

Three colours of Pythagorean triangles

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Summary. For any colouring of the set of all Pythagorean triangles with three colours there exist three Pythagorean triangles $\Delta_1, \Delta_2, \Delta_3$ of the same colour satisfying

$$A(\Delta_1)A(\Delta_2)A(\Delta_3) = t^2,$$

where $A(\Delta)$ denotes the area of the triangle Δ and $t \in \mathbb{N}$. This provides a counterpart to Fermat's right triangle theorem.

The area of a triangle Δ will be denoted by $A(\Delta)$. A right triangle Δ is called *Pythagorean* if all its sides are integers. If Δ is a Pythagorean triangle then by Fermat's famous right triangle theorem, $A(\Delta)$ is not a square of an integer. Here we have a substitute.

THEOREM 1. *For any colouring of the set of all Pythagorean triangles with three colours there exist three Pythagorean triangles $\Delta_1, \Delta_2, \Delta_3$ of the same colour satisfying*

$$(1) \quad A(\Delta_1)A(\Delta_2)A(\Delta_3) = \square.$$

Proof. Recall the classical definition. A natural number d is *congruent* if there exists a right triangle with rational sides and area equal to d . Here and in the sequel we shall use extensively the notion of *square-free part* $\text{sfp}(r)$ of a positive rational number r : we define $\text{sfp}(r)$ as the unique square-free integer d satisfying $r = ds^2$ with some $s \in \mathbb{Q}$. Let

$$M = \{6, 5, 13, 53, 357\}$$

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and consider the set

$$\mathbb{V} := \left\{ v \mid \exists K \subset M, K \neq \emptyset, v = \text{sfp} \left(\prod_{d \in K} d \right) \right\}.$$

It is easy to verify that $\sharp \mathbb{V} = 31$ and with some effort we list all its elements:

$$\mathbb{V} = \{5, 6, 13, 30, 53, 65, 78, 238, 265, 318, 357, 390, 689, 1190, 1590, 1785, 3094, 3445, 4134, 4641, 12614, 15470, 18921, 20670, 23205, 63070, 94605, 163982, 245973, 819910, 1229865\}.$$

Now we shall prove that all elements v of $\mathbb{V}$ are congruent! Namely, each row of the table below starts with $v \in \mathbb{V}$ and columns a, b, c contain values $a, b, c \in \mathbb{Q}$ satisfying

$$a^2 + b^2 = c^2 \quad \text{and} \quad \frac{1}{2}ab = v.$$

v	a	b	c
5	3/2	20/3	41/6
6	3	4	5
13	780/323	323/30	106921/9690
30	5	12	13
53	$a(53)$	$b(53)$	$c(53)$
65	65/6	12	97/6
78	45	52/15	677/15
238	414239/72570	6051600/72570	6065761/72570
265	28/3	795/14	2417/42
318	7685/209	2508/145	1231453/30305
357	84/5	85/2	457/10
390	39/2	40	89/2
689	561/20	27560/561	634721/11220
1190	35/3	204	613/3
1590	12720/247	247/4	79441/988
1785	119/2	60	169/2
3094	429	476/33	14165/33
3445	5738557/36162	249156180/5738557	34141367941801/207517698234
4134	53/3	468	1405/3
4641	273/2	68	305/2
12614	284053/5640	1195680/2387	6777635761/13462680
15470	156	595/3	757/3
18921	3339/22	748/3	19265/66
20670	477/2	520/3	1769/6
23205	2275/6	612/5	11953/30
63070	901/3	420	1549/3
94605	231962059/245836	390879240/1949261	462252622076401/479198527196
163982	195	25228/15	25397/15
245973	31437845/18814	544439532/1849285	59032989045577/34792447990
819910	2764580/2673	2408373/1519	7686181421/4060287
1229865	25493/14	49980/37	1174441/518

where

$$a(53) = \frac{1472112483}{202332130}, \quad b(53) = \frac{21447205780}{1472112483}, \quad c(53) = \frac{4850493897329785961}{297855654284978790}.$$

Now let us consider a complete graph on the set $\mathbb{V}$. For any distinct $v_1, v_2 \in \mathbb{V}$, we colour the edge v_1v_2 using the colour of any chosen Pythagorean triangle Δ which satisfies $\text{sfp}(A(\Delta)) = \text{sfp}(v_1v_2)$. We use here the fact that for any distinct $v_1, v_2 \in \mathbb{V}$ also $\text{sfp}(v_1v_2) \in \mathbb{V}$, and hence $\text{sfp}(v_1v_2)$ is congruent.

