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A note on minimal zero-sum sequences over $\mathbb{Z}$

by

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1. Introduction. We shall follow the notation and definitions in Grynkiewicz's new monograph [15], and refer the reader to it for the definitions that were omitted here.

For all integers x and y with $x \le y$, let $[x,y] = \{i \in \mathbb{Z} : x \le i \le y\}$. Let G_0 be a non-empty subset of an additive abelian group G. Let $\mathcal{F}(G_0)$ denote the free multiplicative abelian monoid with basis G_0, and whose elements are the (unordered) sequences with terms in G_0. The identity element of $\mathcal{F}(G_0)$, also called the *trivial sequence*, is the sequence with no terms. The operation in $\mathcal{F}(G_0)$ is the *sequence concatenation* product that takes $R, T \in \mathcal{F}(G_0)$ to $S = R \cdot T \in \mathcal{F}(G_0)$. In this case, we say that R (and T) is a *subsequence* of S. For every $S = s_1 \cdot \ldots \cdot s_t \in \mathcal{F}(G_0)$, let

$$\begin{aligned} &\text{the } \textit{length} \text{ of } S, \text{ denoted by } |S|, \text{ be } |S| = t;\\ &\text{the } \textit{sum} \text{ of } S, \text{ denoted by } \sigma(S), \text{ be } \sigma(S) = s_1 + \cdots + s_t;\\ &\text{the } \textit{average} \text{ of } S, \text{ denoted by } S_{\mathrm{av}}, \text{ be } S_{\mathrm{av}} = \sigma(S)/|S|;\\ &\text{the } \textit{infinite norm} \text{ of } S, \text{ denoted by } \|S\|_\infty, \text{ be } \|S\|_\infty = \sup_{1\le i\le t} |s_i|. \end{aligned} \tag{1.1}$$

For any $g \in G$ and any integer $d \ge 0$, we let

$$g^{[d]} = \underbrace{g \cdot \ldots \cdot g}_{d},$$

where $g^{[d]}$ denotes the empty sequence if $d = 0$.

A *zero-sum sequence over* G_0 is a sequence $S \in \mathcal{F}(G_0)$ such that $\sigma(S) = 0$. Such a sequence is called *minimal* if it does not contain a proper non-trivial zero-sum subsequence. Then the submonoid

$$\mathcal{B}_0 = \mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) : \sigma(S) = 0\}$$

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of $\mathcal{F}(G_0)$ is a Krull monoid (see e.g. [15]). The set $\mathcal{A}(\mathcal{B}_0)$ of *atoms* of $\mathcal{B}_0$ is the set of all minimal zero-sum sequences in $\mathcal{B}_0$. A characterization of $\mathcal{A}(\mathcal{B}_0)$ would shed some light on the factorization properties of $\mathcal{B}_0$ (see e.g. [12, 13]).

Given a minimal zero-sum sequence $S = s_1 \cdot \ldots \cdot s_t \in \mathcal{A}(\mathcal{B}_0)$, we are interested in bounding its length depending on its terms s_i for $i \in [1,t]$. We are also interested in finding a natural structure for $\mathcal{A}(\mathcal{B}_0)$ when G_0 (and thus $\mathcal{B}_0$) is finite.

The study of zero-sum sequences in $\mathcal{B}(G)$ when G is a finite cyclic group is a very active area of research (see e.g. [2, 5, 6, 9, 18, 19, 22]), with applications to factorization theory (see e.g. [3, 10, 11, 12]). Similar, but less extensive, investigations have been carried out when G is an infinite cyclic group (see e.g. [4, 7, 13, 14]).

For all $S \in \mathcal{B}(\mathbb{Z})$ with $|S|$ finite and $|S| > 1$, there exist positive integers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ with $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_m$ such that

$$S^+ = \prod_{i=1}^{n} a_i^{[x_i]}, \qquad S^- = \prod_{j=1}^{m} (-b_j)^{[y_j]}, \qquad S = S^+ \cdot S^-, \tag{1.2}$$

where x_i and y_j are positive integers for all $i \in [1,n]$ and $j \in [1,m]$.

