A Brun–Titchmarsh inequality for weighted sums over prime numbers

by

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1. Introduction. In this paper we prove upper bounds for sums of the form

$$(1.1) \qquad \sum_{\substack{p \in I \\ p \equiv l \bmod k}} f(p),$$

where $I = [x, x + y] \subset [0, \infty)$ is an interval, k and l are coprime integers, and f is a slowly varying weight function on I. The results generalize the well-known Brun–Titchmarsh inequality for the number of prime numbers in arithmetic progressions [Tit30, Iwa82].

The work is motivated by the following problem. Functions as the Riemann prime counting function $\pi^*(x)$ or the Chebyshev function $\psi(x)$ satisfy certain explicit formulas involving sums over the zeros of the Riemann zeta function [Rie59, vM95]. If one is interested in studying these functions via their explicit formulas, one has to deal with the problem that these sums converge only due to oscillation and are therefore difficult to handle. A natural way to overcome this problem is to study continuous (or smooth) approximations to the functions $\pi^*(x)$ and $\psi(x)$ [FK, Büt]. The problem of estimating the approximation error then leads to sums of the form (1.1). As far as the author knows, this has been carried out in [RS03] for the first time, to obtain short effective intervals containing prime numbers, where the Brun–Titchmarsh inequality is used to estimate such sums.

Apart from that, the results are likely to have other applications. One further application might be to improve the estimates for the error term in the asymptotic formula for the zero counting function of the Riemann zeta function, where similar sums occur [CCM13].

The proofs of the results are elementary and use a simple notion of weighted sieves.

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2. Notations. We use the notations of [MV06] in the context of number theory, and those of [AF03] for function spaces and their norms. In (1.1) we will consider functions $f \in W^{1,1}(I)$, the closure of $C^1(I)$ with respect to the norm

$$\|f\|_{1,1,I} = \|f\|_{1,I} + \|f'\|_{1,I} = \int_I |f(t)| \, dt + \int_I |f'(t)| \, dt,$$

which we regard as a subspace of $C^0(I)$ by the continuous embedding of $C^1(I)$, equipped with the norm $\|\cdot\|_{1,1,I}$, into $C^0(I)$.

Furthermore, we use the notations $I = [x, x+y] \subset [0, \infty)$, $\mathcal{A} = I \cap (l+k\mathbb{Z})$, $\mathcal{A}_d = \{a \in \mathcal{A} \mid d \mid a\}$ and

$$P(z, k) = \prod_{\substack{p \leq z \\ p \nmid k}} p$$

throughout this paper. For explicit estimates, we also use Turing's Θ-notation and say $f(x) = \Theta(g(x))$ for $x \in A$ if $|f(x)| \leq g(x)$ for all $x \in A$.

3. Weighted sieves. The weighted sieves are based on the following lemma.

LEMMA 3.1. *Let $f \in W^{1,1}(I)$ and let $d, k \in \mathbb{N}$ satisfy $(d, k) = 1$. Then*

$$\left| \frac{1}{kd} \int_I f(t) \, dt - \sum_{\substack{n \in I \cap (l+k\mathbb{Z}) \\ d \mid n}} f(n) \right| \leq \|f\|_{\infty,I} + \|f'\|_{1,I}.$$

Proof. This is essentially Gallagher's proof of the large sieve inequality, as noted by the referee. Let

$$(3.1) \qquad\qquad r_d = \sum_{n \in \mathcal{A}_d} f(n) - \frac{1}{kd} \int_I f(t) \, dt.$$

We have to show $|r_d| \leq \|f\|_{\infty,I} + \|f'\|_{1,I}$. By replacing k by kd and adjusting l we may assume $d = 1$, and by replacing f by $f(k \cdot + l)$ and adjusting x and y we may also assume $k = 1$.

Now let $x' = x + [y]$, $I' = [x, x']$ and $I'' = [x', x + y]$. We define

$$r' = \sum_{a \in \mathcal{A} \cap I'} f(a) - \int_{I'} f(t) \, dt, \qquad r'' = \sum_{a \in \mathcal{A} \cap I''} f(a) - \int_{I''} f(t) \, dt.$$

Then we have $r_1 = r' + r''$, since I is the disjoint union of I' and I''.

