Indivisibility of special values of Dedekind zeta functions of real quadratic fields

by

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1. Introduction and statement of results. For a number field \( k \) and a prime number \( p \), we denote by \( h(k) \) the class number of \( k \) and by \( \lambda_p(k) \), \( \mu_p(k) \) the Iwasawa \( \lambda \)-, \( \mu \)-invariants of the cyclotomic \( \mathbb{Z}_p \)-extension of \( k \), where \( \mathbb{Z}_p \) is the ring of \( p \)-adic integers.

Let \( p \) be an odd prime number. Hartung [3] proved, using the Kronecker class number relation for quadratic forms, that there exist infinitely many imaginary quadratic fields \( k \) whose class numbers are not divisible by \( p \). Later, using the idea of Hartung and Eichler’s trace formula combined with the \( p \)-adic Galois representation attached to the Jacobian varieties of certain modular curves, Horie [4] proved that there exist infinitely many imaginary quadratic fields \( k \) such that \( p \) does not split in \( k \) and \( p \) does not divide \( h(k) \). Thus from a theorem of Iwasawa [7], there exist infinitely many imaginary quadratic fields \( k \) with \( \lambda_p(k) = \mu_p(k) = 0 \).

Let \( F \) be a totally real number field. For a prime number \( p \), we denote by \( n(p) \) the maximum value of \( n \) such that the primitive \( p^n \)-th roots \( \zeta_p^n \) of unity are at most of degree 2 over \( F \). If \( F \) is fixed, we have \( n(p) = 0 \) for all but finitely many \( p \). Thus we can put \( \omega_F = 2^{n(2)+1} \prod_{p \neq 2} p^{n(p)} \). Let \( \zeta_F(s) \) be the Dedekind zeta function of \( F \). Serre [11] proved that \( \omega_F \zeta_F(-1) \) is a rational integer. Let \( K \) be a totally imaginary quadratic extension over \( F \). Define

\[
\lambda_p^{-}(K) := \lambda_p(K) - \lambda_p(F), \quad \mu_p^{-}(K) := \mu_p(K) - \mu_p(F).
\]

Using a result of Shimizu about the trace formula of Hecke operators and a result of Ohta about the \( p \)-adic representation of the absolute Galois group over \( F \) related to automorphic forms, Naito [8], [9] generalized the above results of Hartung and Horie to the case of totally imaginary quadratic extensions over a totally real number field and obtained the following theorem.

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**Theorem** (Naito). Let $F$ be a totally real number field. Let $p$ be an odd prime number which does not divide $\omega_F\zeta_F(-1)$. Then there exist infinitely many totally imaginary quadratic extensions $K$ over $F$ such that the relative class number of $K$ is not divisible by $p$ and no prime ideal of $F$ over $p$ splits in $K$, that is, $\lambda_p^-(K) = \mu_p^-(K) = 0$.

Thus it would be interesting to know when or how often $p$ does not divide $\omega_F\zeta_F(-1)$. In this direction, in this note we will show the following theorem.

**Theorem 1.** Let $p$ be an odd prime number. Then there exist infinitely many positive fundamental discriminants $D > 0$ such that $p$ does not divide $\omega_{Q(\sqrt{D})}\zeta_{Q(\sqrt{D})}(-1)$.

Then, from the above theorem of Naito, we immediately have the following theorem.

**Theorem 2.** Let $p$ be an odd prime number. Then there exist infinitely many positive fundamental discriminants $D > 0$ such that the real quadratic field $Q(\sqrt{D})$ has infinitely many totally imaginary quadratic extensions $K$ such that $\lambda_p^-(K) = \mu_p^-(K) = 0$.

**2. Proof of Theorem 1.** Let $D$ be the fundamental discriminant of a quadratic number field and $\chi_D := (\frac{D}{\cdot})$ the usual Kronecker character. Let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight $k$ on $\Gamma_0(N)$ with character $\chi$. Let $r$ and $N$ be nonnegative integers with $r \geq 2$. If $N \not\equiv 0, 1 \pmod{4}$, then let $H(r, N) = 0$. If $N = 0$, then let $H(r, 0) := \zeta(1 - 2r)$. If $Dn^2 = (-1)^r N$, then

$$ H(r, N) := L(1 - r, \chi_D) \sum_{d|n} \mu(d) \chi_D(d)d^{r-1}\sigma_{2r-1}(n/d), $$

where $\sigma_r(n) := \sum_{d|n} d^r$. Cohen [1] proved the following proposition.

**Proposition** (Cohen). Let $D \equiv 0$ or $1 \pmod{4}$ be an integer such that $(-1)^{r-1}D = |D|$. Then for $r \geq 2$,

$$ \sum_{N \geq 0} \left( \sum_{|s| \leq \sqrt{4N}} \sum_{s^2 \equiv 4N \pmod{D}} H\left(r, \frac{4N - s^2}{|D|}\right) \right) q^N \in M_{r+1}(\Gamma_0(D), \chi_D), $$

where $q := e^{2\pi iz}$.

Applying this proposition to the case $r = 2$, Cohen also obtained the following Kronecker–Hurwitz type formula for $H(2, N)$:

$$ -30 \sum_{|s| \leq \sqrt{N}} H(2, N - s^2) = \sum_{d|N} (d^2 + (N/d)^2) \left( -\frac{4}{d} \right). $$
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Lemma. Let $D > 0$ be a positive fundamental discriminant. Then

$$\omega_{\mathbb{Q}(\sqrt{D})} = \begin{cases} 2^3 \cdot 3 & \text{if } D \neq 5, \\ 2^3 \cdot 3 \cdot 5 & \text{if } D = 5. \end{cases}$$

For an odd prime number $p \neq 3$, we can choose $l$ to satisfy the following:

(i) $l$ is an odd prime number,
(ii) $l \equiv 3 \pmod{4}$,
(iii) $l^2 \not\equiv 1 \pmod{p}$,
(iv) $\left(\frac{l}{q}\right) = -1$ for all odd prime numbers $q$ with $3 \leq q \leq X$, where $X > 5$ is an arbitrarily large number.

