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Piatetski-Shapiro sequences via Beatty sequences

by

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1. Introduction. *Piatetski-Shapiro sequences* are sequences of the form $(\lfloor n^c \rfloor)_{n\geq 1}$, where $c>1$ is not an integer. They are named after I. Piatetski-Shapiro, who proved the following Prime Number Theorem (see [19]): If $1<c<12/11$, then

$$|\{n \leq x : \lfloor n^c \rfloor \text{ is prime}\}| \sim \frac{x}{c \log x}. \tag{1}$$

The range for c has been extended several times, the currently best known upper bound being $c < \frac{2817}{2426}$, obtained by Rivat and Sargos [21]. It is expected that the asymptotic formula (1) holds for all $c \in (1,2)$, an expectation backed up by the fact that it is true for almost all $c \in [1,2]$ with respect to the Lebesgue measure (see [12]).

For a collection of arithmetic results on Piatetski-Shapiro sequences see the article [1] by Baker et al. For example in that article it is proved in detail that for $1<c<\frac{149}{87}$ the number of squarefree integers of the form $\lfloor n^c \rfloor$ behaves as expected: for c in this range we have

$$|\{n \leq x : \lfloor n^c \rfloor \text{ is squarefree}\}| = \frac{6}{\pi^2}x + O(x^{1-\varepsilon}).$$

According to [1], this result was sketched before by Cao and Zhai [5].

A more basic question is to ask about the distribution of $\lfloor n^c \rfloor$ in residue classes. In this case it is known that for all noninteger $c>1$, all positive integers m and all $a \in \mathbb{Z}$ we have

$$|\{n \leq x : \lfloor n^c \rfloor \equiv a \bmod m\}| = x/m + O(x^{1-\varepsilon})$$

for some $\varepsilon = \varepsilon(c)$ that can be given explicitly; see Deshouillers [6] and Morgenbesser [18].

Another line of research was initiated by Mauduit and Rivat [13]; it concerns the behaviour of q-multiplicative functions on Piatetski-Shapiro

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sequences. For an integer $q \geq 2$, a function $\varphi : \mathbb{N} \to \mathbb{C}$ is called *q-multiplicative* if for all $a \geq 0$, $k \geq 0$ and for $0 \leq b < q^k$ we have $\varphi(q^k a+b) = \varphi(q^k a)\varphi(b)$. The function $\mathrm{e}(\alpha s_q(n))$, where s_q denotes the sum-of-digits function in base q, and the trigonometric monomial $\mathrm{e}(\alpha n)$ are examples of q-multiplicative functions. Gelfond [9] solved the problem of describing the distribution of the values $s_q(n)$ in residue classes, where n itself is restricted to a residue class, and posed the analogous problem of describing the distribution of $s_q(P(n))$ in residue classes, where P is a polynomial of degree greater than one such that $P(\mathbb{N}) \subseteq \mathbb{N}$. The study of q-multiplicative functions on Piatetski-Shapiro sequences can be seen as a step towards the resolution of this question, in the same way that the Piatetski-Shapiro Prime Number Theorem is an approach to unsolved problems such as proving that there are infinitely many prime numbers of the form $n^2 + 1$. In [14] Mauduit and Rivat proved the following theorem.

THEOREM A (Mauduit and Rivat). *Let $c \in (1, 7/5)$ and $\gamma = 1/c$. For all $\delta \in (0, (7 - 5c)/9)$ there exists a constant $C = C(\gamma, \delta)$ such that for all q-multiplicative functions χ taking values in $\{z \in \mathbb{C} : |z| = 1\}$ and all $x \geq 1$ we have*

$$\Big|\sum_{1\leq n\leq x} \chi(\lfloor n^c\rfloor) - \sum_{1\leq m\leq x^c} \gamma m^{\gamma-1}\chi(m)\Big| \leq C(\gamma,\delta)x^{1-\delta}. \tag{2}$$

Morgenbesser [18] gave a nontrivial bound for the sum $\sum \mathrm{e}(\alpha s_q(\lfloor n^c\rfloor))$ for all noninteger $c > 1$, provided only that q is large enough (depending on c). Deshouillers, Drmota and Morgenbesser [7] investigated subsequences of automatic sequences of the form $\lfloor n^c\rfloor$ for $c < 7/5$ by generalizing the method from [14]. Mauduit and Rivat [15] gave a complete description of the distribution of the sum of digits of squares in residue classes, thus solving the conjecture of Gelfond for the case of $P(X) = X^2$. The problem of proving (2) when $c \geq 7/5$ is not an integer, $\chi(n) = \mathrm{e}(\alpha s_q(n))$ and q is small could not be solved, however.

In the present article we follow a new approach to problems on Piatetski-Shapiro sequences. This approach is based on the idea of approximating the function x^c by a family of tangents $x\alpha+\beta$, each restricted to a small interval. Let $\delta \in (0, 1 - c/2)$ and $\varepsilon > 0$ be given. Then by linear approximation we can choose for $x_0 \geq 1$ some α and β in such a way that $|x^c - x\alpha - \beta| < \varepsilon$ if $|x-x_0| < Cx^\delta$, where C does not depend on x_0. It therefore seems likely that $\lfloor n^c\rfloor = \lfloor n\alpha + \beta\rfloor$ for most integers n in such an interval. These observations are made precise in the lemmas of Section 4.1.

Algebraic properties of the function $x \mapsto x^c$ are not needed for such an approximation. Consequently our method can be adapted to treat functions from a larger class, defined by certain conditions on the derivatives. Functions like $x^c \log^\eta x$ or $x^c \exp(\log^\varepsilon x)$, where $1 < c < 2$, $\eta \in \mathbb{R}$ and $0 \leq \varepsilon < 1$, are contained in this class, as well as linear combinations with positive coefficients of all elements.

A sequence of integers of the form $(\lfloor n\alpha+\beta\rfloor)_{n\geq 1}$, where $\alpha>0$, is called a (*nonhomogeneous*) *Beatty sequence*. They are named after S. Beatty, who posed a problem (concerning the homogeneous case) in the American Mathematical Monthly in 1926 (see [3]), which essentially states that for irrational $\alpha_1,\alpha_2>1$ such that $1/\alpha_1+1/\alpha_2=1$ the sequences $(\lfloor n\alpha_1\rfloor)_{n\geq 1}$ and $(\lfloor n\alpha_2\rfloor)_{n\geq 1}$ form a partition of the set of positive integers. This fact was already found in 1894 by Rayleigh [20, pp. 122–123], and therefore it is called Rayleigh's Theorem or Beatty's Theorem. We refer to [2] for some references to the newer literature concerning Beatty sequences.

We consider a bounded arithmetic function φ and a differentiable function $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $f'>0$ and other conditions on its derivatives, and ask whether

$$\sum_{A<n\leq 2A}\varphi(\lfloor f(n)\rfloor)-\sum_{f(A)<m\leq f(2A)}\varphi(m)(f^{-1})'(m)=o(A) \tag{3}$$

as $A\to\infty$. The two terms on the left hand side resemble those involved in the change of variables in an integral. Heuristically, we expect therefore that "well-behaved" functions φ yield a small error term on the right hand side. This expectation is in general very difficult to verify, which is obvious from the observation that, for instance, (1) can be reduced to a statement of the form (3).

The main result of this paper, based on the method of approximating $\lfloor n^c\rfloor$ by Beatty sequences and the approximation of the periodic Bernoulli polynomial $\psi(x)=x-\lfloor x\rfloor-1/2$ by trigonometric polynomials, is a sufficient condition for (3) to hold. More precisely we give an upper bound on the error term that involves the exponential sum $\sum\varphi(m)\,\mathrm{e}(m\theta)$ over short intervals.

We give several applications of this theorem. One is an improvement of the bound $7/5=1.4$ in Theorem A to the value 1.42 in the case where χ is the Thue–Morse sequence, which expresses the parity of the number of ones in the binary representation of a natural number. In order to prove this result, we use an estimate of the L^1-norm of the corresponding exponential sum (as a function in θ) given by Fouvry and Mauduit [8].

Another application concerns the joint distribution of sum-of-digits functions on Piatetski-Shapiro sequences. It is another problem posed by Gelfond [9] to prove that if $q_1,q_2\geq 2$, $m_1,m_2\geq 1$ and l_1,l_2 are integers such that $(q_1,q_2)=1$, $(m_1,q_1-1)=1$ and $(m_2,q_2-1)=1$, then there exists $\varepsilon>0$ such that

$$\begin{aligned}&|\{n\leq x: s_{q_1}(n)\equiv l_1 \bmod m_1 \text{ and } s_{q_2}(n)\equiv l_2 \bmod m_2\}|\\ &\qquad\qquad=\frac{x}{m_1m_2}+O(x^{1-\varepsilon}).\end{aligned} \tag{4}$$

This statement was proved by Kim [11], but a weaker form of this result, specifically with a nonexplicit error term, had been provided by Bésineau long before (see [4]). To the author's knowledge the problem of proving a result such as (4) for subsequences $\lfloor n^c \rfloor$ of the integers has not been dealt with in the literature. We obtain such a result for all c in the interval $(1, 18/17)$. In the proof we make use (besides the main theorem) of discrete Fourier coefficients related to the sum-of-digits function. These Fourier coefficients have proven to be an excellent tool for treating problems related to the sum of digits (see [15, 16]) and can also be used in our context. We also note that their use leads to an alternative method of proving (4).

As the third application we prove a result on the distribution in residue classes of the Zeckendorf sum-of-digits function s_Z evaluated on Piatetski-Shapiro sequences. By the well-known theorem of Zeckendorf [22] every positive integer n can be uniquely represented as a sum of nonconsecutive Fibonacci numbers. The number of summands in this representation is called the *Zeckendorf sum-of-digits* of n, which we denote by $s_Z(n)$. We prove that for integers $m \geq 1$ and a and for all $c \in (1, 4/3)$ there exists $\varepsilon > 0$ such that

$$|\{n \leq x : s_Z(\lfloor n^c \rfloor) \equiv a \bmod m\}| = x/m + O(x^{1-\varepsilon}).$$

In this article, we denote the set of positive real numbers by $\mathbb{R}^+$ and the set of nonnegative integers by $\mathbb{N}$. For $x \in \mathbb{R}$ we write $\mathrm{e}(x) = e^{2\pi i x}$, $\|x\| = \min_{n\in\mathbb{Z}} |n - x|$ and $\{x\} = x - \lfloor x \rfloor$. Conditions like $i < n$ under a summation or product sign are to be read as $0 \leq i < n$.

2. Main results. The main result is an estimate of the error term in (3) for a special class of functions f.

THEOREM 1. *Assume that f is a twice continuously differentiable real valued function on $\mathbb{R}^+$ such that $f, f', f'' > 0$ and that there exist $c_1 \geq 1/2$ and $c_2 > 0$ such that for $0 < x \leq y \leq 2x$ we have $c_1 f''(x) \leq f''(y) \leq c_2 f''(x)$. Let $A_0 \geq 2$ be such that $f'(A_0) \geq 1$. There exists a constant $C = C(f)$ such that for all complex valued arithmetic functions φ bounded by 1, for all integers $A \geq A_0$ and for all $z > 0$ we have*

$$(5)\qquad \frac{1}{A}\Big|\sum_{A<n\leq 2A} \varphi(\lfloor f(n)\rfloor) - \sum_{f(A)<m\leq f(2A)} \varphi(m)(f^{-1})'(m)\Big| \leq C\biggl(\frac{f''(A)}{f'(A)^2} z^2 + f'(A)(\log A)^3 J(A,z)\biggr),$$

where

$$(6)\qquad J(A,z) = \int_0^1 \sup_{f(A)<x\leq f(2A)} \frac{1}{z}\Big|\sum_{x<m\leq x+z} \varphi(m)\,\mathrm{e}(m\theta)\Big|\, d\theta.$$

Theorem 1 is a consequence of the following result, which provides a way to prove a discrete substitution rule by solving a problem about the behaviour of φ on Beatty sequences.

