v\|_{V(\Omega^t)}^2 & \leq \phi_1(|\varrho_1|_{\infty,S_2^t}, |d_1|_{3,6,S_2^t}, \varrho_*, \varrho^*) \{ \phi(\sup_t \|\tilde{d}\|_{W_{3,\infty}^1(\Omega)}, \varrho^*) \\
& \quad \cdot [\|\tilde{d}\|_{L_2(0,t;W_{3,\infty}^1(\Omega))}^2 + \|\tilde{d}_t\|_{1,6/5,2,\Omega^t}^2 + |f|_{6/5,2,\Omega^t}^2] + \varrho^* |v(0)|_{2,\Omega}^2 \}, \tag{3.33}
\end{aligned}$$

where $\varrho_* \leq \varrho \leq \varrho^*$ and ϱ_*, ϱ^* are described in Lemma 2.4.

Proof. We have to estimate the norm

$$\|v\|_{V(\Omega^t)}^2 \leq \|w\|_{V(\Omega^t)}^2 + \|\delta\|_{V(\Omega^t)}^2, \tag{3.34}$$

where the first norm on the r.h.s. has already been specified in Lemma 3.2, so

$$\begin{aligned}
\|\delta\|_{V(\Omega^t)}^2 & \leq \|b\|_{V(\Omega^t)}^2 + \|\nabla \varphi\|_{V(\Omega^t)}^2 \\
& \leq |b|_{2,\infty,\Omega^t}^2 + |\nabla b|_{2,\Omega^t}^2 + |\nabla \varphi|_{2,\infty,\Omega^t}^2 + |\nabla^2 \varphi|_{2,\Omega^t}^2. \tag{3.35}
\end{aligned}$$

To estimate norms of functions on the r.h.s. of (3.35), we turn to [RZ1, (3.40)]. Then

$$\begin{aligned} \|\delta\|_{V(\Omega^t)}^2 &\leq c \left(|\tilde{d}|_{2,\infty,\Omega^t}^2 + \|\tilde{d}\|_{1,2,2,\Omega^t}^2 + \|\tilde{d}\|_{L_\infty(0,t;W_{3,\infty}^1(\Omega))}^4 \right. \\ &\quad \cdot \exp \left(c \sup_t \|\tilde{d}\|_{L_\infty(0,t;W_{3,\infty}^1(\Omega))} \right) \cdot |\tilde{d}|_{L_2(0,t;L_{2,\infty}(\Omega))}^2 \\ &\quad \left. + \sup_{t' \leq t} |d(t')|_{2,\infty,\Omega}^2 \right). \end{aligned} \quad (3.36)$$

From (3.34), (3.36), and (3.10), we get (3.33). This completes the proof. ■

In summary, we present the theorem regarding weak solutions.

THEOREM 3.4 (see Lemma 3.3). *Assume that*

- *there exist constants ϱ_* , ϱ^* , $0 < \varrho_* < \varrho^*$ such that $\varrho_* \leq \varrho \leq \varrho^*$,*
- *$\varrho_1 \in L_\infty(S_1^t)$, $d_1 \in L_6(0,t;L_3(S_2(-a)))$,*
- *$\tilde{d}_i$ is an extension of d_i into Ω , $i = 1, 2$, and $\tilde{d} = (\tilde{d}_1, \tilde{d}_2)$, $\tilde{d} \in L_\infty(0,t;W_{3,\infty}^1(\Omega)) \cap L_2(0,t;W_{3,\infty}^1(\Omega))$, $\tilde{d}_t \in L_2(0,t;W_{6/5}^1(\Omega))$, $f \in L_2(0,t;L_{6/5}(\Omega))$, $t \leq T$, and $v(0) \in L_2(\Omega)$.*

For the solutions to (1.1), there exist positive increasing functions ϕ and ϕ_1 that satisfy

$$\begin{aligned} \|v\|_{V(\Omega^t)}^2 &\leq \phi_1(|\varrho_1|_{\infty,S_2^t}, |d_1|_{3,6,S_2^t(-a)}, \varrho_*, \varrho^*) \\ &\quad \cdot \{\phi(\|\tilde{d}\|_{L_\infty(0,t;W_{3,\infty}^1(\Omega))})[\|\tilde{d}\|_{L_2(0,t;W_{3,\infty}^1(\Omega))}^2 \\ &\quad + \|\tilde{d}_t\|_{1,6/5,2,\Omega^t}^2 + |f|_{6/5,2,\Omega^t}^2] + |v(0)|_{2,\Omega}^2\} \equiv A_1^2. \end{aligned} \quad (3.37)$$

REMARK 3.5. It is important to stress that no limitations are imposed on the time T .

4. Energy estimate for solutions to (h, q) problem (2.6)

In this section, we examine the function $h = v_{x_3}$ as well as the equations governing h , specifically those in (2.6). To derive an energy-type estimate for the solutions to problem (2.6), it is necessary to render the Dirichlet boundary conditions (2.6)_{4,5} homogeneous. To achieve this, we construct a function $\tilde{h}$ such that, as in [RZ1, (4.14)],

$$\begin{aligned} \operatorname{div} \tilde{h} &= 0 && \text{in } \Omega, \\ \tilde{h} &= 0 && \text{on } S_1, \\ \tilde{h}_i &= -d_{x_i}, \quad i = 1, 2 && \text{on } S_2, \\ \tilde{h}_3 &= 0 && \text{on } S_2. \end{aligned} \quad (4.1)$$

LEMMA 4.1 ([RZ1, Lemma 4.3]). *Assume that $d = (d_1, d_2)$, $d_{x'} \in W_\sigma^1(S_2)$, $d_{x't} \in L_\sigma(S_2)$, $\sigma \in (1, \infty)$. Then there exists a solution to problem (4.1) such that $\tilde{h} \in W_\sigma^1(\Omega)$, $\tilde{h}_t \in L_\sigma(\Omega)$, and*

$$\begin{aligned} \|\tilde{h}\|_{1,\sigma,\Omega} &\leq c \|d_{x'}\|_{1,\sigma,S_2}, \\ |\tilde{h}_t|_{\sigma,\Omega} &\leq c |d_{x't}|_{\sigma,S_2}, \end{aligned} \quad (4.2)$$

where c does not depend on $\tilde{h}$. Define

$$k = h - \tilde{h}. \quad (4.3)$$

Then k is a solution to the problem

$$\begin{aligned}
\varrho k_t - \operatorname{div} \mathbb{T}(h, q) &= -\varrho(v \cdot \nabla h + h \cdot \nabla v) - \varrho \tilde{h}_t + \varrho g \\
&\quad - \varrho_{x_3}(v_t + v \cdot \nabla v - f) \equiv G && \text{in } \Omega^T, \\
\operatorname{div} k &= 0 && \text{in } \Omega^T, \\
\bar{n} \cdot k &= 0, \quad \nu \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha + \gamma h \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\
k_i &= 0, \quad i = 1, 2, \quad h_{3,x_3} = \Delta' d && \text{on } S_2^T, \\
k|_{t=0} &= h(0) - \tilde{h}(0) \equiv k(0) && \text{in } \Omega,
\end{aligned} \tag{4.4}$$

where $g = f_{x_3}$, $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$, and v is a solution to problem (1.1).

Given the decomposition (4.3), we formulate problem (4.4) as

$$\begin{aligned}
\varrho k_t + \varrho v \cdot \nabla k - \operatorname{div} \mathbb{T}(h, q) &= -\varrho(k \cdot \nabla v + \tilde{h} \cdot \nabla v + v \cdot \nabla \tilde{h}) \\
&\quad - \varrho \tilde{h}_t + \varrho g - \varrho_{x_3}(v_t + v \cdot \nabla v - f) \equiv G && \text{in } \Omega^T, \\
\operatorname{div} k &= 0 && \text{in } \Omega^T, \\
\bar{n} \cdot k &= 0, \quad \nu \bar{n} \cdot \mathbb{D}(h) \cdot \bar{\tau}_\alpha + \gamma h \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\
k_i &= 0, \quad i = 1, 2, \quad h_{3,x_3} = \Delta' d && \text{on } S_2^T, \\
k|_{t=0} &= k(0) && \text{in } \Omega.
\end{aligned} \tag{4.5}$$

Projecting $\operatorname{div} k$ on S_2 yields

$$\operatorname{div} k|_{S_2} = k_{3,x_3}|_{S_2} = 0. \tag{4.6}$$

LEMMA 4.2. Assume that $d_{x'} \in L_2(0, t; W_3^1(S_2))$, $d_t \in L_2(0, t; W_2^1(S_2))$, $f_3 \in L_2(0, t; L_{4/3}(S_2))$, $g \in L_2(\Omega^t)$, $v \in L_2(0, t; W_3^1(\Omega))$, $v_t \in L_2(\Omega^t)$, $v \in L_\infty(0, t; L_\infty(\Omega))$, $\varrho_* \leq \varrho \leq \varrho^*$, where ϱ_* , ϱ^* are positive constants from Lemma 2.4. Assume $\varrho_{x_3} \in L_\infty(0, t; L_3(\Omega))$, $k(0) \in L_2(\Omega)$.

Let $\Lambda_1 = \|d_{x'}\|_{1,3,2,S_2^t} + \|d_{x'}\|_{2,\infty,S_2^t} + \|d_t\|_{1,2,S_2^t} + \|f_3\|_{4/3,2,S_2^t} + \|g\|_{2,\Omega^t}$, $D_1 = \|d_1\|_{3,2,S_2^t}$, $V_1 = \|v\|_{1,3,2,\Omega^t}$ and A_1 is defined in (3.37). Then

$$\begin{aligned}
\|k\|_{V(\Omega^t)} &\leq c\phi(\varrho_*, \varrho^*, D_1, V_1, A_1)[\Lambda_1 + |\varrho_{x_3}|_{3,\infty,\Omega^t}(\|v_t\|_{2,\Omega^t} + \|f\|_{2,\Omega^t} \\
&\quad + \|v\|_{\infty,\infty,\Omega^t}) + |k(0)|_{2,\Omega}].
\end{aligned} \tag{4.7}$$

Proof. Multiply (4.5)₁ by k , utilize problem (2.1), and integrate over Ω to get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho k^2 dx + \int_{\Omega} \operatorname{div}(\varrho v k^2) dx - \int_{\Omega} \operatorname{div} \mathbb{T}(h, q) \cdot k dx \\
&= - \int_{\Omega} \varrho k \cdot \nabla v \cdot k dx - \int_{\Omega} \varrho \tilde{h} \cdot \nabla v \cdot k dx - \int_{\Omega} \varrho v \cdot \nabla \tilde{h} \cdot k dx \\
&\quad - \int_{\Omega} \varrho \tilde{h}_t \cdot k dx + \int_{\Omega} \varrho g \cdot k dx - \int_{\Omega} \varrho_{x_3}(v_t + v \cdot \nabla v - f) \cdot k dx \equiv \int_{\Omega} G k dx.
\end{aligned} \tag{4.8}$$

According to the Green theorem, the second term on the l.h.s. of (4.8) equals

$$- \int_{S_2(-a)} \varrho_1 d_1 k_3^2 dS_2 + \int_{S_2(a)} \varrho_2 d_2 k_3^2 dS_2,$$

where the boundary condition (4.5)₄ is used.

By integrating by parts, the third term on the left-hand side of (4.8) becomes

$$- \int_{S_1} \bar{n} \cdot \mathbb{T}(h, q) \cdot k \, dS_1 - \sum_{i=1}^2 \int_{S_2(a_i)} \bar{n} \cdot \mathbb{T}(h, q) \cdot k \, dS_2 + \frac{\nu}{2} \int_{\Omega} \mathbb{D}(h) \cdot \mathbb{D}(k) \, dx \equiv I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= - \int_{S_1} \bar{n} \cdot \mathbb{T}(h, q) \cdot \bar{\tau}_\alpha k \cdot \bar{\tau}_\alpha \, dS_1 = \gamma \int_{S_1} h \cdot \bar{\tau}_\alpha k \cdot \bar{\tau}_\alpha \, dS_1 \\ &= \gamma |k \cdot \bar{\tau}_\alpha|_{2, S_1}^2 + \gamma \int_{S_1} \tilde{h} \cdot \bar{\tau}_\alpha k \cdot \bar{\tau}_\alpha \, dS_1, \\ I_2 &= - \sum_{i=1}^2 \int_{S_2(a_i)} \mathbb{T}_{33}(h, q) k_3 \, dS_2 = - \sum_{i=1}^2 \int_{S_2(a_i)} (2\nu h_{3, x_3} - q) k_3 \, dS_2 \\ &= - \sum_{i=1}^2 \int_{S_2(a_i)} (2\nu \Delta' d_i - q) k_3 \, dS_2 \equiv I_2^1 + I_2^2. \end{aligned}$$

To examine the last integral, we use the third component of $(1.1)_1$ projected on S_2 . On $S_2(-a)$ we have

$$\varrho_1 d_{1,t} + \varrho_1 v' \cdot \nabla' d_1 + \varrho_1 d_1 h_3 - \varrho_1 f_3 = 2\nu \Delta' d_1 - q,$$

where $v' = (v_1, v_2)$, $\nabla' = (\partial_{x_1}, \partial_{x_2})$.

Hence,

$$I_2^1 = \int_{S_2(-a)} (-\varrho_1 d_{1,t} + \varrho_1 f_3) k_3 \, dS_2 - \int_{S_2(-a)} \varrho_1 v' d_{1,x'} k_3 \, dS_2 - \int_{S_2(-a)} \varrho_1 d_1 k_3^2 \, dS_2,$$

where we utilized that $\tilde{h}_3|_{S_2} = 0$.

We estimate the first term in I_2^1 using

$$\varepsilon_1 |k_3|_{4, S_2}^2 + c(1/\varepsilon_1)(\varrho_1^*)^2 (|d_{1,t}|_{4/3, S_2}^2 + |f_3|_{4/3, S_2}^2),$$

the second with

$$\varepsilon_2 |k_3|_{4, S_2}^2 + c(1/\varepsilon_2)(\varrho_1^*)^2 |v'|_{4, S_2}^2 |d_{1,x'}|_{2, S_2}^2$$

and the last one as follows (see [BIN, Ch. 2, Sect. 10]):

$$\begin{aligned} \int_{S_2(-a)} \varrho_1 d_1 k_3^2 \, dS_2 &\leq \varrho_1^* |d_1|_{3, S_2} |k_3|_{3, S_2}^2 \leq (\varepsilon^{1/3} |\nabla k_3|_{2, \Omega}^2 + c\varepsilon^{-5/3} |k_3|_{2, \Omega}^2) \varrho_1^* |d_1|_{3, S_2} \\ &\leq \varepsilon_3^{1/3} |\nabla k_3|_{2, \Omega}^2 + c\varepsilon_3^{-5/3} (\varrho_1^*)^6 |d_1|_{3, S_2}^6 |k_3|_{2, \Omega}^2. \end{aligned}$$

To evaluate I_2^2 , we compute

$$(2\nu \Delta' d_2 - q)|_{S_2} = \varrho d_{1,t} + \varrho v' \cdot \nabla' d_2 + \varrho d_2 k_3 - \varrho f_3.$$

Subsequently,

$$I_2^2 = \int_{S_2(a)} (-\varrho d_{2,t} + \varrho f_3) k_3 \, dS_2 - \int_{S_2(a)} \varrho v' d_{2,x'} k_3 \, dS_2 - \int_{S_2(a)} \varrho d_2 k_3^2 \, dS_2.$$

To estimate I_2^2 , we invoke Lemma 2.4. As in the estimate of I_2^1 , the first term in I_2^2 is bounded by

$$\varepsilon_4 |k_3|_{4,S_2}^2 + c(1/\varepsilon_4)(\varrho^*)^2(|d_{2,t}|_{4/3,S_2}^2 + |f_3|_{4/3,S_2}^2),$$

the second by

$$\varepsilon_5 |k_3|_{4,S_2}^2 + c(1/\varepsilon_5)(\varrho^*)^2 |v'|_{4,S_2}^2 |d_{2,x'}|_{2,S_2}^2$$

and finally the last one by

$$\varepsilon_6^{1/3} |\nabla k_3|_{2,\Omega}^2 + c\varepsilon_6^{-5/3} (\varrho^*)^6 |d_2|_{3,S_2}^6 |k_3|_{2,\Omega}^2.$$

Employing the above estimates in (4.8) with ε_1 - ε_6 sufficiently small, using the Korn inequality (see Lemma 2.5) and exploiting the notation $d = (d_1, d_2)$, we arrive at

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varrho k^2 dx + \nu \|k\|_{1,\Omega}^2 + \gamma |k \cdot \bar{\tau}_\alpha|_{2,S_1}^2 &\leq \int_{S_2(-a)} \varrho_1 d_1 k_3^2 dS_2 \\ &+ c|\tilde{h} \cdot \bar{\tau}_\alpha|_{2,S_2}^2 + c|\mathbb{D}(\tilde{h})|_{2,\Omega}^2 + c(\varrho^*)^2(|d_t|_{4/3,S_2}^2 + |f_3|_{4/3,S_2}^2 \\ &+ |v'|_{4,S_2}^2 |d_{x'}|_{2,S_2}^2) + c(\varrho^*)^6 |d|_{3,S_2}^6 |k_3|_{2,\Omega}^2 + \left| \int_{\Omega} G \cdot k dx \right|, \end{aligned} \quad (4.9)$$

where we incorporated $\varrho_1^* \leq \varrho^*$ and $d = (d_1, d_2)$.

The first term on the r.h.s. of (4.9) can be estimated by the same bound as the third terms in I_2^1 , I_2^2 . In the context of Lemma 4.1, the second and the third terms on the r.h.s. of (4.9) are bounded by

$$c \|d_{x'}\|_{1,2,S_2}^2.$$

Employing the above estimates in (4.9) yields

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varrho k^2 dx + \nu \|k\|_{1,\Omega}^2 + \gamma |k \cdot \bar{\tau}_\alpha|_{2,S-2}^2 &\leq c(\varrho^*)^6 |d_1|_{3,S_2}^6 |k_3|_{2,\Omega}^2 \\ &+ c \|d_{x'}\|_{1,2,S_2}^2 + c(\varrho^*)^2(|d_t|_{4/3,S_2}^2 + |f_3|_{4/3,S_2}^2 + |v'|_{4,S_2}^2 |d_{x'}|_{2,S_2}^2) \\ &+ \left| \int_{\Omega} G \cdot k dx \right|. \end{aligned} \quad (4.10)$$

Finally, we shall estimate the last term on the r.h.s. of (4.10). To this end we utilize the r.h.s. of (4.8). Specifically, we bound the first term by

$$\varrho^* \int_{\Omega} |\nabla v| k^2 dx \leq \varepsilon_1 |k|_{6,\Omega}^2 + c(1/\varepsilon_1)(\varrho^*)^2 |\nabla v|_{3,\Omega}^2 |k|_{2,\Omega}^2,$$

and the second term by

$$\varepsilon_2 |k|_{6,\Omega}^2 + c(1/\varepsilon_2)(\varrho^*)^2 |\nabla v|_{3,\Omega}^2 |\tilde{h}|_{2,\Omega}^2 \leq \varepsilon_2 |k|_{6,\Omega}^2 + c(1/\varepsilon_2)(\varrho^*)^2 |\nabla v|_{3,\Omega}^2 |d_{x'}|_{2,S_2}^2,$$

where Lemma 4.1 was applied.

Continuing, we estimate the third term by

$$\varepsilon_3 |k|_{6,\Omega}^2 + c(1/\varepsilon_3)(\varrho^*)^2 |v|_{2,\Omega}^2 |\nabla \tilde{h}|_{3,\Omega}^2 \leq \varepsilon_3 |k|_{6,\Omega}^2 + c(1/\varepsilon_3)(\varrho^*)^2 A_1^2 \|d_{x'}\|_{1,3,S_2}^2,$$

where Lemmas 3.3 and 4.1 were applied.

Further, the fourth term is controlled by

$$\varepsilon_4 |k|_{2,\Omega}^2 + c/\varepsilon_4 (\varrho^*)^2 |\tilde{h}_t|_{2,\Omega}^2 \leq \varepsilon_4 |k|_{2,\Omega}^2 + c/\varepsilon_4 (\varrho^*)^2 |d_{x't}|_{2,S_2}^2,$$

and the last but one term by

$$\varepsilon_5 |k|_{2,\Omega}^2 + c/\varepsilon_5 (\varrho^*)^2 |g|_{2,\Omega}^2.$$

Finally, the last term on the r.h.s. of (4.8) is bounded by

$$\varepsilon_6 |k|_{6,\Omega}^2 + c(1/\varepsilon_6) |\varrho_{x_3}|_{3,\Omega}^2 (|v_t|_{2,\Omega}^2 + |v|_{\infty,\Omega}^2 |\nabla v|_{2,\Omega}^2 + |f|_{2,\Omega}^2).$$

By applying the preceding estimates with appropriately small $\varepsilon_1 - \varepsilon_6$ in (4.10), we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varrho k^2 dx + \nu \|k\|_{1,\Omega}^2 + \gamma |k \cdot \bar{\tau}_\alpha|_{2,S_1}^2 &\leq c((\varrho^*)^6 |d_1|_{3,S_2}^6 \\ &+ |\nabla v|_{3,\Omega}^2) |k|_{2,\Omega}^2 + c(\varrho^*)^2 (|\nabla v|_{3,\Omega}^2 + |v'|_{4,S_2}^2) |d_{x'}|_{2,S_2}^2 \\ &+ c(\varrho^*)^2 (1 + A_1^2) [\|d_{x'}\|_{1,3,S_2}^2 + \|d_t\|_{1,2,S_2}^2 + |f_3|_{4/3,S_2}^2 + |g|_{2,\Omega}^2] \\ &+ c|\varrho_{x_3}|_{3,\Omega}^2 (|v_t|_{2,\Omega}^2 + |f|_{2,\Omega}^2) + c|\varrho_{x_3}|_{3,\Omega}^2 |v|_{\infty,\Omega}^2 |\nabla v|_{2,\Omega}^2. \end{aligned} \quad (4.11)$$

Let $\nu_1 = \nu/\varrho^*$. Then (4.11) implies the inequality

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varrho k^2 dx + \nu_1 \int_{\Omega} \varrho k^2 dx - \phi(\varrho_*, \varrho^*) (|d_1|_{3,S_2}^6 + \|v\|_{1,3,\Omega}^2) \int_{\Omega} \varrho k^2 dx \\ \leq c(\varrho^*)^2 \|v\|_{1,3,\Omega}^2 |d_{x'}|_{2,S_2}^2 \\ + c(\varrho^*)^2 (1 + A_1^2) [\|d_{x'}\|_{1,3,S_2}^2 + \|d_t\|_{1,2,S_2}^2 + |f_3|_{4/3,S_2}^2 + |g|_{2,\Omega}^2] \\ + c|\varrho_{x_3}|_{3,\Omega}^2 (|v_t|_{2,\Omega}^2 + |f|_{2,\Omega}^2) + c|\varrho_{x_3}|_{3,\Omega}^2 |v|_{\infty,\Omega}^2 |\nabla v|_{2,\Omega}^2. \end{aligned} \quad (4.12)$$

Define

$$D_1(t) = |d_1|_{3,2,S_2^t}, \quad V_1(t) = \|v\|_{1,3,2,\Omega^t}. \quad (4.13)$$