Let $a < b$ and $\xi \in [a, b]$. Then, by choosing an approximating sequence of $C^1(I)$-functions, we can extend the well-known identity

$$\int_a^b g(t) \, dt = (b - a)g(\xi) + \int_a^\xi g'(t)(a - t) \, dt + \int_\xi^b g'(t)(b - t) \, dt$$

to all $g \in W^{1,1}((a,b))$. We therefore obtain the bound

$$(3.2) \qquad \left| g(\xi) - \frac{1}{b-a} \int_a^b g(t)\,dt \right| \leq \int_a^b |g'(t)|\,dt.$$

The interval I' is the disjoint union of half-open intervals $[x+n, x+n+1)$, $n = 0, 1, \ldots, [y] - 1$. Since each such interval contains exactly one element of $\mathcal{A}$, we can apply (3.2), which yields the bound

$$|r'| \leq \int_{I'} |f'(t)|\,dt.$$

It remains to estimate r''. Since $|I''| < 1$, the intersection $\mathcal{A} \cap I''$ is either empty or contains exactly one element a. In the first case we have

$$|r''| \leq \int_{I''} f(t)\,dt \leq \|f\|_{\infty, I''},$$

and in the second case,

$$|r''| \leq \left| \int_{I''} f(t)\,dt - |I''|f(a) \right| + (1 - |I''|)|f(a)|$$

$$\leq |I''| \int_{I''} |f'(t)|\,dt + (1 - |I''|)|f(a)| \leq \|f'\|_{1, I''} + \|f\|_{\infty, I''}.$$

So in both cases $|r''| \leq \|f'\|_{1, I''} + \|f\|_{\infty, I''}$. We thus get

$$|r_1| \leq |r'| + |r''| \leq \|f'\|_{1, I'} + \|f'\|_{1, I''} + \|f\|_{\infty, I''} \leq \|f\|_{\infty, I} + \|f'\|_{1, I}. \qquad \blacksquare$$

PROPOSITION 3.2 (Weighted Selberg sieve). *Let $f \in W^{1,1}(I)$ be nonnegative and let $z \geq 1$. Define*

$$S_k(z) = \sum_{\substack{n \leq z \\ (n,k)=1}} \frac{\mu(n)^2}{\varphi(n)} \quad and \quad H_k(z) = \sum_{\substack{n \leq z \\ (n,k)=1}} \mu(n)^2 \frac{\sigma(n)}{\varphi(n)}.$$

Then

$$\sum_{\substack{n \in I \cap (l+k\mathbb{Z}) \\ (n,P(z,k))=1}} f(n) \leq \frac{\|f\|_{1,I}}{kS_k(z)} + (\|f\|_{\infty,I} + \|f'\|_{1,I}) \frac{H_k(z)^2}{S_k(z)^2}.$$

Proof. The proof is based on Selberg's lambda squared method [Sel47]. Let r_d be as in (3.1) and let

$$\mathcal{N}_k = \{n \in \mathbb{N} \mid 1 \leq n \leq z, \ \mu(n) \neq 0, \ (n,k) = 1\}.$$

Let $(\lambda_n)_{n \in \mathcal{N}_k}$ be a sequence of real numbers satisfying $\lambda_1 = 1$. Then

$$(3.3) \quad \sum_{a \in \mathcal{A}} f(a) \Big(\sum_{\substack{n \in \mathcal{N}_k \\ n|a}} \lambda_n \Big)^2 \geq \sum_{\substack{a \in \mathcal{A} \\ (a,P(z,k))=1}} f(a) \Big(\sum_{\substack{n \in \mathcal{N}_k \\ n|a}} \lambda_n \Big)^2 = \sum_{\substack{a \in \mathcal{A} \\ (a,P(z,k))=1}} f(a).$$