PROPOSITION 1. *Assume that f is a twice continuously differentiable real valued function on $\mathbb{R}^+$ such that $f, f', f'' > 0$, and that there exist $c_1 \ge 1/2$ and $c_2 > 0$ such that for $0 < x \le y \le 2x$ we have $c_1 f''(x) \le f''(y) \le c_2 f''(x)$. There exists $C = C(f)$ such that for all complex valued arithmetic functions φ bounded by 1, for all $A \ge 2$ and $K > 0$ we have*

$$(7)\qquad \frac{1}{A}\Big|\sum_{A<n\le 2A}\varphi(\lfloor f(n)\rfloor) - \sum_{f(A)<m\le f(2A)}\varphi(m)(f^{-1})'(m)\Big| \le C\Big|f''(A)K^2 + \frac{(\log A)^2}{K} + I(A,K)\Big|,$$

where $I(A,K)$ is defined by

$$(8)\qquad I(A,K) = \frac{1}{f'(2A)-f'(A)} \times \int_{f'(A)}^{f'(2A)} \sup_{f(A)<\beta\le f(2A)} \frac{1}{K}\Big|\sum_{0<n\le K}\varphi(\lfloor n\alpha+\beta\rfloor) - \frac{1}{\alpha}\sum_{\beta<m\le\beta+K\alpha}\varphi(m)\Big|\,d\alpha.$$

3. Applications. In the proofs of our applications concerning sum-of-digits functions, we make use of bounds for the exponential sum

$$\sum_{x<m\le x+z}\varphi(m)\,\mathrm{e}(m\theta)$$

that are independent of x. Moreover, for simplicity we concentrate on the case $f(x) = x^c$, although it would be possible to derive analogous results for a larger class of functions, as we noted in the introduction. We state a corollary of Theorem 1 that is adjusted to this situation.

COROLLARY 1. *Let φ be a complex valued arithmetic function bounded by 1. If $a \in (0,1]$ and C are such that*

$$(9)\qquad \int_0^1 \sup_{x\ge 0}\Big|\sum_{x<m\le x+z}\varphi(m)\,\mathrm{e}(m\theta)\Big|\,d\theta \le Cz^a$$

for $z \ge 1$, then for all $c \in (1,2)$ and all $\eta \in \big(0, \frac{2-(a+1)c}{3-a}\big)$ there is a $C_1 = C_1(a,c,C,\eta)$ such that

$$(10)\qquad \frac{1}{N}\Big|\sum_{1\le n\le N}\varphi(\lfloor n^c\rfloor) - \frac{1}{c}\sum_{1\le m\le N^c}\varphi(m)m^{1/c-1}\Big| \le C_1 N^{-\eta}$$

for $N \ge 1$.

Proof. For $A > 0$ we write

$$F(A) = \Bigl|\sum_{A<n\le 2A} \varphi(\lfloor n^c\rfloor) - \frac{1}{c}\sum_{A^c<m\le(2A)^c} \varphi(m)m^{1/c-1}\Bigr|. \tag{11}$$

Let $1 < c < 2$ and set $z = A^{\frac{2c-1}{3-a}}$ for $A \ge 2$. From hypothesis (9) and Theorem 1 it follows by a short calculation that for all integers $A \ge 2$ and all $\varepsilon > 0$ we have

$$F(A) \ll A^{1-\rho+\varepsilon} \tag{12}$$

with the choice $\rho = \frac{2-c(a+1)}{3-a}$. The implied constant in (12) may depend on a, c, C and ε. Altering the summation limits in (11) to $\lfloor A\rfloor < n \le \lfloor 2A\rfloor$ and $\lfloor A\rfloor^c < m \le \lfloor 2A\rfloor^c$ respectively introduces an error term of $O(1)$, which is negligible. Therefore (12) holds for all real $A \ge 2$ and $\varepsilon > 0$. We have $F(A) = 0$ for $A < 1/2$, and it is clear that $F(A)$ is bounded for $0 < A \le 2$. From these observations and (12) it follows that $F(A) \ll A^{1-\rho+\varepsilon}$ for all $A > 0$. Since $\rho - \varepsilon < 1$, we get

$$\begin{aligned}
&\Bigl|\sum_{1\le n\le N} \varphi(\lfloor n^c\rfloor) - \frac{1}{c}\sum_{1\le m\le N^c} \varphi(m)m^{1/c-1}\Bigr| \\
&\qquad = \Bigl|\sum_{i\ge 1}\Bigl(\sum_{N/2^i<n\le N/2^{i-1}} \varphi(\lfloor n^c\rfloor) - \frac{1}{c}\sum_{(N/2^i)^c<n\le(N/2^{i-1})^c} \varphi(m)m^{1/c-1}\Bigr)\Bigr| \\
&\qquad \le \sum_{i\ge 1} F\Bigl(\frac{N}{2^i}\Bigr) \ll CN^{1-\rho+\varepsilon}.
\end{aligned}$$

From this the assertion follows. ∎

3.1. The Thue–Morse sequence. In our first application we are interested in the special case that the function φ is the Thue–Morse sequence of the form $\varphi(n) = (-1)^{s_2(n)}$, where $s_2(n)$ denotes the sum of digits of n in base 2.

THEOREM 2 (The Thue–Morse sequence on $\lfloor n^c\rfloor$). *There exists a in $[0, 0.4076)$ such that for all $c \in (1,2)$ and all $\eta \in \bigl(0, \frac{2-(a+1)c}{3-a}\bigr)$ there is a constant $C = C(c,\eta)$ such that for all $N \ge 2$,*

$$\frac{1}{N}\Bigl|\sum_{1\le n\le N} (-1)^{s_2(\lfloor n^c\rfloor)}\Bigr| \le CN^{-\eta}.$$

In particular, for $1 < c \le 1.42$ there exist $\eta > \max\{0, (7-5c)/9\}$ and C such that this estimate holds.

In order to prove this, we want to apply Corollary 1, and therefore we have to find an estimate for the expression on the left hand side of (9). We use the following statement which follows from Théorème 3 and inequality (1.5) in the paper [8] by Fouvry and Mauduit.

LEMMA 1. *There exists* $\rho \in (0.6543, 0.6632)$ *such that*

$$\int_0^1 \prod_{0 \le k < \lambda} |\sin(2^k \pi\theta)|\, d\theta \asymp \rho^\lambda$$

for all $\lambda \ge 0$.

The number ρ is clearly uniquely determined. No simple representation of ρ seems to be known and in fact the above bounds were obtained with the help of numerical computations. The authors of the cited article also remark that evaluating the numerical value of the integral for about a dozen values of λ (by means of splitting up the interval $[0, 1]$ into 2^λ subintervals of equal length and using the fact that for $k < \lambda$ the function $\sin(2^k\pi\theta)$ has a constant sign on each of them) suggests that $\rho = 0.661\ldots$. From Lemma 1 we deduce the following estimate, which is the main component of the proof of Theorem 2.

PROPOSITION 2. *Let* ρ *be defined as in Lemma* 1. *Then uniformly for* $z \ge 1$ *we have*

$$\int_0^1 \sup_{x \ge 0} \Big| \sum_{x < m \le x+z} (-1)^{s_2(m)}\, \mathrm{e}(m\theta) \Big|\, d\theta \ll z^{1+\frac{\log\rho}{\log 2}}.$$

Proof. If L is an interval of the form $[\ell 2^\lambda, (\ell+1)2^\lambda)$, where ℓ and λ are nonnegative integers, then

$$\Big| \sum_{m \in L} (-1)^{s_2(m)}\, \mathrm{e}(m\theta) \Big| = \prod_{0 \le k < \lambda} |1 - \mathrm{e}(2^k\theta)|. \tag{13}$$

This is clear for $\lambda = 0$. If $\lambda > 0$, then by the relations $s_2(2m) = s_2(m)$ and $s_2(2m+1) = s_2(2m) + 1$ we have

$$\begin{aligned}
&\Big| \sum_{m \in L} (-1)^{s_2(m)}\, \mathrm{e}(m\theta) \Big| \\
&\quad = \Big| \sum_{\ell 2^{\lambda-1} \le m < (\ell+1)2^{\lambda-1}} \big((-1)^{s_2(2m)}\, \mathrm{e}(2m\theta) + (-1)^{s_2(2m+1)}\, \mathrm{e}((2m+1)\theta) \big) \Big| \\
&\quad = |(1 - e(\theta))| \Big| \sum_{\ell 2^{\lambda-1} \le m < (\ell+1)2^{\lambda-1}} (-1)^{s_2(m)}\, \mathrm{e}(2m\theta) \Big|,
\end{aligned}$$

from which (13) follows by induction. Using the trigonometric identity $|1 - \mathrm{e}(\theta)| = 2|\sin(\pi\theta)|$ we get

$$\Big| \sum_{m \in L} (-1)^{s_2(m)}\, \mathrm{e}(m\theta) \Big| = 2^\lambda \prod_{0 \le k < \lambda} |\sin(2^k\pi\theta)|. \tag{14}$$

If L is any finite nonempty interval of nonnegative integers, we use dyadic decomposition of L in the form of the following statement: Let $a < b$ be nonnegative integers. There exists a decomposition $a = a_0 \le \cdots \le a_L = b_L \le \cdots \le b_0 = b$ such that for $j < L$ we have $a_{j+1} - a_j \in \{0, 2^j\}$, $2^j \mid a_j$ and $b_j - b_{j+1} \in \{0, 2^j\}$, $2^j \mid b_j$.