Consequently, (4.12) can be expressed as

$$\begin{aligned} \frac{d}{dt} \left[\int_{\Omega} \varrho k^2 dx \exp(\nu_1 t - \phi(\varrho_*, \varrho^*) (D_1^6(t) + V_1^2(t))) \right] \\ \leq c \{ (\varrho^*)^2 \|v\|_{1,3,\Omega}^2 |d_{x'}|_{2,S_2}^2 \\ + (\varrho^*)^2 (1 + A_1^2) [\|d_{x'}\|_{1,3,S_2}^2 + \|d_t\|_{1,2,S_2}^2 + |f_3|_{4/3,S_2}^2 + |g|_{2,\Omega}^2] \\ + |\varrho_{x_3}|_{3,\Omega}^2 (|v_t|_{2,\Omega}^2 + |f|_{2,\Omega}^2) + |\varrho_{x_3}|_{3,\Omega}^2 |v|_{\infty,\Omega}^2 |\nabla v|_{2,\Omega}^2 \} \\ \cdot \exp[\nu_1 t - \phi(\varrho_*, \varrho^*) (D_1^6(t) + V_1^2(t))]. \end{aligned} \quad (4.14)$$

By integrating (4.14) with respect to time from 0 to $t \leq T$ and using the notation

$$\Lambda_1 \equiv \Lambda_1(t) = \|d_{x'}\|_{1,3,2,S_2^t} + |d_t|_{2,\infty,S_2^t} + \|d_t\|_{1,2,S_2^t} + |f_3|_{4/3,2,S_2^t} + |g|_{2,\Omega^t} \quad (4.15)$$

we derive

$$\begin{aligned} \int_{\Omega} \varrho k^2 dx &\leq \phi(\varrho_*, \varrho^*) (V_1^2(t) + 1 + A_1^2) \cdot \exp[\phi(\varrho_*, \varrho^*) (D_1^6(t) + V_1^2(t))] \\ &\cdot [\Lambda_1^2(t) + |\varrho_{x_3}|_{3,\infty,\Omega^t}^2 (|v_t|_{2,\Omega^t}^2 + |f|_{2,\Omega^t}^2) + |\varrho_{x_3}|_{3,\infty,\Omega^t}^2 |v|_{\infty,\Omega^t}^2 A_1^2] \\ &+ \exp[-\nu_1 t + \phi(\varrho_*, \varrho^*) (D_1^6(t) + V_1^2(t))] \int_{\Omega} \varrho(0) k^2(0) dx. \end{aligned} \quad (4.16)$$

Further, by integrating (4.11) over time from 0 to $t \leq T$ and applying (4.16) we infer

$$\begin{aligned}
\|k\|_{V(\Omega^t)}^2 &\leq c\phi(\varrho_*, \varrho^*)(D_1^6 + V_1^2)(1 + V_1^2 + A_1^2) \left\{ \exp[\phi(\varrho_*, \varrho^*)(D_1^6 + V_1^2)] \right. \\
&\quad \cdot [\Lambda_1^2 + |\varrho_{x_3}|_{3,\infty,\Omega^t}^2 (|v_t|_{2,\Omega^t}^2 + |f|_{2,\Omega^t}^2 + |v|_{\infty,\Omega^t}^2 A_1^2)] \\
&\quad + \exp[-\nu_1 t + \phi(\varrho_*, \varrho^*)(D_1^6 + V_1^2)] \int_{\Omega} \varrho(0) k^2(0) dx \Big\} \\
&\quad + \phi(\varrho_*, \varrho^*)(1 + V_1^2 + A_1^2) [\Lambda_1^2 + |\varrho_{x_3}|_{3,\infty,\Omega^t}^2 (|v_t|_{2,\Omega^t}^2 + |f|_{2,\Omega^t}^2 + |v|_{\infty,\Omega^t}^2 A_1^2)] \\
&\quad + \int_{\Omega} \varrho(0) k^2(0) dx.
\end{aligned} \tag{4.17}$$

Simplifying (4.17) results in

$$\begin{aligned}
\|k\|_{V(\Omega^t)}^2 &\leq c\phi(\varrho_*, \varrho^*, D_1, V_1, A_1) [\Lambda_1^2 + |\varrho_{x_3}|_{3,\infty,\Omega^t}^2 (|v_t|_{2,\Omega^t}^2 \\
&\quad + |f|_{2,\Omega^t}^2 + |v|_{\infty,\Omega^t}^2) + |k(0)|_{2,\Omega}^2].
\end{aligned} \tag{4.18}$$

The inequality (4.18) leads to (4.7) and completes the proof of Lemma 4.2. ■

COROLLARY 4.3. *Given that*

$$\begin{aligned}
\|\tilde{h}\|_{V(\Omega^t)} &\leq c(|d_{x'}|_{2,\infty,S_2^t} + \|d_{x'}\|_{1,2,S_2^t}), \\
|\tilde{h}(0)|_{2,\Omega} &\leq c|d_{x'}(0)|_{2,S_2} \leq c|d_{x'}|_{2,\infty,S_2^t}
\end{aligned}$$

from (4.7), we derive

$$\begin{aligned}
\|h\|_{V(\Omega^t)} &\leq c\phi(\varrho_*, \varrho^*, D_1, V_1, A_1) [\Lambda_1 + |\varrho_{x_3}|_{3,\infty,\Omega^t} (|v_t|_{2,\Omega^t} \\
&\quad + |f|_{2,\Omega^t} + |v|_{\infty,\Omega^t})] + |h(0)|_{2,\Omega}.
\end{aligned} \tag{4.19}$$

5. A priori estimates for first derivatives of ϱ

Let $x' = (x_1, x_2)$ and $X_r(t) = (|\varrho_{x'}(t)|_{r,\Omega}^r + |\varrho_t(t)|_{r,\Omega}^r)^{1/r}$ for ϱ the solution to (2.1).

LEMMA 5.1. *Assume that $X_r(0) < \infty$, $r \in (1, \infty)$. Suppose that*

$$|\varrho_{1,x'}|_{r,S_2^t(-a)} + |\varrho_{1,t}|_{r,S_2^t(-a)} < \infty,$$

$v \in L_\infty(\Omega^t)$, $v_x, v_t \in L_1(0, t; L_\infty(\Omega))$, $d_1 \in L_\infty(S_2^t(-a))$, $t \leq T$, and there exists a positive constant d_ satisfying $v_3 \geq d_* > 0$. Then there is a positive increasing function ϕ such that the solutions to (2.1) fulfill*

$$\begin{aligned}
X_r(t) &\leq \phi(|v|_{\infty,\Omega^t}, |v_x|_{\infty,1,\Omega^t}, |v_t|_{\infty,1,\Omega^t}) \\
&\quad \cdot [d_1|_{\infty,S_2^t} (|\varrho_{1,x'}|_{r,S_2^t(-a)} + |\varrho_{1,t}|_{r,S_2^t(-a)}) + X_r(0)].
\end{aligned} \tag{5.1}$$

Proof. Differentiate (2.1)₁ with respect to x_α , $\alpha = 1, 2$, multiply by $\varrho_{x_\alpha} |\varrho_{x_\alpha}|^{r-2}$, $r \geq 2$, and integrate over Ω to get

$$\frac{1}{r} \frac{d}{dt} |\varrho_{x_\alpha}|_{r,\Omega}^r + \frac{1}{r} \int_{\Omega} v \cdot \nabla |\varrho_{x_\alpha}|^r dx + \int_{\Omega} v_{x_\alpha} \cdot \nabla \varrho \varrho_{x_\alpha} |\varrho_{x_\alpha}|^{r-2} dx = 0. \tag{5.2}$$

Using that v is divergence-free, the second term in (5.2) equals

$$\frac{1}{r} \int_{\Omega} \operatorname{div}(v |\varrho_{x_\alpha}|^r) dx = -\frac{1}{r} \int_{S_2(-a)} d_1 |\varrho_{1,x_\alpha}|^r dS_2 + \frac{1}{r} \int_{S_2(a)} d_2 |\varrho_{x_\alpha}|^r dS_2.$$

The last integral in (5.2) has the form

$$\sum_{\beta=1}^2 \int_{\Omega} v_{\beta,x_\alpha} \varrho_{x_\beta} \varrho_{x_\alpha} |\varrho_{x_\alpha}|^{r-2} dx + \int_{\Omega} v_{3,x_\alpha} \varrho_{x_3} \varrho_{x_\alpha} |\varrho_{x_\alpha}|^{r-2} dx.$$

In view of the above expressions, we deduce from (5.2) the inequality

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} |\varrho_{x_\alpha}|_{r,\Omega}^r &\leq \frac{1}{r} \int_{S_2(-a)} d_1 |\varrho_{1,x_\alpha}|^r dS_2 + \sum_{\beta=1}^2 \int_{\Omega} |v_{\beta,x_\alpha}| |\varrho_{x_\beta}| |\varrho_{x_\alpha}|^{r-1} dx \\ &\quad + \int_{\Omega} |v_{3,x_\alpha}| |\varrho_{x_3}| |\varrho_{x_\alpha}|^{r-1} dx, \end{aligned} \quad (5.3)$$

where $\alpha = 1, 2$.

Introduce the notation

$$\begin{aligned} x' &= (x_1, x_2), \quad \varrho_{x'} = (\varrho_{x_1}, \varrho_{x_2}), \quad |\varrho_{x'}| = |\varrho_{x_1}| + |\varrho_{x_2}|, \quad v' = (v_1, v_2), \\ v'_{x'} &= (v_{1,x_1}, v_{1,x_2}, v_{2,x_1}, v_{2,x_2}), \quad |v'_{x'}| = |v_{1,x_1}| + |v_{1,x_2}| + |v_{2,x_1}| + |v_{2,x_2}|. \end{aligned}$$

Then (5.3) can be written in the form

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} |\varrho_{x'}|_{r,\Omega}^r &\leq \frac{1}{r} \int_{S_2(-a)} d_1 |\varrho_{1,x'}|^r dS_2 + |v'_{x'}|_{\infty,\Omega} |\varrho_{x'}|_{r,\Omega}^r \\ &\quad + |v_{3,x'}|_{\infty,\Omega} |\varrho_{x_3}|_{r,\Omega} |\varrho_{x'}|_{r,\Omega}^{r-1}. \end{aligned} \quad (5.4)$$

Differentiate (2.1)₁ with respect to t , multiply by $\varrho_t |\varrho_t|^{r-2}$, and integrate over Ω to get

$$\frac{1}{r} \frac{d}{dt} |\varrho_t|_{r,\Omega}^r - \frac{1}{r} \int_{S_2(-a)} d_1 |\varrho_{1,t}|^r dS_2 + \frac{1}{r} \int_{S_2(a)} d_2 |\varrho_t|^r dS_2 + \int_{\Omega} v_t \cdot \nabla \varrho \varrho_t |\varrho_t|^{r-2} dx = 0. \quad (5.5)$$

Simplifying, we write (5.5) in the form

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} |\varrho_t|_{r,\Omega}^r &\leq \frac{1}{r} \int_{S_2(-a)} d_1 |\varrho_{1,t}|^r dS_2 + |v'_t|_{\infty,\Omega} |\varrho_{x'}|_{r,\Omega} |\varrho_t|_{r,\Omega}^{r-1} \\ &\quad + |v_{3,t}|_{\infty,\Omega} |\varrho_{x_3}|_{r,\Omega} |\varrho_t|_{r,\Omega}^{r-1}. \end{aligned} \quad (5.6)$$

Inequalities (5.4) and (5.6) result in

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} (|\varrho_{x'}|_{r,\Omega}^r + |\varrho_t|_{r,\Omega}^r) &\leq \frac{1}{r} \int_{S_2(-a)} d_1 (|\varrho_{1,x'}|^r + |\varrho_{1,t}|^r) dS_2 \\ &\quad + |v'_{x'}|_{\infty,\Omega} |\varrho_{x'}|_{r,\Omega}^r + |v_{3,x'}|_{\infty,\Omega} |\varrho_{x_3}|_{r,\Omega} |\varrho_{x'}|_{r,\Omega}^{r-1} \\ &\quad + |v'_t|_{\infty,\Omega} |\varrho_{x'}|_{r,\Omega} |\varrho_t|_{r,\Omega}^{r-1} + |v_{3,t}|_{\infty,\Omega} |\varrho_{x_3}|_{r,\Omega} |\varrho_t|_{r,\Omega}^{r-1}. \end{aligned}$$

Together with the Hölder and Young inequalities, we get

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} (|\varrho_{x'}|_{r,\Omega}^r + |\varrho_t|_{r,\Omega}^r) &\leq \frac{1}{r} \int_{S_2(-a)} d_1 (|\varrho_{1,x}|^r + |\varrho_{1,t}|^r) dS_2 \\ &\quad + |v_{x'}|_{\infty,\Omega} \left(\frac{2r-1}{r} |\varrho_{x'}|_{r,\Omega}^r + \frac{1}{r} |\varrho_{x_3}|_{r,\Omega}^r \right) \\ &\quad + |v_t|_{\infty,\Omega} \left(\frac{1}{r} |\varrho_{x'}|_{r,\Omega}^r + \frac{1}{r} |\varrho_{x_3}|_{r,\Omega}^r + \frac{2(r-1)}{r} |\varrho_t|_{r,\Omega}^r \right). \end{aligned}$$

Simplifying again yields

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} (|\varrho_{x'}|_{r,\Omega}^r + |\varrho_t|_{r,\Omega}^r) &\leq \frac{1}{r} \int_{S_2(-a)} d_1 (|\varrho_{1,x'}|^r + |\varrho_{1,t}|^r) dS_2 \\ &\quad + 2(|v_{x'}|_{\infty,\Omega} + |v_t|_{\infty,\Omega}) \left(|\varrho_{x'}|_{r,\Omega}^r + \frac{1}{r} |\varrho_{x_3}|_{r,\Omega}^r + \frac{r-1}{r} |\varrho_t|_{r,\Omega}^r \right). \end{aligned} \quad (5.7)$$

We write the continuity equation in the form

$$\varrho_t + v_3 \varrho_{x_3} + v_\alpha \varrho_{x_\alpha} = 0, \quad (5.8)$$

where $\alpha = 1, 2$ and the summation convention with respect to repeated α is assumed. We use the fact that

$$v_3 \geq d_* > 0 \quad (5.9)$$

holds in the whole domain Ω and d_* is a constant. Incorporating (5.9) in (5.8) yields

$$\varrho_{x_3} = -\frac{1}{v_3} (\varrho_t + v_\alpha \varrho_{x_\alpha}). \quad (5.10)$$

Using (5.10) in (5.7) implies that

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} (|\varrho_{x'}|_{r,\Omega}^r + |\varrho_t|_{r,\Omega}^r) &\leq \frac{1}{r} \int_{S_2(-a)} d_1 (|\varrho_{1,x'}|^r + |\varrho_{1,t}|^r) dS_2 \\ &\quad + 2(|v_{x'}|_{\infty,\Omega} + |v_t|_{\infty,\Omega}) \left[|\varrho_{x'}|_{r,\Omega}^r + \frac{1}{rd_*^r} |\varrho_t + v_\alpha \varrho_{x_\alpha}|_{r,\Omega}^r + \frac{r-1}{r} |\varrho_t|_{r,\Omega}^r \right]. \end{aligned} \quad (5.11)$$

Hence, (5.11) takes the form

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} (|\varrho_{x'}|_{r,\Omega}^r + |\varrho_t|_{r,\Omega}^r) &\leq \frac{1}{r} \int_{S_2(-a)} d_1 (|\varrho_{1,x'}|^r + |\varrho_{1,t}|^r) dS_2 \\ &\quad + 2(|v_{x'}|_{\infty,\Omega} + |v_t|_{\infty,\Omega}) \left[\left(1 + \frac{2^r}{rd_*^r} |v'|_{\infty,\Omega}^r \right) |\varrho_{x'}|_{r,\Omega}^r \right. \\ &\quad \left. + \left(\frac{2^r}{rd_*^r} + \frac{r-1}{r} \right) |\varrho_t|_{r,\Omega}^r \right]. \end{aligned} \quad (5.12)$$

For any finite r , (5.12) implies

$$\frac{d}{dt} X_r^r \leq \int_{S_2(-a)} d_1 (|\varrho_{1,x}|^r + |\varrho_{1,t}|^r) dS_2 + c(|v_{x'}|_{\infty,\Omega} + |v_t|_{\infty,\Omega}) (1 + |v'|_{\infty,\Omega}^r) X_r^r. \quad (5.13)$$

Integrating with respect to time yields

$$\begin{aligned} X_r^r(t) \leq & \exp \left[c \int_0^t (|v_x|_{\infty, \Omega} + |v_t|_{\infty, \Omega})(1 + |v'|_{\infty, \Omega}^r) dt' \right] \\ & \cdot \left[\int_0^t dt' \int_{S_2(-a)} d_1 (|\varrho_{1,x'}|^r + |\varrho_{1,t}|^r) dS_2 + X_r^r(0) \right]. \end{aligned} \quad (5.14)$$

The above inequality gives (5.1) and concludes the proof. ■

REMARK 5.2. From (5.10) we obtain

$$|\varrho_{x_3}|_{r, \infty, \Omega^t} \leq \frac{1}{d_*} (1 + |v'|_{\infty, \Omega^t}) X_r(t). \quad (5.15)$$

COROLLARY 5.3. *The inequalities (5.1) and (5.15) imply*

$$\begin{aligned} |\varrho_x(t)|_{r, \Omega} + |\varrho_t(t)|_{r, \Omega} \leq & \phi(1/d_*, |v|_{\infty, \Omega^t}, |v_x|_{\infty, 1, \Omega^t}, |v_t|_{\infty, 1, \Omega^t}) \\ & \cdot [|\varrho_{1,x'}|_{r, S_2^t(-a)} + |\varrho_{1,t}|_{r, S_2^t(-a)} + |\varrho_{x'}(0)|_{r, \Omega} + |\varrho_t(0)|_{r, \Omega}]. \end{aligned} \quad (5.16)$$

6. A lower bound for v_3

To demonstrate Lemma 5.1, it is necessary to establish that there exists a positive constant d_* such that

$$v_3(x, t) \geq d_*. \quad (6.1)$$

We can anticipate that (6.1) is valid, as shown by the conditions

$$v_3|_{x_3=-a} = d_1 > 0, \quad v_3|_{x_3=a} = d_2 > 0.$$

In this section, we will prove (6.1). From (1.1), we get the following problem for v_3 :

$$\begin{aligned} \varrho v_{3,t} + \varrho v \cdot \nabla v_3 - \nu \Delta v_3 + q &= \varrho f_3, \\ v_3|_{x_3=a_\alpha} &= d_\alpha, \quad \alpha = 1, 2, \\ v_3|_{t=0} &= v_3(0), \end{aligned} \quad (6.2)$$

where $q = p_{x_3}$.

To determine (6.1) for the solutions of (6.2), it is essential to observe that ϱ constitutes a solution to the problem

$$\begin{aligned} \varrho_t + v \cdot \nabla \varrho &= 0 \quad \text{in } \Omega^T, \\ \varrho &= \varrho_1 \quad \text{on } S_2^T, \\ \varrho|_{t=0} &= \varrho_0. \end{aligned} \quad (6.3)$$

LEMMA 6.1. *Assume that $\bar{d}_0 \geq v_3(0) \geq d_0$, where $d_0, \bar{d}_0$ are positive constants. Furthermore, let $d_i \geq d_\infty$, $i = 1, 2$, where d_∞ is also a positive constant. Assume that $p_{x_3}, f_3 \in L_1(0, t; L_\infty(\Omega))$. Then the quantity*

$$d_* = \phi \left(\exp \left[-\frac{1}{\varrho_*} (|p_{x_3}|_{\infty, 1, \Omega^t} + |f_3|_{\infty, 1, \Omega^t}) \right], \frac{d_\infty d_0}{3\bar{d}_0 + d_\infty} \right) > 0$$

satisfies

$$v_3 \geq d_*. \quad (6.4)$$

Proof. Multiply (6.2)₁ by $\frac{v_3}{|v_3|^{s+1}}$ and integrate over Ω . Formally, we should use the function

$$v_3(\varepsilon) = \begin{cases} v_3, & v_3 > \varepsilon, \\ \varepsilon, & v_3 \leq \varepsilon, \end{cases}$$

and let $\varepsilon \rightarrow 0$ in the final estimate. For simplicity, we will omit the notation $v_3(\varepsilon)$. Then, we consider

$$\begin{aligned} \int_{\Omega} \varrho v_{3,t} \frac{v_3}{|v_3|^{s+1}} dx + \int_{\Omega} \varrho v \cdot \nabla v_3 \frac{v_3}{|v_3|^{s+1}} dx \\ - \nu \int_{\Omega} \Delta v_3 \frac{v_3}{|v_3|^{s+1}} dx + \int_{\Omega} q \frac{v_3}{|v_3|^{s+1}} dx = \varrho \int_{\Omega} f_3 \frac{v_3}{|v_3|^{s+1}} dx. \end{aligned} \quad (6.5)$$

Subsequently, we analyze the specific terms in (6.5),

$$\begin{aligned} v_{3,t} \frac{v_3}{|v_3|^{s+1}} &= \frac{1}{2} \frac{\partial_t |v_3|^2}{|v_3|^{s+1}} = \partial_t |v_3| \frac{1}{|v_3|^s} = \frac{1}{-s+1} \partial_t |v_3|^{-s+1}, \\ \nabla v_3 \frac{v_3}{|v_3|^{s+1}} &= \frac{1}{-s+1} \nabla |v_3|^{-s+1}. \end{aligned}$$

Therefore, the sum of the first two terms in the l.h.s. of (6.5) becomes

$$I_1 = \frac{1}{-s+1} \int_{\Omega} (\varrho \partial_t |v_3|^{-s+1} + \varrho v \cdot \nabla |v_3|^{-s+1}) dx.$$

By applying the continuity equation

$$\varrho_t + \operatorname{div}(v\varrho) = 0$$

in I_1 , we can derive the expression

$$I_1 = \frac{1}{-s+1} \frac{d}{dt} \int_{\Omega} \varrho |v_3|^{-s+1} dx + \frac{1}{-s+1} \int_{\Omega} \operatorname{div}(\varrho v |v_3|^{-s+1}) dx.$$

Given that $v \cdot \bar{n}|_{S_1} = 0$, the second term in I_1 becomes

$$\begin{aligned} \frac{1}{-s+1} \int_{S_2(-a)} \varrho v \cdot \bar{n} |v_3|^{-s+1} dS_2 + \frac{1}{-s+1} \int_{S_2(a)} \varrho v \cdot \bar{n} |v_3|^{-s+1} dS_2 \\ = -\frac{1}{-s+1} \int_{S_2(-a)} \varrho_1 d_1 |v_3|^{-s+1} dS_2 + \frac{1}{-s+1} \int_{S_2(a)} \varrho_2 d_2 |v_3|^{-s+1} dS_2, \end{aligned}$$

where $i = 1, 2$. The third term on the l.h.s. of (6.5) gives

$$\begin{aligned} -\nu \int_{\Omega} \operatorname{div}(\nabla v_3 v_3 |v_3|^{-s-1}) dx + \nu \int_{\Omega} |\nabla v_3|^2 |v_3|^{-s-1} dx + \nu \int_{\Omega} v_3 \nabla v_3 \nabla |v_3|^{-s-1} dx \\ \equiv I_2 + I_3 + I_4. \end{aligned}$$

In view of the assumptions,

$$\begin{aligned}
I_2 &= -\nu \int_{\Omega} \operatorname{div}(\nabla v_3 v_3^{-s}) dx = -\frac{\nu}{-s+1} \int_{\Omega} \operatorname{div}(\nabla v_3^{-s+1}) dx \\
&= -\frac{\nu}{-s+1} \int_S \bar{n} \cdot \nabla v_3^{-s+1} dS = -\nu \int_S v_3^{-s} \bar{n} \cdot \nabla v_3 dS \\
&= \nu \int_{S_2(-a)} d_1^{-s} v_{3,x_3} dS_2 - \nu \int_{S_2(a)} d_2^{-s} v_{3,x_3} dS_2 - \nu \int_{S_1} v_3^{-s} \bar{n} \cdot \nabla v_3 dS_1 \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}$$

Since v is divergence free, it follows that

$$\begin{aligned}
J_1 &= -\nu \int_{S_2(-a)} d_1^{-s} v_{\alpha,x_\alpha} dS_2 \\
&= -\nu \int_{S_2(-a)} (d_1^{-s} v_\alpha)_{,x_\alpha} dS_2 - \nu s \int_{S_2(-a)} d_1^{-s-1} d_{1,x_\alpha} v_\alpha dS_2 \\
&= -\nu \int_{\partial S_2(-a)} d_1^{-s} v_\alpha \cdot \bar{n} v_\alpha|_{S_1} dL_1 - \nu s \int_{S_2(-a)} d_1^{-s-1} d_{1,x_\alpha} v_\alpha dS_2,
\end{aligned}$$

The first integral vanishes because $L_1 \subset S_1$ and $v \cdot \bar{n}|_{S_1} = 0$.

