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Indivisibility of special values of Dedekind zeta functions of real quadratic fields

by

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1. Introduction and statement of results. For a number field k and a prime number p, we denote by $h(k)$ the class number of k and by $\lambda_p(k)$, $\mu_p(k)$ the Iwasawa λ-, μ-invariants of the cyclotomic $\mathbb{Z}_p$-extension of k, where $\mathbb{Z}_p$ is the ring of p-adic integers.

Let p be an odd prime number. Hartung [3] proved, using the Kronecker class number relation for quadratic forms, that there exist infinitely many imaginary quadratic fields k whose class numbers are not divisible by p.

Later, using the idea of Hartung and Eichler's trace formula combined with the p-adic Galois representation attached to the Jacobian varieties of certain modular curves, Horie [4] proved that there exist infinitely many imaginary quadratic fields k such that p does not split in k and p does not divide $h(k)$. Thus from a theorem of Iwasawa [7], there exist infinitely many imaginary quadratic fields k with $\lambda_p(k) = \mu_p(k) = 0$.

Let F be a totally real number field. For a prime number p, we denote by $n(p)$ the maximum value of n such that the primitive p^nth roots ζ_{p^n} of unity are at most of degree 2 over F. If F is fixed, we have $n(p) = 0$ for all but finitely many p. Thus we can put $\omega_F = 2^{n(2)+1}\prod_{p\neq 2} p^{n(p)}$. Let $\zeta_F(s)$ be the Dedekind zeta function of F. Serre [11] proved that $\omega_F\zeta_F(-1)$ is a rational integer. Let K be a totally imaginary quadratic extension over F. Define

$$\lambda_p^-(K) := \lambda_p(K) - \lambda_p(F), \qquad \mu_p^-(K) := \mu_p(K) - \mu_p(F).$$

Using a result of Shimizu about the trace formula of Hecke operators and a result of Ohta about the p-adic representation of the absolute Galois group over F related to automorphic forms, Naito [8], [9] generalized the above results of Hartung and Horie to the case of totally imaginary quadratic extensions over a totally real number field and obtained the following theorem.

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Theorem (Naito). *Let F be a totally real number field. Let p be an odd prime number which does not divide $\omega_F\zeta_F(-1)$. Then there exist infinitely many totally imaginary quadratic extensions K over F such that the relative class number of K is not divisible by p and no prime ideal of F over p splits in K, that is, $\lambda_p^-(K) = \mu_p^-(K) = 0$.*

Thus it would be interesting to know when or how often p does not divide $\omega_F\zeta_F(-1)$. In this direction, in this note we will show the following theorem.

Theorem 1. *Let p be an odd prime number. Then there exist infinitely many positive fundamental discriminants $D > 0$ such that p does not divide $\omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$.*

Then, from the above theorem of Naito, we immediately have the following theorem.

Theorem 2. *Let p be an odd prime number. Then there exist infinitely many positive fundamental discriminants $D > 0$ such that the real quadratic field $\mathbb{Q}(\sqrt{D})$ has infinitely many totally imaginary quadratic extensions K such that $\lambda_p^-(K) = \mu_p^-(K) = 0$.*

2. Proof of Theorem 1. Let D be the fundamental discriminant of a quadratic number field and $\chi_D := \left(\frac{D}{\cdot}\right)$ the usual Kronecker character. Let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight k on $\Gamma_0(N)$ with character χ. Let r and N be nonnegative integers with $r \geq 2$. If $N \not\equiv 0, 1 \pmod 4$, then let $H(r, N) = 0$. If $N = 0$, then let $H(r, 0) := \zeta(1 - 2r)$. If $Dn^2 = (-1)^r N$, then

$$H(r, N) := L(1 - r, \chi_D) \sum_{d|n} \mu(d)\chi_D(d)d^{r-1}\sigma_{2r-1}(n/d),$$

where $\sigma_\nu(n) := \sum_{d|n} d^\nu$. Cohen [1] proved the following proposition.

Proposition (Cohen). *Let $D \equiv 0$ or $1 \pmod 4$ be an integer such that $(-1)^{r-1}D = |D|$. Then for $r \geq 2$,*

$$\sum_{N\geq 0} \Bigg(\sum_{\substack{|s|\leq\sqrt{4N} \\ s^2\equiv 4N \,(\mathrm{mod}\, D)}} H\left(r, \frac{4N - s^2}{|D|}\right)\Bigg) q^N \in M_{r+1}(\Gamma_0(D), \chi_D),$$

where $q := e^{2\pi i z}$.

Applying this proposition to the case $r = 2$, Cohen also obtained the following Kronecker–Hurwitz type formula for $H(2, N)$:

$$-30 \sum_{|s|\leq\sqrt{N}} H(2, N - s^2) = \sum_{d|N} (d^2 + (N/d)^2)\left(\frac{-4}{d}\right). \tag{1}$$

Lemma. *Let $D > 0$ be a positive fundamental discriminant. Then*

$$\omega_{\mathbb{Q}(\sqrt{D})} = \begin{cases} 2^3 \cdot 3 & \text{if } D \neq 5, \\ 2^3 \cdot 3 \cdot 5 & \text{if } D = 5. \end{cases}$$

For an odd prime number $p \neq 3$, we can choose l to satisfy the following:

(i) l is an odd prime number,
(ii) $l \equiv 3 \pmod 4$,
(iii) $l^2 \not\equiv 1 \pmod p$,
(iv) $\left(\frac{l}{q}\right) = -1$ for all odd prime numbers q with $3 \leq q \leq X$, where $X > 5$ is an arbitrarily large number.