Now let $R_r(3)$ be the least natural number R for which the following statement holds:

- For any r -colouring of edges of a complete graph on R vertices there exists at least one monochromatic triangle.

By the classical Ramsey theorem, $R_r(3)$ is a finite number and

$$R_r(3) \leq 3r!$$

by [3, Lemma 8.7, pp. 225–226]. In our case $r = 3$ and $R_3(3) \leq 18 < 31$. Hence we obtain a monochromatic triangle with vertices $v_1, v_2, v_3 \in \mathbb{V}$. So there exist Pythagorean triangles $\Delta_1, \Delta_2, \Delta_3$ of the same colour such that $\text{sfp}(A(\Delta_1)) = \text{sfp}(v_2v_3)$, $\text{sfp}(A(\Delta_2)) = \text{sfp}(v_1v_3)$, $\text{sfp}(A(\Delta_3)) = \text{sfp}(v_1v_2)$.

This yields (1). ■

REMARK. Being a congruent number depends only on $\text{sfp}(d)$; here are the details. A Pythagorean triangle Δ with sides $a < b < c$ is similar to the triangle Δ_1 with sides $a/c, b/c, 1$; it follows that

$$A(\Delta) = c^2 \cdot A(\Delta_1).$$

By a well known rational parametrization of the unit circle there exists $x \in \mathbb{Q}$ such that

$$\frac{a}{c} = \frac{2x}{x^2 + 1}, \quad \frac{b}{c} = \frac{x^2 - 1}{x^2 + 1},$$

hence

$$A(\Delta) = \left(\frac{c}{x^2 + 1} \right)^2 (x^3 - x).$$

Concluding, the numbers $\text{sfp}(x^3 - x)$ for rational $x > 1$ are exactly the square-free congruent numbers. It follows that Theorem 1 can also be formulated in the following way.

THEOREM 2. *For any colouring of $\mathbb{Q}_{>1}$ with three colours there exists a monochromatic triple $x, y, z \in \mathbb{Q}_{>1}$ satisfying*

$$(2) \quad (x^3 - x)(y^3 - y)(z^3 - z) = \square.$$

Proof. For each rational $x > 1$ let us consider the following rational right triangle $\Delta(x)$:

$$a = 2x, \quad b = x^2 - 1, \quad c = x^2 + 1$$

with area $A(\Delta(x)) = x^3 - x$. Now let us colour all Pythagorean triangles similar to $\Delta(x)$ the same colour the number x is coloured. Now the assertion of Theorem 2 follows from Theorem 1 when we recognize that scaling the sides of a triangle by a rational factor $k > 0$ leads to scaling its area by k^2 . ■

For properly restricted kinds of colourings we give here a strengthened variant of Theorem 2 but only for two colours. We need the following definition. A colouring of $\mathbb{N}$ or $\mathbb{Q}^+$ will be called a *square respecting colouring* (breifly, an *SR-colouring*) if $\text{sfp}(x) = \text{sfp}(y)$ implies that x and y are of the same colour.

THEOREM 3. *For any SR-colouring of $\mathbb{Q}^+$ with two colours there exist $x, y, z \in \mathbb{Q}_{>1}$ such that*

$$(3) \quad (x-1)x(x+1)(y-1)y(y+1)(z-1)z(z+1) = \square$$

and all nine factors on the left-hand side are of the same colour.

Proof. As in the proof of Theorem 1, we start with

$$M = \{6, 5, 13\}$$

and relevant

$$\mathbb{V} = \{5, 6, 13, 30, 65, 78, 390\}.$$

Not only are all numbers in $\mathbb{V}$ congruent but we can easily write down explicit equations satisfied by points on the relevant elliptic curves $x^3 - x = dy^2$:

$$\begin{aligned} 9^3 - 9 &= 5 \cdot 12^2, & 2^3 - 2 &= 6 \cdot 1^2, & \left(\frac{325}{36}\right)^3 - \frac{325}{36} &= 13 \cdot \left(\frac{1615}{216}\right)^2, \\ 5^3 - 5 &= 30 \cdot 2^2, & \left(\frac{9}{4}\right)^3 - \frac{9}{4} &= 65 \cdot \left(\frac{3}{8}\right)^2, & 26^3 - 26 &= 78 \cdot 15^2, \\ \left(\frac{13}{3}\right)^3 - \frac{13}{3} &= 390 \cdot \left(\frac{4}{9}\right)^2 \end{aligned}$$

(consult [2] for more information on the relation between congruent numbers and elliptic curves).