In a similar manner,

$$J_2 = \nu s \int_{S_2(a)} d_2^{-s-1} d_{2,x_\alpha} v_\alpha dS_2.$$

To analyze J_3 , we recall that condition (1.1)₅ for $\bar{\tau}_\alpha = \bar{e}_3$ on S_1 is expressed as

$$\nu v_{3,n} + \gamma v_3 = 0.$$

Hence,

$$J_3 = \gamma \int_{S_1} v_3^{-s+1} dS_1.$$

Summarizing,

$$I_2 = -\nu s \int_{S_2(-a)} d_1^{-s-1} d_{1,x_\alpha} v_\alpha dS_2 + \nu s \int_{S_2(a)} d_2^{-s-1} d_{2,x_\alpha} v_\alpha dS_2 + \gamma \int_{S_1} v_3^{-s+1} dS_1.$$

Next, we proceed to calculate

$$I_3 = \nu \int_{\Omega} |\nabla v_3|^2 v_3^{-s-1} dx = \nu \int_{\Omega} |v_3^{-s/2-1/2} \nabla v_3|^2 dx = \frac{4\nu}{(-s+1)^2} \int_{\Omega} |\nabla v_3^{-s/2+1/2}|^2 dx.$$

Additionally, we find

$$I_4 = -(s+1)\nu \int_{\Omega} |\nabla v_3|^2 v_3^{-s-1} dx = -\frac{4\nu(s+1)}{(-s+1)^2} \int_{\Omega} |\nabla v_3^{-s/2+1/2}|^2 dx.$$

Therefore,

$$I_3 + I_4 = -\frac{4\nu s}{(-s+1)^2} \int_{\Omega} |\nabla v_3^{-s/2+1/2}|^2 dx.$$

Applying the results in (6.5) leads to

$$\begin{aligned}
& \frac{1}{-s+1} \frac{d}{dt} \int_{\Omega} \varrho v_3^{-s+1} dx - \frac{1}{-s+1} \int_{S_2(-a)} \varrho_1 d_1^{-s+2} dS_2 \\
& + \frac{1}{-s+1} \int_{S_2(a)} \varrho d_2^{-s+2} dS_2 - \nu s \int_{S_2(-a)} d_1^{-s-1} d_{1,x_\alpha} v_\alpha dS_2 \\
& + \nu s \int_{S_2(a)} d_2^{-s-1} d_{2,x_\alpha} v_\alpha dS_2 - \frac{4\nu s}{(-s+1)^2} \int_{\Omega} |\nabla v_3^{-s/2+1/2}|^2 dx \\
& + \int_{\Omega} q v_3^{-s} dx = \varrho \int_{\Omega} f_3 v_3^{-s} dx. \tag{6.6}
\end{aligned}$$

In view of the assumptions of this lemma, the last term on the l.h.s. of (6.6) is bounded by

$$|q|_{\infty, \Omega} \frac{1}{\varrho_* d_*} \int_{\Omega} \varrho v_3^{-s+1} dx \tag{6.7}$$

and the r.h.s. term by

$$|f_3|_{\infty, \Omega} \frac{1}{\varrho_* d_*} \int_{\Omega} \varrho v_3^{-s+1} dx, \tag{6.8}$$

where $d_* = \min_{\Omega} v_3$. The existence of this quantity has not been proved yet; it will be established at the end of this proof. We introduce the notation

$$X^s = \int_{\Omega} \varrho v_3^{-s+1} dx, \tag{6.9}$$

then we multiply (6.6) by $-s+1$ and exploit estimates (6.7) and (6.8) to arrive at

$$\begin{aligned}
\frac{d}{dt} X^s - \frac{4\nu s}{-s+1} \int_{\Omega} |\nabla v_3^{-s/2+1/2}|^2 dx & \leq (|q|_{\infty, \Omega} + |f_3|_{\infty, \Omega})(s-1) \frac{1}{\varrho_* d_*} X^s + \int_{S_2(-a)} \varrho_1 d_1^{-s+2} dS_2 \\
& - \int_{S_2(a)} \varrho d_2^{-s+2} dS_2 - \nu s(s-1) \int_{S_2(-a)} d_1^{-s-1} d_{1,x_\alpha} v_\alpha dS_2 \\
& + \nu s(s-1) \int_{S_2(a)} d_2^{-s-1} d_{2,x_\alpha} v_\alpha dS_2. \tag{6.10}
\end{aligned}$$

Let

$$\alpha(t) = (|q(t)|_{\infty, \Omega} + |f_3(t)|_{\infty, \Omega}) \frac{s-1}{d_* \varrho_*}.$$

Then (6.10) implies

$$\begin{aligned}
& \frac{d}{dt} \left(X^s \exp \left(- \int_0^t \alpha(t') dt' \right) \right) \\
& \leq \left[\int_{S_2(-a)} \varrho_1 d_1^{-s+2} dS_2 + \nu s(s-1) \left(\int_{S_2(-a)} d_1^{-s-1} |d_{1,x'}| |v'| dS_2 \right. \right. \\
& \quad \left. \left. + \int_{S_2(a)} d_2^{-s-1} |d_{2,x'}| |v'| dS_2 \right) \right] \exp \left(- \int_0^t \alpha(t') dt' \right), \tag{6.11}
\end{aligned}$$

where $d_{x'} = (d_{x_1}, d_{x_2})$, $v' = (v_1, v_2)$.

Integrating (6.11) with respect to time yields

$$\begin{aligned}
X^s \leq & \exp\left(\int_0^t \alpha(t') dt'\right) \int_0^t \left[\int_{S_2(-a)} \varrho_1 d_1^{-s+2} dS_2 \right. \\
& + \nu s(s-1) \left(\int_{S_2(-a)} d_1^{-s-1} |d_{1,x'}| |v'| dS_2 + \int_{S_2(a)} d_2^{-s-1} |d_{2,s'}| |v'| dS_2 \right) \Big] \\
& \cdot \exp\left(-\int_0^{t'} \alpha(t'') dt''\right) dt' + \exp\left(\int_0^t \alpha(t') dt'\right) X^s(0).
\end{aligned} \tag{6.12}$$

Accordingly,

$$\begin{aligned}
X \leq & \exp\left(\frac{1}{s} \int_0^t \alpha(t') dt'\right) \frac{1}{d_\infty} \left[\left(\int_{S_2^t(-a)} \varrho_1 d_1^2 dS_2 dt' \right)^{1/s} \right. \\
& + (\nu s(s-1))^{1/s} \left(\left(\int_{S_2^t(-a)} d_1^{-1} |d_{1,x'}| |v'| dS_2 dt' \right)^{1/s} \right. \\
& \left. \left. + \left(\int_{S_2^t(a)} d_2^{-1} |d_{2,x'}| |v'| dS_2 dt' \right)^{1/s} \right) \right] \\
& + \exp\left(\frac{1}{s} \int_0^t \alpha(t') dt'\right) X(0).
\end{aligned} \tag{6.13}$$

Letting $s \rightarrow \infty$ results in

$$\sup_{x \in \Omega} \left| \frac{1}{v_3} \right| \leq \exp \left[\frac{1}{d_* \varrho_*} (|q|_{\infty,1,\Omega^t} + |f_3|_{\infty,1,\Omega^t}) \right] \left(\frac{3}{d_\infty} + \left| \frac{1}{v_3(0)} \right|_{\infty,\Omega} \right).$$

Therefore, in view of the properties of the local solution, we obtain

$$\begin{aligned}
v_3 & \geq \inf_{x \in \Omega} v_3(x) \\
& \geq \exp \left(- \left[\frac{1}{d_* \varrho_*} (|q|_{\infty,1,\Omega^t} + |f_3|_{\infty,1,\Omega^t}) \right] \right) \frac{d_\infty \inf_{x \in \Omega} |v_3(0)|}{3 \inf_{x \in \Omega} |v_3(0)| + d_\infty} \\
& \geq \exp \left(- \left[\frac{1}{d_* \varrho_*} (|q|_{\infty,1,\Omega^t} + |f_3|_{\infty,1,\Omega^t}) \right] \right) \frac{d_\infty d_0}{3d_0 + d_\infty} \\
& \equiv d_*.
\end{aligned} \tag{6.14}$$

We start with the notation

$$\begin{aligned}
a(t) &= \frac{1}{\varrho_*} (|q|_{\infty,1,\Omega^t} + |f_3|_{\infty,1,\Omega^t}), \\
b &= \frac{d_0 d_\infty}{3d_0 + d_\infty}.
\end{aligned}$$

It is clear that $a(t)$ and b are positive, and $a(t)$ can be estimated by

$$a(t) \leq a'(t) = \phi(t^{\bar{a}})(\Lambda + \Lambda_3(4));$$

see Remark 10.4. Moreover, $a(t)$ is small for small $\phi(t^{\bar{a}})(\Lambda + \Lambda_3(4))$. Inequality (6.14)

implies the following equation for d_* ,

$$\exp\left(-\frac{1}{d_*}a(t)\right)b = d_*. \quad (6.15)$$

Therefore

$$\exp(-a(t)) = \left(\frac{d_*}{b}\right)^{d_*}. \quad (6.16)$$

We now prove the existence of d_* . Let us define the function

$$h(x) = \left(\frac{x}{b}\right)^x,$$

equal to 1 for $x = 0$ and $x = b$.

In the interval $(0, b)$, the function $h(x) < 1$ and attains a minimum at $x = be^{-b} = b_*$. Moreover, $\frac{dh}{dx} < 0$ for $x \in (0, b_*)$ and $\frac{dh}{dx} > 0$ for $x \in (b_*, b)$.

At the minimum

$$h(be^{-b}) = e^{-b^2e^{-b}} \equiv h_*.$$

Given that (6.16) holds, we have the restriction

$$e^{-b^2e^{-b}} \leq e^{-a(t)}.$$

Hence

$$e^b a(t) \leq b^2. \quad (6.17)$$

Therefore, (6.17) indicates that $t^{\bar{a}}(\Lambda + \Lambda_3(4))$ needs to be sufficiently small. We have to emphasize that considerations in this section have been made for a local solution that can be extended in time if the long time estimate holds.

Assuming $h_* < e^{-a(t)} < 1$, there exists a point $x_* \in (0, b)$ such that

$$\left(\frac{x_*}{b}\right)^{x_*} = e^{-a(t)}.$$

Hence $x_* = d_*$, which is a solution to (6.15).

Moreover,

$$d_* = \phi(e^{-a(t)}, b) = \phi\left(\exp\left[-\frac{|q|_{\infty,1,\Omega^t} + |f_3|_{\infty,1,\Omega^t}}{\varrho_*}\right], \frac{d_0 d_\infty}{3d_0 + d_\infty}\right).$$

Thus, we need to find estimates for $\|q\|_{L_1(0,t;L_\infty(\Omega))} = \|p_{x_3}\|_{L_1(0,t;L_\infty(\Omega))}$ to conclude the proof, and we accomplish this through the long time estimate (1.9), established in Section 11. ■

7. Estimates for χ , the third component of the vorticity

In order to obtain the energy-type estimate, we analyze system (2.8) for the function χ , which represents the third component of the vorticity. To achieve this, it is essential to

derive an energy estimate for the problem with homogenous Dirichlet boundary conditions on S_1 . Consequently, we define $\tilde{\chi}$ as a solution to the problem

$$\begin{aligned}\tilde{\chi}_t - \nu \Delta \tilde{\chi} &= 0 && \text{in } \Omega^T, \\ \tilde{\chi} &= \chi_* && \text{on } S_1^T, \\ \tilde{\chi}_{,x_3} &= 0 && \text{on } S_2^T, \\ \tilde{\chi}|_{t=0} &= \chi(0) && \text{in } \Omega,\end{aligned}\tag{7.1}$$

where $\chi(0) = v_2(0)_{,x_1} - v_1(0)_{,x_2}$ and χ_* is described by (2.8)₂.

To justify the existence of solutions to (7.1) and obtain appropriate estimates, we must fulfill the following compatibility conditions:

$$\chi(0)|_{S_1} = \chi_*|_{t=0}\tag{7.2}$$

and

$$\chi_{*,x_3} = 0 \quad \text{on } \bar{S}_1 \cap \bar{S}_2.\tag{7.3}$$

Calculating (7.2) explicitly, we have

$$\begin{aligned}v_{2,x_1}(0) - v_{1,x_2}(0) \\ = v_i(0)(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) + v(0) \cdot \bar{\tau}_1(\tau_{12,x_1} - \tau_{11,x_2}) + \frac{\gamma}{\nu}v_j(0)\tau_{1j}.\end{aligned}\tag{7.4}$$

To satisfy (7.3), we differentiate χ_* with respect to x_3 . This is possible because S_1 is the part of the boundary of Ω that is parallel to the x_3 -axis. Moreover, the vectors $\bar{n}|_{S_1}$ and $\bar{\tau}_1|_{S_1}$ do not depend on x_3 . Therefore, we need to differentiate only the components of the velocity. In χ_* , only two components of the velocity, v_1 and v_2 , appear. Differentiating them with respect to x_3 , projecting on S_2 , and employing (2.6)_{4,5}, we obtain the compatibility condition (7.3) in the form

$$\begin{aligned}\chi_{*,x_3}|_{\bar{S}_1 \cap \bar{S}_2} \\ = - \sum_{i,j=1}^2 \left[d_{x_i}(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) + \frac{\gamma}{\nu}d_{x_j}\tau_{1j} + d_{x_i}\tau_{1i}(\tau_{12,x_1} - \tau_{11,x_2}) \right] = 0.\end{aligned}\tag{7.5}$$

Then, we can introduce the new function $\chi' = \chi - \tilde{\chi}$, satisfying the following system:

$$\begin{aligned}\varrho(\chi'_t + v \cdot \nabla \chi') - \nu \Delta \chi' &= \varrho F + \varrho h_3 \chi - \varrho v \cdot \nabla \tilde{\chi} \\ &\quad - \varrho(h_2 v_{3,x_1} - h_1 v_{3,x_2}) + \varrho_{x_1}(v_{2,t} + v \cdot \nabla v_2 + f_2) \\ &\quad - \varrho_{x_2}(v_{1,t} + v \cdot \nabla v_1 + f_1) && \text{in } \Omega^T, \\ \chi' &= 0 && \text{on } S_1^T, \\ \chi'_{,x_3} &= 0 && \text{on } S_2^T, \\ \chi'|_{t=0} &= 0 && \text{in } \Omega.\end{aligned}\tag{7.6}$$

First, we describe the properties of solutions to problem (7.1).

LEMMA 7.1 ([RZ1, Lemma 4.7]). *For the solutions to problem (7.1), we get*

$$\begin{aligned} |\tilde{\chi}|_{3,\infty,\Omega^t} &\leq c|\chi_*|_{3,\infty,S_1^t} + |\chi(0)|_{3,\Omega} \\ &\leq c\|v'\|_{5/6,2,\infty,\Omega^t} + |\chi(0)|_{3,\Omega}, \end{aligned} \quad (7.7)$$

$$\begin{aligned} \|\tilde{\chi}\|_{1,2,\Omega^t} &\leq c(\|\chi_*\|_{W_2^{1/2,1/4}(S_1^t)} + |\chi(0)|_{2,\Omega}) \\ &\leq c(\|v'\|_{W_2^{1/2,1/4}(S_1^t)} + |\chi(0)|_{2,\Omega}) \\ &\leq c(\|v'\|_{W_2^{1,1/2}(\Omega^t)} + |\chi(0)|_{2,\Omega}). \end{aligned} \quad (7.8)$$

Proof. As (7.8) follows from potential theory, we only prove (7.7).

We write $\tilde{\chi} = \chi_1 + \chi_2$, where

$$\chi_{1,t} - \nu\Delta\chi_1 = 0, \quad \chi_1|_{S_1} = \chi_*, \quad \chi_{1,x_3}|_{S_2} = 0, \quad \chi_1|_{t=0} = 0 \quad (7.9)$$

and

$$\chi_{2,t} - \nu\Delta\chi_2 = 0, \quad \chi_2|_{S_1} = 0, \quad \chi_{2,x_3}|_{S_2} = 0, \quad \chi_1|_{t=0} = \chi(0). \quad (7.10)$$

For solutions to (7.9), we deduce (7.7) and (7.8) for $\chi(0) = 0$ (see [RZ1, Lemma 4.7]).

Multiplying (7.10) by χ_2 and integrating over Ω^t yields

$$|\chi_2|_{2,\infty,\Omega^t}^2 + \nu|\nabla\chi_2|_{2,\Omega}^2 = |\chi_2(0)|_{2,\Omega}^2.$$

Multiplying (7.10) by χ_2^2 and integrating over Ω^t gives

$$|\chi_2|_{3,\infty,\Omega^t}^3 + \nu|\nabla|\chi_2|^{3/2}|_{2,\Omega}^2 = |\chi_2(0)|_{3,\Omega}^3.$$

Assuming that $\tilde{\chi} = \chi_1 + \chi_2$, we derive (7.7) and (7.8). This concludes the proof. ■

Next, we take into account problem (2.8):

LEMMA 7.2. *Assume that there exists a local solution to problem (1.1) described by Theorem 1.1. Suppose that there exist positive constants ϱ_* , ϱ^* , and D^* such that $\varrho_* \leq \varrho \leq \varrho^*$, $d_i \leq \bar{d}$, $i = 1, 2$. Assume that*

$$\begin{aligned} v' &\in W_2^{1,1/2}(\Omega^t) \cap L_\infty(0, t; W_2^{5/6}(\Omega)), \quad h \in L_\infty(0, t; L_3(\Omega)), \\ F &= (\text{rot } f)_3 \in L_2(0, t; L_{6/5}(\Omega)), \quad \chi(0) \in L_3(\Omega), \\ \nabla\varrho &\in L_\infty(0, t; L_r(\Omega)), \quad (f_1, f_2) \in L_2(0, t; L_{6r/(5r-6)}(\Omega)), \quad r > 3. \end{aligned}$$

In addition, let $v'_t \in L_2(0, t; L_{6r/(5r-6)}(\Omega))$, $v \in L_\infty(0, t; L_{3r/(r-3)}(\Omega))$, and $t \leq T$. Then the solutions of (2.8) fulfill

$$\begin{aligned} |\chi|_{2,\infty,\Omega^t}^2 + \|\chi\|_{1,2,\Omega^t}^2 &\leq \phi(1/\varrho_*, \varrho^*, \bar{d}, A_1)[\|v'\|_{W_2^{1,1/2}(\Omega^t)}^2 \\ &\quad + \|v'\|_{5/6,2,\infty,\Omega^t}^2 + A_1^2 + |h|_{3,\infty,\Omega^t}^2 + |F|_{6/5,2,\Omega^t}^2 + |\chi(0)|_{3,\Omega}^2 \\ &\quad + |\nabla\varrho|_{r,\infty,\Omega^t}^2(\|v'\|_{5/6,2,\infty,\Omega^t}^2 + \|v'\|_{W_2^{1,1/2}(\Omega^t)}^2 + |\chi(0)|_{3,\Omega}^2) \\ &\quad + |\varrho_x|_{r,\infty,\Omega^t}^2(|v'_t|_{6r/(5r-6),2,\Omega^t}^2 + |v|_{3r/(r-3),\infty,\Omega^t}^2 \\ &\quad + |(f_1, f_2)|_{6r/(5r-6),2,\Omega^t}^2)]. \end{aligned} \quad (7.11)$$

Proof. We will first tackle problem (7.6). Multiply (7.6)₁ by χ' , then integrate over Ω , applying the boundary conditions and problem (2.1), to get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho \chi'^2 dx + \frac{1}{2} \int_{\Omega} \operatorname{div}(\varrho v \chi'^2) dx + \nu |\nabla \chi'|_{2,\Omega}^2 \\
&= \int_{\Omega} \varrho F \chi' dx + \int_{\Omega} \varrho h_3 \chi \chi' dx - \int_{\Omega} \varrho v \cdot \nabla \tilde{\chi} \chi' dx \\
&\quad - \int_{\Omega} \varrho (h_2 v_{3,x_1} - h_1 v_{3,x_2}) \chi' dx \\
&\quad + \int_{\Omega} [\varrho_{x_1} (v_{2,t} + v \cdot \nabla v_2 + f_2) - \varrho_{x_2} (v_{1,t} + v \cdot \nabla v_1 + f_1)] \chi' dx. \tag{7.12}
\end{aligned}$$

We can bound the first term on the r.h.s. of (7.12) by

$$\varepsilon_1 |\chi'|_{6,\Omega}^2 + c(1/\varepsilon_1)(\varrho^*)^2 |F|_{6/5,\Omega}^2,$$

and the second by

$$\varepsilon_2 |\chi'|_{6,\Omega}^2 + c(1/\varepsilon_2)(\varrho^*)^2 |h_3|_{3,\Omega}^2 |\chi|_{2,\Omega}^2.$$

Integrating by parts in the third term on the r.h.s. of (7.12) results in

$$- \int_{\Omega} \varrho v \cdot \nabla (\tilde{\chi} \chi') dx + \int_{\Omega} \varrho v \cdot \nabla \chi' \tilde{\chi} dx \equiv I_1 + I_2,$$

where

$$|I_2| \leq \varepsilon |\nabla \chi'|_{2,\Omega}^2 + c(1/\varepsilon)(\varrho^*)^2 |v|_{6,\Omega}^2 |\tilde{\chi}|_{3,\Omega}^2.$$

Integrating by parts in I_1 leads to

$$I_1 = - \int_{\Omega} \operatorname{div}(\varrho v \tilde{\chi} \chi') dx + \int_{\Omega} \varrho \cdot \nabla v \tilde{\chi} \chi' dx \equiv I_1^1 + I_1^2,$$

where

$$|I_1^2| \leq \varepsilon |\chi'|_{6,\Omega}^2 + c(1/\varepsilon) |\nabla \varrho|_{r,\Omega}^2 |v|_{6,\Omega}^2 |\tilde{\chi}|_{3r/(2r-3),\Omega}^2.$$

Next, according to the Green theorem, we obtain

$$I_1^1 = \int_{S_2(-a)} \varrho_1 d_1 \tilde{\chi} \chi' dS_2 - \int_{S_2(a)} \varrho d_2 \tilde{\chi} \chi' dS_2.$$

Hence,

$$|I_1^1| \leq 2\varrho^* \bar{d} \int_{S_2} |\tilde{\chi} \chi'| dS_2 \leq \varepsilon |\chi'|_{4,S_2}^2 + c(1/\varepsilon, \varrho^*, \bar{d}) |\tilde{\chi}|_{4/3,S_2}^2.$$

Summarizing, the third term on the r.h.s. of (7.12) is bounded by

$$\varepsilon_3 \|\chi'\|_{1,\Omega}^2 + c(1/\varepsilon_3, \varrho^*, \bar{d}) [|v|_{6,\Omega}^2 |\tilde{\chi}|_{3,\Omega}^2 + |\nabla \varrho|_{r,\Omega}^2 (|v|_{6,\Omega}^2 |\tilde{\chi}|_{3r/(2r-3),\Omega}^2 + |\tilde{\chi}|_{4/3,S_2}^2)].$$

We proceed to evaluate the fourth term on the r.h.s. of (7.12) by

$$\varepsilon_4 |\chi'|_{6,\Omega}^2 + c(1/\varepsilon_4, \varrho^*) |h|_{3,\Omega}^2 |v_{3,x'}|_{2,\Omega}^2.$$

Finally, the last term on the r.h.s. of (7.12) is bounded by

$$\begin{aligned} |\varrho_{x'}|_{\lambda_1, \Omega} (|v'_t|_{\lambda_2, \Omega} + |v|_{\lambda_2 \mu_1, \Omega} |\nabla v'|_{\lambda_2 \mu_2, \Omega} + |(f_1, f_2)|_{\lambda_2, \Omega}) |\chi'|_{6, \Omega} \\ \leq \varepsilon_5 |\chi'|_{6, \Omega}^2 + c(1/\varepsilon_5) |\varrho_{x'}|_{\lambda_1, \Omega}^2 (|v'_t|_{\lambda_2, \Omega}^2 + |v|_{\lambda_2 \mu_1, \Omega}^2 |\nabla v'|_{\lambda_2 \mu_2, \Omega}^2 + |(f_1, f_2)|_{\lambda_2, \Omega}^2), \end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 + 1/6 = 1$, $1/\mu_1 + 1/\mu_2 = 1$.