In his work on Diophantine linear equations, Lambert [17] proved the following theorem.

Theorem 1.1 (Lambert [17]). *Let S be a minimal zero-sum sequence over $\mathbb{Z}$ with $|S|$ finite and $|S| > 1$. If S is as in* (1.2), *then*

$$|S^+| \le \|S^-\|_\infty = b_m \quad \text{and} \quad |S^-| \le \|S^+\|_\infty = a_n.$$

This was reformulated and reproved in the language of sequences by Baginski et al. [4]. Perhaps due to inconsistent notation across various areas, Theorem 1.1 has been independently rediscovered by Diaconis et al. [8] and Sahs et al. [21]. Currently, the best bounds for $|S^+|$ and $|S^-|$ are due to Henk–Weismantel [16]. They proved the following theorem of which Theorem 1.1 is a special case upon setting $\ell = m$ and $k = n$.

Theorem 1.2 (Henk–Weismantel [16]). *Let S be a minimal zero-sum sequence over $\mathbb{Z}$ with $|S|$ finite and $|S| > 1$. If S is as in* (1.2), *then*

$$|S^+| \le b_\ell - \sum_{j=1}^{\ell-1} \left\lfloor \frac{b_\ell - b_j}{a_n} \right\rfloor y_j + \sum_{j=\ell+1}^{m} \left\lceil \frac{b_j - b_\ell}{a_1} \right\rceil y_j \quad \text{for all } \ell \in [1,m], \tag{J_ℓ}$$

$$|S^-| \le a_k - \sum_{i=1}^{k-1} \left\lfloor \frac{a_k - a_i}{b_m} \right\rfloor x_i + \sum_{i=k+1}^{n} \left\lceil \frac{a_i - a_k}{b_1} \right\rceil x_i \quad \text{for all } k \in [1,n]. \tag{I_k}$$

In this paper, we improve on Theorem 1.2 by proving the following theorem.

THEOREM 1.3. *Let S be a minimal zero-sum sequence over $\mathbb{Z}$ with $|S|$ finite and $|S| > 1$. If S is as in* (1.2), *then*

$$|S^+| \le \lfloor -S^-_{\mathrm{av}} \rfloor = \left\lfloor \frac{\sum_{j=1}^m b_j y_j}{\sum_{j=1}^m y_j} \right\rfloor \quad \textit{and} \quad |S^-| \le \lfloor S^+_{\mathrm{av}} \rfloor = \left\lfloor \frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n x_i} \right\rfloor .$$

The bounds in Theorems 1.1–1.3 are all tight for the minimal zero-sum sequences

$$S = a^{\left[\frac{b}{\gcd(a,b)}\right]} \cdot (-b)^{\left[\frac{a}{\gcd(a,b)}\right]},$$

for all positive integers a and b. On the other hand, if we consider the minimal zero-sum sequence $S = 3^{[1]} \cdot 4^{[2]} \cdot (-1)^{[2]} \cdot (-9)^{[1]}$, then Theorem 1.1 yields $|S^+| \le 9$ and $|S^-| \le 4$, Theorem 1.2 yields $|S^+| \le 4$ and $|S^-| \le 4$, while Theorem 1.3 yields the tight bounds $|S^+| \le 3$ and $|S^-| \le 3$.

In Section 2, we prove Theorem 1.3 by refining the method of Sahs et al. [21]. In Section 3, we define a natural partial order on the set $\mathcal{A}(\mathcal{B}_0)$ of minimal zero-sum sequences and discuss its relevance. In Section 4, we show that the bounds in Theorem 1.3 are always sharper than or equivalent to the bounds in Theorem 1.2.