By a simple calculation we get

$$(3.4) \qquad \sum_{\substack{a \in \mathcal{A} \\ }} f(a) \Big(\sum_{\substack{n \in \mathcal{N}_k \\ n \mid a}} \lambda_n \Big)^2 = \sum_{(n_1,n_2) \in \mathcal{N}_k^2} \lambda_{n_1} \lambda_{n_2} \sum_{a \in \mathcal{A}_{[n_1,n_2]}} f(a)$$

$$= \frac{\|f\|_{1,I}}{k} \sum_{(n_1,n_2) \in \mathcal{N}_k^2} \frac{\lambda_{n_1} \lambda_{n_2}}{[n_1,n_2]} + \sum_{(n_1,n_2) \in \mathcal{N}_k^2} \lambda_{n_1} \lambda_{n_2} r_{[n_1,n_2]}.$$

Now (3.3), (3.4) and Lemma 3.1 together imply

$$\sum_{\substack{n \in I \cap (l+k\mathbb{Z}) \\ (n,P(z,k))=1}} f(n) \leq \frac{\|f\|_{1,I}}{k} \sum_{(n_1,n_2) \in \mathcal{N}_k^2} \frac{\lambda_{n_1} \lambda_{n_2}}{[n_1,n_2]} + (\|f\|_{\infty,I} + \|f\|_1') \Big(\sum_{n \in \mathcal{N}_k} |\lambda_n| \Big)^2.$$

Here we choose the well-known minimizing sequence

$$\lambda_n = \frac{\mu(n)n}{S_k(z)} \sum_{\substack{m \in \mathcal{N}_k \\ n \mid m}} \frac{1}{\varphi(n)}$$

for the quadratic form in (3.4), which gives

$$\sum_{(n_1,n_2) \in \mathcal{N}_k^2} \frac{\lambda_{n_1} \lambda_{n_2}}{[n_1,n_2]} = \frac{1}{S_k(z)} \quad \text{and} \quad \sum_{n \in \mathcal{N}_k} |\lambda_n| = \frac{H_k(z)}{S_k(z)}$$

(see [vLR65]). ∎

PROPOSITION 3.3 (Weighted Eratosthenes sieve). *Let* $f \in W^{1,1}(I)$ *be non-negative and let* $z \geq 1$. *Then*

$$\sum_{\substack{n \in I \cap (l+k\mathbb{Z}) \\ (n,P(z,k))=1}} f(n) \leq \frac{\|f\|_{1,I}}{k} \prod_{\substack{p \leq z \\ p \nmid k}} \Big(1 - \frac{1}{p} \Big) + (\|f\|_{\infty,I} + \|f'\|_{1,I}) \, 2^{\pi(z)}.$$

Proof. We have

$$\sum_{\substack{n \in I \cap (l+k\mathbb{Z}) \\ (n,P(z,k))=1}} f(n) = \sum_{a \in \mathcal{A}} f(a) \sum_{d \mid (a,P(z,k))} \mu(d) = \sum_{d \mid P(z,k)} \mu(d) \sum_{a \in \mathcal{A}_d} f(a)$$

$$= \frac{\|f\|_{1,I}}{k} \sum_{d \mid P(z,k)} \frac{\mu(d)}{d} + \sum_{d \mid P(z,k)} \mu(d) r_d$$

$$\leq \frac{\|f\|_{1,I}}{k} \prod_{p \mid P(z,k)} \Big(1 - \frac{1}{p} \Big) + (\|f\|_{\infty,I} + \|f'\|_{1,I}) \, 2^{\pi(z)}. \quad \blacksquare$$

4. The weighted Brun–Titchmarsh inequality. We first prove a general version of the weighted Brun–Titchmarsh inequality based on the

considerations in [vLR65]. We then give a stronger result for the case $k = 1$, which is of special interest for the prime counting function. The first result implies the Brun–Titchmarsh inequality as stated in [vLR65], and the second result implies the stronger version in [MV73].

DEFINITION 4.1. We define the functional $\rho_I \colon W^{1,1}(I) \to [0, \infty)$ by

$$\rho_I(f) = \frac{\|f\|_{1,I}}{\|f\|_{\infty,I} + \|f'\|_{1,I}}$$

for $f \in W^{1,1}(I) \setminus \{0\}$ and $\rho_I(0) = 0$.