Let $\lambda_1 = r$. Then $\lambda_2 = \frac{6r}{5r-6}$. Additionally, it is required that $\lambda_2 \mu_2 = 2$, which leads to $\frac{6r}{5r-6} \mu_2 = 2$ and $\mu_2 = \frac{5r-6}{3r}$. Then $\mu_1 = \frac{5r-6}{2r-6}$ and $\lambda_2 \mu_1 = \frac{3r}{r-3}$.

The middle term on the l.h.s. of (7.12) equals

$$\begin{aligned} \int_{\Omega} \operatorname{div}(\varrho v \chi'^2) dx &= \int_{S_2(-a)} \varrho_1 v \cdot \bar{n} \chi'^2 dS_2 + \int_{S_2(a)} \varrho v \cdot \bar{n} \chi'^2 dS_2 \\ &= - \int_{S_2(-a)} \varrho_1 d_1 \chi'^2 dS_2 + \int_{S_2(a)} \varrho d_2 \chi'^2 dS_2. \end{aligned}$$

By applying the estimations mentioned above in (7.12) and assuming that $\varepsilon_1 - \varepsilon_5$ remain sufficiently small, we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \varrho \chi'^2 dx + \nu \|\chi'\|_{1, \Omega}^2 &\leq \int_{S_2(-a)} \varrho_1 d_1 \chi'^2 dS_2 \\ &\quad + c(\varrho^*)^2 [|F|_{6/5, \Omega}^2 + |h|_{3, \Omega}^2 (|\chi|_{2, \Omega}^2 + |v_{3, x'}|_{3, \Omega}^2)] \\ &\quad + \phi(\varrho^*, \bar{d}) [|v|_{6, \Omega}^2 |\tilde{\chi}|_{3, \Omega}^2 + |\nabla \varrho|_{r, \Omega}^2 (|v|_{6, \Omega}^2 |\tilde{\chi}|_{3r/(2r-3), \Omega}^2 + |\tilde{\chi}|_{4/3, 2, S_2})] \\ &\quad + c|\varrho_{x'}|_{r, \Omega}^2 [|v'_t|_{6r/(5r-6), \Omega}^2 + |v|_{3r/(r-3), \Omega} |\nabla v'|_{2, \Omega}^2 + |(f_1, f_2)|_{6r/(5r-6), \Omega}^2], \end{aligned} \quad (7.13)$$

where $r > 3$.

Integrating (7.13) with respect to time and applying Theorem 3.4 gives

$$\begin{aligned} \int_{\Omega} \varrho \chi^2 dx + \nu \|\chi\|_{1, 2, \Omega^t}^2 &\leq \phi(\varrho^*, \bar{d}, A_1) [|\chi'|_{2, S_2^t}^2 \\ &\quad + |\tilde{\chi}|_{2, \infty, \Omega^t}^2 + \|\tilde{\chi}\|_{1, 2, \Omega^t}^2 + |h|_{3, \infty, \Omega^t}^2 + |F|_{6/5, 2, \Omega^t}^2 \\ &\quad + |\tilde{\chi}|_{3, \infty, \Omega^t}^2 + |\nabla \varrho|_{r, \infty, \Omega^t}^2 (|\tilde{\chi}|_{3r/(2r-3), \infty, \Omega^t}^2 + |\tilde{\chi}|_{4/3, 2, S_2^t}^2) \\ &\quad + c|\varrho_{x'}|_{r, \infty, \Omega^t}^2 (|v'_t|_{6r/(5r-6), 2, \Omega^t}^2 + |v|_{3r/(r-3), \infty, \Omega^t}^2 \\ &\quad + |(f_1, f_2)|_{6r/(5r-6), 2, \Omega^t}^2)], \end{aligned} \quad (7.14)$$

where $r > 3$ and $A_1 \geq 1$.

By using the equation $\chi' = \chi - \tilde{\chi}$, we obtain

$$\begin{aligned} |\chi'|_{2, S_2^t}^2 &= |\chi - \tilde{\chi}|_{2, S_2^t}^2 \leq |\chi|_{2, S_2^t}^2 + |\tilde{\chi}|_{2, S_2^t}^2 \\ &\leq \varepsilon |\nabla \chi|_{2, \Omega^t}^2 + c(1/\varepsilon) A_1^2 + \|\tilde{\chi}\|_{1, 2, \Omega^t}^2. \end{aligned} \quad (7.15)$$

Additionally, Lemma 7.1 implies

$$\begin{aligned} |\tilde{\chi}|_{2, S_2^t}^2 + |\tilde{\chi}|_{4/3, 2, S_2^t}^2 &\leq c(\|v'\|_{W_2^{1, 1/2}(\Omega^t)} + |\chi(0)|_{2, \Omega}^2), \\ |\tilde{\chi}|_{2, \infty, \Omega^t}^2 + |\tilde{\chi}|_{3, \infty, \Omega^t}^2 + |\tilde{\chi}|_{3r/(2r-3), \infty, \Omega^t}^2 \\ &\leq c|\tilde{\chi}|_{3, \infty, \Omega^t}^2 \leq c(\|v'\|_{5/6, 2, \infty, \Omega^t}^2 + |\chi(0)|_{3, \Omega}^2). \end{aligned} \quad (7.16)$$

Utilizing (7.15) and (7.16) on the r.h.s. of (7.14) provides

$$\begin{aligned}
|\chi|_{2,\infty,\Omega^t}^2 + \|\chi\|_{1,2,\Omega^t}^2 &\leq \phi(1/\varrho_*, \varrho^*, \bar{d}, A_1) \\
&\cdot [\|v'\|_{W_2^{1,1/2}(\Omega^t)}^2 + \|v'\|_{5/6,2,\infty,\Omega^t}^2 + A_1^2 + |h|_{3,\infty,\Omega^t}^2 \\
&+ |F|_{6/5,2,\Omega^t}^2 + |\chi(0)|_{3,\Omega}^2 \\
&+ |\nabla \varrho|_{r,\infty,\Omega^t}^2 (\|v'\|_{5/6,2,\infty,\Omega^t}^2 + \|v'\|_{W_2^{1,1/2}(\Omega^t)}^2 + |\chi(0)|_{3,\Omega}^2) \\
&+ |\varrho_{x'}|_{r,\infty,\Omega^t}^2 (|v'_t|_{6r/(5r-6),2,\Omega^t}^2 + |v|_{3r/(r-3),\infty,\Omega^t}^2 + |(f_1, f_2)|_{6r/(5r-6),2,\Omega^t}^2)], \quad (7.17)
\end{aligned}$$

given that $r > 3$. This leads to (7.11) and completes the proof. ■

8. Auxiliary results

In this section, we establish several relationships involving the norms of h , χ , and ϱ that are useful for considerations of Sections 9 and 11. To begin, we recall the positive constants ϱ_* , ϱ^* , d_* , and $\bar{d}$, defined as follows:

$$\varrho_* \leq \varrho \leq \varrho^*, \quad d_* \leq d_i \leq \bar{d}, \quad i = 1, 2. \quad (8.1)$$

Let

$$D_1(t) = |d_1|_{3,2,S_2^t}. \quad (8.2)$$

The following quantity is defined in (4.15):

$$\Lambda_1 = \|d_{x'}\|_{1,3,2,S_2^t} + |d_{x'}|_{2,\infty,S_2^t} + \|d_t\|_{1,2,S_2^t} + |f_3|_{4/3,2,S_2^t} + |g|_{2,\Omega^t}. \quad (8.3)$$

In (1.5), we have introduced another quantity assumed to be small:

$$\Lambda_2 = |\varrho_{1,x'}|_{r,S_2^t} + |\varrho_{1,t}|_{r,S_2^t} + |\varrho_{0,x}|_{r,\Omega}. \quad (8.4)$$

The inequality (5.14) can be expressed as follows:

$$|\varrho_{x'}|_{r,\infty,\Omega^t} + |\varrho_t|_{r,\infty,\Omega^t} \leq \Phi_1 \cdot \phi_1 \cdot \Lambda_2, \quad (8.5)$$

where

$$\begin{aligned}
\Phi_1 &= (1 + |d_1|_{\infty,S_2^t(a_1)})(1 + |v_0|_{\infty,\Omega}), \\
\phi_1 &= \exp \left[\frac{c}{r} (|v_x|_{\infty,1,\Omega^t} + |v_t|_{\infty,1,\Omega^t})(1 + |v'|_{\infty,\Omega^t}) \right]. \quad (8.6)
\end{aligned}$$

From (5.15) it can be inferred that

$$\begin{aligned}
|\varrho_{x_3}|_{r,\infty,\Omega^t} &\leq \frac{1}{d_*} (1 + |v'|_{\infty,\Omega^t}) \cdot \Phi_1 \cdot \phi_1 \cdot \Lambda_2 \\
&\equiv \Phi_1 \cdot \phi_2 \cdot \Lambda_2, \quad (8.7)
\end{aligned}$$

with

$$\phi_2 = \frac{1}{d_*} (1 + |v'|_{\infty,\Omega^t}) \phi_1.$$

LEMMA 8.1. *Suppose that $v \in W_\sigma^{2+s,1+s/2}(\Omega^t)$, $s \in (0,1)$ and $\sigma > 3/s$, $t \leq T$. Then there exist increasing positive functions*

$$\begin{aligned}\Phi_2 &= \Phi_2(\varrho_*, \varrho^*, A_1, D_1(t), |d_1|_{\infty, S_2^t(a_1)}, |v_0|_{\infty, \Omega}, |f|_{2, \Omega^t}), \\ \phi_3 &= \phi_3(t^{\bar{a}} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}),\end{aligned}$$

where $\bar{a} > 0$, such that

$$\|h\|_{V(\Omega^t)} \leq \Phi_2 \cdot \phi_3 \cdot (\Lambda_1 + \Lambda_2 + |h(0)|_{2,\Omega}) \quad (8.8)$$

and

$$|\varrho_x|_{r,\infty,\Omega^t} + |\varrho_t|_{r,\infty,\Omega^t} \leq \Phi_1 \cdot \phi_2 \cdot \Lambda_2, \quad (8.9)$$

with $t \leq T$.

Proof. Inequality (4.19) implies

$$\begin{aligned}\|h\|_{V(\Omega^t)} &\leq \Phi'_1(\varrho_*, \varrho^*, A_1, D_1) \cdot \Phi'_1(\|v\|_{1,3,2,\Omega^t}) \\ &\quad \cdot [\Lambda_1 + |\varrho_{x3}|_{3,\infty,\Omega^t} \cdot (|v_t|_{2,\Omega^t} + |v|_{\infty,\infty,\Omega^t} + |f|_{2,\Omega^t}) \\ &\quad + |h(0)|_{2,\Omega}].\end{aligned} \quad (8.10)$$

Utilizing (8.7) results in

$$\begin{aligned}\|h\|_{V(\Omega^t)} &\leq \Phi'_1[\Lambda_1 + \Phi_1 \cdot \phi_2 \cdot (|v_t|_{2,\Omega^t} + |v|_{\infty,\infty,\Omega^t} \\ &\quad + |f|_{2,\Omega^t})\Lambda_2 + |h(0)|_{2,\Omega}].\end{aligned} \quad (8.11)$$

Consider the embeddings

$$\begin{aligned}|v|_{3,2,\Omega^t} &\leq ct^{\frac{\sigma_1-2}{2\sigma_1}} \|v\|_{W_{\sigma_1}^{2,1}(\Omega^t)}, & \sigma_1 &\geq 2, \\ |v|_{\infty,\Omega^t} &\leq c\|v\|_{W_{\sigma_2}^{2,1}(\Omega^t)}, & \sigma_2 &> 5/2, \\ |v_x|_{\infty,1,\Omega^t} &\leq t^{1-1/\sigma_3} |v_x|_{\infty,\sigma_3,\Omega^t} \\ &\leq t^{1-1/\sigma_3} \|v\|_{W_{\sigma_3}^{2,1}(\Omega^t)}, & \sigma_3 &> 3, \\ |v_t|_{\infty,1,\Omega^t} &\leq t^{1-1/\sigma_4} |v_t|_{\infty,\sigma_4,\Omega^t} \\ &\leq t^{1-1/\sigma_4} \|v\|_{W_{\sigma_4}^{2+s,1+s/2}(\Omega^t)}, & \sigma_4 &> 3/s, s \in (0,1).\end{aligned} \quad (8.12)$$

In view of these embeddings and the properties of ϕ_2 , there exists a function ϕ_3 such that

$$\phi_2 \cdot (|v_t|_{2,\Omega^t} + |v|_{\infty,\infty,\Omega^t}) \leq \phi_3(t^{\bar{a}} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \quad (8.13)$$

for $\sigma > 3/s$, $s \in (0,1)$.

Furthermore,

$$\Phi'_1 \Phi_1 \leq \Phi_2. \quad (8.14)$$

Therefore, (8.8) has been proved. Based on (8.5), (8.7) and previous considerations, we establish (8.9). This concludes the proof. ■

LEMMA 8.2. *Assume that $v \in W_\sigma^{2+s,1+s/2}(\Omega^t)$, $s \in (0,1)$ and $\sigma > 3/s$, $t \leq T$. Let*

$$\begin{aligned}D_2 &= A_1 + |F|_{6/5,2,\Omega^t} + |\chi(0)|_{3,\Omega}, \\ D_3 &= |(f_1, f_2)|_{6r/(5r-6),2,\Omega^t} + |\chi(0)|_{3,\Omega}.\end{aligned} \quad (8.15)$$

Then

$$\begin{aligned} \|\chi\|_{V(\Omega^t)} &\leq \Phi_2 \cdot [\|v'\|_{W_2^{1,1/2}(\Omega^t)} + \|v'\|_{5/6,2,\infty,\Omega^t} \\ &\quad + |h|_{3,\infty,\Omega^t} + D_2 + (\phi_3 + D_3\phi_2)\Lambda_2], \end{aligned} \quad (8.16)$$

where $v' = (v_1, v_2)$, ϕ_3 is defined in (8.20), Φ_2 is (8.21), and ϕ_2 appears in (8.9).

Proof. Applying (8.9) in (7.11) results in

$$\begin{aligned} \|\chi\|_{V(\Omega^t)} &\leq \Phi(1/\varrho_*, \varrho^*, \bar{d}, A_1) \cdot [\|v'\|_{W_2^{1,1/2}(\Omega^t)} \\ &\quad + \|v'\|_{5/6,2,\infty,\Omega^t} + |h|_{3,\infty,\Omega^t} + D_2 \\ &\quad + \Phi_1 \cdot \phi_2 \Lambda_2 \cdot (\|v'\|_{W_2^{1,1/2}(\Omega^t)} + \|v'\|_{5/6,2,\infty,\Omega^t} \\ &\quad + |v'_t|_{6r/(5r-6),2,\Omega^t} + |v|_{3r/(r-3),\infty,\Omega^t} + D_3)]. \end{aligned} \quad (8.17)$$

The following embeddings are valid:

$$\|v\|_{5/6,2,\infty,\Omega^t} + \|v\|_{W_2^{1,1/2}(\Omega^t)} + |v|_{3r/(r-3),\infty,\Omega^t} \leq ct^{\bar{a}} \|v\|_{W_{\sigma}^{2,1}(\Omega^t)}, \quad (8.18)$$

where $\bar{a} > 0$, $\sigma > 3$, and

$$\|v_t\|_{6r/(5r-6),2,\Omega^t} \leq t^{1/2-1/\sigma_4} \|v\|_{W_{\sigma_4}^{2+s,1+s/2}(\Omega^t)}, \quad (8.19)$$

where $\sigma_4 \geq 3/s$, $s \in (0, 1)$. Considering (8.18) and (8.19), there exists an increasing positive function ϕ_3 satisfying

$$\begin{aligned} \phi_2 \cdot (\|v'\|_{5/6,2,\infty,\Omega^t} + \|v'\|_{W_2^{1,1/2}(\Omega^t)} + |v'_t|_{6r/(5r-6),2,\Omega^t} \\ + |v|_{3r/(r-3),\infty,\Omega^t}) \leq c\phi_2 \cdot t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)} \\ \leq \phi_3(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}), \quad \sigma > 3/s, \quad s \in (0, 1). \end{aligned} \quad (8.20)$$

Taking into account

$$\Phi\Phi_1 \leq \Phi_2(1/\varrho_*, \varrho^*, d_*, A_1), \quad \Phi \sim \Phi', \quad (8.21)$$

we infer (8.16) from (8.17). This concludes the proof. ■

LEMMA 8.3. Assume that the r.h.s. of (8.9) is finite. Let $\varrho_* \leq \varrho \leq \varrho^*$. Then

$$\|\varrho\|_{C^\alpha(\Omega^t)} \leq c\|\varrho\|_{W_{r,1}^{1,1}(\Omega^t)} \leq \Phi_1 \cdot \phi_2 \cdot \Lambda_2(t, r) + c\varrho^*, \quad (8.22)$$

provided that

$$\frac{3}{r} + \alpha < 1, \quad r > 3,$$

and

$$\|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)} = |\varrho|_{r,\infty,\Omega^t} + |\varrho_x|_{r,\infty,\Omega^t} + |\varrho_t|_{r,\infty,\Omega^t}.$$

9. Increasing regularity for the velocity and pressure

In this section, we enhance the regularity of v and p step by step, primarily by utilizing findings related to the Stokes system, discussed in the Appendix (Lemmas 9.2–9.5). To begin, we apply the rot-div problem (1.20) to end up with an estimate for v' in higher norms.

LEMMA 9.1. Assume D_2, D_3 are defined in (8.15), ϕ_3 in (8.20) and Φ_2 in (8.21). Additionally, Λ_1 is introduced in (8.3) and Λ_2 in (8.4). Assume that $h \in L_\infty(0, t; L_3(\Omega))$, $h(0) = v_{0,x_3} \in L_2(\Omega)$, $v' \in L_2(\Omega; H^{1/2}(0, t))$, where $v' = (v_1, v_2)$, $t \leq T$. Then

$$\begin{aligned} \|v'\|_{V^1(\Omega^t)} &\leq \Phi_2[\|v'\|_{L_2(\Omega; H^{1/2}(0, t))} \\ &\quad + |h|_{3, \infty, \Omega^t} + D_2 + (1 + D_3)\phi_3(\Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)})]. \end{aligned} \quad (9.1)$$

Proof. Let Ω' represent the cross-section of Ω obtained by intersecting it with a plane that is perpendicular to the x_3 -axis and contains the point $x_3 \in (-a, a)$. Define S'_1 as the intersection of S_1 with this plane. Hence, S'_1 serves as the boundary for Ω' . As a result, the elliptic system (rot, div) confined to Ω' leads to the following problem:

$$\begin{aligned} v_{2,x_1} - v_{1,x_2} &= \chi && \text{in } \Omega', \\ v_{1,x_1} + v_{2,x_2} &= -h_3 && \text{in } \Omega', \\ v' \cdot \bar{n}' &= 0 && \text{on } S'_1, \end{aligned} \quad (9.2)$$

where x_3 is treated as a parameter, $v' = (v_1, v_2)$ and $\bar{n}'$ denotes the unit outward vector normal to S_1 at a given point of S'_1 .