2. Proof of Theorem 1.3. Let G be an additive abelian group, and let $S = s_1 \cdot \ldots \cdot s_t \in \mathcal{F}(G)$. For all $i, j \in [1, t]$ such that $i \ne j$, let S' be the sequence obtained by removing the terms s_i and s_j from S and inserting (anywhere) the term $s_i + s_j$. We call this process an (s_i, s_j)-*derivation* and say that S' is (s_i, s_j)-*derived* from S. We also say that S' is *derived* from S without specifying the pair (s_i, s_j). For instance, if $S = 2^{[3]} \cdot (-3)^{[2]}$, then $S' = 2^{[2]} \cdot (-3) \cdot (-1)$ is $(2, -3)$-derived from S, and $S' = 4^{[1]} \cdot 2^{[1]} \cdot (-3)^{[2]}$ is $(2, 2)$-derived from S.

We will use the following lemma, which is a special case of Lemma 2 in Sahs et al. [21]. For completeness, we include a very short proof.

LEMMA 2.1. *Let G be an additive abelian group. Let $S = s_1 \cdot \ldots \cdot s_t$ be a minimal zero-sum sequence over G, and let $i, j \in [1, t]$ be such that $i \ne j$. If S' is (s_i, s_j)-derived from S, then S' is also a minimal zero-sum sequence over G.*

Proof. By definition S' is a zero-sum sequence over G since $s_i + s_j \in G$ and

$$\sigma(S') = \sigma(s) - s_i - s_j + (s_i + s_j) = \sigma(S) = 0.$$

Suppose that S' is not minimal. Then there exist non-trivial zero-sum subsequences R and T such that $S' = R \cdot T$, and the specific term $s_i + s_j$ (there may be other copies of $s_i + s_j$ in S' and S) is a subsequence of either R or T, and not both. Thus, either R or T is a proper zero-sum subsequence of S.

This would contradict the minimality of S. Thus, S' is minimal zero-sum sequence. ■

We now prove our main theorem.

Proof of Theorem 1.3. Let S be a minimal zero-sum sequence over $\mathbb{Z}$ with $|S|$ finite and $|S|>1$. Then there exist positive integers $a_1,\dots,a_n$ and $b_1,\dots,b_m$ with $a_1\le\dots\le a_n$ and $b_1\le\dots\le b_m$ such that

$$S^+=\prod_{i=1}^{n} a_i^{[x_i]},\qquad S^-=\prod_{j=1}^{m}(-b_j)^{[y_j]},\qquad S=S^+\cdot S^-,$$

where x_i and y_j are positive integers for all $i\in[1,n]$ and $j\in[1,m]$.

We shall prove by induction on $|S|\ge 2$ that

$$|S^+|\le -S^-_{\mathrm{av}}\quad\text{and}\quad |S^-|\le S^+_{\mathrm{av}}. \tag{2.1}$$

If $|S|=2$, then we must have $m=n=1$, $S=a_1\cdot(-b_1)$, and $a_1-b_1=0$. Since $a_1,b_1>0$, the statement (2.1) clearly holds. Assume that $|S|\ge 2$ and (2.1) holds for all minimal zero-sum sequences R such that $2\le|R|<|S|$.

If $a_i=b_j$ for some $i\in[1,n]$ and $j\in[1,m]$, then $S=a_i\cdot(-b_j)$, as otherwise $S'=a_i\cdot(-b_j)$ would be a proper zero-sum subsequence of S, which would contradict the minimality of S. Thus, we may assume that

$$\{a_1,\dots,a_n\}\cap\{b_1,\dots,b_m\}=\emptyset.$$

Without loss of generality, we also assume that $a_n=\|S^+\|_\infty>\|S^-\|_\infty=b_m$.

To prove the inductive step, we first show that $|S^+|\le -S^-_{\mathrm{av}}$. Since $x_n>0$, $y_m>0$, and $a_n-b_m>0$, we can use Lemma 2.1 to perform an $(a_n,-b_m)$-derivation from S, and obtain the minimal zero-sum sequence

$$R=(a_n-b_m)^{[1]}\cdot a_n^{[x_n-1]}\cdot\prod_{i=1}^{n-1}a_i^{[x_i]}\cdot(-b_m)^{[y_m-1]}\prod_{j=1}^{m}(-b_j)^{[y_j]},$$

where we omit the term a_n if $x_n=1$ and the term $(-b_m)$ if $y_m=1$.