To prove this, one first establishes the special case that $a < 2^K \le b < 2^{K+1}$ for some K, and obtains the general case by adding a multiple of 2^{K+1}. We skip the details of the proof since we will return to a very similar problem in Section 3.3. We can therefore decompose L into intervals of the form $[\ell 2^\lambda, (\ell+1)2^\lambda)$ in such a way that for each λ there are at most two such intervals of length 2^λ. From this, using (14) we obtain

$$\Big|\sum_{m\in L}(-1)^{s_2(m)}\,\mathrm{e}(m\theta)\Big| \ll \sum_{0\le\lambda\le\frac{\log|L|}{\log 2}} 2^\lambda \prod_{0\le k<\lambda} |\sin(2^k\pi\theta)|.$$

By Lemma 1 (note that in particular $2\rho > 1$) this implies

$$\begin{aligned}\int_0^1 \sup_{x\ge 0}\Big|\sum_{x<m\le x+z}(-1)^{s_2(m)}\,\mathrm{e}(m\theta)\Big|\,d\theta &\ll \sum_{\lambda\le\frac{\log(z+1)}{\log 2}} 2^\lambda \int_0^1 \prod_{k<\lambda}|\sin(2^k\pi\theta)|\,d\theta \\ &\ll \sum_{\lambda\le\frac{\log(z+1)}{\log 2}} 2^\lambda\rho^\lambda \ll (2\rho)^{\frac{\log(z+1)}{\log 2}+1} \ll (2\rho)^{\frac{\log z}{\log 2}} = z^{1+\frac{\log\rho}{\log 2}}\end{aligned}$$

for all $z \ge 1$. ∎

Proof of Theorem 2. Note first that $1 + \frac{\log\rho}{\log 2} < 0.4076$ according to the estimate $\rho < 0.6632$. Combining Proposition 2 and Corollary 1 we get the following statement: there exists $a < 0.4076$ such that for all $c \in (1,2)$ and all $\eta \in \big(0, \frac{2-(a+1)c}{3-a}\big)$ there exists C such that for all $N \ge 2$ we have

$$\frac{1}{N}\Big|\sum_{1\le n\le N}(-1)^{s_2(\lfloor n^c\rfloor)} - \frac{1}{c}\sum_{1\le m\le N^c}(-1)^{s_2(m)}m^{1/c-1}\Big| \le CN^{-\eta}. \tag{15}$$

To prove the main statement, it remains to eliminate the second sum in this inequality. For all nonnegative integers K we have $\sum_{m<2K}(-1)^{s_2(m)} = 0$, therefore it follows by partial summation that

$$\begin{aligned}\frac{1}{N}\sum_{1\le m\le N^c}(-1)^{s_2(m)}m^{1/c-1} &\ll \frac{1}{N}(N^c)^{1/c-1}\sup_{1\le u\le N^c}\Big|\sum_{1\le m\le u}(-1)^{s_2(m)}\Big| \\ &\ll N^{-c}.\end{aligned}$$

This quantity is dominated by the error term, so we may remove the second sum in (15). To finish the proof, we note that $2-(a+1)c > 0$ and $\frac{7-5c}{9} < \frac{2-(a+1)c}{3-a}$ for $c \le 1.42$ and $a < 0.4076$. ∎

We remark that our method even yields a value around 1.425 for the upper bound on c, if indeed ρ is around 0.661 as the computations suggest. In [8, p. 579], an analogous remark on the dependence of a parameter on ρ is made.

3.2. The joint distribution of sum-of-digits functions. For integers $q \ge 2$ and $n \ge 0$ we denote by $s_q(n)$ the sum of digits of n in base q. In this section we prove the following independence result for sum-of-digits functions with respect to coprime bases q_1 and q_2.

THEOREM 3 (Joint distribution of sum-of-digits functions on $\lfloor n^c \rfloor$). *Let $q_1, q_2 \ge 2$, $m_1, m_2 \ge 1$, and let l_1, l_2 be integers such that $(q_1, q_2) = 1$, $(m_1, q_1 - 1) = 1$ and $(m_2, q_2 - 1) = 1$. Let $1 < c < 18/17$. There exists $\varepsilon > 0$ such that*

$$\begin{aligned} (16)\qquad &|\{n \le x : s_{q_1}(\lfloor n^c \rfloor) \equiv l_1 \bmod m_1 \text{ and } s_{q_2}(\lfloor n^c \rfloor) \equiv l_2 \bmod m_2\}| \\ &\qquad = \frac{x}{m_1 m_2} + O(x^{1-\varepsilon}). \end{aligned}$$

Generalizing this theorem (and its proof) to more than two bases is straightforward, however the upper bound on c that we can obtain using our method has then to be adjusted. In order to prove Theorem 3, we estimate the relevant integral as well as the integrand at $\theta = 0$.

PROPOSITION 3. *Let $q_1, q_2 \ge 2$ be relatively prime integers. There exists $C = C(q_1, q_2)$ such that for all $\alpha, \beta \in \mathbb{R}$ and $z \ge 1$ we have*

$$(17)\qquad \int_0^1 \sup_{x \ge 0} \Big| \sum_{x < n \le x+z} \mathrm{e}(\alpha s_{q_1}(n) + \beta s_{q_2}(n) + n\theta) \Big| \, d\theta \le C z^{8/9}.$$

Moreover,

$$(18)\qquad \sup_{x \ge 0} \Big| \sum_{x < n \le x+z} \mathrm{e}(\alpha s_{q_1}(n) + \beta s_{q_2}(n)) \Big| \le C_1 z^{1-\eta(\alpha)}$$

for $z \ge 1$, where $\eta(\alpha) = \frac{\|(q_1 - 1)\alpha\|^2}{15 \log q_1}$ and C_1 may depend on α, β, q_1 and q_2.

In the proof of this proposition we make use of the truncated sum-of-digits function $s_{q,\lambda}$, which adds up the first λ digits of the base-q representation of a nonnegative integer n. That is, if $n = \sum_{i \ge 0} \varepsilon_i q^i$ and ε_i is in $\{0, 1, \ldots, q-1\}$ for all i, then

$$s_{q,\lambda}(n) = \sum_{0 \le i < \lambda} \varepsilon_i = s_q(n \bmod q^\lambda).$$

For convenience we extend $s_{q,\lambda}$ to a q^λ-periodic function on $\mathbb{Z}$. By periodicity, we can represent the function $\mathrm{e}(\alpha s_{q,\lambda}(n))$ with the aid of the discrete Fourier transform. For integers $q \geq 2$, $\lambda \geq 0$ and n we have

$$\mathrm{e}(\alpha s_{q,\lambda}(n)) = \sum_{h<q^\lambda} \mathrm{e}(hnq^{-\lambda}) F_{q,\lambda}(h,\alpha), \tag{19}$$

$$\mathrm{e}(-\alpha s_{q,\lambda}(n)) = \sum_{h<q^\lambda} \mathrm{e}(hnq^{-\lambda}) \overline{F_{q,\lambda}(-h,\alpha)}, \tag{20}$$

where

$$F_{q,\lambda}(h,\alpha) = \frac{1}{q^\lambda} \sum_{u<q^\lambda} \mathrm{e}(\alpha s_{q,\lambda}(u) - huq^{-\lambda}).$$

The Fourier coefficients $F_{q,\lambda}(h,\alpha)$ may be estimated uniformly in h using the following lemma [15, Lemme 9].

Lemma 2. *Let $q, \lambda \geq 2$ and h be integers and $\alpha \in \mathbb{R}$. Then*

$$|F_{q,\lambda}(h,\alpha)| \leq e^{\pi^2/48} q^{-c_q \|(q-1)\alpha\|^2 \lambda},$$

where

$$c_q = \frac{\pi^2}{12 \log q} \left(1 - \frac{2}{q+1}\right).$$

We prove the following lemma on the truncated sum-of-digits function, which is a way of expressing the idea that addition of an integer r to n should only change digits at low positions in most cases.

Lemma 3. *Let $q \geq 2$, $\lambda \geq 0$ and r be integers, and let I be a finite interval in $\mathbb{N}$ such that $I + r \subseteq \mathbb{N}$. Then*

$$|\{n \in I : s_q(n+r) - s_q(n) \neq s_{q,\lambda}(n+r) - s_{q,\lambda}(n)\}| \leq |I| \frac{|r|}{q^\lambda} + |r|.$$

Proof. It is sufficient to assume that r is nonnegative, since the other case then follows by shifting the interval I.

For a nonnegative integer n, there exist unique t and u such that $n = tq^\lambda + u$, where $u < q^\lambda$. Clearly we have $s_q(n) = s_q(t) + s_q(u)$ and $s_{q,\lambda}(n) = s_q(u)$. If $n \equiv k \bmod q^\lambda$ for some k such that $0 \leq k < q^\lambda - r$, then $s_q(n+r) = s_q(t) + s_q(u+r)$ and $s_{q,\lambda}(n+r) = s_q(u+r)$, therefore $s_q(n+r) - s_q(s) = s_{q,\lambda}(n+r) - s_{q,\lambda}(n)$. It therefore remains to show that

$$|\{n \in I : q^\lambda - r \leq n \bmod q^\lambda < q^\lambda\}| \leq |I| r/q^\lambda + r,$$

which is not difficult. ∎

The inequality of van der Corput is well known. For our purposes, we will employ it in the following form.

Lemma 4. *Let I be a finite interval in $\mathbb{Z}$ and let $a_n \in \mathbb{C}$ for $n \in I$. Then*

$$\Big|\sum_{n\in I} a_n\Big|^2 \leq \frac{|I| - 1 + R}{R} \sum_{0 \leq |r| < R} \left(1 - \frac{|r|}{R}\right) \sum_{\substack{n\in I\\ n+r\in I}} a_{n+r}\overline{a_n}$$

for all integers $R \geq 1$.

Proof of Proposition 3. To estimate the left hand side of (17), we introduce two parameters to be chosen later, λ_1 and λ_2. Rounding off z to the nearest multiple M of $q_1^{\lambda_1} q_2^{\lambda_2}$ introduces an error term $O(q_1^{\lambda_1} q_2^{\lambda_2})$. Let $x \geq 0$, $z \geq 1$ and let $R \in [1, z]$ be an integer. Then by van der Corput's inequality we get

$$\Big| \sum_{x<n\leq x+M} \mathrm{e}(\alpha s_{q_1}(n) + \beta s_{q_2}(n) + n\theta)\Big|^2 \ll \frac{z}{R} \sum_{|r|<R} \left(1 - \frac{|r|}{R}\right)$$
$$\times \sum_{x<n,\, n+r\leq x+M} \mathrm{e}\big(\alpha(s_{q_1}(n+r) - s_{q_1}(n)) + \beta(s_{q_2}(n+r) - s_{q_2}(n)) + r\theta\big).$$

Applying Lemma 3 in order to replace s_{q_1} and s_{q_2} by s_{q_1,λ_1} and s_{q_2,λ_2} respectively and omitting the summation condition $x < n+r \leq x+M$ afterwards we get an error term $O(zR + z^2 R(1/q_1^{\lambda_1} + 1/q_2^{\lambda_2}))$, and after inserting equations (19) and (20) it remains to estimate the quantity

$$\text{(21)} \qquad \frac{z}{R^2} \sum_{\substack{h_1,k_1<q_1^{\lambda_1} \\ h_2,k_2<q_2^{\lambda_2}}} F_{q_1,\lambda_1}(h_1,\alpha)\, \overline{F_{q_1,\lambda_1}(-k_1,\alpha)}\, F_{q_2,\lambda_2}(h_2,\beta)\, \overline{F_{q_2,\lambda_2}(-k_2,\beta)}$$
$$\times \sum_{x<n\leq x+M} \mathrm{e}\left(n\left(\frac{h_1+k_1}{q_1^{\lambda_1}} + \frac{h_2+k_2}{q_2^{\lambda_2}}\right)\right) \sum_{|r|<R} (R-|r|)\, \mathrm{e}\left(r\left(\frac{h_1}{q_1^{\lambda_1}} + \frac{h_2}{q_2^{\lambda_2}} + \theta\right)\right).$$

By our choice of M and by the Chinese Remainder Theorem, the contribution of the case that $(h_1+k_1, h_2+k_2) \not\equiv (0,0) \bmod (q_1^{\lambda_1}, q_2^{\lambda_2})$ is 0. Using the identity

$$\sum_{|r|<R} (R - |r|)\, \mathrm{e}(rx) = \Big|\sum_{r<R} \mathrm{e}(rx)\Big|^2,$$

we see that (21) is bounded by the expression

$$\text{(22)} \quad \frac{z^2}{R^2} \sum_{\substack{h_1<q_1^{\lambda_1} \\ h_2<q_2^{\lambda_2}}} |F_{q_1,\lambda_1}(h_1,\alpha)|^2 |F_{q_2,\lambda_2}(h_2,\beta)|^2 \Big| \sum_{|r|<R} \mathrm{e}\left(r\left(\frac{h_1}{q_1^{\lambda_1}} + \frac{h_2}{q_2^{\lambda_2}} + \theta\right)\right)\Big|^2,$$

which is independent of x.