THEOREM 4.2. *Let* $I = [x, x + y] \subset [0, \infty)$, *and let* $l, k \in \mathbb{N}$ *be coprime numbers. Then for all non-negative* $f \in W^{1,1}(I)$ *satisfying* $\rho_I(f) > k$,

$$\sum_{\substack{p \in I \\ p \equiv l \bmod k}} f(p) < 2 \frac{\|f\|_{1,I}}{\varphi(k) \log(\rho_I(f)/k)} \left(1 + \frac{8}{\log(\rho_I(f)/k)} \right),$$

$$\sum_{\substack{p \in I \\ p \equiv l \bmod k}} f(p) < 3 \frac{\|f\|_{1,I}}{\varphi(k) \log(\rho_I(f)/k)}.$$

Proof. We reduce the proof to a situation in the proof of the ordinary Brun–Titchmarsh inequality in [vLR65].

Let $g = f/(\|f\|_{\infty,I} + \|f'\|_{1,I})$. Then $\rho_I(g) = \rho_I(f)$ and $\|g\|_{1,I} = \rho_I(f)$. We define $Y = \rho_I(f)$. It suffices to prove the inequalities

$$(4.1) \qquad \sum_{\substack{p \in I \\ p \equiv l \bmod k}} g(p) < 2 \frac{Y}{\varphi(k) \log(Y/k)} \left(1 + \frac{8}{\log(Y/k)} \right),$$

$$(4.2) \qquad \sum_{\substack{p \in I \\ p \equiv l \bmod k}} g(p) < 3 \frac{Y}{\varphi(k) \log(Y/k)}.$$

Let $k_1 = [k, 2]$ and let l_1 be a representative of the odd pre-image (i.e. the pre-image containing only odd representatives) of the residue class of l under the projection $\mathbb{Z}/k_1\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$. Then, since $\|g\|_{\infty,I} \leq 1$, we have

$$\sum_{\substack{p \in I \\ p \equiv l \bmod k}} g(p) \leq \sum_{\substack{p \in I \\ p \equiv l_1 \bmod k_1}} g(p) + 1.$$

Now Proposition 3.2 gives the bound

$$(4.3) \qquad \sum_{\substack{p \in I \\ p \equiv l \bmod k}} g(p) \leq \frac{Y}{\varphi(k) S_1(z)} + \frac{H_{k_1}^2(z)}{S_{k_1}^2(z)} + \pi(z, k_1, l_1) + 1.$$

We are now in the situation where we have to show that the right hand side of (4.3) is bounded by either (4.1) or (4.2) for a suitable choice of z. But this is carried out in [vLR65]. ∎

From Theorem 4.2 one recovers the ordinary Brun–Titchmarsh inequality in the form

$$\pi(x+y;k,l) - \pi(x;k,l) < \frac{2y}{\varphi(k)\log(y/k)}\left(1 + \frac{8}{\log(y/k)}\right)$$

by taking $f \equiv 1$ on $[x, x+y] \subset [0,\infty)$, which is slightly weaker than the strongest version proved in [MV73].

THEOREM 4.3. *Let* $I = [x, x+y] \subset [0,\infty)$. *Then*

$$\sum_{p\in I} f(p) < 2\frac{\|f\|_{1,I}}{\log(\rho_I(f))}$$

for all non-negative $f \in W^{1,1}(I)$ *satisfying* $\rho_I(f) > 1$.

The proof is based on sharper estimates for H_1 and S_1, provided by the following two lemmas. The second lemma is an explicit version of a result of Ward [War27].

LEMMA 4.4. *We have*

$$H_1(z) = \frac{15}{\pi^2}z + \Theta(47\sqrt{z}) \quad \text{*for all* } z \geq 1.$$

Proof. We will be needing the following well-known identities for the Riemann zeta function (see e.g. [Tit51]):

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \frac{\zeta(s)}{\zeta(2s)} = \prod_p (1 + p^{-s}) = \sum_n \frac{\mu(n)^2}{n^s},$$

(4.4)
$$\sum_n \frac{2^{\omega(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}, \quad \frac{1}{\zeta(s)} = \sum_n \frac{\mu(n)}{n^s}.$$