Since $|R|=|S|-1$, it follows from the induction hypothesis that

$$\begin{aligned}|R^+|&=1+(x_n-1)+\sum_{i=1}^{n-1}x_i=\sum_{i=1}^{n}x_i\\&\le -R^-_{\mathrm{av}}=\frac{(y_m-1)b_m+\sum_{j=1}^{m-1}y_jb_j}{(y_m-1)+\sum_{j=1}^{m-1}y_j}.\end{aligned}\tag{2.2}$$

Since $b_m=\|S^-\|_\infty\ge\|R^-\|_\infty$, it follows from (2.2) that

$$|R^+|=\sum_{i=1}^{n}x_i\le\frac{b_m+(y_m-1)b_m+\sum_{j=1}^{m-1}y_jb_j}{1+(y_m-1)+\sum_{j=1}^{m-1}y_j}=\frac{-\sigma(S^-)}{|S^-|}=-S^-_{\mathrm{av}}.$$

Thus,

$$(2.3)\qquad S^+ = \sum_{i=1}^{n} x_i = |R^+| \le -S^-_{\mathrm{av}}.$$

Next, we show that $|S^-| \le S^+_{\mathrm{av}}$. Since $\sigma(S) = 0$, we have $\sigma(S^+) = -\sigma(S^-)$. This observation and (2.3) yield

$$(2.4)\qquad |S^+| \le -S^-_{\mathrm{av}} = \frac{-\sigma(S^-)}{|S^-|} = \frac{\sigma(S^+)}{|S^-|}, \quad \text{so} \quad |S^-| \le \frac{\sigma(S^+)}{|S^+|} = S^+_{\mathrm{av}}.$$

Since $|S^+|$ and $|S^-|$ are integers, the theorem follows from (2.3) and (2.4) by taking the floors of S^+_{av} and $-S^-_{\mathrm{av}}$. ∎

REMARK 2.2. Let S be as in (1.2) and suppose that there exists $t \in [1, m]$ such that

$$(2.5)\qquad a_n > b_t > -S^-_{\mathrm{av}} = \frac{\sum_{j=1}^{m} b_j y_j}{\sum_{j=1}^{m} y_j}.$$

Then the $(a_n, -b_t)$-derivation on S yields the minimal zero-sum sequence

$$R = (a_n - b_t)^{[1]} \cdot a_n^{[x_n-1]} \cdot \prod_{i=1}^{n-1} a_i^{[x_i]} \cdot (-b_t)^{[y_t-1]} \prod_{j=1,\, j\ne t}^{m} (-b_j)^{[y_j]}.$$

Thus, by applying Theorem 1.3 to R, we obtain

$$(2.6)\qquad |S^+| = \sum_{i=1}^{n} x_i = |R^+| \le \lfloor -R^-_{\mathrm{av}} \rfloor.$$

Since $-R^-_{\mathrm{av}} < -S^-_{\mathrm{av}}$ (by the definition of R and (2.5)), the bound for $|S^+|$ in (2.6) is sometimes better than $|S^+| \le \lfloor -S^-_{\mathrm{av}} \rfloor$ given by Theorem 1.3. By symmetry, we may sometimes obtain a better bound for $|S^-|$ in a similar manner.