Then from (1) and (i), (ii), we have

$$\sum_{|s| \leq \sqrt{4l}} (-2H(2, 4l - s^2)) = l^2 - 1.$$ 

From (ii), (iv), for $|s| \leq \sqrt{4l}$, we have

$$4l - s^2 = D_{l,s}n^2,$$

where $D_{l,s} > X > 5$ is a positive fundamental discriminant.

From the above lemma, for $|s| \leq \sqrt{4l}$, we have

$$-2H(2, 4l - s^2) = \omega_{\mathbb{Q}(\sqrt{D_{l,s}})}\zeta_{\mathbb{Q}}(-1)H(2, 4l - s^2)$$

$$= \omega_{\mathbb{Q}(\sqrt{D_{l,s}})}\zeta_{\mathbb{Q}}(-1)L(-1, \chi_{D_{l,s}}) \sum_{d|n} \mu(d)\chi_{D_{l,s}}(d)d\sigma_3(n/d)$$

$$= \omega_{\mathbb{Q}(\sqrt{D_{l,s}})}\zeta_{\mathbb{Q}(\sqrt{D_{l,s}})}(-1) \sum_{d|n} \mu(d)\chi_{D_{l,s}}(d)d\sigma_3(n/d) \in \mathbb{Z}.$$ 

Finally from (iii), we see that there exist $s$ such that $|s| \leq \sqrt{4l}$ and

$$-2H(2, 4l - s^2) \not\equiv 0 \pmod{p}, \quad \text{i.e.,} \quad \omega_{\mathbb{Q}(\sqrt{D_{l,s}})}\zeta_{\mathbb{Q}(\sqrt{D_{l,s}})}(-1) \not\equiv 0 \pmod{p}.$$ 

Since $D_{l,s} > X$ and $X$ is arbitrarily large, for an odd prime number $p \neq 3$, there exist infinitely many positive fundamental discriminants $D$ satisfying $p\mid \omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1).$ 

For the case of $p = 3$, we cannot choose $l$ satisfying the above (iii). However we can choose $u, v$ to satisfy the following:

(i) $u, v$ are odd prime numbers,
(ii) $u \equiv 1 \pmod{4}$ and $v \equiv 3 \pmod{4}$,
(iii) $u^2v^2 \not\equiv -1 \pmod{3}$,
(iv) $\left(\frac{uv}{q}\right) = -1$ for all odd prime numbers $q$ with $3 \leq q \leq X$, where $X > 5$ is an arbitrarily large number.
Then by the same method we can easily show that there exist $s$ such that $|s| \leq \sqrt{4uv}$ and $-2H(2, 4uv-s^2) \not\equiv 0 \pmod{3}$ and there exist infinitely many positive fundamental discriminants $D$ satisfying $3^j \omega_{Q(\sqrt{D})} \zeta_{Q(\sqrt{D})}(-1)$.

3. Remarks. For the case $p = 3$ or 5, by a different method, we can obtain stronger results. From the construction of the Kubota–Leopoldt $p$-adic $L$-function $L_p(s, \chi_D)$, the Kummer congruence and the $p$-adic class number formula, we have the following two congruence relations for $\omega_{Q(\sqrt{D})} \zeta_{Q(\sqrt{D})}(-1)$, when $D \neq 5$:

$$(2) \quad \omega_{Q(\sqrt{D})} \zeta_{Q(\sqrt{D})}(-1) = -2L(-1, \chi_D)$$

$$\equiv -2L_3(-1, \chi_D) \pmod{3}$$

$$\equiv -2L_3(1, \chi_D) \pmod{3}$$

$$\equiv -\frac{4h(Q(\sqrt{D})) R_3(Q(\sqrt{D}))}{\sqrt{D}} \left(1 - \frac{\chi_D(3)}{3}\right) \pmod{3},$$

$$(3) \quad \omega_{Q(\sqrt{D})} \zeta_{Q(\sqrt{D})}(-1) = -2L(-1, \chi_D)$$

$$\equiv -2L_5(-1, \chi_5D) \pmod{5}$$

$$\equiv -2L_5(1, \chi_5D) \pmod{5}$$

$$\equiv -\frac{4h(Q(\sqrt{5D})) R_5(Q(\sqrt{5D}))}{\sqrt{5D}} \pmod{5}.$$

Thus from (2) and a theorem of Davenport and Heilbronn [2], as refined by Horie and Nakagawa [6], we know that a positive proportion of positive fundamental discriminants $D > 0$ satisfy $3^j \omega_{Q(\sqrt{D})} \zeta_{Q(\sqrt{D})}(-1)$ and from (3) and a result of Ono [10], we have

$$\# \{0 < D < X \mid 3^j \omega_{Q(\sqrt{D})} \zeta_{Q(\sqrt{D})}(-1) \} \gg \sqrt{X}/\log X.$$ 

Finally, we mention that Horie and Kimura [5] recently showed that there always exist infinitely many totally imaginary quadratic extensions $K$ over a totally real number field $F$ such that $\lambda_3^F(K) = \mu_3^F(K) = 0$ whether $\omega_F \zeta_F(-1)$ is divisible by 3 or not.

References


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