In order to prove the first part of Proposition 3, we use the Cauchy–Schwarz inequality, Parseval's identity and the identity

$$\int_0^1 \Big|\sum_{r\in I} \mathrm{e}(r(t+\theta))\Big|^2\, d\theta = |I|,$$

and collect the error terms to arrive at the estimate

$$\int_0^1 \sup_{x\ge 0}\Big|\sum_{x<n\le x+z} \mathrm{e}(\alpha s_{q_1}(n)+\beta s_{q_2}(n)+n\theta)\Big|\,d\theta \tag{23}$$
$$= O\big(q_1^{\lambda_1}q_2^{\lambda_2}+z^{1/2}R^{1/2}+zR^{1/2}(q_1^{-\lambda_1/2}+q_2^{-\lambda_2/2})+zR^{-1/2}\big),$$

which is valid for all real α,β and $z\ge 1$ and all integers $R\in[1,z]$ and $\lambda_1,\lambda_2\ge 0$. The implied constant is an absolute one. This estimate is also valid for real R,λ_1 and λ_2, however the implied constant may then depend on q_1 and q_2. We set

$$\lambda_1=\frac{4\log z}{9\log q_1},\qquad \lambda_2=\frac{4\log z}{9\log q_2}\quad\text{and}\quad R=z^{2/9}.$$

Then clearly $R\in[1,z]$, and a short calculation shows that all four summands in the error term are $\ll z^{8/9}$, which proves the first part.

For the second part we make use of Lemma 2 and Parseval's identity to estimate (22) by

$$\frac{z^2}{R^2}\sup_{h\in\mathbb{Z}}|F_{q_1,\lambda_1}(h,\alpha)|^2\sup_{t\in\mathbb{R}}\Big|\sum_{h_1<q_1^{\lambda_1}}\min\{R^2,\|h_1/q_1^{\lambda_1}+t\|^{-2}\}\Big| \tag{24}$$
$$\times\sum_{h_2<q_2^{\lambda_2}}|F_{q_2,\lambda_2}(h_2,\beta)|^2\ll z^2q_1^{-2c\lambda_1}\frac{q_1^{\lambda_1}}{R},$$

where $c=c_{q_1}\|(q_1-1)\alpha\|^2$. Therefore for some constant C the following holds for all $x,z\ge 0$ and all integers $R\in[1,z]$:

$$\Big|\sum_{x<n\le x+z}\mathrm{e}(\alpha s_{q_1}(n)+\beta s_{q_2}(n))\Big|$$
$$\le C\big(q_1^{\lambda_1}q_2^{\lambda_2}+z^{1/2}R^{1/2}+zR^{1/2}(q_1^{-\lambda_1/2}+q_2^{-\lambda_2/2})+zq_1^{\lambda_1(1/2-c)}R^{-1/2}\big).$$

Again we may assume that R,λ_1 and λ_2 are real numbers. We set

$$\lambda_1=\frac{2\log z}{(4+c)\log q_1},\qquad \lambda_2=\frac{2\log z}{(4+c)\log q_2}\quad\text{and}\quad R=z^{\frac{2-2c}{4+c}}.$$

With these choices, after a short calculation we get

$$\sum_{x<n\le x+z}\mathrm{e}(\alpha s_{q_1}(n)+\beta s_{q_2}(n))\ll z^{1-c/(4+c)}.$$

To get a convenient form of the exponent, we note that $q_1\ge 2$, which implies $c_{q_1}\ge \pi^2/(36\log q_1)$. By the same condition and monotonicity of $x/(4+x)$ we get

$$\frac{c}{4+c}\ge\frac{\pi^2\|(q_1-1)\alpha\|^2}{36\log q_1\big(4+\frac{\pi^2\|(q_1-1)\alpha\|^2}{36\log q_1}\big)}\ge\frac{\|(q_1-1)\alpha\|^2}{\frac{144\log q_1}{\pi^2}+\frac14}\ge\frac{\|(q_1-1)\alpha\|^2}{15\log q_1}.\ \blacksquare$$

By Corollary 1 and (17) we see that for all real α and β the function $\varphi(m) = \mathrm{e}(\alpha s_{q_1}(m) + \beta s_{q_2}(m))$ admits a "change of variables" as long as $2 - (8/9 + 1)c > 0$, that is, $c < 18/17$. We assume now that $(q_1 - 1)\alpha \notin \mathbb{Z}$ or $(q_2 - 1)\beta \notin \mathbb{Z}$. Then by partial summation and equation (18) the second sum in (10) can be eliminated, leading to the following statement:

Let $q_1, q_2 \geq 2$ be relatively prime and $\alpha, \beta \in \mathbb{R}$ such that $(q_1 - 1)\alpha \notin \mathbb{Z}$ or $(q_2 - 1)\beta \notin \mathbb{Z}$. Then for all $c \in (1, 18/17)$ there exist $\varepsilon > 0$ and C such that for $N \geq 1$ we have

$$\sum_{1 \leq n \leq N} \mathrm{e}\big(\alpha s_{q_1}(\lfloor n^c \rfloor) + \beta s_{q_2}(\lfloor n^c \rfloor)\big) \leq C N^{1-\varepsilon}.$$

From this exponential sum estimate we get the statement of Theorem 3 by an orthogonality argument.

Note that by the same orthogonality argument, (4) can be deduced from from (18), which gives an alternative to Kim's proof [11].

3.3. The Zeckendorf sum-of-digits function. In our third application we study the distribution in residue classes of the values of the Zeckendorf sum-of-digits function on $\lfloor n^c \rfloor$.

For $k \geq 0$ let F_k be the kth Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. By Zeckendorf's Theorem [22] every positive integer n admits a unique representation

$$n = \sum_{i \geq 2} \varepsilon_i F_i,$$

where $\varepsilon_i \in \{0, 1\}$ and $\varepsilon_i = 1 \Rightarrow \varepsilon_{i+1} = 0$. By this theorem we may write the ith coefficient ε_i as a function of n. The Zeckendorf sum-of-digits of n is then defined as

$$s_Z(n) = \sum_{i \geq 2} \varepsilon_i(n).$$

We set $s_Z(0) = 0$. We note that $s_Z(n)$ is the least k such that n is the sum of k Fibonacci numbers.

THEOREM 4 (The Zeckendorf sum-of-digits function on $\lfloor n^c \rfloor$). *Let $m \geq 1$ and a be integers. Then for all $c \in (1, 4/3)$ there exists $\varepsilon > 0$ such that uniformly for $x \geq 1$ we have*

$$|\{n \leq x : s_Z(\lfloor n^c \rfloor) \equiv a \bmod m\}| = \frac{x}{m} + O(x^{1-\varepsilon}).$$

The proof of this statement is based on the following proposition.

PROPOSITION 4. *There exists C such that for all $\alpha \in \mathbb{R}$ and $z \geq 1$ we have*

$$\int_0^1 \sup_{x \geq 0} \Big| \sum_{x < n \leq x+z} \mathrm{e}(\alpha s_Z(n) + n\theta) \Big| \, d\theta \leq C z^{1/2}. \tag{25}$$

Moreover, for $\alpha \notin \mathbb{Z}$ there exist $\eta > 0$ and C_1 such that for all $z \geq 1$,

$$\sup_{x\geq 0}\Big|\sum_{x<n\leq x+z} \mathrm{e}(\alpha s_Z(n))\Big| \leq C_1 z^{1-\eta}. \tag{26}$$

Proof. For $k \geq 0$ we define

$$G_k(\alpha,\theta) = \sum_{0\leq u<F_k} \mathrm{e}(\alpha s_Z(u) + \theta u).$$

By the Cauchy–Schwarz inequality and the formula $F_k \asymp \varphi^k$, where $\varphi = (\sqrt{5}+1)/2$, we clearly have

$$\int_0^1 \Big|\sum_{n<F_k} G_k(\alpha,\theta)\Big|\, d\theta \leq F_k^{1/2} \ll \varphi^{k/2}. \tag{27}$$

Moreover, by the relation $s_Z(u+F_k) = 1 + s_Z(u)$ that holds for $k \geq 2$ and $0 \leq u < F_{k-1}$ the terms $G_k(\alpha,0)$ satisfy the linear recurrence relation

$$G_{k+1}(\alpha,0) = G_k(\alpha,0) + \mathrm{e}(\alpha)G_{k-1}(\alpha,0).$$

Its characteristic polynomial has the roots $\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\,\mathrm{e}(\alpha)}$, whose absolute values are bounded by $\frac{1}{2} + \frac{1}{2}(17+8\cos(2\pi\alpha))^{1/4}$. This expression is equal to φ if $\alpha \in \mathbb{Z}$, and strictly less than φ otherwise. Consequently, if $\alpha \notin \mathbb{Z}$, then there is some $\eta > 0$ such that

$$G_k(\alpha,0) \ll \varphi^{k(1-\eta)}. \tag{28}$$

The expression for $G_k(\alpha,\theta)$ involves a sum over the interval $[0,F_k)$. In order to deal with arbitrary finite intervals I in $\mathbb{N}$, we decompose the interval I according to the Zeckendorf representation of its endpoints. This procedure is analogous to the decomposition of an interval into dyadic intervals, which we used in the proof of Theorem 2.

LEMMA 5. *Let $0 \leq A < B$ be integers. There exist integers $L \geq 2$ and a_j, b_j for $2 \leq j \leq L$ such that $A = a_2 \leq \cdots \leq a_L = b_L \leq \cdots \leq b_2 = B$ having the properties that $\varepsilon_i(a_j) = \varepsilon_i(b_j) = 0$ for $2 \leq i < j \leq L$ and that $a_{j+1} - a_j \in \{0, F_{j-1}\}$ and $b_j - b_{j+1} \in \{0, F_j\}$ for $2 \leq j < L$.*

Proof. We first show that it is sufficient to assume that $0 \leq A < F_K \leq B < F_{K+1}$ for some $K \geq 2$. Let $K = \max\{i : \varepsilon_i(A) \neq \varepsilon_i(B)\}$ and $C = \sum_{i>K} \varepsilon_i(A)F_i = \sum_{i>K} \varepsilon_i(B)F_i$. Then $0 \leq A - C < F_K \leq B - C < F_{K+1}$ and by our assumption we get a decomposition $A - C = a_2 \leq \cdots \leq a_L = b_L \leq \cdots \leq b_2 = B - C$ as in the lemma. We have $\varepsilon_i(a_j) = \varepsilon_i(b_j) = 0$ for $2 \leq j \leq L$ and $i > K$, and since $\varepsilon_K(B) = 1$, we have $\varepsilon_i(C) = 0$ for $i \leq K+1$. Therefore $A = a_2 + C \leq \cdots \leq a_L + C = b_L + C \leq \cdots \leq b_2 + C = B$ is a valid decomposition of the interval $[A,B]$.