3. The structure of the minimal zero-sum sequences. Let G_0 be a finite subset of $\mathbb{Z}$. We are interested in finding a natural structure on the set $\mathcal{A}(\mathcal{B}_0)$ of minimal zero-sum sequences in $\mathcal{B}_0 = \mathcal{B}(G_0)$. As mentioned in the introduction, $\mathcal{A}(\mathcal{B}_0)$ is also the set of atoms of the Krull monoid $\mathcal{B}_0$. There are other interesting interpretations of $\mathcal{A}(\mathcal{B}_0)$. In the context of Diophantine linear equations (see e.g. [16, 17, 20]), $\mathcal{A}(\mathcal{B}_0)$ corresponds to the union of all *Hilbert bases* (1), which are minimal generating sets of all the solutions. In the context of integer partitions, each sequence $S = a_1 \cdot \ldots \cdot a_p \cdot (-b_1) \cdot \ldots \cdot (-b_q) \in \mathcal{A}(\mathcal{B}_0)$ such that $p + q \ge 3$, $a_i > 0$ for $i \in [1, p]$, and $b_j > 0$ for $j \in [1, q]$, corresponds to the *primitive partition identity*

(1) This union is also known as the *Graver basis* of the corresponding *toric ideal* (see e.g. [24]).

$a_1+\cdots+a_p=b_1+\cdots+b_q$ (see [8, p. 1]). Primitive partition identities were studied by Diaconis et al. [8] who were motivated by applications in Gröbner bases, computational statistics, and integer programming (see e.g. [23, 24]).

In the process of characterizing $\mathcal{A}(\mathcal{B}_0)$, we assume that $S=s_1\cdot\ldots\cdot s_t \in \mathcal{A}(\mathcal{B}_0)$ is equivalent to $-S=(-s_1)\cdot\ldots\cdot(-s_t)\in\mathcal{A}(\mathcal{B}_0)$ and we only include one of them in $\mathcal{A}(\mathcal{B}_0)$. For any positive integer n, define the *n-derived set*, $\mathcal{D}_n(S)$, of $S=s_1\cdot\ldots\cdot s_t\in\mathcal{B}(\mathbb{Z})$ by

$$\mathcal{D}_n(S)=\{S' : i,j\in[1,t],\ i\neq j,\ S' \text{ is } (s_i,s_j)\text{-derived, and } \|S'\|_\infty\le n\}.$$

Given $R,S\in\mathcal{B}(\mathbb{Z})$, we write $R\prec_n S$ if and only if $R=S$ or $R\in\mathcal{D}_n(S)$.

The following proposition is a direct consequence of Lemma 2.1.

Proposition 3.1. *Let n be a positive integer, $G_0=[-n,n]$, and $\mathcal{B}_0=\mathcal{B}(G_0)$.*

(i) *If $S\in\mathcal{A}(\mathcal{B}_0)$, then $\mathcal{D}_n(S)\subseteq\mathcal{A}(\mathcal{B}_0)$.*
(ii) *$\mathcal{P}_n=(\mathcal{A}(\mathcal{B}_0),\prec_n)$ is a poset.*

For instance, if $S=2^{[3]}\cdot(-3)^{[2]}$, then Figure 1 shows the poset $\mathcal{P}_3$. Note that $S'=2^{[3]}\cdot(-6)$ is $(-3,-3)$-derived from S, but $S'\notin\mathcal{D}_3(S)$ since $\|S'\|_\infty=6>3$.

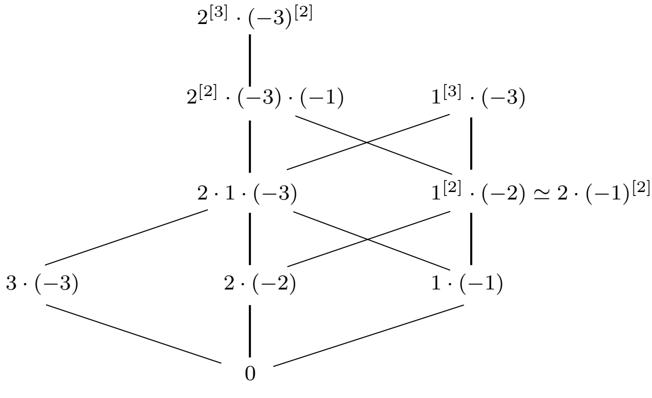

Fig. 1. The poset $\mathcal{P}_3$

Let $\mathcal{M}_n$ be the set of maximal elements of the poset $\mathcal{P}_n$ of Proposition 3.1, i.e., $\mathcal{M}_n$ contains all minimal sequences $R\in\mathcal{A}(\mathcal{B}_0)$ that cannot be derived from any $S\in\mathcal{A}(\mathcal{B}_0)$. Then the following proposition is immediate.