Solutions to (9.2) satisfy

$$\sup_t \|v'\|_{L_2(-a, a; H^1(\Omega'))} + \|v'\|_{L_2(0, t; L_2(-a, a; H^2(\Omega'))} \leq c(\|\chi\|_{V(\Omega^t)} + \|h_3\|_{V(\Omega^t)}). \quad (9.3)$$

From (8.8) it can be derived that

$$\|v'_{x_3}\|_{V(\Omega^t)} \leq \Phi_2 \cdot \phi_3 \cdot (\Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)}). \quad (9.4)$$

Therefore, (9.3) and (9.4) lead to

$$\|v'\|_{V^1(\Omega^t)} \leq c[\|\chi\|_{V(\Omega^t)} + \Phi_2 \cdot \phi_3 \cdot (\Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)})]. \quad (9.5)$$

Utilizing (8.16) results in

$$\begin{aligned} \|v'\|_{V^1(\Omega^t)} &\leq \Phi_2 \cdot [\|v'\|_{W_2^{1,1/2}(\Omega^t)} + \|v'\|_{6/5, 2, \infty, \Omega^t} \\ &\quad + |h|_{3, \infty, \Omega^t} + D_2 + (\phi_3 + D_3\phi_2)\Lambda_2 + \phi_3 \cdot (\Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)})]. \end{aligned} \quad (9.6)$$

Drawing upon

$$\begin{aligned} \|v'\|_{W_2^{1,1/2}(\Omega^t)} &\leq \|v'\|_{L_2(0, t; H^1(\Omega))} + \|v'\|_{L_2(\Omega; H^{1/2}(0, t))} \\ &\leq A_1 + \|v'\|_{L_2(\Omega; H^{1/2}(0, t))} \end{aligned}$$

and the interpolation

$$\begin{aligned} \|v'\|_{5/6, 2, \infty, \Omega^t} &\leq \varepsilon \|v'\|_{1, 2, \infty, \Omega^t} + c(1/\varepsilon) \|v'\|_{2, \Omega^t} \\ &\leq \varepsilon \|v'\|_{1, 2, \infty, \Omega^t} + c(1/\varepsilon) A_1, \end{aligned}$$

as mentioned in (9.6), results in the following inequality for sufficiently small ε :

$$\begin{aligned} \|v'\|_{V^1(\Omega^t)} &\leq \Phi_2 \cdot [\|v'\|_{L_2(\Omega; H^{1/2}(0, t))} + |h|_{3, \infty, \Omega^t} \\ &\quad + D_2 + (\phi_3 + D_3\phi_2)\Lambda_2 + \phi_3(\Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)})]. \end{aligned} \quad (9.7)$$

Given that

$$\phi_2 \leq \phi_3$$

we can derive from (9.7) the inequality stated in (9.1). This completes the proof. ■

LEMMA 9.2. *Let the assumptions of Lemma 9.1 hold. Suppose that $v \in W_\sigma^{2+s,1+s/2}(\Omega^t)$, $p \in L_{5/3}(\Omega^t)$, $\sigma > 3/5$, $\bar{a} > 0$,*

$$H = |h|_{3,\infty,\Omega^t} + |h|_{10/3,\Omega^t} + \|h(0)\|_{L_2(\Omega)}, \quad (9.8)$$

$$D_4 = |f|_{5/3,\Omega^t} + \|d\|_{W_{5/3}^{7/5,7/10}(S_2^t)} + \|v_0\|_{W_{5/3}^{4/5}(\Omega)}. \quad (9.9)$$

Define D_2 and D_3 as in (8.15) and let

$$\phi_5 = \phi_5(\varrho^*, A_1, t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}), \quad (9.10)$$

where ϕ_5 is a combination of ϕ_2 , ϕ_3 , and the function ϕ from Lemma A.1. Additionally, $\phi_5(\varrho^*, A_1, 0) = 0$. Then

$$\begin{aligned} \|v\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5/3}(\Omega^t)} &\leq \phi_5(\varrho^*, A_1, t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot (\Lambda_1 + \Lambda_2) \cdot [\|v\|_{W_{5/3}^{1,1/2}(\Omega^t)} + |p|_{5/3,\Omega^t} + D_3] + c(\varrho^*, A_1) \\ &\quad \cdot [H + D_2 + D_4 + \phi_3\|h(0)\|_{L_2(\Omega)}] \end{aligned} \quad (9.11)$$

and

$$\begin{aligned} \|v'\|_{V^1(\Omega^t)} &\leq \phi_5(\varrho^*, A_1, t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \cdot (\Lambda_1 + \Lambda_2) \\ &\quad \cdot [\|v\|_{W_{5/3}^{1,1/2}(\Omega^t)} + |p|_{5/3,\Omega^t} + D_3] + c(\varrho^*, A_1)[H + D_2 + D_4 \\ &\quad + \phi_3\|h(0)\|_{L_2(\Omega)}]. \end{aligned} \quad (9.12)$$

Proof. Examine problem (1.1) expressed as two separate problems

$$\begin{aligned} \varrho v_t - \operatorname{div} \mathbb{T}(v, p) &= -\varrho v' \cdot \nabla v - \varrho v_3 h + \varrho f && \text{in } \Omega^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ v \cdot \bar{n} &= 0, \quad \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\ v \cdot \bar{n} &= d, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 && \text{on } S_2^T, \\ v|_{t=0} &= v_0 && \text{in } \Omega, \end{aligned} \quad (9.13)$$

where ϱ is treated as given and ϱ satisfies

$$\begin{aligned} \varrho_t + v \cdot \nabla \varrho &= 0 && \text{in } \Omega^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ \varrho &= \varrho_1 && \text{on } S_2^T(a_1), \\ \varrho|_{t=0} &= \varrho_0 && \text{in } \Omega, \end{aligned} \quad (9.14)$$

with v treated as given.

The proof of Lemma 3.4 in [Z4] gives us

$$\|v'\|_{L_{10}(\Omega^T)} \leq c\|v'\|_{V^1(\Omega^T)}. \quad (9.15)$$

Thus,

$$\begin{aligned} |v' \cdot \nabla v|_{5/3,\Omega^t} &\leq cA_1\|v'\|_{V^1(\Omega^t)}, \\ |v_3 h|_{5/3,\Omega^t} &\leq cA_1|h|_{10/3,\Omega^t}. \end{aligned} \quad (9.16)$$

Considering (9.16) along with Lemma A.1 provides

$$\begin{aligned} & \|v\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5/3}(\Omega^t)} \\ & \leq \phi_4(\varrho^*, t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))})\Lambda_2 \\ & \quad \cdot [\|v\|_{W_{5/3}^{1,1/2}(\Omega^t)} + |p|_{5/3,\Omega^t}] + c(\varrho^*)[A_1\|v'\|_{V_2^1(\Omega^t)} + A_1|h|_{10/3,\Omega^t} + D_4]. \end{aligned} \quad (9.17)$$

Employing (9.7) and the interpolation inequality

$$\begin{aligned} \|v'\|_{L_2(\Omega;H^{1/2}(0,t))} & \leq \varepsilon\|v'\|_{W_{5/3}^{2,1}(\Omega^t)} + c(1/\varepsilon)|v'|_{2,\Omega^t} \\ & \leq \varepsilon\|v'\|_{W_{5/3}^{2,1}(\Omega^t)} + c(1/\varepsilon)A_1 \end{aligned} \quad (9.18)$$

allows us to find that

$$\begin{aligned} & \|v\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5/3}(\Omega^t)} \leq \phi_4(\varrho^*, t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ & \quad \cdot \Lambda_2[\|v\|_{W_{5/3}^{1,1/2}(\Omega^t)} + |p|_{5/3,\Omega^t}] + c(\varrho^*, A_1) \\ & \quad \cdot [H + D_2 + D_4 + (2\phi_3 + D_3\phi_2)\Lambda_2 + \phi_3(\Lambda_1 + \|h(0)\|_{L_2(\Omega)})] \\ & \leq \phi_5(\varrho^*, A_1, t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \cdot (\Lambda_1 + \Lambda_2) \cdot [\|v\|_{W_{5/3}^{1,1/2}(\Omega^t)} \\ & \quad + |p|_{5/3,\Omega^t} + D_3] + c(\varrho^*, A_1)(H + D_2 + D_4 + \phi_3\|h(0)\|_{L_2(\Omega)}), \end{aligned} \quad (9.19)$$

where ϕ_5 is a combination of ϕ_2 , ϕ_3 and ϕ from (A.1).

Applying (9.19) in (9.7) yields

$$\begin{aligned} \|v'\|_{V^1(\Omega^t)} & \leq \phi_5(\varrho^*, A_1, t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ & \quad \cdot (\Lambda_1 + \Lambda_2) \cdot [\|v\|_{W_{5/3}^{1,1/2}(\Omega^t)} + |p|_{5/3,\Omega^t} + D_3] \\ & \quad + c(\varrho^*, A_1)(H + D_2 + D_4 + \phi_3\|h(0)\|_{L_2(\Omega)}). \end{aligned} \quad (9.20)$$

The inequalities found in (9.19) and (9.20) lead to the conclusions (9.11) and (9.12), respectively. This ends the proof. ■

To increase regularity, we need

LEMMA 9.3. *Let the assumptions of Lemma 9.2 hold. Define*

$$D_5 = |f|_{2,\Omega^t} + \|d\|_{W_2^{3/2,3/4}(S_2^t)} + \|v_0\|_{H^1(\Omega)}. \quad (9.21)$$

Then

$$\begin{aligned} & \|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \leq \phi_6(\varrho^*, A_1, t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ & \quad \cdot (\Lambda_1 + \Lambda_2) \cdot [\|v\|_{W_2^{1,1/2}(\Omega^t)}^2 + \|v\|_{W_2^{1,1/2}(\Omega^t)} + |p|_{2,\Omega^t}^2 + |p|_{2,\Omega^t} \\ & \quad + (H + D_2 + D_4 + \phi_3 \cdot \|h(0)\|_{L_2(\Omega)} + 1)^2] + c(\varrho^*, A_1)[H + D_2 + D_4 \\ & \quad + D_5 + \phi_3\|h(0)\|_{L_2(\Omega)}], \end{aligned} \quad (9.22)$$

where ϕ_6 is a combination of ϕ from (A.2) and ϕ_5 and Λ_1 , Λ_2 are assumed to be small.

Proof. It follows that

$$|v|_{5,\Omega^t} + |\nabla v|_{5/2,\Omega^t} \leq c\|v\|_{W_{5/3}^{2,1}(\Omega^t)}.$$

Therefore,

$$\begin{aligned} |v' \cdot \nabla v|_{2,\Omega^t} &\leq |v'|_{10,\Omega^t} |\nabla v|_{5/2,\Omega^t} \leq c \|v'\|_{V^1(\Omega^t)} \|v\|_{W_{5/3}^{2,1}(\Omega^t)}, \\ |v_3 h|_{2,\Omega^t} &\leq |v_3|_{5,\Omega^t} |h|_{10/3,\Omega^t} \leq c |h|_{10/3,\Omega^t} \|v\|_{W_{5/3}^{2,1}(\Omega^t)}. \end{aligned} \quad (9.23)$$

By applying Lemma A.1 to problem (9.13) with $r = 2$ and utilizing (9.23) we arrive at

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} &\leq \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot \Lambda_2 \cdot [\|v\|_{W_2^{1,1/2}(\Omega^t)} + |p|_{2,\Omega^t}] \\ &\quad + c(\varrho^*) [\|v'\|_{V^1(\Omega^t)} \|v\|_{W_{5/3}^{2,1}(\Omega^t)} + |h|_{10/3,\Omega^t} \|v\|_{W_{5/3}^{2,1}(\Omega^t)} + D_5]. \end{aligned} \quad (9.24)$$

Incorporating (9.11) and (9.12) in (9.24), we derive a qualitatively equivalent inequality

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} &\leq \phi_6(\varrho^*, A_1, t^{\bar{a}} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot (\Lambda_1 + \Lambda_2) \cdot [\|v\|_{W_2^{1,1/2}(\Omega^t)}^2 + \|v\|_{W_2^{1,1/2}(\Omega^t)} + |p|_{2,\Omega^t}^2 \\ &\quad + |p|_{2,\Omega^t} + (H + D_2 + D_4 + \phi_3 \cdot \|h(0)\|_{L_2(\Omega)} + 1)^2] \\ &\quad + c(\varrho^*, A_1)(H + D_2 + D_4 + \phi_3 \|h(0)\|_{L_2(\Omega)} + D_5), \end{aligned}$$

in which ϕ_6 represents the combination of ϕ from (A.2) along with ϕ_5 , while Λ_1, Λ_2 are assumed to be small. The inequality above leads to (9.22) and concludes the proof. ■

LEMMA 9.4. *Let the assumptions of Lemma 9.3 hold. Let*

$$\begin{aligned} D_6 &= |f|_{5/2,\Omega^t} + \|d\|_{W_{5/2}^{8/5,4/5}(S_2^t)} + \|v_0\|_{W_{5/2}^{6/5}(\Omega)}, \\ A_1 + |F|_{6/5,2,\Omega^t} + |\chi(0)|_{3,\Omega} + D_6 &\leq D_7. \end{aligned} \quad (9.25)$$

Then

$$\begin{aligned} \|v\|_{W_{5/2}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5/2}(\Omega^t)} &\leq \phi(\varrho^*, A_1, t^{\bar{a}} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot (\Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)}) \cdot [\|v\|_{W_{5/2}^{1,1/2}(\Omega^t)}^4 + |p|_{5/2,\Omega^t}^4 \\ &\quad + \|v\|_{W_{5/2}^{1,1/2}(\Omega^t)} + |p|_{5/2,\Omega^t} + (H + D_7 + 1)^4] \\ &\quad + c(\varrho^*, A_1)(H + D_7 + 1)^2. \end{aligned} \quad (9.26)$$

Proof. We examine problem (9.13) in the form

$$\begin{aligned} \varrho v_t - \operatorname{div} \mathbb{T}(v, p) &= -\varrho v \cdot \nabla v + \varrho f && \text{in } \Omega^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ v \cdot \bar{n} &= 0, \quad \nu \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha + \gamma v \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S_1^T, \\ v \cdot \bar{n} &= d, \quad \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 && \text{on } S_2^T, \\ v|_{t=0} &= v_0 && \text{in } \Omega. \end{aligned} \quad (9.27)$$

Employing the estimate

$$|\varrho v \cdot \nabla v|_{5/2,\Omega^t} \leq c \varrho^* \|v\|_{W_2^{2,1}(\Omega^t)}^2, \quad (9.28)$$

we apply Lemma A.1 for $r = 5/2$ to problem (9.27). Consequently,

$$\begin{aligned} \|v\|_{W_{5/2}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5/2}(\Omega^t)} &\leq \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot \Lambda_2 \cdot [\|v\|_{W_{5/2}^{1,1/2}(\Omega^t)} + |p|_{5/2,\Omega^t}] \\ &\quad + c(\varrho^*, A_1) [\|v\|_{W_2^{2,1}(\Omega^t)} + D_7]. \end{aligned} \quad (9.29)$$

Substituting (9.22) into (9.29) results in

$$\begin{aligned} \|v\|_{W_{5/2}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5/2}(\Omega^t)} &\leq \phi(\varrho^*, A_1, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot (\Lambda_1 + \Lambda_2) \cdot [\|v\|_{W_{5/2}^{1,1/2}(\Omega^t)}^4 + |p|_{5/2,\Omega^t}^4 + \|v\|_{W_{5/2}^{1,1/2}(\Omega^t)} \\ &\quad + |p|_{5/2,\Omega^t} + (H + D_2 + D_4 + \phi_3 \cdot \|h(0)\|_{L_2(\Omega)} + D_5 + 1)^4] \\ &\quad + c(\varrho^*, A_1) [(H + D_2 + D_4 + \phi_3 \cdot \|h(0)\|_{L_2(\Omega)} + D_5 + 1)^2 + D_6]. \end{aligned} \quad (9.30)$$

Reformulating (9.30) yields (9.26). This concludes the proof. ■

LEMMA 9.5. Assume $\sigma > 3/s$. Let the assumptions of Lemma 9.4 hold. Let $5' < 5$ be a number close to 5, and let

$$D_8 = |f|_{5',\Omega^t} + \|d\|_{W_{5'}^{2-1/5,1-1/10}(S_2^t)} + \|v(0)\|_{W_{5'}^{2-2/5}(\Omega)} \quad (9.31)$$

be finite. Then

$$\begin{aligned} &\|v\|_{W_{5'}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5'}(\Omega^t)} \\ &\leq \phi(\varrho^*, A_1, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot (\Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)}) [\|v\|_{W_{5'}^{1,1/2}(\Omega^t)}^8 \\ &\quad + \|p\|_{L_{5'}^8(\Omega^t)} + \|v\|_{W_{5'}^{1,1/2}(\Omega^t)} \\ &\quad + \|p\|_{L_{5'}(\Omega^t)} + (H + D_7 + 1)^8] + c(\varrho^*, A_1) [(H + D_7 + 1)^4 + D_8]. \end{aligned} \quad (9.32)$$

Proof. To prove the lemma, we make use of Lemma A.2. Since

$$|v \cdot \nabla v|_{5',\Omega^t} \leq c \|v\|_{W_{5/2}^{2,1}(\Omega^t)}^2,$$

inequality (A.2) takes the form

$$\begin{aligned} \|v\|_{W_{5'}^{2,1}(\Omega^t)} + \|\nabla p\|_{L_{5'}(\Omega^t)} &\leq \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\quad \cdot \Lambda_2 \cdot [\|v\|_{W_{5'}^{1,1/2}(\Omega^t)} + |p|_{5',\Omega^t}] \\ &\quad + c(\varrho^*) [\|v\|_{W_{5/2}^{2,1}(\Omega^t)}^2 + D_8]. \end{aligned} \quad (9.33)$$

Taking advantage of (9.26) in (9.33) yields (9.32). This concludes the proof. ■

LEMMA 9.6. Assume that $\sigma > 3/s$, $v \in W_{5'}^{2,1}(\Omega^t)$, and $5' < 5$ but close to 5. Assume

$$D_9 = \|f\|_{W_{\sigma}^{s,s/2}(\Omega^t)} + \|d\|_{W_{\sigma}^{2+s-1/\sigma,1+s/2-1/2\sigma}(S_2^t)} + \|v(0)\|_{W_{\sigma}^{2+s-2/\sigma}(\Omega)}$$

is finite. Then

$$\begin{aligned}
& \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s,s/2}(\Omega^t)} \\
& \leq \phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\
& \quad \cdot \Lambda_2[\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)} + \|p\|_{W_\sigma^{s,s/2}(\Omega^t)} + \|v\|_{W_\sigma^{2,1}(\Omega^t)}^2 \\
& \quad + \|f\|_{W_\sigma^{s,s/2}(\Omega^t)}] + c(\varrho^*)[\|v\|_{W_{5'}^{2,1}(\Omega^t)} + D_9 + A_1], \tag{9.34}
\end{aligned}$$

where $\bar{a} > 2$.

Proof. We rely on inequality (A.7). Next, we have to estimate the first term under the second square bracket on the r.h.s. of (A.7) in terms of Corollary 5.3 and Lemma 9.5. Therefore, we take a closer look at

$$\begin{aligned}
J &= \|\varrho v \cdot \nabla v\|_{W_\sigma^{s,s/2}(\Omega^t)} \\
&= \|\varrho v \cdot \nabla v\|_{L_\sigma(0,t;W_\sigma^s(\Omega))} + \|\varrho v \cdot \nabla v\|_{L_\sigma(\Omega;W_\sigma^{s/2}(0,t))} \equiv J_1 + J_2.
\end{aligned}$$

Let us focus on J_1 :

$$\begin{aligned}
J_1 &= \left(\int_0^t dt \int_\Omega \int_\Omega dx' dx'' \frac{|\varrho(x',t)v(x',t)\nabla v(x',t) - \varrho(x'',t)v(x'',t)\nabla v(x'',t)|^\sigma}{|x' - x''|^{3+s\sigma}} \right)^{1/\sigma} \\
&\leq \left(\int_0^t dt \int_\Omega \int_\Omega dx' dx'' \frac{|\varrho(x',t) - \varrho(x'',t)|^\sigma |v(x',t)|^\sigma |\nabla v(x',t)|^\sigma}{|x' - x''|^{3+s\sigma}} \right)^{1/\sigma} \\
&\quad + \varrho^* \left(\int_0^t dt \int_\Omega \int_\Omega dx' dx'' \frac{|v(x',t) - v(x'',t)|^\sigma}{|x' - x''|^{3+s\sigma}} |\nabla v(x',t)|^\sigma \right)^{1/\sigma} \\
&\quad + \varrho^* \left(\int_0^t dt \int_\Omega \int_\Omega dx' dx'' \frac{|v(x'',t)|^\sigma |\nabla v(x',t) - \nabla v(x'',t)|^\sigma}{|x' - x''|^{3+s\sigma}} \right)^{1/\sigma} \\
&\equiv J_1^1 + J_1^2 + J_1^3.
\end{aligned}$$

and analyze J_1^1 . According to the Hölder inequality, we find that

$$\begin{aligned}
J_1^1 &\leq \left(\int dt \int_\Omega \int_\Omega dx' dx'' \frac{|\varrho(x',t) - \varrho(x'',t)|^{\sigma\lambda_1}}{|x' - x''|^{3+\sigma\lambda_1} [\frac{1}{\sigma\lambda_1}(\frac{3}{2}\lambda_1 - 3) + s]} \right)^{1/\sigma\lambda_1} \\
&\quad \cdot \left(\int dt \int_\Omega \int_\Omega dx' dx'' \frac{|v(x',t)|^{\sigma\lambda_2} |\nabla v(x',t)|^{\sigma\lambda_2}}{|x' - x''|^{(3/2)\lambda_2}} \right)^{1/\sigma\lambda_2} \equiv L_1 L_2,
\end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\lambda_2 < 2$ and $\lambda_1 > 2$.

Given that

$$L_1 \leq \|\varrho\|_{L_{\sigma\lambda_1}(0,t;W_{\sigma\lambda_1}^{s'}(\Omega))} \equiv L_1^1,$$

where $s' = \frac{1}{\sigma\lambda_1}(\frac{3}{2}\lambda_1 - 3) + s$, we apply the embedding

$$L_1^1 \leq c\|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)}$$

which is valid for

$$\frac{3}{r} - \frac{5}{\sigma\lambda_1} + s' \leq 1. \quad (9.35)$$

Next,

$$L_2 \leq c|v|_{\infty, \Omega^t} |\nabla v|_{\sigma\lambda_2, \Omega^t} \equiv L_2^1.$$

To evaluate L_2^1 , we employ the embedding

$$|\nabla v|_{\sigma\lambda_2, \Omega^t} \leq \|v\|_{W_{5'}^{2,1}(\Omega^t)}$$

which holds for

$$\frac{5}{5'} - \frac{5}{\sigma\lambda_2} \leq 1. \quad (9.36)$$

The inequalities in (9.35) and (9.36) imply that

$$\frac{3}{r} - \frac{8}{\sigma\lambda_1} + \frac{3}{2\sigma} + s \leq 1, \quad \frac{5}{5'} - \frac{5}{\sigma\lambda_2} \leq 1.$$

By multiplying the second inequality by $8/5$ and combining it with the first, we arrive at

$$\frac{3}{r} - \frac{13}{2\sigma} + s \leq 1, \quad (9.37)$$

indicating that there is no restriction.

The first component in L_2^1 is evaluated using the inequality

$$|v|_{\infty, \Omega^t} \leq c\|v\|_{W_5^{2,1}(\Omega^t)}.$$

Using Corollary 5.3, we find

$$J_1^1 \leq \phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s, 1+s/2}(\Omega^t)}) \cdot \Lambda_2 \cdot \|v\|_{W_5^{2,1}(\Omega^t)}. \quad (9.38)$$

We proceed to analyze J_1^2 . The Hölder inequality yields

$$J_1^2 \leq \varrho^* \|v\|_{L_{\sigma\lambda_1}(0,t; W_{\sigma\lambda_1}^{s'}(\Omega))} |\nabla v|_{\sigma\lambda_2, \Omega^t} = N_1 N_2,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\lambda_1 > 2$, $\lambda_2 < 2$ and $s' = \frac{1}{\sigma\lambda_1}(\frac{3}{2}\lambda_1 - 3) + s$.