Now, denoting the x -coordinate of the double point on $x^3 - x = dy^2$ by x_d we obtain

$$\begin{aligned} x_5 &= \frac{1681}{720}, & x_6 &= \frac{25}{24}, & x_{13} &= \frac{11432100241}{4882597200}, & x_{30} &= \frac{169}{120}, \\ x_{65} &= \frac{9409}{9360}, & x_{78} &= \frac{458329}{70200}, & x_{390} &= \frac{7921}{6240}. \end{aligned}$$

and finally

$$x_5 - 1 = 5 \left(\frac{31}{60}\right)^2, \quad x_5 = 5 \left(\frac{41}{60}\right)^2 = \frac{1681}{720}, \quad x_5 + 1 = 5 \left(\frac{49}{60}\right)^2,$$

$$\begin{aligned}
x_6 - 1 &= 6 \left(\frac{1}{12} \right)^2, & x_6 &= 6 \left(\frac{5}{12} \right)^2 = \frac{25}{24}, & x_6 + 1 &= 6 \left(\frac{7}{12} \right)^2, \\
x_{13} - 1 &= 13 \left(\frac{80929}{251940} \right)^2, & x_{13} &= 13 \left(\frac{106921}{251940} \right)^2 = \frac{11432100241}{4882597200}, \\
x_{13} + 1 &= 13 \left(\frac{127729}{251940} \right)^2, \\
x_{30} - 1 &= 30 \left(\frac{7}{60} \right)^2, & x_{30} &= 30 \left(\frac{13}{60} \right)^2 = \frac{169}{120}, & x_{30} + 1 &= 30 \left(\frac{17}{60} \right)^2, \\
x_{65} - 1 &= 65 \left(\frac{7}{780} \right)^2, & x_{65} &= 65 \left(\frac{97}{780} \right)^2 = \frac{9409}{9360}, & x_{65} + 1 &= 65 \left(\frac{137}{780} \right)^2, \\
x_{78} - 1 &= 78 \left(\frac{623}{2340} \right)^2, & x_{78} &= 78 \left(\frac{677}{2340} \right)^2 = \frac{458329}{70200}, & x_{78} + 1 &= 78 \left(\frac{727}{2340} \right)^2, \\
x_{390} - 1 &= 390 \left(\frac{41}{1560} \right)^2, & x_{390} &= 390 \left(\frac{89}{1560} \right)^2 = \frac{7921}{6240}, \\
x_{390} + 1 &= 390 \left(\frac{119}{1560} \right)^2.
\end{aligned}$$

Consider now the complete graph on $\mathbb{V}$, colouring the edge uv with the colour of $\text{sfp}(uv)$. Since

$$R_2(3) = 6 < 7 = |\mathbb{V}|,$$

there is a monochromatic triangle with vertices v_1, v_2, v_3 . Put

$$d_1 = \text{sfp}(v_2 v_3), \quad d_2 = \text{sfp}(v_1 v_3), \quad d_3 = \text{sfp}(v_1 v_2).$$

Then d_1, d_2, d_3 have the same colour, while $d_1 d_2 d_3$ is a square. Take $x = x_{d_1}, y = x_{d_2}, z = x_{d_3}$; then the SR property and the displayed identities imply that all nine factors

$$x - 1, x, x + 1, y - 1, y, y + 1, z - 1, z, z + 1$$

have the same colour, and their product is a square. ■

CONJECTURE. *For any colouring of $\mathbb{Q}_{>1}$ with $r \geq 2$ colours there exists a monochromatic triple $x, y, z \in \mathbb{Q}_{>1}$ satisfying (2). Moreover, for any SR-colouring of $\mathbb{Q}^+$ with $r \geq 2$ colours there exist $x, y, z \in \mathbb{Q}_{>1}$ satisfying (3) and such that all nine factors on its left-hand side are of the same colour.*

REMARK. On the one hand, values of many integer polynomials $f(x)$ cannot be squares, but simultaneously the equation $f(x)f(y)f(z) = t^2$ has many solutions in integers (or rationals) x, y, z, t . On the other hand, the product of three numbers being a square is an interesting combinatorial property of subsets of numbers. In [1] Szemerédi-type results are provided, and in [4] some general Ramsey-type issues are dealt with.

References

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