We define

$$h(s) = \prod_p \left(1 + \frac{2}{(1+p^s)(p-1)}\right), \quad \tilde{h}(s) = \prod_p \left(1 + \frac{2}{(p^s-1)(p-1)}\right).$$

In $\Re(s) > 0$ these products converge normally and we have

$$h(s) = \sum_n c_n n^{-s} \quad \text{and} \quad \tilde{h}(s) = \sum_n |c_n| n^{-s}$$

for suitable c_n. From (4.4) we get

$$\frac{\zeta(s)}{\zeta(2s)}h(s) = \prod_p (1 + p^{-s}) \prod_p \left(1 + \frac{2}{(1 + p^s)(p - 1)}\right)$$

$$= \prod_p \left(1 + p^{-s}\frac{p + 1}{p - 1}\right) = \sum_n \frac{\mu(n)^2\sigma(n)}{\varphi(n)}n^{-s},$$

which gives

$$(4.5) \qquad \sum_{n \leq z} \frac{\mu(n)^2\sigma(n)}{\varphi(n)} = \sum_{d \leq z} c_d \sum_{m \leq z/d} \mu(m)^2.$$

Combining [CDEM07, Theorem 3] with a short computer calculation, we obtain the bound

$$(4.6) \qquad Q(z) := \sum_{n \leq z} \mu(n)^2 = \frac{6}{\pi^2}z + \Theta(0.68\sqrt{z})$$

for $z \geq 1$. This and (4.5) together imply

$$(4.7) \qquad H_1(z) = \sum_{d \leq z} c_d\left(\frac{6z}{\pi^2 d} + \Theta\left(0.68\sqrt{\frac{z}{d}}\right)\right)$$

$$= \frac{6z}{\pi^2}\sum_d \frac{c_d}{d} + \Theta\left(0.68\sqrt{z}\sum_d \frac{|c_d|}{\sqrt{d}} + \frac{6z}{\pi^2}\sum_{d>z}\frac{|c_d|}{d}\right)$$

$$= \frac{6}{\pi^2}h(1)z + \Theta(1.3\tilde{h}(1/2)\sqrt{z}).$$

The value $h(1)$ is given by

$$h(1) = \prod_p\left(1 + \frac{2}{p^2 - 1}\right) = \sum_n \frac{2^{\omega(n)}}{n^2} = \frac{\zeta(2)^2}{\zeta(4)} = \frac{5}{2}.$$

Next we estimate $\tilde{h}(1/2)$. For $t \geq 16$ we have $(\sqrt{t} - 1)(t - 1) \geq \frac{2}{3}t^{3/2}$ and therefore

$$(4.8) \qquad \tilde{h}(1/2) \leq \left(\prod_{p<10000}\frac{1 + 2(\sqrt{p} - 1)^{-1}(p - 1)^{-1}}{(1 + p^{-3/2})^3}\right)\frac{\zeta(3/2)^3}{\zeta(3)^3}.$$

Here, the product on the right hand side is bounded by 3.5 and $\zeta(3/2)^3/\zeta(3)^3 \leq 10.27$. Therefore, $\tilde{h}(1/2) \leq 36$. Inserting this bound in (4.7) yields the assertion. ∎

LEMMA 4.5. *Let $z \geq 10^9$. Then*

$$(4.9) \qquad S_1(z) = \log(z) + C_0 + \sum_p \frac{\log(p)}{p(p - 1)} + \Theta\left(\frac{58}{\sqrt{z}}\right).$$

Here, $C_0 = 0.5772156\ldots$ denotes the Euler–Mascheroni constant.

Proof. The proof is similar to that of Lemma 4.4. Let $Q(z)$ be as in (4.6) and let $R(z) = Q(z) - \frac{6}{\pi^2}z$. Then partial summation yields

$$(4.10) \qquad \sum_{n \leq z} \frac{\mu(n)^2}{n} = \frac{6}{\pi^2} \log(z) + \frac{Q(z)}{z} + \int_1^\infty \frac{R(t)}{t^2}\, dt - \int_z^\infty \frac{R(t)}{t^2}\, dt.$$

Consequently,

$$(4.11) \qquad \sum_{n \leq z} \frac{\mu(n)^2}{n} = \frac{\log(z)}{\zeta(2)} + A + \Theta\left(\frac{2.04}{\sqrt{z}}\right),$$

where the constant is given by

$$A = \frac{C_0}{\zeta(2)} - 2\sum_d \frac{\mu(d)\log(d)}{d^2} = \frac{C_0}{\zeta(2)} - 2\frac{\zeta'(2)}{\zeta(2)^2}$$

(see e.g. [Brü95, Lemma 5.4.2]).