It remains to prove the simplified statement. In the case that $A = 0$ we set $a_2 = \cdots = a_{K+1} = 0$ and $b_j = \sum_{i\geq j} \varepsilon_i(B)F_i$ for $2 \leq j \leq K+1$.

Otherwise we set $b_j = \sum_{i\ge j} \varepsilon_i(B)F_i$ for $2 \le j \le K$, and to choose a_j, we use the following assertion which we prove by (downward) induction on k:

- Let $K \ge 2$. Assume that $0 < A \le F_K$ and $k = \min\{i : \varepsilon_i(A) = 1\}$. There exist integers $A = a_k \le \cdots \le a_K = F_K$ such that for $k \le j < K$ and $2 \le i < j$ we have $\varepsilon_i(a_j) = 0$ and $a_{j+1} - a_j \in \{0, F_{j-1}\}$.

If $k = K$, then $A = F_K$ and we choose $a_K = A$. Otherwise $2 \le k < K$ and we set $A' = A + F_{k-1}$ and $k' = \min\{i : \varepsilon_i(A') = 1\}$. Then $k' > k$. We choose $a_{k'}, \ldots, a_K$ according to the assumption, $a_k = A$ and $a_{k+1} = \cdots = a_{k'-1} = A'$. This choice gives an admissible decomposition of the interval $[A, F_K]$ and the simplified statement is proved. Setting $a_2 = \cdots = a_{k-1} = A$ completes the proof of Lemma 5. ∎

By this lemma we can decompose an arbitrary finite interval in $\mathbb{N}$ into intervals of the form $[A, A + F_j)$, where $\varepsilon_i(A) = 0$ for $i \le j$, in such a way that for each $j \ge 1$ there are at most two intervals of this form. Noting also that $s_Z(n) = s_Z(A) + s_Z(n - A)$ for all n in such an interval and using the formula $F_k \asymp \varphi^k$, one can easily derive (25) and (26) from (27) and (28). ∎

We plug (25) into Corollary 1 and eliminate the second sum in (10) by partial summation and (26), which results in the statement that for $\alpha \in \mathbb{R}\backslash\mathbb{Z}$ and $c \in (1, 4/3)$ there exist $\eta > 0$ and C such that

$$\sum_{1\le n\le N} \mathrm{e}(\alpha s_Z(\lfloor n^c \rfloor)) \le C N^{1-\eta}$$

for $N \ge 1$. By transferring this to a statement about residue classes, we obtain Theorem 4.

4. Proofs of the main results. We start with a couple of lemmas that we need in the proofs of Theorem 1 and Proposition 1. The first one will allow proving that the left hand sides of (5) and (7) are always $O(A)$.

Lemma 6. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be differentiable and assume that f' is increasing and positive. Then*

$$\sum_{f(A)<m\le f(2A)} (f^{-1})'(m) \ll A \quad \textit{for } A > 0.$$

Proof. If $g : \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing and $0 < s \le t$, we have

$$\sum_{s<m\le t} g(m) = \sum_{\lfloor s\rfloor+1\le m\le \lfloor t\rfloor} g(m) = \int_{\lfloor s\rfloor}^{\lfloor t\rfloor} g(\lfloor x\rfloor + 1)\,dx$$
$$\le g(\lfloor s\rfloor + 1) + \int_{\lfloor s\rfloor+1}^{\lfloor t\rfloor} g(x)\,dx \le g(s) + \int_s^t g(x)\,dx.$$

We apply this to the function $g(x) = (f^{-1})'(x)$, noting also that there is some $a > 0$ such that the sum in the lemma is equal to 0 for $A < a$. For $A \ge a$ we have

$$\sum_{f(A)<m\le f(2A)} (f^{-1})'(m) \le \frac{1}{f'(A)} + f^{-1}(x)|_{f(A)}^{f(2A)} \le \frac{1}{f'(a)} + A \ll A. \ \blacksquare$$

In the next lemma we study properties of functions f as in Theorem 1 and Proposition 1.

LEMMA 7. *Assume that $f : \mathbb{R}^+ \to \mathbb{R}$ is twice continuously differentiable, $f, f', f'' > 0$ and there exist $c_1 \ge 1/2$ and $c_2 > 0$ such that for $0 < x \le y \le 2x$ we have $c_1 f''(x) \le f''(y) \le c_2 f''(x)$. Then:*

$$xf''(x) \ll yf''(y) \qquad \text{for } 0 < x \le y, \tag{29}$$

$$xf''(x) \ll f'(x) \ll xf''(x)\log x \qquad \text{for } x \ge 2, \tag{30}$$

$$f'(x) \le f'(y) \ll f'(x) \qquad \text{for } 0 < x \le y \le 2x, \tag{31}$$

$$\log x \ll f'(x) \ll x^{\delta} \qquad \text{for some } \delta \ge 0 \text{ and all } x \ge 2. \tag{32}$$

Moreover for $0 < x \le a \le b \le 2x$ we have

$$f(b) - f(a) \asymp f'(x)(b-a), \tag{33}$$

$$f'(b) - f'(a) \asymp f''(x)(b-a). \tag{34}$$

Proof. In order to prove (29), we show the equivalent statement that

$$f''(x) \ll af''(ax)$$

for $a \ge 1$ and $x > 0$. This is clear for $a = 2^k$ by the inequalities $c_1 f''(x) \le f''(2x)$ and $c_1 \ge 1/2$. If $2^k \le a < 2^{k+1}$, we have $f''(ax) \ge c_1 f''(2^k x) \ge c_1 2^{-k} f''(x) \gg 1/af''(x)$. We turn to the first inequality in (30). By the Mean Value Theorem there exists some $\xi \in [x/2, x]$ such that $f'(x) \ge f'(x) - f'(x/2) = (x/2)f''(\xi) \ge (x/(2c_2))f''(x)$. For the proof of the second inequality in (30), let $x \ge 2$. For $t \le x$ we have $tf''(t) \ll xf''(x)$ by (29), and therefore

$$f'(x) = f'(2) + \int_2^x f''(t)\,dt \ll f'(2) + xf''(x)\int_2^x \frac{1}{t}\,dt \le f'(2) + xf''(x)\log x.$$

For $x \ge 2$ we have $xf''(x)\log x \gg f''(2) \gg f'(2)$ by (29) and $f', f'' > 0$, therefore $f'(x) \ll xf''(x)\log x$. The first inequality of (31) is obvious since f' is increasing. By applying the Mean Value Theorem it follows that there exists $\xi \in [x, 2x]$ such that $f'(2x) - f'(x) = xf''(\xi) \ll xf''(x)$. Together with (30) this gives $f'(2x) \ll f'(x)$. We prove (32). The first estimate follows from (29) if we set $x = 1$ and integrate in y. By (31) there exists $c > 0$ such that $f'(2z) \le cf'(z)$ for all $z > 0$, from which we get $f'(x) \ll c^{\frac{\log x}{\log 2}} f'(1)$ for all $x \ge 1$. Let $0 < x \le a \le b \le 2x$. By the Mean Value Theorem there is some $\xi \in [a, b]$ such that $f(b) - f(a) = f'(\xi)(b-a)$. From the monotonicity of f' and (31) we get (33). Analogously, (34) is proved via the assumption $c_2 f''(x) \le f''(y) \le c_2 f''(x)$. ∎

In the following lemma we integrate over a well-known estimate for the exponential sum $\sum \mathrm{e}(nx)$, where the sum extends over an interval containing B integers.

Lemma 8. *Let $a \le b$ be real numbers and $B \ge 2$. Then*

$$\int_a^b \min\{B, \|x\|^{-1}\}\, dx \le 2(b-a+1)(1+\log B).$$

Proof. Since the integrand is 1-periodic and symmetric with respect to 1/2, we have

$$\begin{aligned}\int_a^b \min\{B, \|x\|^{-1}\}\, dx &\le 2(b-a+1) \int_0^{1/2} \min\{B, \|x\|^{-1}\}\, dx \\ &\le 2(b-a+1)\Big(\int_0^{1/B} B\, dx + \int_{1/B}^{1/2} x^{-1}\, dx\Big) \\ &\le 2(b-a+1)\big(1+\log(1/2)-\log(1/B)\big) \le 2(b-a+1)(1+\log B).\end{aligned}$$ ∎

4.1. Proof of Proposition 1. We prepare for the proof by giving some results on the approximation of a twice differentiable function by an affine linear function.

Lemma 9. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable and $|f''| \le M$. For all $\alpha \in f'([a,b])$ and $a \le x \le b$ we have*

$$|x\alpha + f(a) - a\alpha - f(x)| \le M(b-a)^2.$$

Proof. By the Mean Value Theorem there exists some $\xi_1 \in [a,x]$ such that $f(x) - f(a) = f'(\xi_1)(x-a)$, that is, such that $|x\alpha + f(a) - a\alpha - f(x)| = (x-a)|f'(\xi_1) - \alpha|$. There exists some $y \in [a,b]$ such that $\alpha = f'(y)$. By applying the Mean Value Theorem to f', we get some ξ_2 between ξ_1 and y such that $|f'(\xi_1) - \alpha| = |f'(\xi_1) - f'(y)| = |(\xi_1 - y)f''(\xi_2)|$. From this the statement follows easily. ∎

The following result will permit us to replace the function $\lfloor f(n)\rfloor$ by a Beatty sequence on an interval $(a,b]$.

Lemma 10. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable and $|f''| \le M$. For all $\alpha \in f'([a,b])$ and $a \le x \le b$ such that $\|x\alpha + f(a) - a\alpha\| > M(b-a)^2$ we have*

$$\lfloor f(x)\rfloor = \lfloor x\alpha + f(a) - a\alpha\rfloor.$$

Proof. We write $\beta = f(a) - a\alpha$ and $d = M(b-a)^2$. The condition $\|x\alpha+\beta\| > d$ in the statement of the lemma implies $\lfloor x\alpha+\beta-d\rfloor = \lfloor x\alpha+\beta\rfloor =$

$\lfloor x\alpha+\beta+d\rfloor$. Moreover by Lemma 9 we get $x\alpha+\beta-d\le f(x)\le x\alpha+\beta+d$. Combining these observations yields the claim. ∎

We estimate the number of integers in an interval for which such an approximation fails.

Lemma 11. *Let $a\le b$ be integers and let $f:[a,b]\to\mathbb{R}$ be twice differentiable. Assume that $|f''|\le M$. For all $\alpha\in f'([a,b])$ and all $R\ge 1$ we have the estimate*

$$\begin{aligned}&|\{n\in(a,b]:\lfloor f(n)\rfloor\ne\lfloor n\alpha+f(a)-a\alpha\rfloor\}|\\&\qquad\le 2M(b-a)^3+\frac{(b-a)}{R}+\sum_{1\le r\le R}\frac{1}{r}\Big|\sum_{a<n\le b}\mathrm{e}(nr\alpha)\Big|.\end{aligned}$$

Proof. Write $d=M(b-a)^2$ and $\beta=f(a)-a\alpha$. If $d\ge 1/2$ or $a=b$, the statement follows immediately since the left hand side is bounded by $b-a$. Otherwise it suffices by Lemma 10 to estimate the quantity

$$|\{n\in(a,b]:\|n\alpha+\beta\|\le d\}|.$$

To do this, we apply the inequality of Erdős and Turán to the sequence $(\{n\alpha+\beta+d\})_{a<n\le b}$ in $[0,1)$. According to [17, Lemma 1], the discrepancy of any real valued finite sequence $(x_1,\ldots,x_N)$ in $[0,1)$, where $N\ge 1$, satisfies

$$\begin{aligned}D_N(x_1,\ldots,x_N)&=\sup_{0\le r\le s<1}\left|\frac{1}{N}|\{1\le n\le N:r\le x_n\le s\}|-(s-r)\right|\\&\le\frac{1}{H+1}+\sum_{1\le h\le H}\frac{1}{h}\left|\frac{1}{N}\sum_{1\le n\le N}\mathrm{e}(hx_n)\right|\end{aligned}$$

for all $H\ge 1$. This is the classical inequality of Erdős and Turán with an improved constant, equal to 1.