Consequently,

$$N_1 \leq c\|v\|_{W_{5'}^{2,1}(\Omega^t)} \quad \text{for} \quad \frac{5}{5'} - \frac{5}{\sigma\lambda_1} + \frac{3}{2\sigma} - \frac{3}{\sigma\lambda_1} + s \leq 2$$

and

$$N_2 \leq c\|v\|_{W_5^{2,1}(\Omega^t)} \quad \text{for} \quad \frac{5}{5'} - \frac{5}{\sigma\lambda_2} \leq 1.$$

Hence,

$$J_1^2 \leq c\|v\|_{W_{5'}^{2,1}(\Omega^t)}$$

is valid for

$$s - \frac{13}{2\sigma} < 1. \quad (9.39)$$

Ultimately, it follows that

$$J_1^3 \leq c|v|_{\sigma\lambda_1, \Omega^t} \|\nabla v\|_{L_{\sigma\lambda_2}(0,t; W_{\sigma\lambda_2}^{s'}(\Omega))} \leq c\|v\|_{W_{5'}^{2,1}(\Omega^t)}^2,$$

where the last inequality holds for

$$s - \frac{13}{2\sigma} \leq \frac{8}{5}. \quad (9.40)$$

In summary,

$$J_1 \leq \phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s, 1+s/2}(\Omega^t)}) \cdot \Lambda_2 \cdot \|v\|_{W_5^{2,1}(\Omega^t)} + c \|v\|_{W_5^{2,1}(\Omega^t)}. \quad (9.41)$$

We now examine J_2 :

$$\begin{aligned} J_2 &\leq \left(\int_{\Omega} dx \int_0^t \int_0^t \frac{dt' dt'' |\varrho(x, t') v(x, t') \nabla v(x, t') - \varrho(x, t'') v(x, t'') \nabla v(x, t'')|^{\sigma}}{|t' - t''|^{1+s\sigma/2}} \right)^{1/\sigma} \\ &\leq \left(\int_{\Omega} dx \int_0^t \int_0^t dt' dt'' \frac{|\varrho(x, t') - \varrho(x, t'')|^{\sigma}}{|t' - t''|^{1+s\sigma/2}} |v(x, t')|^{\sigma} |\nabla v(x, t')|^{\sigma} \right)^{1/\sigma} \\ &\quad + \varrho^* \left(\int_{\Omega} dx \int_0^t \int_0^t dt' dt'' \frac{|v(x, t') - v(x, t'')|^{\sigma} |\nabla v(x, t')|^{\sigma}}{|t' - t''|^{1+s\sigma/2}} \right)^{1/\sigma} \\ &\quad + \varrho^* \left(\int_{\Omega} dx \int_0^t \int_0^t dt' dt'' \frac{|v(x, t'')|^{\sigma} |\nabla v(x, t') - \nabla v(x, t'')|^{\sigma}}{|t' - t''|^{1+s\sigma/2}} \right)^{1/\sigma} \\ &\equiv J_2^1 + J_2^2 + J_2^3. \end{aligned}$$

According to the Hölder inequality, we find that

$$J_2^1 \leq c \|v\|_{\infty, \Omega^t} \|\varrho\|_{L_{\sigma\lambda_1}(\Omega; W_{\sigma\lambda_1}^{s'/2}(0, t))} \|\nabla v\|_{\sigma\lambda_2, \Omega^t} \equiv J_2^{11},$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\lambda_2 < 2$, $s' = \frac{1}{\sigma\lambda_1}(\lambda_1 - 2) + s$. Let $\sigma\lambda_1 = r$. Then $\sigma\lambda_2 = \frac{\sigma r}{r-\sigma}$. By means of the embedding

$$\|\varrho\|_{L_r(\Omega; W_r^{s'/2}(0, t))} \leq c \|\varrho\|_{W_{r, \infty}^{1,1}(\Omega^t)}$$

which holds for $s' \leq 2$, and

$$\begin{aligned} \|v\|_{\infty, \Omega^t} &\leq c \|v\|_{W_{5'}^{2,1}(\Omega^t)}, \quad \frac{5}{5'} < 2, \\ \|\nabla v\|_{\frac{\sigma r}{r-\sigma}, \Omega^t} &\leq c \|v\|_{W_{5'}^{2,1}(\Omega^t)} \end{aligned}$$

which is true for $5/5' - 5/\sigma + 5/r < 1$, with the last inequality valid for $r > \sigma$, we infer

$$J_2^{11} \leq \phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s, 1+s/2}(\Omega^t)}) \Lambda_2 \|v\|_{W_5^{2,1}(\Omega^t)}^2.$$

Applying the Hölder inequality with respect to space variables, we get

$$J_2^2 \leq \varrho^* \left(\int_0^t \int_0^t \frac{|v(t') - v(t'')|^{\sigma\lambda_1, \Omega} |\nabla v(t')|^{\sigma\lambda_2, \Omega}}{|t' - t''|^{1+s\sigma/2}} dt' dt'' \right)^{1/\sigma} \equiv J_2^{21},$$

where $1/\lambda_1 + 1/\lambda_2 = 1$.

On the other hand, applying the Hölder inequality with respect to time yields

$$\begin{aligned} J_2^{21} &\leq \varrho^* \left(\int_0^t \int_0^t \frac{|v(t') - v(t'')|^{\sigma\lambda_1, \Omega}}{|t' - t''|^{1+\sigma\lambda_1 s'/2}} dt' dt'' \right)^{1/\sigma\lambda_1} \\ &\quad \cdot \left(\int_0^t \int_0^t \frac{|\nabla v(t')|^{\sigma\lambda_2, \Omega}}{|t' - t''|^{\lambda_2/2}} dt' dt'' \right)^{1/\sigma\lambda_2} \equiv J_2^{22}, \end{aligned}$$

where $s' = s + \frac{1}{\sigma\lambda_1}(\lambda_1 - 2)$. For $\lambda_2 < 2$, we can integrate with respect to t'' in the second factor of J_2^{22} to obtain

$$J_2^{22} \leq \varrho^* \|v\|_{L_{\sigma\lambda_1}(\Omega; W_{\sigma\lambda_1}^{s'}(0,t))} \|\nabla v\|_{\sigma\lambda_2, \Omega^t} \equiv J_2^{23}.$$

We use the embeddings

$$\|v\|_{L_{\sigma\lambda_1}(\Omega; W_{\sigma\lambda_1}^{s'/2}(0,t))} \leq c \|v\|_{W_{5'}^{2,1}(\Omega^t)}$$

for $\frac{5}{5'} - \frac{5}{\sigma\lambda_1} + s' \leq 2$ so $\frac{5}{5'} - \frac{5}{\sigma\lambda_1} + s + \frac{1}{\sigma} - \frac{2}{\sigma\lambda_1} \leq 2$, and

$$\|\nabla v\|_{L_{\sigma\lambda_2}(\Omega^t)} \leq c \|v\|_{W_{5'}^{2,1}(\Omega^t)},$$

which holds for $\frac{5}{5'} - \frac{5}{\sigma\lambda_2} \leq 1$.

The above restrictions are satisfied. Hence

$$J_2^{23} \leq c \varrho^* \|v\|_{W_{5'}^{2,1}(\Omega^t)}^2.$$

At last,

$$J_2^3 \leq \varrho^* |v|_{\infty, \Omega^t} \|\nabla v\|_{L_\sigma(\Omega; W_\sigma^{s/2}(0,t))}$$

where we use the interpolation

$$\|\nabla v\|_{L_\sigma(\Omega; W_\sigma^{s/2}(0,t))} \leq c \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}^\theta \|v\|_{L_\infty(\Omega^t)}^{1-\theta}$$

with $\theta = \frac{1+s-5/\sigma}{2+s-5/\sigma}$. This implies

$$J_2^3 \leq \varepsilon \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + c(1/\varepsilon, \varrho^*) \|v\|_{L_\infty(\Omega^t)}^{\frac{2-\theta}{1-\theta}}.$$

In summary, the aforementioned estimates produce

$$\begin{aligned} J_2 &\leq \phi(t^{\bar{a}} \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t; W_4^1(\Omega^t))}) \Lambda_2 \|v\|_{W_{5'}^{2,1}(\Omega^t)} \\ &\quad + \varepsilon \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + c(1/\varepsilon, \varrho^*) \|v\|_{W_{5'}^{2,1}(\Omega^t)}^{\frac{2-\theta}{1-\theta}}. \end{aligned} \quad (9.42)$$

From (9.41) and (9.42), we have

$$\begin{aligned} J &\leq J_1 + J_2 \leq \phi(t^{\bar{a}} \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t; W_4^1(\Omega^t))}) \\ &\quad \cdot \Lambda_2 \|v\|_{W_{5'}^{2,1}(\Omega^t)}^2 + \varepsilon \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + c(1/\varepsilon, \varrho^*) \|v\|_{W_{5'}^{2,1}(\Omega^t)}^{\frac{2-\theta}{1-\theta}}. \end{aligned} \quad (9.43)$$

In order to use (A.7) we need the estimate

$$\|\varrho\|_{C^\alpha(\Omega^t)} \leq \|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)} \quad (9.44)$$

which holds for

$$\frac{3}{r} + \alpha < 1.$$

Exploiting (A.7) and estimates (9.43), (9.44) we infer (9.34). This ends the proof. ■

Define

$$\begin{aligned} X_1 &= \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s, s/2}(\Omega^t)}, \\ X_2 &= \|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla q\|_{L_{5/3}(\Omega^t)} + \|q\|_{L_4(0,t; W_4^1(\Omega^t))}, \\ \bar{D}_1 &= \Phi(D_7, D_8, D_9). \end{aligned}$$

Then (9.1) assumes the form

$$X_1 \leq \Phi(t^{\bar{a}} X_1, t^{\bar{a}} X_2) \Lambda + c X_2^8 + \bar{D}_1. \quad (9.45)$$

Inequality (9.3) means

$$X_2 \leq \Phi(t^{\bar{a}} X_1, t^{\bar{a}} X_2, \Phi(\bar{D}_1)) \Lambda + c B_1 \quad (9.46)$$

where

$$B_1 = D_{10} = \|g\|_{L_{5/3}(\Omega^t)} + \|d_{x'}\|_{W_{5/3}^{7/5, 7/10}(S_2^t)} + \|h(0)\|_{W_{5/3}^{4/5}(\Omega)}.$$

10. Estimates for h and q

The long time estimate is inferred in this section by estimating H (using the Stokes regularity theory) and simplifying formulas for the constants D_i .

REMARK 10.1. Assume that $\Lambda = \Lambda_1 + \Lambda_2 + \|h(0)\|_{L_2(\Omega)}$ and $\sigma > 3/s, s \in (0, 1)$. Utilizing (9.32) in (9.34) yields

$$\begin{aligned} & \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \leq \phi(t^{\bar{a}} \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{a}} \|\nabla p\|_{W_\sigma^{s, s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0, t; W_4^1(\Omega))}, H, D_7) \cdot \Lambda \\ & \quad + c(\varrho^*, A_1)[(H + D_7 + 1)^8 + D_8^2 + D_9 + 1], \end{aligned} \quad (10.1)$$

where $\bar{a} > 0$ and

$$H = |h|_{3, \infty, \Omega^t} + \|h\|_{1, 2, \Omega^t} + |h|_{10/3, \Omega^t} \leq c \|h\|_{W_{5/3}^{2, 1}(\Omega^t)}. \quad (10.2)$$

Next, we have to find an estimate for $\|h\|_{W_{5/3}^{2, 1}(\Omega^t)}$.

LEMMA 10.2. Assume that D_2, D_4, D_{10} are finite and defined in (8.15), (9.9) and (10.6), respectively, $s \in (0, 1)$, $h \in W_{5/3}^{1, 1/2}(\Omega^t)$, $q \in L_{5/3}(\Omega^t)$, $v \in W_2^{1, 1/2}(\Omega^t)$, $p \in L_2(\Omega^t)$, $f \in W_\sigma^{s, s/2}(\Omega^t)$, A_1 and Λ are defined in (1.8) and (1.5). Then (h, q) solving problem (2.6) satisfy

$$\begin{aligned} & \|h\|_{W_{5/3}^{2, 1}(\Omega^t)} + \|\nabla q\|_{L_{5/3}(\Omega^t)} \\ & \leq \phi(\varrho^*, A_1, t^{\bar{a}} \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{a}} \|\nabla p\|_{W_\sigma^{s, s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0, t; W_4^1(\Omega))}) \\ & \quad \cdot \Lambda \cdot [\|h\|_{W_{5/3}^{1, 1/2}(\Omega^t)} + \|q\|_{L_{5/3}(\Omega^t)} + \|v\|_{W_2^{1, 1/2}(\Omega^t)}^2 \\ & \quad + \|v\|_{W_2^{1, 1/2}(\Omega^t)} + \|p\|_{L_2(\Omega^t)}^2 + \|p\|_{L_2(\Omega^t)} + (H + D_2 + D_4 + 1)^2 \\ & \quad + H + D_2 + D_4 + 1 + \|f\|_{W_\sigma^{s, s/2}(\Omega^t)}] + c(\varrho^*, A_1) D_{10}. \end{aligned} \quad (10.3)$$

Proof. Lemma A.1 applied to problem (2.6) yields

$$\begin{aligned}
& \|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla q\|_{L_{5/3}(\Omega^t)} \leq \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}) \\
& \quad \cdot \Lambda \cdot [\|h\|_{W_{5/3}^{1,1/2}(\Omega^t)} + \|q\|_{L_{5/3}(\Omega^t)}] \\
& \quad + c(\varrho^*, A_1) [\|v \cdot \nabla h\|_{L_{5/3}(\Omega^t)} + \|h \cdot \nabla v\|_{L_{5/3}(\Omega^t)} + \|g\|_{L_{5/3}(\Omega^t)} \\
& \quad + \|\varrho_{x_3}(v_t + v \cdot \nabla v - f)\|_{L_{5/3}(\Omega^t)} + \|d_{x'}\|_{W_{5/3}^{7/5,7/10}(S_2^t)} \\
& \quad + \|h(0)\|_{W_{5/3}^{4/5}(\Omega)}].
\end{aligned} \tag{10.4}$$

As a consequence of (8.8), we consider

$$\begin{aligned}
|h \cdot \nabla v|_{5/3, \Omega^t} & \leq |h|_{10/3, \Omega^t} |\nabla v|_{10/3, \Omega^t} \leq c \|h\|_{V(\Omega^t)} \|v\|_{W_2^{2,1}(\Omega^t)} \\
& \leq \phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}) \cdot \Lambda, \\
|v \cdot \nabla h|_{5/3, \Omega^t} & \leq |v|_{10, \Omega^t} |\nabla h|_{2, \Omega^t} \leq c \|h\|_{V(\Omega^t)} \|v\|_{W_2^{2,1}(\Omega^t)} \\
& \leq \phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}) \cdot \Lambda, \\
|\varrho_{x_3}(v_t + v \cdot \nabla v - f)|_{5/3, \Omega^t} & \leq |\varrho_{x_3}|_{r, \infty, \Omega^t} (|v_t|_{\frac{5r}{3r-5}, \frac{5}{3}, \Omega^t} \\
& \quad + |v|_{\infty, \Omega^t} |\nabla v|_{\frac{5r}{3r-5}, \frac{5}{3}, \Omega^t} + |f|_{\frac{5r}{3r-5}, \frac{5}{3}, \Omega^t}) \\
& \leq \phi(t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}) \cdot \Lambda \cdot (1 + \|f\|_{W_{\sigma}^{s,s/2}(\Omega^t)}).
\end{aligned}$$

Based on these estimates and inequality (9.22), the inequality in (10.4) can be rewritten as follows:

$$\begin{aligned}
& \|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla q\|_{5/3(\Omega^t)} \\
& \leq \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}) \cdot \Lambda \cdot [\|h\|_{W_{5/3}^{1,1/2}(\Omega^t)} + \|q\|_{L_{5/3}(\Omega^t)}] \\
& \quad + c(\varrho^*, A_1) \|h\|_{V(\Omega^t)} \|v\|_{W_2^{2,1}(\Omega^t)} \\
& \quad + \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}) \cdot \Lambda \cdot (1 + \|f\|_{W_{\sigma}^{s,s/2}(\Omega^t)}) \\
& \quad + c(\varrho^*, A_1) D_{10}.
\end{aligned} \tag{10.5}$$

The definitions imply

$$D_{10} = |g|_{5/3, \Omega^t} + \|d_{x'}\|_{W_{5/3}^{7/5,7/10}(S_2^t)} + \|h(0)\|_{W_{5/3}^{4/5}(\Omega)}. \tag{10.6}$$

From (8.8), it can be deduced that

$$\|h\|_{V(\Omega^t)} \leq \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}) \cdot \Lambda. \tag{10.7}$$

In addition, (9.22) leads to

$$\begin{aligned}
\|v\|_{W_2^{2,1}(\Omega^t)} & \leq \phi(\varrho^*, t^{\bar{a}} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}) \cdot \Lambda \\
& \quad \cdot [\|v\|_{W_2^{1,1/2}(\Omega^t)}^2 + \|v\|_{W_2^{1,1/2}(\Omega^t)} + |p|_{2, \Omega^t}^2 + |p|_{2, \Omega^t} + (H + D_2 + D_4 + 1)^2] \\
& \quad + c(\varrho^*, A_1) [H + D_2 + D_4 + D_5].
\end{aligned} \tag{10.8}$$

By substituting (10.7) and (10.8) into (10.5), we arrive at (10.3):

$$\begin{aligned} \|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla q\|_{L_{5/3}(\Omega^t)} &\leq \phi(\varrho^*, A_1, t^{\bar{a}}\|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega))}) \\ &\quad \cdot \Lambda \cdot [\|h\|_{W_{5/3}^{1,1/2}(\Omega^t)} + \|q\|_{L_{5/3}(\Omega^t)} + \|v\|_{W_2^{1,1/2}(\Omega^t)}^2 \\ &\quad + \|v\|_{W_2^{1,1/2}(\Omega^t)} + \|p\|_{L_2(\Omega^t)}^2 + \|p\|_{L_2(\Omega^t)} + (H + D_2 + D_4 + 1)^2 \\ &\quad + H + D_2 + D_4 + 1 + \|f\|_{W_{\sigma}^{s,s/2}(\Omega^t)}] + c(\varrho^*, A_1)D_{10}. \blacksquare \end{aligned}$$

LEMMA 10.3. *Let the assumptions of Lemma 10.2 be fulfilled and*

$$v \in W_{\sigma}^{2+s,1+s/2}(\Omega^t), \nabla p \in W_{\sigma}^{s,s/2}(\Omega^t), h \in W_{5/3}^{2,1}(\Omega^t), \nabla q \in L_{5/3}(\Omega^t).$$

Then the solutions to (2.6) satisfy

$$\begin{aligned} \|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega))} \\ \leq \phi(t^{\bar{a}}X_1, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega))}, \|f\|_{L_2(0,t;L_4(\Omega))})\Lambda + c\Lambda_3(4) \end{aligned} \quad (10.9)$$

where

$$\Lambda_3(4) = \sum_{i,j=1}^2 \|d_{j,x_i}\|_{W_4^{7/4,7/8}(S_2^t(a_j))} + \|g\|_{L_4(\Omega^t)} + \|h(0)\|_{W_4^{3/2}(\Omega)} \quad (10.10)$$

and Λ is specified in (1.5).

Proof. We utilize the partition of unity described in Appendix A, and we adopt the notation

$$\tilde{h}^{(k,l)} = h\zeta^{(k,l)}, \quad \tilde{q}^{(k,l)} = q\zeta^{(k,l)}. \quad (10.11)$$

The localized problem (2.6) is restated as

$$\begin{aligned} \varrho(\xi^{(k)}, \xi_0^{(l)})\tilde{h}_t^{(k,l)} - \nu\Delta\tilde{h}^{(k,l)} + \nabla\tilde{q}^{(k,l)} &= [\varrho(\xi^{(k)}, \xi_0^{(l)}) \\ &\quad - \varrho(x, t)]\tilde{h}_t^{(k,l)} + \varrho v\zeta_t^{(k,l)} + q\nabla\zeta^{(k,l)} - 2\nu\nabla h \cdot \nabla\zeta^{(k,l)} \\ &\quad - \nu h\Delta\zeta^{(k,l)} - \varrho(v \cdot \nabla h + h \cdot \nabla v - g)\zeta^{(k,l)} \\ &\quad - \varrho_{x_3}(v_t + v \cdot \nabla v - f)\zeta^{(k,l)} \quad \text{in } \Omega^T, \\ \operatorname{div} \tilde{h}^{(k,l)} &= h \cdot \nabla\zeta^{(k,l)} \quad \text{in } \Omega^T, \\ \bar{n} \cdot \tilde{h}^{(k,l)} &= 0 \quad \text{on } S_1^T, \\ \nu\bar{n} \cdot \mathbb{D}(\tilde{h}^{(k,l)}) \cdot \bar{\tau}_{\alpha} \\ &= \nu n_i(h_i\zeta_{x_j}^{(k,l)} + h_j\zeta_{x_i}^{(k,l)})\tau_{\alpha j} - \gamma\tilde{h}^{(k,l)} \cdot \bar{\tau}_{\alpha}, \quad \alpha = 1, 2 \quad \text{on } S_1^T, \\ \tilde{h}_i^{(k,l)} &= -d_{j,x_i}\zeta^{(k,l)}, \quad i = 1, 2 \quad \text{on } S_2^T(a_j), \quad j = 1, 2, \\ \tilde{h}_{3,x_3}^{(k,l)} &= h_3\zeta_{x_3}^{(k,l)} + \Delta' d_j\zeta^{(k,l)} \quad \text{on } S_2^T(a_j), \quad j = 1, 2, \\ \tilde{h}^{(k,l)}|_{t=0} &= \tilde{h}^{(k,l)}(0). \end{aligned} \quad (10.12)$$

Applying [RZ4, Theorem 6.1], we establish

$$\begin{aligned}
& \|\tilde{h}^{(k,l)}\|_{W_p^{2,1}(\Omega^t)} + \|\nabla \tilde{q}^{(k,l)}\|_{L_p(\Omega^t)} \\
& \leq c(\|\varrho(\xi^{(k)}, \xi_0^{(l)}) - \varrho(x, t)\|_{L_p(\Omega^t)} \|\tilde{h}_t^{(k,l)}\|_{L_p(\Omega^t)} \\
& \quad + c\|\varrho h \zeta_t^{(k,l)}\|_{L_p(\Omega^t)} + c\|q \nabla \zeta^{(k,l)}\|_{L_p(\Omega^t)} + c\|\nabla h \cdot \nabla \zeta^{(k,l)}\|_{L_p(\Omega^t)} \\
& \quad + c\|h \Delta \zeta^{(k,l)}\|_{L_p(\Omega^t)} + c\|\varrho(v \cdot \nabla h + h \cdot \nabla v - g)\zeta^{(k,l)}\|_{L_p(\Omega^t)} \\
& \quad + c\|h \cdot \nabla \zeta^{(k,l)}\|_{W_p^{1,1/2}(\Omega^t)} + c\|\varrho_{x_3}(v_t + v \cdot \nabla v - f)\zeta^{(k,l)}\|_{L_p(\Omega^t)} \\
& \quad + c\|h \cdot \nabla \zeta^{(k,l)}\|_{W_p^{2-3/p, 1-3/2p}(S_1^t)} + c \sum_{i=1}^2 \|d_{i,x'} \zeta^{(k,l)}\|_{W_p^{2-1/p, 1-1/2p}(S_2^t)} \\
& \quad + c\|h_3 \zeta_{x_3}^{(k,l)}\|_{W_p^{2-3/p, 1-3/2p}(S_2^t)} + c \sum_{i=1}^2 \|\Delta d_i \zeta^{(k,l)}\|_{W_p^{2-3/p, 1-3/2p}(S_2^t)} \\
& \quad + c\|\tilde{h}^{(k,l)}(0)\|_{W_p^{2-2/p}(\Omega)}), \tag{10.13}
\end{aligned}$$

Now, we evaluate the particular components on the r.h.s. of (10.13) based on the following assumptions:

$$v \in W_\sigma^{2+s, 1+s/2}(\Omega^t), \quad \nabla p \in W_\sigma^{s, s/2}(\Omega^t), \quad h \in W_{5/3}^{2,1}(\Omega^t), \quad \nabla q \in L_{5/3}(\Omega^t).$$

The first term on the r.h.s. of (10.13) can be estimated by

$$\frac{1}{2} \|\tilde{h}_t^{(k,l)}\|_{L_p(\Omega^t)},$$

where we have used the fact that for $\alpha \in (0, 1)$,

$$\|[\varrho(\xi^{(k)}, \xi_0^{(l)}) - \varrho(x, t)]\tilde{\zeta}^{(k,l)}\|_{L_\infty(\Omega^t)} \leq \phi(t^{\bar{\alpha}}\|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{\alpha}}\|q\|_{L_4(0,t;W_4^1(\Omega))})\lambda^\alpha \leq 1/2.$$

The second term is estimated by

$$c\varrho^*\phi(t^{\bar{\alpha}}\|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{\alpha}}\|q\|_{L_4(0,t;W_4^1(\Omega))})\|h\|_{L_p(\Omega^t \cap \text{supp } \zeta^{(k,l)})},$$

the third by

$$c\phi(t^{\bar{\alpha}}\|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{\alpha}}\|q\|_{L_4(0,t;W_4^1(\Omega))})\|q\|_{L_p(\Omega^t \cap \text{supp } \zeta^{(k,l)})},$$

and the fourth, fifth, seventh, ninth, and eleventh by

$$\begin{aligned}
& \phi(t^{\bar{\alpha}}X_1, t^{\bar{\alpha}}\|q\|_{L_4(0,t;W_4^1(\Omega))})(\|\nabla h\|_{L_p(\Omega^t \cap \text{supp } \zeta^{(k,l)})} + \|h\|_{L_p(\Omega^t \cap \text{supp } \zeta^{(k,l)})}) \\
& \leq c\phi(t^{\bar{\alpha}}X_1, t^{\bar{\alpha}}\|q\|_{L_4(0,t;W_4^1(\Omega))})\|h\|_{W_{5/3}^{2,1}(\Omega^t \cap \text{supp } \zeta^{(k,l)})}
\end{aligned}$$

for $p = 5/2$ and

$$X_1 = \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s, s/2}(\Omega^t)}.$$

Similarly, the sixth term can be bounded by

$$\begin{aligned}
& \varrho^*\|v\|_{L_\infty(\Omega^t)}\|\nabla h\|_{L_p(\Omega^t \cap \text{supp } \zeta^{(k,l)})} + \|g\|_{L_p(\Omega^t \cap \text{supp } \zeta^{(k,l)})} \\
& \quad + \|\nabla v\|_{L_{p\lambda_1}(\Omega^t \cap \text{supp } \zeta^{(k,l)})}\|h\|_{L_{p\lambda_2}(\Omega^t \cap \text{supp } \zeta^{(k,l)})} \quad \text{for } p = 5/2.
\end{aligned}$$

The eighth term is estimated by

$$\begin{aligned} \sup_t \|\varrho_{x_3}\|_{L_{p\lambda_1}(\Omega)} (\|v_t\|_{L_p(0,t;L_{p\lambda_2}(\Omega^t \cap \text{supp } \zeta^{(k,l)}))} + \|v \cdot \nabla v\|_{L_p(0,t;L_{p\lambda_2}(\Omega \cap \text{supp } \zeta^{(k,l)}))} \\ + \|f\|_{L_p(0,t;L_{p\lambda_2}(\Omega \cap \text{supp } \zeta^{(k,l)}))}, \end{aligned}$$

where $1/\lambda_1 + 1/\lambda_2 = 1$.