Now let

$$h(s) = \prod_p \left(1 + \frac{1}{(1+p^s)(p-1)}\right), \quad \tilde{h}(s) = \prod_p \left(1 + \frac{1}{(p^s-1)(p-1)}\right).$$

Then we have

$$(4.12) \qquad \frac{\zeta(s)}{\zeta(2s)}h(s) = \sum_n \frac{n\mu(n)^2}{\varphi(n)} n^{-s}$$

and there are c_n such that

$$h(s) = \sum_n c_n n^{-s} \quad \text{and} \quad \tilde{h}(s) = \sum_n |c_n| n^{-s}$$

in $\Re(s) > 0$. Combining (4.12) and (4.11) we now get

$$(4.13) \qquad \sum_{n \leq z} \frac{\mu(n)^2}{\varphi(n)} = \sum_{d \leq z} \frac{c_d}{d} \sum_{m \leq z/d} \frac{\mu(m)^2}{m}$$

$$= \sum_{d \leq z} \left(\frac{\log(z/d)}{\zeta(2)} + A\right)\frac{c_d}{d} + \Theta\left(\tilde{h}\left(\frac{1}{2}\right)\frac{2.04}{\sqrt{z}}\right).$$

Since

$$h(1) = \prod_p \left(1 + p^{-2}\frac{1}{1-p^{-2}}\right) = \zeta(2),$$

we have

$$(4.14) \qquad \sum_{d \leq z} \left(\frac{\log(z/d)}{\zeta(2)} + A\right)\frac{c_d}{d} = \log(z) + B + \Theta\left(\sum_{d > z} \frac{|c_d|}{d}\left(\frac{\log(dz)}{\zeta(2)} + |A|\right)\right)$$

for a suitable constant B.

To estimate the Θ-term we note that $\log(z) \le 1.56 z^{1/8}$ for $z \ge 10^9$ and therefore

$$(4.15) \quad \sum_{d>z} \frac{|c_d|}{d}\left(\frac{\log(z)+\log(d)}{\zeta(2)}+|A|\right) \le \frac{1}{\sqrt{z}}\left(\frac{2 \cdot 1.56}{\zeta(2)}+10^{-9/8}|A|\right)\tilde{h}\left(\frac{3}{8}\right)$$

for such z.

Next we estimate $\tilde{h}(\sigma)$. For $\sigma \ge 1/4$ and $t \ge 20$ we observe that

$$t^{1+\sigma}-t^{\sigma}-t+1 \ge t^{1+\sigma}/2,$$

hence

$$(4.16) \quad \tilde{h}(\sigma) \le \left(\prod_{p<10000} \frac{1+(p^{\sigma}-1)^{-1}(p-1)^{-1}}{(1+p^{-1-\sigma})^2}\right)\frac{\zeta(1+\sigma)^2}{\zeta(2+2\sigma)^2}$$

for $\sigma \ge 1/4$. This gives the bounds $\tilde{h}(3/8) \le 19$ and $\tilde{h}(1/2) \le 9.4$. For $|A|$ we use the bound

$$|A| \le \frac{C_0}{\zeta(2)}+2\left|\sum_{d\le 100}\frac{\mu(d)\log(d)}{d^2}\right|+2\int_{100}^{\infty}\frac{\log(t)}{t^2}\,dt \le 1.8.$$

The error term in (4.14) is thus bounded by $(1.9+0.14) \cdot 19/\sqrt{z} \le 38.8/\sqrt{z}$ for $z \ge 10^9$. If we use this in (4.13), we arrive at

$$\sum_{n\le z}\frac{\mu(n)^2}{\varphi(n)} = \log(z)+B+\Theta\left(\frac{58}{\sqrt{z}}\right).$$