Considering the interval $[0,2d]$, we obtain from this the estimate

$$\begin{aligned}&\left|\frac{1}{b-a}|\{n\in(a,b]:\|n\alpha+\beta\|\le d\}|-2d\right|\\&\qquad=\left|\frac{1}{b-a}|\{n\in(a,b]:\{n\alpha+\beta+d\}\in[0,2d]\}|-2d\right|\\&\qquad\le\frac{1}{R}+\frac{1}{b-a}\sum_{1\le r\le R}\frac{1}{r}\Big|\sum_{a<n\le b}\mathrm{e}(nr\alpha+r\beta+rd)\Big|,\end{aligned}$$

from which the claim follows. ∎

The rough idea of the proof of Proposition 1 is to relate the two sums in (7) to each other in three steps, introducing the expression (8). We replace the function $\lfloor f(n)\rfloor$ by a Beatty sequence $\lfloor n\alpha+\beta\rfloor$ on small subintervals of $(A,2A]$. Analogously, we replace the expression $(f^{-1})'(m)$ by the constant value $1/\alpha$ on the corresponding subintervals of $(f(A), f(2A)]$. To link the two expressions thus obtained we insert (8), which expresses the error that arises when we replace the sum of $\varphi(n)$ over a Beatty sequence by the sum of $\varphi(n)$ over all integers in an interval. Afterwards we collect the error terms and we are done.

Proof of Proposition 1. Let $A \geq 2$. It is sufficient to concentrate on the case that K is an integer and $2 \leq K \leq A$, for the following reasons. If $K < 2$, then $(\log A)^2/K \gg 1$, and if $K > A$, then $f''(A)K^2 \geq Af''(A)A \gg 2f''(2) \gg 1$ by (29). Therefore the right hand side of (7) is bounded below for these cases, while the left hand side of (7) is always bounded above by Lemma 6. For general K in $[2, A]$ we have $|I(A, \lfloor K \rfloor) - I(A, K)| \ll 1/K$, which can be deduced from the inequality $|ab - a'b'| \leq |a - a'||b| + |a'||b - b'|$ and the estimate $\alpha \geq f'(2) \gg 1$ that is valid for $\alpha \in [f'(A), f'(2A)]$. This error is absorbed by the term $(\log A)^2/K$, therefore the general case can easily be accounted for by adjusting the implied constant C.

To guarantee that all expressions involving φ are well defined, we set $\varphi(n) = 0$ for $n \leq 0$. For K an integer and $2 \leq K \leq A$ we partition $(A, 2A]$ into smaller intervals of length at most K as follows. Define integral partition points $a_i = \lceil A \rceil + iK$ for $i \geq 0$ and set $L = \max\{i : a_i \leq 2A\}$, which is well defined since $K > 0$. The integer L satisfies the estimate $L \leq A/K$. We have the decomposition

$$(A, 2A] = (A, \lceil A \rceil] \cup \bigcup_{0 \leq i < L} (a_i, a_{i+1}] \cup (a_L, 2A]. \tag{35}$$

Let $\alpha \in \mathbb{R}$. Then by the triangle inequality and the relation $a_{i+1} - a_i = K$, for $i < L$ we have

$$\Big| \sum_{a_i < n \leq a_{i+1}} \varphi(\lfloor f(n) \rfloor) - \sum_{f(a_i) < m \leq f(a_{i+1})} \varphi(m)(f^{-1})'(m) \Big| \\ \leq T_1(\alpha, i) + T_2(\alpha, i) + T_3(\alpha, i) + T_4(\alpha, i), \tag{36}$$

where

$$T_1(\alpha, i) = \Big| \sum_{a_i < n \leq a_{i+1}} \big(\varphi(\lfloor f(n) \rfloor) - \varphi(\lfloor n\alpha + f(a_i) - a_i\alpha \rfloor) \big) \Big|,$$

$$T_2(\alpha, i) = \Big| \sum_{0 < n \leq K} \varphi(\lfloor n\alpha + f(a_i) \rfloor) - \frac{1}{\alpha} \sum_{f(a_i) < m \leq f(a_i) + K\alpha} \varphi(m) \Big|,$$

$$T_3(\alpha, i) = \Big| \frac{1}{\alpha} \sum_{f(a_i) < m \leq a_{i+1}\alpha + f(a_i) - a_i\alpha} \varphi(m) - \frac{1}{\alpha} \sum_{f(a_i) < m \leq f(a_{i+1})} \varphi(m) \Big|,$$

$$T_4(\alpha, i) = \Big| \sum_{f(a_i) < m \leq f(a_{i+1})} \varphi(m) \Big(\frac{1}{\alpha} - (f^{-1})'(m) \Big) \Big|.$$

We integrate (36) in α from $f'(a_i)$ to $f'(a_{i+1})$, divide by the length of the integration range, and take the sum over i from 0 to $L-1$, obtaining

$$\begin{aligned}(37)\qquad &\Bigl|\sum_{\lceil A\rceil<n\le a_L}\varphi(\lfloor f(n)\rfloor)-\sum_{f(\lceil A\rceil)<m\le f(a_L)}\varphi(m)(f^{-1})'(m)\Bigr|\\ &\le\sum_{0\le i<L}\frac{1}{f'(a_{i+1})-f'(a_i)}\int_{f'(a_i)}^{f'(a_{i+1})}\bigl(T_1(\alpha,i)+T_2(\alpha,i)+T_3(\alpha,i)+T_4(\alpha,i)\bigr)\,d\alpha.\end{aligned}$$

The first summand will be estimated with the help of Lemma 11, the second by $AI(A,K)$, and the third and fourth terms will be estimated trivially.

We estimate the first summand in (37). If R is a positive integer, $0\le i<L$ and $\alpha\in f'([a_i,a_{i+1}])$, then Lemma 11 gives

$$(38)\qquad T_1(\alpha,i)\le 2f''(A)K^3+\frac{K}{R}+\sum_{1\le r\le R}\frac{1}{r}\Bigl|\sum_{a_i<n\le a_{i+1}}\mathrm{e}(nr\alpha)\Bigr|.$$

By (34) we have $f'(2A)-f'(A)\ll Af''(A)$ and $f'(a_{i+1})-f'(a_i)\gg f''(A)K$ for $0\le i<L$. Note also that $Af''(A)\gg 2f''(2)>0$ for all $A\ge 2$ by (29) and $f''>0$. From Lemma 8 it follows that for $2\le K\le A$ and $r\ge 1$,

$$\begin{aligned}(39)\qquad &\sum_{0\le i<L}\frac{1}{f'(a_{i+1})-f'(a_i)}\int_{f'(a_i)}^{f'(a_{i+1})}\Bigl|\sum_{a_i<n\le a_{i+1}}\mathrm{e}(nr\alpha)\Bigr|\,d\alpha\\ &\ll\frac{1}{f''(A)K}\sum_{0\le i<L}\frac{1}{r}\int_{rf'(a_i)}^{rf'(a_{i+1})}\Bigl|\sum_{a_i<n\le a_{i+1}}\mathrm{e}(xn)\Bigr|\,dx\\ &\le\frac{1}{f''(A)K}\,\frac{1}{r}\int_{rf'(A)}^{rf'(2A)}\min\{K,\|x\|^{-1}\}\,dx\\ &\ll\frac{1}{f''(A)K}\,\frac{1}{r}2(rAf''(A)+1)(1+\log K)\ll A\frac{\log K}{K}.\end{aligned}$$

From (38) and (39) and the estimates $L\le A/K$ and $\sum_{r=1}^{R}\frac{1}{r}\le\log R+1$ it follows that for $2\le K\le A$ and $R\ge 2$ we have

$$\begin{aligned}(40)\qquad &\sum_{0\le i<L}\frac{1}{f'(a_{i+1})-f'(a_i)}\int_{f'(a_i)}^{f'(a_{i+1})}T_1(\alpha,i)\,d\alpha\\ &\ll\frac{A}{K}\Bigl(f''(A)K^3+\frac{K}{R}\Bigr)+\frac{A\log K\,(\log R+1)}{K}\\ &\ll A\Bigl(f''(A)K^2+\frac{1}{R}+\frac{\log K\log R}{K}\Bigr),\end{aligned}$$

which concludes our treatment of the first term in (37).

We turn to the second summand. Again we use (34) and obtain the estimates

$$\frac{1}{f'(a_{i+1})-f'(a_i)} \ll \frac{1}{f''(A)K} = \frac{A}{K}\,\frac{1}{Af''(A)} \ll A\frac{1}{f'(2A)-f'(A)}\,\frac{1}{K}$$

for $0 \le i < L$. By this and the definition of $T_2(\alpha, i)$, we easily obtain

$$\sum_{0\le i<L} \frac{1}{f'(a_{i+1})-f'(a_i)} \int\limits_{f'(a_i)}^{f'(a_{i+1})} T_2(\alpha,i)\,d\alpha \ll AI(A,K). \tag{41}$$

To estimate the third term in (37), assume that $0 \le i < L$ and α is in $[f'(a_i), f'(a_{i+1})]$. We use Lemma 9 (setting $x = a_{i+1}$) to get

$$|a_{i+1}\alpha + f(a_i) - a_i\alpha - f(a_{i+1})| \le c_2 f''(A)K^2,$$

therefore the two sums in the definition of $T_3(\alpha, i)$ differ by not more than $c_2 f''(A)K^2 + 1$ summands. Moreover, $L \le A/K$. Estimating $1/\alpha \le 1/f'(A)$ we get

$$\begin{aligned}\sum_{0\le i<L} \frac{1}{f'(a_{i+1})-f'(a_i)} &\int\limits_{f'(a_i)}^{f'(a_{i+1})} T_3(\alpha,i)\,d\alpha \\ &\ll \frac{A}{K}\,\frac{1}{f'(A)}(f''(A)K^2+1) = A\biggl(\frac{f''(A)K}{f'(A)} + \frac{1}{f'(A)K}\biggr).\end{aligned} \tag{42}$$

Finally let $0 \le i < L$, $\alpha \in f'([a_i, a_{i+1}])$ and $f(a_i) < m \le f(a_{i+1})$. Choose $x, y \in [a_i, a_{i+1}]$ in such a way that $\alpha = f'(x)$ and $m = f(y)$. Then by (34) and the monotonicity of f' we have

$$\begin{aligned}\left|\frac{1}{\alpha} - (f^{-1})'(m)\right| &= \left|\frac{1}{f'(x)} - \frac{1}{f'(y)}\right| = \left|\frac{f'(y)-f'(x)}{f'(x)f'(y)}\right| \\ &\le \frac{f'(a_{i+1})-f'(a_i)}{f'(a_i)^2} \ll \frac{f''(A)K}{f'(A)^2}.\end{aligned}$$