Proposition 3.2. *Let n be a positive integer, $G_0=[-n,n]$, and $\mathcal{B}_0=\mathcal{B}(G_0)$. If $\mathcal{Q}$ is a set such that $\mathcal{M}_n\subseteq\mathcal{Q}\subseteq\mathcal{A}(\mathcal{B}_0)$, then*

$$\mathcal{A}(\mathcal{B}_0)=\mathcal{Q}\cup\bigcup_{S\in\mathcal{Q}}\mathcal{D}_n(S),$$

where we assume that $S\in\mathcal{A}(\mathcal{B}_0)$ is equivalent to $-S\in\mathcal{A}(\mathcal{B}_0)$.

For instance, Figure 1 shows that

$$\mathcal{M}_3 = \{2^{[3]} \cdot (-3)^{[2]}, 1^{[3]} \cdot (-3)^{[1]}\}.$$

We also verified that

$$\mathcal{M}_n \subseteq \{a^{\left[\frac{b}{\gcd(a,b)}\right]} \cdot (-b)^{\left[\frac{a}{\gcd(a,b)}\right]} : a, b \in [1, n]\} \qquad \text{for } n \in [1, 5]. \tag{3.1}$$

However, by using the 4ti2-software package [1], we found that (3.1) does not hold for $n = 6$. In particular,

$$\begin{aligned}\mathcal{M}_6 - \{a^{\left[\frac{b}{\gcd(a,b)}\right]} \cdot (-b)^{\left[\frac{a}{\gcd(a,b)}\right]} : a, b \in [1, 6]\} \\ = \{2^{[2]} \cdot 3^{[1]} \cdot 5^{[1]} \cdot (-6)^{[2]}, 1^{[1]} \cdot 3^{[1]} \cdot 4^{[2]} \cdot (-6)^{[2]}\}.\end{aligned}$$

Determining $\mathcal{M}_n$ (or a small enough superset of $\mathcal{M}_n$), for all $n > 0$, would directly yield an algorithm for generating $\mathcal{P}_n$, and an approach for computing the cardinality of $\mathcal{A}(\mathcal{B}_0)$ (e.g., by studying the Möbius function of $\mathcal{P}_n$).

4. Comparison of the bounds in Theorems 1.2 & 1.3. In this section, we show that the bounds in Theorem 1.3 are in general sharper than or equivalent to the bounds in Theorem 1.2. To do this, we will show that it is enough to compare those two theorems for sequences S (where S is as in (1.2)) such that

$$a_1 \le |S^-| = \sum_{j=1}^{m} y_j \le a_n \quad \text{and} \quad b_1 \le |S^+| = \sum_{i=1}^{n} x_i \le b_m. \tag{4.1}$$

First, note that it follows from Theorem 1.1 that

$$\sum_{j=1}^{m} y_j = |S^-| \le a_n \quad \text{and} \quad \sum_{i=1}^{n} x_i = |S^+| \le b_m. \tag{4.2}$$

Let $\ell \in [1, m]$, $k \in [1, n]$, and consider the upper bounds

$$U_{J_\ell} = b_\ell - \sum_{j=1}^{\ell-1} \left\lfloor \frac{b_\ell - b_j}{a_n} \right\rfloor y_j + \sum_{j=\ell+1}^{m} \left\lfloor \frac{b_\ell - b_j}{a_1} \right\rfloor y_j, \tag{4.3}$$

$$U_{I_k} = a_k - \sum_{i=1}^{k-1} \left\lfloor \frac{a_k - a_i}{b_m} \right\rfloor x_i + \sum_{i=k+1}^{n} \left\lceil \frac{a_i - a_k}{b_1} \right\rceil x_i, \tag{4.4}$$

in the inequalities (J_ℓ) and (I_k) of Theorem 1.2, where $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_m$.