Utilizing the above estimates in (10.13) and summing over the partition of unity, we derive

$$\begin{aligned} \|h\|_{W_p^{2,1}(\Omega^t)} + \|q\|_{L_p(0,t;W_p^1(\Omega))} &\leq \phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \\ &\cdot [\|h\|_{L_p(\Omega^t)} + \|q\|_{L_p(\Omega^t)} + \|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \varepsilon\|h\|_{W_p^{1,1}(\Omega^t)} + c(1/\varepsilon)\|h\|_{W_{5/3}^{2,1}(\Omega^t)}] \\ &+ \varrho^*\|v\|_{W_p^{2+s,1+s/2}(\Omega^t)}\|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \|g\|_{L_p(\Omega^t)} \\ &+ \phi(t^{\bar{a}}X_1, t^{\bar{a}}X_2)\Lambda_2(\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)} + \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}^2 + \|f\|_{L_\sigma(\Omega^t)}) \\ &+ c \sum_{i,j=1}^2 \|d_{j,x_i}\|_{W_p^{2-1/p,1-1/2p}(S_2^t(a_j))} + \|h(0)\|_{W_p^{2-2/p}(\Omega)}, \end{aligned} \quad (10.14)$$

where we have applied the Poincaré inequality

$$\|\tilde{q}^{(k,l)}\|_{L_p(\Omega)} \leq c\|\nabla \tilde{q}^{(k,l)}\|_{L_p(\Omega)}.$$

After simplifying, we find that

$$\begin{aligned} \|h\|_{W_p^{2,1}(\Omega^t)} + \|q\|_{L_p(0,t;W_p^1(\Omega))} &\leq \phi(t^{\bar{a}}X_1, t^{\bar{a}}X_2)\Lambda + \phi(t^{\bar{a}}X_1)\|q\|_{L_p(\Omega^t)} \\ &+ c \sum_{i,j=1}^2 \|d_{j,x_i}\|_{W_p^{2-1/p,1-1/2p}(S_2^t(a_j))} + \|h(0)\|_{W_p^{2-2/p}(\Omega)}, \end{aligned} \quad (10.15)$$

where $p = 5/2$ and $X_2 = \|h\|_{W_{5/3}^{2,1}(\Omega^t)} + \|\nabla q\|_{L_{5/3}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega^t))}$. Now, we examine the second term on the r.h.s. of (10.15):

$$\begin{aligned} \left(\int_{\Omega^t} |q|^p dx dt \right)^{1/p} &= \left(\int_{\Omega^t} |q|^{p\mu_1} |q|^{p\mu_2} dx dt' \right)^{1/p} \\ &\leq \left(\int_{\Omega^t} |q|^{p\mu_1\lambda_1} dx dt' \right)^{1/p\lambda_1} \left(\int_{\Omega^t} |q|^{p\mu_2\lambda_2} dx dt' \right)^{1/p\lambda_2} \equiv I, \end{aligned}$$

where $\mu_1 + \mu_2 = 1, 1/\lambda_1 + 1/\lambda_2 = 1$. We set

$$p\mu_1\lambda_1 = \sigma, \quad p\mu_2\lambda_2 = 2, \quad 2 < p < \sigma.$$

Hence

$$\begin{aligned} \frac{p\mu_1}{\sigma} + \frac{p\mu_2}{2} &= 1, \quad \frac{p\mu_1}{\sigma} + \frac{p}{2}(1 - \mu_1) = 1, \quad \left(\frac{p}{\sigma} - \frac{p}{2} \right) \mu_1 = 1 - \frac{p}{2}, \\ \mu_1 &= \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{\sigma}}, \quad \mu_2 = \frac{\frac{1}{p} - \frac{1}{\sigma}}{\frac{1}{2} - \frac{1}{\sigma}} \end{aligned}$$

We consider the case where $p = \frac{5}{2}, \sigma = 5$. We then have $\mu_1 = \frac{1}{3}$ and $\mu_2 = \frac{2}{3}$, and we can

write

$$I \leq \|q\|_{L_\sigma(\Omega^t)}^{1/3} \|q\|_{L_2(\Omega^t)}^{2/3} \leq X_1^{1/3} \|q\|_{L_2(\Omega^t)}^{2/3}.$$

Now, we focus on $\|q\|_{L_2(\Omega^t)}$. Let φ be the function satisfying

$$\begin{aligned} \operatorname{div} \varphi &= q, \\ \varphi|_S &= 0, \end{aligned}$$

which leads to $\|\varphi\|_{H_1(\Omega)} \leq c\|q\|_{L_2(\Omega)}$. By multiplying (2.6) by φ and integrating over Ω^t , we arrive at

$$\begin{aligned} \|q\|_{L_2(\Omega^t)}^2 &\leq c(\|h_t\|_{L_2(\Omega^t)}^2 + \|\nabla h\|_{L_2(\Omega^t)}^2) \\ &\quad + \varrho^*(\|v \cdot \nabla h\|_{L_2(\Omega^t)}^2 + \|h \cdot \nabla v\|_{L_2(\Omega^t)}^2 + \|g\|_{L_2(\Omega^t)}^2) \\ &\quad + \|\varrho_{x_3}\|_{L_\infty(0,t;L_4(\Omega))}^2 (\|v_t\|_{L_2(0,t;L_4(\Omega))}^2 + \|v \cdot \nabla v\|_{L_2(0,t;L_4(\Omega))}^2 + \|f\|_{L_2(0,t;L_4(\Omega))}^2) \\ &\leq \phi(t^{\bar{a}} X_1) \|h\|_{W_p^{2,1}(\Omega^t)}^{2\theta} \|h\|_{L_2(\Omega^t)}^{2(1-\theta)} + \|g\|_{L_2(\Omega^t)}^2 \\ &\quad + (\phi(t^{\bar{a}} X_1, t^{\bar{a}} X_2) + \|f\|_{L_2(0,t;L_4(\Omega))}^2) \cdot \Lambda \end{aligned} \quad (10.16)$$

We now interpolate in the first term on the r.h.s. of (10.16), namely, we apply the relation

$$\|h_t\|_{L_2(\Omega^t)} \leq \|h\|_{W_p^{2,1}(\Omega^t)}^\theta \|h\|_{L_2(\Omega^t)}^{(1-\theta)},$$

where

$$\frac{5}{2} - 2 = (1 - \theta) \frac{5}{2} + \theta \left(\frac{5}{p} - 2 \right).$$

Given that $p = \frac{5}{2}$, we can deduce $\frac{1}{2} = (1 - \theta) \frac{5}{2}$, which results in $\theta = \frac{4}{5}$. According to (4.19), we derive

$$\|h\|_{V(\Omega^t)} \leq \phi(t^{\bar{a}} X_1, t^{\bar{a}} X_2) \Lambda. \quad (10.17)$$

Incorporating this into (10.16) leads to

$$\begin{aligned} \|q\|_{L_2(\Omega^t)} &\leq \phi(t^{\bar{a}} X_1, t^{\bar{a}} X_2) \|h\|_{W_p^{2,1}(\Omega^t)}^\theta \Lambda^{(1-\theta)} \\ &\quad + \|g\|_{L_2(\Omega^t)} + (\phi(t^{\bar{a}} X_1, t^{\bar{a}} X_2) + \|f\|_{L_2(0,t;L_4(\Omega))}^2) \Lambda. \end{aligned} \quad (10.18)$$

Set

$$\Lambda_3(p) = \sum_{i,j=1}^2 \|d_{j,x_i}\|_{W_p^{2-1/p, 1-1/2p}(S_2^i(a_j))} + \|g\|_{L_p(\Omega^t)} + \|h(0)\|_{W_p^{2-2/p}(\Omega)}. \quad (10.19)$$

By utilizing (10.17)–(10.19) in (10.15) and applying the Young inequality to eliminate $\|h\|_{W_\sigma^{2,1}(\Omega^t)}$ from the r.h.s. of (10.15), we obtain

$$\|h\|_{W_p^{2,1}(\Omega^t)} + \|q\|_{L_p(0,t;W_p^1(\Omega))} \leq [\phi(t^{\bar{a}} X_1, t^{\bar{a}} X_2) + \|f\|_{L_2(0,t;L_4(\Omega))}] \Lambda + c\Lambda_3(p) \quad (10.20)$$

where $p = 5/2$. Next, we will analyze (10.12), assuming $p = 4$. Then we obtain the results in (10.20) under the condition $p = 4$. Hence (10.20) for $p = 4$ and (10.18) together lead to (10.9), which states that

$$\|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega))} \leq \phi(t^{\bar{a}} X_1, t^{\bar{a}} \|q\|_{L_4(0,t;W_4^1(\Omega))}, \|f\|_{L_2(0,t;L_4(\Omega))}) \Lambda + c\Lambda_3(4).$$

This completes the proof. ■

REMARK 10.4. Estimate (10.9) is crucial in description of d_* (see (6.14)) which is a minimum of v_3 ,

$$v_3 \geq d_*. \quad (10.21)$$

In d_* the norm $\|q\|_{L_1(0,t;L_\infty(\Omega))}$ appears. Therefore, from (10.9) we can derive

$$\|q\|_{L_1(0,t;L_\infty(\Omega))} \leq t^{\bar{a}}(\phi\Lambda + \Lambda_3(4)).$$

Consequently, (6.17) leads to the following requirement:

$$t^{\bar{a}}(\phi\Lambda + \Lambda_3(4) + \|f_3\|_{L_1(0,t;L_\infty(\Omega))}) \leq b^2 e^{-b}. \quad (10.22)$$

11. Long time estimate

REMARK 11.1. We reduce the number of constants D_i , $i = 1, \dots, 11$. Recall that

$$\begin{aligned} D_1 &= |d_1|_{3,2,S_2^t}, \quad D_2 = A_1 + |F|_{6/5,2,\Omega^t} + |\chi(0)|_{3,\Omega}, \\ D_3 &= |(f_1, f_2)|_{\frac{6r}{5r-6},2,\Omega^t} + |\chi(0)|_{3,\Omega}, \\ D_4 &= |f|_{5/3,\Omega^t} + \|d\|_{W_{5/3}^{7/5,7/10}(S_2^t)} + \|v(0)\|_{W_{5/3}^{4/5}(\Omega)}, \\ D_5 &= |f|_{2,\Omega^t} + \|d\|_{W_2^{3/2,3/4}(S_2^t)} + \|v(0)\|_{1,\Omega}. \end{aligned}$$

Therefore, for $r \geq 3$ we have

$$D_1 + D_3 + D_4 \leq D_2 + D_5.$$

Subsequently, we find that

$$D_6 = \|f\|_{L_{5/2}(\Omega^t)} + \|d\|_{W_{5/2}^{8/5,4/5}(S_2^t)} + \|v(0)\|_{W_{5/2}^{6/5}(\Omega)},$$

where $r \geq \sigma$, $s > 5/r$, $r > 3$.

We find that $D_5 \leq D_6$ and

$$\begin{aligned} D_7 &= D_2 + D_6 = A_1 + |F|_{6/5,2,\Omega^t} + |\chi(0)|_{3,\Omega} + \|f\|_{L_{5/2}(\Omega^t)} \\ &\quad + \|d\|_{W_{5/2}^{8/5,4/5}(S_2^t)} + \|v(0)\|_{W_{5/2}^{6/5}(\Omega)}, \\ D_8 &= |f|_{5,\Omega^t} + \|d\|_{W_5^{9/5,9/10}(S_2^t)} + \|v(0)\|_{W_5^{8/5}(\Omega)}. \end{aligned}$$

Moreover,

$$D_7 \leq |F|_{6/5,2,\Omega^t} + A_1 + D_8 \equiv \bar{D}_8$$

since

$$\|\chi(0)\|_{L_3(\Omega)} \leq \|v(0)\|_{W_3^1(\Omega)} \leq \|v(0)\|_{W_5^{8/5}(\Omega)}.$$

We introduce

$$D_9 = \|f\|_{W_\sigma^{s,s/2}(\Omega^t)} + \|d\|_{W_\sigma^{2+s-1/\sigma,1+s/2-1/2\sigma}(S_2^t)} + \|v(0)\|_{W_\sigma^{2+s-2/\sigma}(\Omega)}$$

and

$$\bar{D}_8 \leq A_1 + |F|_{6/5,2,\Omega^t} + D_9.$$

Furthermore, we define

$$D_{10} = \|g\|_{L_{5/3}(\Omega^t)} + \|d_{x'}\|_{W_{5/3}^{7/5,7/10}(S_2^t)} + \|h(0)\|_{W_{5/3}^{4/5}(\Omega)}.$$

Finally, we introduce our last constant that encompasses all previously mentioned constants:

$$D_{11} = D_9 + D_{10} + |F|_{6/5,2,\Omega^t} + A_1.$$

LEMMA 11.2 (Long time estimate). *Assume that $r > \sigma$, $3/r < s$, $5/\sigma < 1 + s$,*

$$D_9 = \|f\|_{W_\sigma^{s,s/2}(\Omega^t)} + \|d\|_{W_\sigma^{2+s-1/\sigma,1+s/2-1/2\sigma}(S_2^t)} + \|v(0)\|_{W_\sigma^{2+s-2/\sigma}(\Omega)} < \infty,$$

$$D_{10} = \|g\|_{L_{5/3}(\Omega^t)} + \|d_{x'}\|_{W_{5/3}^{7/5,7/10}(S_2^t)} + \|h(0)\|_{W_{5/3}^{4/5}(\Omega)} < \infty,$$

$$\|F\|_{L_2(0,t;L_{6/5}(\Omega))} < \infty, \quad D_{11} = D_9 + D_{10} + \|F\|_{L_2(0,t;L_{6/5}(\Omega))} + A_1.$$

Let

$$\begin{aligned} \Lambda_1 &= \|d_{x'}\|_{L_2(0,t;W_3^1(S_2))} + \|d_{x'}\|_{L_\infty(0,t;L_2(S_2))} + \|d_t\|_{L_2(0,t;H^1(S_2))} \\ &\quad + \|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|f_{x_3}\|_{L_4(\Omega^t)}, \end{aligned}$$

$$\Lambda_2 = \|\varrho_{1,x'}\|_{L_r(S_2^t(-a))} + \|\varrho_{1,t}\|_{L_r(S_2^t(-a))} + \|\varrho_{0,x}\|_{L_r(\Omega)},$$

Let $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3(4) + \|h(0)\|_{L_2(\Omega)}$ and $\phi(t)\Lambda$, where ϕ is an increasing positive function, be sufficiently small. Then

$$\begin{aligned} &\|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s,s/2}(\Omega^t)} + \|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega))} \\ &\leq \phi(\varrho^*, A_1, D_{11}). \end{aligned} \quad (11.1)$$

Proof. We introduce the quantity

$$X(t) = \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s,s/2}(\Omega^t)} + \|h\|_{W_4^{2,1}(\Omega^t)} + \|q\|_{L_4(0,t;W_4^1(\Omega))}. \quad (11.2)$$

Inequalities (10.1), (10.3) and (10.9) together imply

$$X \leq \phi(t^{\bar{a}}X) \cdot \Lambda + \phi(\varrho^*, A_1, D_{11}).$$

Therefore, for sufficiently small $\phi(t)\Lambda$, inequality (11.1) holds. ■

REMARK 11.3. Suppose we wish to consider the limit $t \rightarrow \infty$ to deduce a global estimate and global existence. In this scenario, $\phi(t)\Lambda$ in Lemma 11.2 must remain small for all $t \in \mathbb{R}_+$, requiring $\Lambda = 0$ as $t = \infty$ because the limit of $\phi(t)\Lambda$ as $t \rightarrow \infty$ does not exist. Therefore, we consider consequences of the fact that $\Lambda = 0$ and how problem (1.1) degenerates.

The definition of Λ indicates that at $t = \infty$, the following variables are all equal to zero:

$$\begin{aligned} \varrho_{0,x}, \varrho_{0,t}, \varrho_{1,x'}, \varrho_{1,t}, d_{i,x'}, d_{i,t} &= 0, \quad i = 1, 2, \\ v_{0,x_3}, f_{x_3}, f_3 &= 0. \end{aligned}$$

Therefore, $\varrho_0 = \text{const}$, $\varrho_1 = \text{const}$. From (5.1), we deduce that

$$\varrho_{x'} = 0, \varrho_t = 0 \quad \text{as } t = \infty.$$

The continuity equation leads to

$$v_3 \varrho_{x_3} = 0.$$

Given that $v_3|_{S_2(a_i)} = d_i$, $i = 1, 2$ and $d_i \neq 0$, we conclude that $\varrho_{x_3} = 0$. Therefore, (4.19) implies $\|h\|_{V(\Omega^t)} = 0$, meaning $\nabla h = 0$ and $h = 0$ for $t = \infty$. Problem (2.6) can be

expressed as

$$\begin{aligned} h_t - \operatorname{div} \mathbb{T}(h, q) &= 0, \\ \operatorname{div} h &= 0. \end{aligned} \tag{11.3}$$

Multiplying (11.3) by h_t , integrating over Ω and using boundary conditions yields

$$\int_0^t \int_{\Omega} |h_t|^2 dx dt' + \int_{\Omega} |\nabla h(t)|^2 dx = \int_{\Omega} |\nabla h(0)|^2 dx.$$

This results in $h_t = 0$. Further, multiplying (11.3) by η such that

$$\begin{aligned} \operatorname{div} \eta &= q, \\ \eta|_S &= 0 \end{aligned}$$

we obtain $\int_{\Omega} q^2 dx = 0$ so also $q = 0$. Since $v_{x_3} = 0$, it follows that v_3 depends solely on x_1, x_2 and $d_1 = d_2$, $v_3 = d = \operatorname{const}$, $d_1 = d_2 = d$. Then (6.2) transforms to

$$v_{3t} + v \cdot v_3 - \nu \Delta v_3 = 0, \quad v_3|_{S_2} = d.$$

Consequently, we have $v_3 = d + w$, where $w = 0$.

Then problem (1.1) related to $v' = (v_1(x', t), v_2(x', t))$, with $x' = (x_1, x_2)$ and $p' = p(x', t)$, transforms to

$$\begin{aligned} v'_t + v' \cdot \nabla v' - \operatorname{div} \mathbb{T}(v', p') &= f', \\ \operatorname{div} v' &= 0, \\ v' \cdot \bar{n}|_{S_1} &= 0, \\ \nu \bar{n} \cdot \mathbb{D}(v') \cdot \bar{\tau}_1 + \gamma v' \cdot \bar{\tau}_1 &= 0, \quad \bar{\tau}_1 \cdot \bar{e}_3 = 0, e_3 = (0, 0, 1), \\ v'|_{t=0} &= v'(0) \end{aligned}$$

reducing to a two-dimensional problem.

12. Appendix: Stokes problem

This section examines the Stokes problem corresponding to system (1.1) and provides some estimates for potential solutions. The general theory of the Stokes system and parabolic equations can be found in [LSU, S1, S2]. Solvability of the Stokes system was extensively studied in [ZZ1, ZZ2].

The partition of unity introduced in the proof of Lemma 5.2 for [RZ3] is $\zeta^{(k,l)}$, where $\zeta^{(k,l)} = \zeta^{(k)}(x)\zeta_0^{(l)}(t)$, $k, l \in \mathbb{N}$. Let

$$\begin{aligned} \sup_k \operatorname{diam} \operatorname{supp} \zeta^{(k)}(x) &\leq \lambda, \\ \sup_l \operatorname{diam} \operatorname{supp} \zeta_0^{(l)}(t) &\leq \lambda, \end{aligned}$$

with λ to be chosen later. Let $\xi^{(k)}, \xi_0^{(l)}$ be interior points of $\operatorname{supp} \zeta^{(k)}$ and $\operatorname{supp} \zeta_0^{(l)}$, respectively. This indicates that $\xi^{(k)}$ is an interior point of $\operatorname{supp} \zeta^{(k)} \cap S$.