Moreover, the length of summation in the definition of $T_4(\alpha, i)$ can be estimated using (33), giving $f(a_{i+1}) - f(a_i) + 1 \ll f'(A)K + 1$. It follows that

$$\begin{aligned}\sum_{0\le i<L} \frac{1}{f'(a_{i+1})-f'(a_i)} &\int\limits_{f'(a_i)}^{f'(a_{i+1})} T_4(\alpha,i)\,dx \\ &\ll \frac{A}{K}(f'(A)K+1)\biggl(\frac{f''(A)K}{f'(A)^2}\biggr) \ll A\biggl(\frac{f''(A)K}{f'(A)} + \frac{f''(A)}{f'(A)^2}\biggr).\end{aligned} \tag{43}$$

We still have to take care of the first and the last intervals in (35). To do this, we take any interval $(a, b]$ such that $A \le a \le b \le a + K \le 2A$. For all $m \in (f(a), f(b)]$ we have $(f^{-1})'(m) = 1/f'(f^{-1}(m)) \le 1/f'(A)$ since f'

is monotonic, moreover $f(b)-f(a)+1 \ll f'(A)K+1 \ll f'(A)K$ by (33) and the relation $f'(A) \geq f'(2) > 0$, and finally $b-a+1 \ll K$. Therefore

$$\Big|\sum_{a<n\leq b} \varphi(\lfloor f(n)\rfloor) - \sum_{f(a)<m\leq f(b)} \varphi(m)(f^{-1})'(m)\Big| \ll K + f'(A)K\frac{1}{f'(A)} \ll K. \tag{44}$$

Combining (37) and (40)–(44) we get

$$\begin{aligned}\Big|\sum_{A<n\leq 2A} \varphi(\lfloor f(n)\rfloor) &- \sum_{f(A)<m\leq f(2A)} \varphi(m)(f^{-1})'(m)\Big| \\ &\ll A\biggl(f''(A)K^2 + \frac{1}{R} + \frac{\log K \log R}{K} + I(A,K) \\ &\qquad + \frac{f''(A)K}{f'(A)} + \frac{1}{f'(A)K} + \frac{f''(A)}{f'(A)^2} + \frac{K}{A}\biggr)\end{aligned}$$

for $A, K, R \geq 2$. Since $f'(A) \geq f'(2) \gg 1$, the first term dominates the fifth and seventh terms and the third term dominates the sixth. Since $Af''(A) \gg 2f''(2) \gg 1$ by (29), we have $f''(A) \gg 1/A$, and therefore the first term also dominates the last term. We choose $R = A$. Then the third term dominates the second, and the error is

$$\ll A\bigl(f''(A)K^2 + (\log A)^2/K + I(A,K)\bigr). \ \blacksquare$$

4.2. Proof of Theorem 1. We want to find an estimate for (8); more precisely, we want to treat the expression

$$\sum_{a<n\leq b} \varphi(\lfloor n\alpha + \beta\rfloor)$$

with the help of exponential sums. To do this, we resort to the following useful approximation of the sawtooth function $x \mapsto \{x\} - 1/2$ by trigonometric polynomials that was given by Vaaler. (See [10, Theorem A.6].)

LEMMA 12. *Assume that H is a positive integer. There exist real numbers $a_H(h) \in [0,1]$ for $1 \leq |h| \leq H$ such that*

$$|\psi(t) - \psi_H(t)| \leq \kappa_H(t) \tag{45}$$

for all real t, where

$$\psi(x) = \{x\} - \frac{1}{2}, \qquad \psi_H(t) = -\frac{1}{2\pi i}\sum_{1\leq|h|\leq H} \frac{a_H(h)}{h}\,\mathrm{e}(ht)$$

and

$$\kappa_H(t) = \frac{1}{2(H+1)}\sum_{0\leq|h|\leq H}\biggl(1 - \frac{|h|}{H+1}\biggr)\mathrm{e}(ht).$$

Note that $\kappa_H(t)$ is a nonnegative real number since for all H we have

$$\sum_{0\le |h|<H} (H-|h|)\,\mathrm{e}(hx) = \Big|\sum_{0\le h<H} \mathrm{e}(hx)\Big|^2.$$

Let α and β be real numbers and suppose that $\alpha \ge 1$. An elementary argument shows that for all integers m we have

$$\left\lfloor -\frac{m-\beta}{\alpha}\right\rfloor - \left\lfloor -\frac{m+1-\beta}{\alpha}\right\rfloor = \begin{cases} 1 & \text{if } m = \lfloor n\alpha+\beta\rfloor \text{ for some integer } n,\\ 0 & \text{otherwise.}\end{cases} \tag{46}$$

With the help of this characterization of the elements of a Beatty sequence we prove the following statement, which allows us to deduce Theorem 1 from Proposition 1.

Proposition 5. *Let $\varphi : \mathbb{N} \to \mathbb{C}$ be a function bounded by 1. For all real $\alpha \ge 1$, $\beta \ge 0$, $K \ge 0$ and $H \ge 1$ we have*

$$\begin{aligned}\Big|\sum_{0<n\le K} \varphi(\lfloor n\alpha+\beta\rfloor) - \frac{1}{\alpha}\sum_{\beta<m\le\beta+K\alpha}\varphi(m)\Big| &\le \sum_{1\le |h|\le H} \min\Big\{\frac{1}{\alpha},\frac{1}{|h|}\Big\}\Big|\sum_{\beta<m\le\beta+K\alpha}\varphi(m)\,\mathrm{e}\Big(-m\frac{h}{\alpha}\Big)\Big| \\ &\quad + \frac{1}{H}\sum_{0\le|h|\le H}\Big|\sum_{\beta<m\le\beta+K\alpha}\mathrm{e}\Big(-m\frac{h}{\alpha}\Big)\Big| + O(1).\end{aligned}$$

The implied constant is an absolute one.

Proof. We write $\psi(x) = \{x\} - 1/2 = x - \lfloor x\rfloor - 1/2$. Since $\alpha \ge 1$, the function $n \mapsto \lfloor n\alpha+\beta\rfloor$ is injective. Using this fact and (46), we see that

$$\begin{aligned}\sum_{0<n\le K}\varphi(\lfloor n\alpha+\beta\rfloor) &= \sum_{m\in\mathbb{Z}}\varphi(m)\cdot\begin{cases}1, & m=\lfloor n\alpha+\beta\rfloor \text{ for some } 0<n\le K,\\ 0, & \text{otherwise}\end{cases}\\ &= \sum_{\lfloor\beta\rfloor<m\le\lfloor\beta+K\alpha\rfloor}\varphi(m)\cdot\begin{cases}1, & m=\lfloor n\alpha+\beta\rfloor \text{ for some } n,\\ 0, & \text{otherwise}\end{cases}\\ &= \sum_{\lfloor\beta\rfloor<m\le\lfloor\beta+K\alpha\rfloor}\varphi(m)\Big(\Big\lfloor -\frac{m-\beta}{\alpha}\Big\rfloor - \Big\lfloor -\frac{m+1-\beta}{\alpha}\Big\rfloor\Big)\\ &= \frac{1}{\alpha}\sum_{\beta<m\le\beta+K\alpha}\varphi(m)\\ &\quad + \sum_{\beta<m\le\beta+K\alpha}\varphi(m)\Big(\psi\Big(-\frac{m+1-\beta}{\alpha}\Big) - \psi\Big(-\frac{m-\beta}{\alpha}\Big)\Big) + O(1).\end{aligned}$$

It remains to treat the second sum. For brevity, write

$$L = \{m \in \mathbb{Z} : \beta < m \le \beta + K\alpha\}.$$

Let $H \ge 1$ be an integer. For each m we replace ψ by ψ_H with the help of (45) to get

$$\begin{aligned}
&\Bigl|\sum_{m\in L} \varphi(m)\Bigl(\psi\Bigl(-\frac{m+1-\beta}{\alpha}-\gamma\Bigr)-\psi\Bigl(-\frac{m-\beta}{\alpha}-\gamma\Bigr)\Bigr)\\
&\qquad -\frac{-1}{2\pi i}\sum_{m\in L}\varphi(m)\sum_{1\le|h|\le H}\frac{a_H(h)}{h}\Bigl(\mathrm{e}\Bigl(-h\frac{m+1-\beta}{\alpha}\Bigr)-\mathrm{e}\Bigl(-h\frac{m-\beta}{\alpha}\Bigr)\Bigr)\Bigr|\\
&\quad\le \frac{1}{2H+2}\sum_{m\in L}\sum_{|h|\le H}\Bigl(1-\frac{|h|}{H+1}\Bigr)\Bigl(\mathrm{e}\Bigl(-h\frac{m+1-\beta}{\alpha}\Bigr)+\mathrm{e}\Bigl(-h\frac{m-\beta}{\alpha}\Bigr)\Bigr)\\
&\quad\le \frac{1}{H+1}\sum_{0\le|h|\le H}\Bigl|\sum_{m\in L}\mathrm{e}\Bigl(-h\frac{m}{\alpha}\Bigr)\Bigr|.
\end{aligned}$$

Finally we use the inequalities $|a_H(h)| \le 1$ and $|\,\mathrm{e}(x)-1| \le \min\{2, 2\pi x\}$ to calculate:

$$\begin{aligned}
&\Bigl|\frac{1}{2\pi i}\sum_{m\in L}\varphi(m)\sum_{1\le|h|\le H}\frac{a_H(h)}{h}\Bigl(\mathrm{e}\Bigl(-h\frac{m+1-\beta}{\alpha}\Bigr)-\mathrm{e}\Bigl(-h\frac{m-\beta}{\alpha}\Bigr)\Bigr)\Bigr|\\
&\qquad = \Bigl|\frac{1}{2\pi}\sum_{1\le|h|\le H}\frac{a_H(h)}{h}\,\mathrm{e}\Bigl(-\frac{\beta}{\alpha}\Bigr)\Bigl(\mathrm{e}\Bigl(-\frac{h}{\alpha}\Bigr)-1\Bigr)\sum_{m\in L}\varphi(m)\Bigl(-h\frac{m}{\alpha}\Bigr)\Bigr|\\
&\qquad \le \sum_{1\le|h|\le H}\min\Bigl\{\frac{1}{\alpha},\frac{1}{|h|}\Bigr\}\Bigl|\sum_{m\in L}\varphi(m)\Bigl(-h\frac{m}{\alpha}\Bigr)\Bigr|.
\end{aligned}$$

If $H \ge 1$ is a real number, we apply these calculations to $\lfloor H\rfloor$. Note that in this process the summations over h remain unchanged and $1/(\lfloor H\rfloor+1) \le 1/H$, therefore the assertion follows. ∎

We will use the following standard lemma to extend the range of a summation in exchange for a controllable factor.