Without loss of generality, assume that $b_m \ge a_n$. Then $\left\lfloor \frac{a_k - a_i}{b_m} \right\rfloor = 0$ for $1 \le i < k \le n$, and it follows from (4.4) that

$$U_{I_k} \ge a_k \ge a_1 \quad \text{for all } k \in [1, n]. \tag{4.5}$$

Thus, it follows from (4.2), (4.5), and the fact that $S^+_{\mathrm{av}} \geq a_1$ that Theorems 1.2 and 1.3 can only give meaningful upper bounds for $|S^-|$ if

$$a_1 \leq |S^-| = \sum_{j=1}^{m} y_j \leq a_n. \tag{4.6}$$

Next, it follows from the definition of $-S^-_{\mathrm{av}}$ in (1.1) that

$$\begin{aligned} -S^-_{\mathrm{av}} &= \frac{-\sigma(S^-)}{|S^-|} = \frac{\sum_{j=1}^m b_j y_j}{\sum_{j=1}^m y_j} \\ &= \frac{\sum_{j=1}^m b_\ell y_j - \sum_{j=1}^{\ell-1}(b_\ell - b_j) y_j + \sum_{j=\ell+1}^m (b_j - b_\ell) y_j}{\sum_{j=1}^m y_j} \\ &= b_\ell - \frac{\sum_{j=1}^{\ell-1}(b_\ell - b_j) y_j}{\sum_{j=1}^m y_j} + \frac{\sum_{j=\ell+1}^m (b_j - b_\ell) y_j}{\sum_{j=1}^m y_j}. \end{aligned} \tag{4.7}$$

Since $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_m$, it follows from (4.6) and (4.7) that

$$\begin{aligned} -S^-_{\mathrm{av}} &\leq b_\ell - \sum_{j=1}^{\ell-1} \frac{(b_\ell - b_j) y_j}{a_n} + \sum_{j=\ell+1}^{m} \frac{(b_j - b_\ell) y_j}{a_1} \\ &\leq b_\ell - \sum_{j=1}^{\ell-1} \left\lfloor \frac{b_\ell - b_j}{a_n} \right\rfloor y_j + \sum_{j=\ell+1}^{m} \left\lceil \frac{b_j - b_\ell}{a_1} \right\rceil y_j = U_{J_\ell}. \end{aligned} \tag{4.8}$$

Thus, Theorem 1.3 and (4.8) yield

$$|S^+| \leq \lfloor -S^-_{\mathrm{av}} \rfloor \leq -S^-_{\mathrm{av}} \leq U_{J_\ell}, \tag{4.9}$$

which implies inequality (J_ℓ) of Theorem 1.2.

Moreover, it follows from (4.9) and the definition of $-S^-_{\mathrm{av}}$ that

$$b_1 \leq -S^-_{\mathrm{av}} \leq U_{J_\ell}. \tag{4.10}$$

Thus, it follows from (4.2) and (4.10) that Theorems 1.2 and 1.3 can only give meaningful upper bounds for $|S^+|$ if

$$b_1 \leq |S^+| = \sum_{i=1}^{n} x_i \leq b_m. \tag{4.11}$$

Similarly to the proof of (4.9), we can now use (4.11) to show (although we omit the details here) that Theorem 1.3 implies inequality (I_k) of Theorem 1.2, i.e.

$$|S^-| \leq \lfloor S^+_{\mathrm{av}} \rfloor \leq S^+_{\mathrm{av}} \leq U_{I_k}. \tag{4.12}$$

Finally, it follows from (4.9) and (4.12) that the bounds in Theorem 1.3 are in general sharper than or equivalent to the bounds in Theorem 1.2.

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