Set $\tilde{v}^{(k,l)} = v\zeta^{(k,l)}$, $\tilde{p}^{(k,l)} = p\zeta^{(k,l)}$, $\tilde{f}^{(k,l)} = f\zeta^{(k,l)}$. The localized problem (1.1) then reads

$$\begin{aligned}
& \varrho(\xi^{(k)}, \xi_0^{(l)})\tilde{v}_t^{(k,l)} - \nu\Delta\tilde{v}^{(k,l)} + \nabla\tilde{p}^{(k,l)} = [\varrho(\xi^{(k)}, \xi_0^{(l)}) \\
& \quad - \varrho(x, t)]\tilde{v}_t^{(k,l)} + \varrho v\zeta_t^{(k,l)} + p\nabla\zeta^{(k,l)} - 2\nu\nabla v\nabla\zeta^{(k,l)} \\
& \quad - \nu v\Delta\zeta^{(k,l)} - \varrho v \cdot \nabla v\zeta^{(k,l)} + \varrho\tilde{f}^{(k,l)} \quad \text{in } \Omega^T, \\
& \operatorname{div} \tilde{v}^{(k,l)} = v \cdot \nabla\zeta^{(k,l)} \quad \text{in } \Omega^T, \\
& \bar{n} \cdot \tilde{v}^{(k,l)} = 0 \quad \text{on } S_1, \\
& \nu\bar{n} \cdot \mathbb{D}(\tilde{v}^{(k,l)}) \cdot \bar{\tau}_\alpha = \nu n_i (v_i \zeta_{x_j}^{(k,l)} + v_j \zeta_{x_i}^{(k,l)}) \tau_{\alpha j} - \gamma \tilde{v}^{(k,l)} \cdot \bar{\tau}_\alpha \quad \text{on } S_1, \\
& \bar{n} \cdot \tilde{v}^{(k,l)}| = \tilde{d}^{(k,l)} \quad \text{on } S_2, \\
& \bar{n} \cdot \mathbb{D}(\tilde{v}^{(k,l)}) \cdot \bar{\tau}_\alpha = n_i (v_i \zeta_{x_j}^{(k,l)} + v_j \zeta_{x_i}^{(k,l)}) \cdot \tau_{\alpha j} \quad \text{on } S_2, \\
& \tilde{v}^{(k,l)}|_{t=0} = \tilde{v}^{(k,l)}(0).
\end{aligned} \tag{A.1}$$

LEMMA A.1. Assume that

$$\begin{aligned}
& v \in W_r^{2+\sigma, 1+\sigma/2}(\Omega^t), \quad v \in W_r^{1, 1/2}(\Omega^t), \quad p \in L_r(\Omega^t), \quad q \in L_4(0, t; W_4^1(\Omega^t)), \\
& v \cdot \nabla v \in L_r(\Omega^t), \quad f \in L_r(\Omega^t), \quad d \in W_r^{2-1/r, 1-1/2r}(S_2^t), \quad v(0) \in W_r^{2-2/r}(\Omega), \\
& 3/r < \sigma, \quad r > 3, \quad \alpha < 1 - 3/r, \\
& \Lambda_2 = \|\varrho_{1, x'}\|_{L_r(S_2^t)} + \|\varrho_{1, t}\|_{L_r(S_2^t)} + \|\varrho_{0, x}\|_{L_r(\Omega)}.
\end{aligned}$$

Additionally, let ϕ be an increasing positive function such that $\phi(0) = 0$. Then

$$\begin{aligned}
& \|v\|_{W_r^{2,1}(\Omega^t)} + \|\nabla p\|_{L_r(\Omega^t)} \\
& \leq c(\varrho^*)\phi(t^{\bar{\alpha}}\|v\|_{W_r^{2+\sigma, 1+\sigma/2}(\Omega^t)}, t^{\bar{\alpha}}\|q\|_{L_4(0, t; W_4^1(\Omega^t))}) \cdot \Lambda_2 \cdot [\|v\|_{W_r^{1, 1/2}(\Omega^t)} + \|p\|_{L_r(\Omega^t)}] \\
& \quad + c(\varrho^*)[\|v \cdot \nabla v\|_{L_r(\Omega^t)} + \|f\|_{L_r(\Omega^t)} + \|d\|_{W_r^{2-1/r, 1-1/2r}(S_2^t)} + \|v(0)\|_{W_r^{2-2/r}(\Omega)}]. \tag{A.2}
\end{aligned}$$

Proof. Theorem 6.1 in [RZ4] gives us

$$\begin{aligned}
& \|\tilde{v}^{(k,l)}\|_{W_r^{2,1}(\Omega^t)} + \|\nabla\tilde{p}^{(k,l)}\|_{L_r(\Omega^t)} \\
& \leq c(\|(\varrho(\xi^{(k)}, \xi_0^{(l)}) - \varrho(x, t))\tilde{v}_t^{(k,l)}\|_{L_r(\Omega^t)} + \|\varrho v\zeta_t^{(k,l)}\|_{L_r(\Omega^t)} \\
& \quad + \|p\nabla\zeta^{(k,l)}\|_{L_r(\Omega^t)} + \|\nabla v\nabla\zeta^{(k,l)}\|_{L_r(\Omega^t)} + \|v\Delta\zeta^{(k,l)}\|_{L_r(\Omega^t)} \\
& \quad + \|\varrho v \cdot \nabla v\|_{L_r(\Omega^{(k,l)} \times (0, t))} + \|\varrho\tilde{f}^{(k,l)}\|_{L_r(\Omega^t)} \\
& \quad + \|v\nabla\zeta^{(k,l)}\|_{W_r^{1-1/r, 1/2-1/2r}(S_2^t)} + \|\tilde{d}^{(k,l)}\|_{W_r^{2-1/r, 1-1/2r}(S_2^t)} \\
& \quad + \|\tilde{v}^{(k,l)}(0)\|_{W_r^{2-2/r}(\Omega)}), \tag{A.3}
\end{aligned}$$

where $\Omega^{(k,l)} = \Omega^t \cap \operatorname{supp} \zeta^{(k,l)}$.

The first term on the r.h.s. of (A.3) is bounded by

$$\begin{aligned}
& \left(\sup_{x, x', t \in \Omega^{(k,l)}} \frac{|\varrho(x, t) - \varrho(x', t)|}{|x - x'|^\alpha} \lambda^\alpha + \sup_{x, t, t' \in \Omega^{(k,l)}} \frac{|\varrho(x, t) - \varrho(x, t')|}{|t - t'|^\alpha} \lambda^\alpha \right) \|\tilde{v}_t^{(k,l)}\|_{L_r(\Omega^t)} \\
& \leq c\|\varrho\|_{\dot{C}^\alpha(\Omega^{(k,l)})} \lambda^\alpha \|\tilde{v}_t^{(k,l)}\|_{L_r(\Omega^t)}.
\end{aligned}$$

Summing inequalities (A.3) over all neighborhoods of the partition of unity, taking into account that

$$c\|\varrho\|_{\dot{C}^\alpha(\Omega \times (0,t))} \lambda^\alpha \leq \frac{1}{2} \quad (\text{A.4})$$

and noting that $|\nabla \zeta^{(k,l)}| \leq \frac{c}{\lambda}$ etc. results in the inequality

$$\begin{aligned} & \|v\|_{W_r^{2,1}(\Omega^t)} + \|\nabla p\|_{L_r(\Omega^t)} \\ & \leq c(\varrho^*)\phi(\|\varrho\|_{\dot{C}^\alpha(\Omega^t)}) \cdot (\|v\|_{W_r^{1,1/2}(\Omega^t)} + \|p\|_{L_r(\Omega^t)}) \\ & \quad + c(\varrho^*)(\|v \cdot \nabla v\|_{L_r(\Omega^t)} + \|f\|_{L_r(\Omega^t)} + \|d\|_{W_r^{2-1/r, 1-1/2r}(S_2^t)} + \|v(0)\|_{W_r^{2-2/r}(\Omega)}). \end{aligned} \quad (\text{A.5})$$

Assume $r > 3$. Then for $\alpha < 1 - 3/r$ we conclude that

$$\|\varrho\|_{\dot{C}^\alpha(\Omega^t)} \leq c\|\varrho\|_{\dot{W}_{r,\infty}^{1,1}(\Omega^t)} \leq \phi(t^{\bar{a}}\|v\|_{W_r^{2+\sigma, 1+\sigma/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \cdot \Lambda_2, \quad (\text{A.6})$$

where $3/r < \sigma$. This ends the proof. ■

LEMMA A.2. Assume that $r > \sigma$, $3/r < s$, $5/\sigma < 1+s$. Suppose also $v \in W_\sigma^{2+s, 1+s/2}(\Omega^t)$, $p \in W_\sigma^{s, s/2}(\Omega^t)$, $q \in L_4(0, t; W_4^1(\Omega^t))$, $\varrho \in W_{r,\infty}^{1,1}(\Omega^t)$, $d \in W_\sigma^{2+s-1/\sigma, 1+s/2-1/2\sigma}(\Omega^t)$, $f \in W_\sigma^{s, s/2}(\Omega^t)$, $v(0) \in W_\sigma^{2+s-2/\sigma}(\Omega)$, and A_1 is introduced in (1.8). Then

$$\begin{aligned} & \|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + \|\nabla p\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \leq [\phi(t^{\bar{a}}\|v\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)}, t^{\bar{a}}\|q\|_{L_4(0,t;W_4^1(\Omega^t))}) \cdot \Lambda_2 + c\varrho^*] \\ & \quad \cdot (\|v\|_{W_\sigma^{1+s, 1/2+s/2}(\Omega^t)} + \|p\|_{W_\sigma^{s, s/2}(\Omega^t)}) \\ & \quad + \phi(\varrho^*)[\|\varrho v \cdot \nabla v\|_{W_\sigma^{s, s/2}(\Omega^t)} + \|\varrho\|_{C^\alpha(\Omega^t)}\|f\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \quad + A_1 + \|d\|_{W_\sigma^{2+s-1/\sigma, 1+s/2-1/2\sigma}(S_2^t)} + \|v(0)\|_{W_\sigma^{2+s-2/\sigma}(\Omega)}]. \end{aligned} \quad (\text{A.7})$$

Proof. Applying Theorem 1.1 from [RZ4] to (A.1) results in

$$\begin{aligned} & \|\tilde{v}^{(k,l)}\|_{W_\sigma^{2+s, 1+s/2}(\Omega^t)} + \|\nabla \tilde{p}^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \leq c[\|(\varrho(\xi^{(k)}, \xi_0^{(l)}) - \varrho(x, t))\tilde{v}_t^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \quad + \|\varrho v \zeta_t^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} + \|p \nabla \zeta^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \quad + \|\nabla v \nabla \zeta^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} + \|v \Delta \zeta^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \quad + \|\varrho v \cdot \nabla v \zeta^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} + \|\varrho \tilde{f}^{(k,l)}\|_{W_\sigma^{s, s/2}(\Omega^t)} \\ & \quad + \|v \nabla \zeta^{(k,l)}\|_{W_\sigma^{1+s-1/\sigma, 1/2+s/2-1/2\sigma}(S_2^t)} \\ & \quad + \|\tilde{v}^{(k,l)}\|_{W_\sigma^{1+s-1/\sigma, 1/2+s/2-1/2\sigma}(S_2^t)} \\ & \quad + \|\tilde{d}^{(k,l)}\|_{W_\sigma^{2+s-1/\sigma, 1+s/2-1/2\sigma}(S_2^t)} + \|\tilde{v}^{(k,l)}(0)\|_{W_\sigma^{2+s-2/\sigma}(\Omega)}]. \end{aligned} \quad (\text{A.8})$$

Considering $u, v \in L_\infty(\Omega^t)$, we get

$$\|uv\|_{W_\sigma^{s, s/2}(\Omega^t)} \leq \|u\|_{L_\infty(\Omega^t)}\|v\|_{W_\sigma^{s, s/2}(\Omega^t)} + \|v\|_{L_\infty(\Omega^t)}\|u\|_{W_\sigma^{s, s/2}(\Omega^t)}. \quad (\text{A.9})$$

Define $\Omega_{k,l}^t = \Omega^t \cap \text{supp } \zeta^{(k,l)}$.

By utilizing (A.9), the second term on the r.h.s. of (A.8) is bounded by

$$\frac{c}{\lambda} (\|\varrho\|_{L_\infty(\Omega^t)} \|v\|_{W_\sigma^{s,s/2}(\Omega_{k,l}^t)} + \|\varrho\|_{W_\sigma^{s,s/2}(\Omega_{k,l}^t)} \|v\|_{L_\infty(\Omega^t)} + \|\varrho\|_{L_\infty(\Omega^t)} \|v\|_{L_\infty(\Omega_{k,l}^t)})$$

and we apply the embeddings

$$\begin{aligned} \|\varrho\|_{W_\sigma^{s,s/2}(\Omega^t)} &\leq c \|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)} && \text{for } \frac{3}{r} - \frac{5}{\sigma} + s < 1, \\ \|\varrho\|_{L_\infty(\Omega^t)} &\leq c \|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)} && \text{for } \frac{3}{r} < 1, \\ \|v\|_{L_\infty(\Omega^t)} &\leq c \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)} && \text{for } \frac{5}{\sigma} < 2 + s. \end{aligned}$$

In a similar way, the fifth term is bounded by

$$\frac{c}{\lambda} \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega_{k,l}^t)}.$$

We assume that

$$\|\nabla v\|_{L_\infty(\Omega^t)} \leq c \|v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}$$

which is valid for $5/\sigma < 1 + s$. Consequently, the fourth term is bounded by

$$\frac{c}{\lambda} \|\nabla v\|_{W_\sigma^{2+s,1+s/2}(\Omega^t)}.$$

The seventh term on the r.h.s. of (A.8) equals

$$I \equiv \|\varrho \tilde{f}^{(k,l)}\|_{L_\sigma(0,t;W_\sigma^s(\Omega))} + \|\varrho \tilde{f}^{(k,l)}\|_{L_\sigma(\Omega;W_\sigma^{s/2}(0,t))} \equiv I_1 + I_2,$$

where I_1 is bounded by

$$\begin{aligned} &\left(\frac{\int_0^t dt \int_\Omega \int_\Omega dx' dx'' |\varrho(x',t) - \varrho(x'',t)|^\sigma |\tilde{f}^{(k,l)}(x',t)|^\sigma}{|x' - x''|^{3+s\sigma/2}} \right)^{1/\sigma} \\ &+ \left(\int_0^t dt \int_\Omega \int_\Omega dx' dx'' \frac{|\varrho(x'',t)|^\sigma |\tilde{f}^{(k,l)}(x',t) - \tilde{f}^{(k,l)}(x'',t)|^\sigma}{|x' - x''|^{3+s\sigma/2}} \right)^{1/\sigma} \equiv I_1^1 + I_1^2. \end{aligned}$$

Then we get

$$I_1^2 \leq c \|\varrho\|_{L_\infty(\Omega^t)} \|\tilde{f}^{(k,l)}\|_{L_\sigma(0,t;W_\sigma^s(\Omega))}.$$

To estimate I_1^1 , we utilize the Hölder inequality. This leads to

$$\begin{aligned} I_1^1 &\leq \left[\int_0^t dt \left(\int_\Omega \int_\Omega dx' dx'' \frac{|\varrho(x',t) - \varrho(x'',t)|^{\sigma\lambda_1}}{|x' - x''|^{3+\sigma\lambda_1 s'}} \right) \right]^{1/\sigma\lambda_1} \\ &\cdot \left[\int_0^t dt \left(\int_\Omega \int_\Omega \frac{|\tilde{f}^{(k,l)}(x',t)|^{\sigma\lambda_2}}{|x' - x''|^{3/2\lambda_2}} dx' dx'' \right) \right]^{1/\sigma\lambda_2} \equiv I_1^{11}, \end{aligned}$$

where $s' = \frac{1}{\sigma\lambda_1} (\frac{3}{2}\lambda_1 - 3) + s$ and $1/\lambda_1 + 1/\lambda_2 = 1$. By employing the fact that $\lambda_2 < 2$, we obtain

$$I_1^{11} \leq \|\varrho\|_{L_{\sigma\lambda_1}(0,t;W_{\sigma\lambda_1}^{s'}(\Omega))} \|\tilde{f}^{(k,l)}\|_{L_{\sigma\lambda_2}(0,t;L_{\sigma\lambda_2}(\Omega))} \equiv I_1^{12}.$$

We apply the previously mentioned embedding

$$\|\varrho\|_{L_{\sigma\lambda_1}(0,t;W_{\sigma\lambda_1}^{s'}(\Omega))} \leq c \|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)},$$

which holds under the condition $\frac{3}{r} - \frac{5}{\sigma\lambda_1} + s' \leq 1$ and

$$\|\tilde{f}^{(k,l)}\|_{L_{\sigma\lambda_2}(\Omega^t)} \leq c\|\tilde{f}^{(k,l)}\|_{W_{\sigma}^{s,s/2}(\Omega^t)}$$

for $\frac{5}{\sigma} - \frac{5}{\sigma\lambda_2} \leq s$.

The constraints above lead to

$$\frac{3}{r} + \frac{3}{2\sigma} - \frac{3}{\sigma\lambda_1} + s \leq 1 + s \quad \text{so} \quad \frac{3}{r} + \frac{3}{2\sigma} \leq 1 + \frac{3}{\sigma\lambda_1} \leq 1 + \frac{3}{2\sigma}$$

since $1/\lambda_1 < 1/2$. Hence we get $\frac{3}{r} \leq 1$.

In summary, we obtain

$$I_1 \leq c\|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)}\|\tilde{f}^{(k,l)}\|_{W_{\sigma}^{s,s/2}(\Omega^t)} \quad \text{for} \quad \frac{3}{r} \leq 1.$$

Now, let us examine I_2 , where

$$\begin{aligned} I_2 &= \left(\int_{\Omega} dx \int_0^t \int_0^t dt' dt'' \frac{|\varrho(x, t')\tilde{f}^{(k,l)}(x, t') - \varrho(x, t'')\tilde{f}^{(k,l)}(x, t'')|^{\sigma}}{|t' - t''|^{1+\sigma s/2}} \right)^{1/\sigma} \\ &\leq \left(\int_{\Omega} dx \int_0^t \int_0^t dt' dt'' \frac{|\varrho(x, t') - \varrho(x, t'')|^{\sigma} |\tilde{f}^{(k,l)}(x, t')|^{\sigma}}{|t' - t''|^{1+\sigma s/2}} \right)^{1/\sigma} \\ &\quad + \left(\int_{\Omega} dx \int_0^t \int_0^t dt' dt'' \frac{|\varrho(x, t'')|^{\sigma} |\tilde{f}^{(k,l)}(x, t') - \tilde{f}^{(k,l)}(x, t'')|^{\sigma}}{|t' - t''|^{1+\sigma s/2}} \right)^{1/\sigma} \\ &\equiv I_2^1 + I_2^2, \end{aligned}$$

with

$$I_2^2 \leq \|\varrho\|_{L_{\infty}(\Omega^t)}\|\tilde{f}^{(k,l)}\|_{L_{\sigma}(\Omega; W_{\sigma}^{s/2}(0,t))}$$

and

$$\begin{aligned} I_2^1 &\leq \left(\int_{\Omega} dx \int_0^t \int_0^t dt' dt'' \frac{|t' - t''|^{\sigma} \sup_t |\varrho_t(x, t)|^{\sigma} |\tilde{f}^{(k,l)}(x, t')|^{\sigma}}{|t' - t''|^{1+\sigma s/2}} \right)^{1/\sigma} \\ &\leq c \sup_t |\varrho_t|_{\lambda_1\sigma, \Omega} |\tilde{f}^{(k,l)}|_{\lambda_2\sigma, \sigma, \Omega^t} \equiv I_2^{11}, \end{aligned}$$

given that $1/\lambda_1 + 1/\lambda_2 = 1$, $\lambda_1\sigma = r$, $\frac{5}{\sigma} - \frac{3}{\lambda_2\sigma} - \frac{2}{\sigma} \leq s$. This means that $\frac{3}{r} \leq s$ and $r > \sigma$. Next,

$$I_2^{11} \leq c\|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)}\|\tilde{f}^{(k,l)}\|_{W_{\sigma}^{s,s/2}(\Omega^t)}.$$

To summarize, we have

$$I \leq c\|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)}\|\tilde{f}^{(k,l)}\|_{W_{\sigma}^{s,s/2}(\Omega^t)}$$

for $r > \sigma$, $\frac{3}{r} \leq s$.

In a similar manner, the third component on the r.h.s. of (A.8) can be estimated by

$$\frac{c}{\lambda} \|p\|_{W_{\sigma}^{s,s/2}(\Omega_{k,t}^t)}$$

and the eighth term by

$$\frac{c}{\lambda} \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega_{k,t}^t)}.$$

At last, we estimate the first term on the right side of (A.8) by

$$\begin{aligned}
& \sup_{\Omega_{k,l}^t} |\varrho(x, t) - \varrho(\xi^{(k)}, \xi_0^{(l)})| \|\tilde{v}_t^{(k,l)}\|_{W_{\sigma}^{s,s/2}(\Omega^t)} \\
& + \left(\int_{\Omega} dt \int_{\Omega} \frac{|\varrho(x', t) - \varrho(x'', t)|^{\sigma} |\tilde{v}_t^{(k,l)}(x', t)|^{\sigma}}{|x' - x''|^{3+s\sigma}} dx' dx'' \right)^{1/\sigma} \\
& + \left(\int dx \int_0^t \int_0^t \frac{|\varrho(x, t') - \varrho(x, t'')|^{\sigma} |\tilde{v}_t^{(k,l)}(x, t')|^{\sigma}}{|t' - t''|^{1+s\sigma/2}} dt' dt'' \right)^{1/\sigma} \\
& \equiv J_1 + J_2 + J_3,
\end{aligned}$$

where

$$\begin{aligned}
J_1 & \leq \|\varrho\|_{\dot{C}^{\alpha}(\Omega_{k,l}^t)} \lambda^{\alpha} \|\tilde{v}_t^{(k,l)}\|_{W_{\sigma}^{s,s/2}(\Omega^t)}, \\
J_2 + J_3 & \leq c \|\varrho\|_{W_{r,\infty}^{1,1}(\Omega^t)} \|\tilde{v}_t^{(k,l)}\|_{W_{\sigma}^{s/2,s/4}(\Omega^t)},
\end{aligned}$$

and the final inequality is derived similarly to the estimation of the seventh term.

By summing over all neighborhoods from the partition of unity, we arrive at

$$\begin{aligned}
& \|v\|_{W_{\sigma}^{2+s,1+s/2}(\Omega^t)} + \|\nabla p\|_{W_{\sigma}^{s,s/2}(\Omega^t)} \\
& \leq \phi(\|\varrho\|_{\dot{C}^{\alpha}(\Omega^t)}) [\|v\|_{W_{\sigma}^{1+s,1/2+s/2}(\Omega^t)} + \|p\|_{W_{\sigma}^{s,s/2}(\Omega^t)}] \\
& + c(\varrho^*) [\|\varrho v \cdot \nabla v\|_{W_{\sigma}^{s,s/2}(\Omega^t)} + \|\varrho\|_{C^{\alpha}(\Omega^t)} \|f\|_{W_{\sigma}^{s,s/2}(\Omega^t)}] \\
& + \|v\|_{W_{\sigma}^{1+s,1/2+s/2}(\Omega^t)} + \|d\|_{W_{\sigma}^{2+s-1/\sigma,1+s/2-1/2\sigma}(S_2^t)} \\
& + \|v(0)\|_{W_{\sigma}^{2+s-2/\sigma}(\Omega^t)}, \tag{A.10}
\end{aligned}$$

where $\phi(0) = 0$.

Through various interpolations and employing Corollary 5.3, we achieve (A.7). This concludes the proof. ■

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