Lemma 13. *Let $x \le y \le z$ be real numbers and $a_n \in \mathbb{C}$ for $x < n \le z$. Then*

$$\Bigl|\sum_{x<n\le y} a_n\Bigr| \le \int_0^1 \min\{y-x+1, \|\xi\|^{-1}\}\Bigl|\sum_{x<n\le z} a_n\,\mathrm{e}(n\xi)\Bigr|\,d\xi.$$

Proof. Since $\int_0^1 \mathrm{e}(k\xi)\,d\xi = \delta_{k,0}$ for $k \in \mathbb{Z}$, it follows that

$$\sum_{x<n\le y} a_n = \sum_{x<n\le z} a_n \sum_{x<m\le y}\delta_{n-m,0} = \int_0^1 \sum_{x<m\le y}\mathrm{e}(-m\xi)\sum_{x<n\le z} a_n\,\mathrm{e}(n\xi)\,d\xi,$$

from which the statement follows. ∎

Finally, to obtain the correct error term in the theorem, we will use the following lower bound on the L^1-norm of an exponential sum.

Lemma 14. *Let $a < b$ be real numbers and x_m a complex number for $a < m \le b$. Then*

$$\int_0^1 \Big| \sum_{a<m\le b} x_m \,\mathrm{e}(m\theta) \Big|\, d\theta \ge \max_{a<m\le b} |x_m|.$$

Proof. For $a < n \le b$ we have

$$\begin{aligned}\int_0^1 \Big| \sum_{a<m\le b} x_m \,\mathrm{e}(m\theta) \Big|\, d\theta &= \int_0^1 \Big| \sum_{a<m\le b} x_m \,\mathrm{e}((m-n)\theta) \Big|\, d\theta \\ &\ge \Big| \sum_{a<m\le b} x_m \int_0^1 \mathrm{e}((m-n)\theta)\, d\theta \Big| = x_n. \ \blacksquare\end{aligned}$$

Proof of Theorem 1. Note first that by (32) we have $f'(x) \to \infty$, therefore there exists $A_0 \ge 2$ such that $f'(A) \ge 1$ for $A \ge A_0$. Let $z > 0$. By an argument similar to that at the beginning of the proof of Proposition 1 we may restrict ourselves to the case that $z \le Af'(A)$. Also, we may assume that there exists an m in the range $f(A) < m \le f(2A) + z$ such that $|\varphi(m)| = 1$, since the general case follows from this one by rescaling both sides of (5). To see this, we note that $A \ge 2$ is an integer and $f'(x) \ge 1$ for all $x \ge A$, and therefore the relation (5) only depends on integers m in the range $f(A) < m \le f(2A) + z$. By Lemma 14, this restriction implies

$$\begin{aligned}(47) \qquad &\int_0^1 \sup_{f(A)<x\le f(2A)} \Big| \sum_{x<m\le x+z} \varphi(m) \,\mathrm{e}(m\theta) \Big|\, d\theta \\ &\quad\ge \sup_{f(A)<x\le f(2A)} \int_0^1 \Big| \sum_{x<m\le x+z} \varphi(m) \,\mathrm{e}(m\theta) \Big|\, d\theta \\ &\quad\ge \sup_{f(A)<x\le f(2A)} \ \sup_{x<m\le x+z} |\varphi(m)| = \sup_{f(A)<m\le f(2A)+z} |\varphi(m)| \ge 1.\end{aligned}$$

If $z < \max\{2, f'(2A)\}$, this lower bound implies $f'(A)(\log A)^3 J(A,z) \gg 1$, and since by Lemma 6 the left hand side of (5) is bounded, this proves the assertion in this case. For the remaining part of the proof we assume therefore that $\max\{2, f'(2A)\} \le z \le A\, f'(A)$. Moreover, we assume throughout that $1 \le K \le A$ and $H \ge 2$. We want to apply Proposition 1 and therefore we have to find an estimate for $I(A,K)$. We apply Proposition 5 to the expression in the absolute value in equation (8), which is possible since $\alpha \ge f'(A) \ge 1$ for all α in question, and obtain the estimate

$$I(A,K) \ll \frac{1}{f'(2A)-f'(A)}\,\frac{1}{K}\int\limits_{f'(A)}^{f'(2A)}\biggl(\sum_{1\le|h|\le H}\min\Bigl\{\frac{1}{\alpha},\frac{1}{|h|}\Bigr\}S_1(\alpha,h) + \frac{1}{H}S_2(\alpha,0)+\frac{1}{H}\sum_{1\le|h|\le H}S_2(\alpha,h)+O(1)\biggr)\,d\alpha, \tag{48}$$

where

$$S_1(\alpha,h) = \sup_{f(A)<x\le f(2A)}\Bigl|\sum_{x<m\le x+K\alpha}\varphi(m)\,\mathrm{e}\Bigl(-m\frac{h}{\alpha}\Bigr)\Bigr|,$$
$$S_2(\alpha,h) = \sup_{f(A)<x\le f(2A)}\Bigl|\sum_{x<m\le x+K\alpha}\mathrm{e}\Bigl(-m\frac{h}{\alpha}\Bigr)\Bigr|.$$

The four summands in (48) are arranged according to their importance. We estimate them in the order of increasing importance, the treatment of the fourth term being trivial:

$$\int\limits_{f'(A)}^{f'(2A)} O(1)\,d\alpha \ll f'(2A)-f'(A). \tag{49}$$

To estimate the third term, it is sufficient to consider the sum over $1\le h\le H$, since $S_2(\alpha,-h)=S_2(\alpha,h)$. We interchange the integration and the summation and substitute $\theta=-h/\alpha$ to obtain

$$\int\limits_{f'(A)}^{f'(2A)}\frac{1}{H}\sum_{1\le h\le H}S_2(\alpha,h)\,d\alpha \ll \frac{1}{H}\sum_{1\le h\le H}h\int\limits_{-h/f'(A)}^{-h/f'(2A)}\frac{1}{\theta^2}\min\{f'(2A)K+1,\|\theta\|^{-1}\}\,d\theta.$$

We note some simple estimates before applying Lemma 8. We have $0< f'(1)\le f'(2A)\ll A^\delta$ for some $\delta\ge 0$ since f' is monotone and by (32), and therefore $f'(2A)K+1\ll A^{\delta+1}$. By (31) we have $0<-1/\theta\le f'(2A)/h\ll f'(A)/h$ for all θ under consideration. Moreover, the length of the integration range is $h/f'(A)-h/f'(2A)\le h/f'(A)$, and finally from (30) and (34) it follows that $f'(A)\ll(f'(2A)-f'(A))\log A$. Hence Lemma 8 gives

$$\int\limits_{f'(A)}^{f'(2A)}\frac{1}{H}\sum_{1\le h\le H}S_2(\alpha,h)\,d\alpha \ll f'(A)\frac{1}{H}\sum_{1\le h\le H}\frac{f'(A)}{h}\Bigl(\frac{h}{f'(A)}+1\Bigr)(1+\log A^{\delta+1}) \tag{50}$$

$$
\begin{aligned}
&\ll f'(A)\biggl(1+\frac{f'(A)\log H}{H}\biggr)\log A\\
&\ll \bigl(f'(2A)-f'(A)\bigr)(\log A)^2\biggl(1+\frac{f'(A)\log H}{H}\biggr).
\end{aligned}
$$

The contribution of the second term in (48) is easily determined: the sum occurring in the definition of S_2 comprises not more than $f'(2A)K+1 \ll f'(A)K$ summands, therefore

$$
\int\limits_{f'(A)}^{f'(2A)} \frac{1}{H} S_2(\alpha,0)\,d\alpha \ll (f'(2A)-f'(A))K\frac{f'(A)}{H}. \tag{51}
$$

Now we turn to the treatment of the main term in (48). We concentrate on the case $h>0$. We exchange the integral and the sum and apply the substitution $-h/\alpha=\theta$. The factor $\min\{1/\alpha,1/h\}$ then transforms into $\min\{-1/\theta,1/\theta^2\}$, which is $\ll (f'(A)/h)\min\{1,f'(A)/h\}$ by (31). We obtain

$$
\begin{aligned}
&\int\limits_{f'(A)}^{f'(2A)} \sum_{1\le h\le H} \min\{1/\alpha,1/h\} S_1(\alpha,h)\,d\alpha\\
&\qquad\ll f'(A)\sum_{1\le h\le H}\frac{1}{h}\min\{1,f'(A)/h\}\int\limits_{-h/f'(A)}^{-h/f'(2A)} S_1(-h/\theta,h)\,d\theta,
\end{aligned}
$$

and to estimate the integral we use Lemma 13:

$$
\begin{aligned}
&\int\limits_{-h/f'(A)}^{-h/f'(2A)} S_1(-h/\theta,h)\,d\theta \ll \int\limits_0^1 \min\{f'(2A)K+1,\|\xi\|^{-1}\}\\
&\qquad\times\int\limits_{\xi-h/f'(A)}^{\xi-h/f'(2A)} \sup_{f(A)<x\le f(2A)}\Bigl|\sum_{x<m\le x+f'(2A)K}\varphi(m)\,\mathrm{e}(m\theta)\Bigr|\,d\theta\,d\xi.
\end{aligned}
$$

The length of integration of the inner integral is bounded trivially by $h/f'(A)$ and the integrand is 1-periodic, so that we may replace this integral, using the definition (6) of J, by the upper bound

$$
\biggl(\frac{h}{f'(A)}+1\biggr)f'(2A)KJ(A,f'(2A)K),
$$

which is independent of ξ. We use the estimate $f'(2A)K+1\ll A^{\delta+1}$, which we mentioned before, and Lemma 8, to obtain

$$
\int\limits_0^1 \min\{f'(2A)K+1,\|\xi\|^{-1}\}\,d\xi \ll \log A.
$$

Splitting the summation over h at $f'(A)$ we get

$$\sum_{1\le h\le H}\frac{1}{h}\min\left\{1,\frac{f'(A)}{h}\right\}\left(\frac{h}{f'(A)}+1\right)$$
$$\ll \sum_{1\le h\le f'(A)}\frac{1}{h}+\sum_{f'(A)<h\le H}\frac{1}{h}\,\frac{f'(A)}{h}\,\frac{h}{f'(A)} \ll \sum_{1\le h\le H}\frac{1}{h}\ll \log H.$$

Collecting the terms and using the estimate $f'(2A)\ll (f'(2A)-f'(A))\log A$, which follows from Lemma 7, we arrive at

$$\int_{f'(A)}^{f'(2A)}\sum_{1\le h\le H}\min\{1/\alpha,1/h\}S_1(\alpha,h)\,d\alpha \ll f'(A)\big(f'(2A)-f'(A)\big)K(\log A)^2(\log H)J\big(A,f'(2A)K\big). \tag{52}$$

By analogous reasoning the sum over $-H\le h\le -1$ can be estimated by the same expression.

We choose
$$H=z \quad\text{and}\quad K=\frac{z}{f'(2A)}.$$

From the restrictions $\max\{2,f'(2A)\}\le z\le Af'(A)$ it easily follows that $1\le K\le A$ and $H\ge 2$, therefore this is an admissible choice. Note also that $\log H\ll \log A$ by (32). We combine (48)–(52) to get the estimate
$$I\left(A,\frac{z}{f'(2A)}\right)\ll \frac{f'(A)(\log A)^3}{z}+f'(A)(\log A)^3J(A,z).$$

Applying Proposition 1 we see that the left hand side of (5) is bounded by a constant times
$$\frac{f''(A)}{f'(A)^2}z^2+\frac{f'(A)(\log A)^3}{z}+f'(A)(\log A)^3J(A,z).$$

By (47) the second term in this expression is dominated by the third, which completes the proof. ■

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