For a proof that B coincides with the constant in (4.9) see e.g. [Brü95, Lemma 5.4.2]. ∎

Proof of Theorem 4.3. Let g and Y be as in the proof of Theorem 4.2. First, we consider the case $Y > 4 \cdot 10^{18}$. From Proposition 3.2 we get

$$\sum_{p\in I}g(p) \le \frac{Y}{S_1(z)}+\frac{H_1^2(z)}{S_1^2(z)}+\pi(z)$$

for every $z \ge 1$. Here we choose $z = \sqrt{Y}/2$. Then $z > 10^9$ and Lemma 4.5 gives

$$S_1(z) \ge \log(z)+C_0+\sum_{p<1000}\frac{\log(p)}{p(p-1)}-0.002$$

$$\ge \log(z)+1.32 \ge \tfrac{1}{2}(\log(Y)+1.25).$$

From Lemma 4.4 we get

$$H_1(z) \le \left(\frac{15}{\pi^2}+\frac{47}{\sqrt{10^9}}\right)z \le 1.53z,$$

hence

$$\frac{Y}{S_1(z)} \le 2\frac{Y}{\log(Y)+1.25} = 2\frac{Y}{\log(Y)}-2.5\frac{Y}{\log(Y)(\log(Y)+1.25)}$$

and
$$\frac{H_1^2(z)}{S_1^2(z)} \leq 2.4 \frac{Y}{(\log(Y) + 1.25)^2}.$$

Using the trivial bound $\pi(z) \leq z$ we thus arrive at
$$\frac{Y}{S_1(z)} + \frac{H_1^2(z)}{S_1^2(z)} + \pi(z) \leq 2\frac{Y}{\log(Y)} + \frac{\sqrt{Y}}{2} - \frac{0.1Y}{\log(Y)(\log(Y) + 1.25)}$$
$$< 2\frac{Y}{\log(Y)}.$$

Next, we consider the case $1 < Y < 20000$. From Proposition 3.3 we obtain the bound
$$\sum_{p \in I} g(p) \leq Y \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + 2^{\pi(z)} + \pi(z).$$

It thus suffices to show that for any such Y there exists a number z with
$$Y \prod_{p \leq z} \left(1 - \frac{1}{p}\right) + 2^{\pi(z)} + \pi(z) < 2\frac{Y}{\log(Y)};$$

but this is carried out in [MV73, p. 130].

It remains to treat the case $20000 \leq Y \leq 4 \cdot 10^{18}$. This can be done numerically: the function $t \mapsto t/\log(t)$ is concave for $t > e^2$. So if, for fixed z,

(4.17)
$$\frac{Y}{S_1(z)} + \frac{H_1^2(z)}{S_1^2(z)} + \pi(z) < 2\frac{Y}{\log(Y)}$$

holds for $Y \in \{Y_0, Y_1\}$, it holds for all $Y \in [Y_0, Y_1]$. The inequality (4.17) was verified with a simple C-routine for all $z \in [50, 2 \cdot 10^9) \cap \mathbb{Z}$ and $Y \in \{4z^2, 4(z+1)^2\}$. Therefore, (4.17) is true for every $Y \in [4 \cdot 50^2, 1.6 \cdot 10^{19}]$ with $z = \sqrt{Y}/2$. ∎

5. Improvements. The results in Theorem 4.2 can be improved by sharpening the estimates for S_k and H_k, as pointed out in [vLR65]. It can also be seen that the equivalent of the stronger result in [MV73],
$$\sum_{\substack{p \in I \\ p \equiv l \bmod k}} f(p) < \frac{2}{\varphi(k)} \frac{\|f\|_{1,I}}{\log(\rho_I(f)/k)},$$

holds at least for $\rho_I(f) \gg_\varepsilon k^{1+\varepsilon}$ and every $\varepsilon > 0$. However, it is not clear whether it can be shown for $\rho_I(f) > k$.

his/her helpful comments, especially for his/her suggestions to improve the error terms in Lemmas 4.4 and 4.5.

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