Then from (1) and (i), (ii), we have

$$\sum_{|s| \leq \sqrt{4l}} (-2H(2, 4l - s^2)) = l^2 - 1.$$

From (ii), (iv), for $|s| \leq \sqrt{4l}$, we have

$$4l - s^2 = D_{l,s} n^2,$$

where $D_{l,s} > X > 5$ is a positive fundamental discriminant.

From the above lemma, for $|s| \leq \sqrt{4l}$, we have

$$\begin{aligned} -2H(2, 4l - s^2) &= \omega_{\mathbb{Q}(\sqrt{D_{l,s}})} \zeta_{\mathbb{Q}}(-1) H(2, 4l - s^2) \\ &= \omega_{\mathbb{Q}(\sqrt{D_{l,s}})} \zeta_{\mathbb{Q}}(-1) L(-1, \chi_{D_{l,s}}) \sum_{d|n} \mu(d) \chi_{D_{l,s}}(d) d \sigma_3(n/d) \\ &= \omega_{\mathbb{Q}(\sqrt{D_{l,s}})} \zeta_{\mathbb{Q}(\sqrt{D_{l,s}})}(-1) \sum_{d|n} \mu(d) \chi_{D_{l,s}}(d) d \sigma_3(n/d) \in \mathbb{Z}. \end{aligned}$$

Finally from (iii), we see that there exist s such that $|s| \leq \sqrt{4l}$ and

$$-2H(2, 4l - s^2) \not\equiv 0 \pmod p, \quad \text{i.e.,} \quad \omega_{\mathbb{Q}(\sqrt{D_{l,s}})} \zeta_{\mathbb{Q}(\sqrt{D_{l,s}})}(-1) \not\equiv 0 \pmod p.$$

Since $D_{l,s} > X$ and X is arbitrarily large, for an odd prime number $p \neq 3$, there exist infinitely many positive fundamental discriminants D satisfying $p \nmid \omega_{\mathbb{Q}(\sqrt{D})} \zeta_{\mathbb{Q}(\sqrt{D})}(-1)$.

For the case of $p = 3$, we cannot choose l satisfying the above (iii). However we can choose u, v to satisfy the following:

(i) u, v are odd prime numbers,
(ii) $u \equiv 1 \pmod 4$ and $v \equiv 3 \pmod 4$,
(iii) $u^2 v^2 \not\equiv -1 \pmod 3$,
(iv) $\left(\frac{uv}{q}\right) = -1$ for all odd prime numbers q with $3 \leq q \leq X$, where $X > 5$ is an arbitrarily large number.

Then by the same method we can easily show that there exist s such that $|s| \le \sqrt{4uv}$ and $-2H(2, 4uv - s^2) \not\equiv 0 \pmod 3$ and there exist infinitely many positive fundamental discriminants D satisfying $3 \nmid \omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$.

3. Remarks. For the case $p = 3$ or 5, by a different method, we can obtain stronger results. From the construction of the Kubota–Leopoldt p-adic L-function $L_p(s, \chi_D)$, the Kummer congruence and the p-adic class number formula, we have the following two congruence relations for $\omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$, when $D \ne 5$:

$$\begin{aligned}
(2)\quad \omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1) &= -2L(-1, \chi_D) \\
&\equiv -2L_3(-1, \chi_D) \pmod 3 \\
&\equiv -2L_3(1, \chi_D) \pmod 3 \\
&\equiv -\frac{4h(\mathbb{Q}(\sqrt{D}))R_3(\mathbb{Q}(\sqrt{D}))}{\sqrt{D}}\left(1 - \frac{\chi_D(3)}{3}\right) \pmod 3,
\end{aligned}$$

$$\begin{aligned}
(3)\quad \omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1) &= -2L(-1, \chi_D) \\
&\equiv -2L_5(-1, \chi_{5D}) \pmod 5 \\
&\equiv -2L_5(1, \chi_{5D}) \pmod 5 \\
&\equiv -\frac{4h(\mathbb{Q}(\sqrt{5D}))R_5(\mathbb{Q}(\sqrt{5D}))}{\sqrt{5D}} \pmod 5.
\end{aligned}$$

Thus from (2) and a theorem of Davenport and Heilbronn [2], as refined by Horie and Nakagawa [6], we know that a positive proportion of positive fundamental discriminants $D > 0$ satisfy $3 \nmid \omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1)$ and from (3) and a result of Ono [10], we have

$$\sharp\{0 < D < X \mid 5 \nmid \omega_{\mathbb{Q}(\sqrt{D})}\zeta_{\mathbb{Q}(\sqrt{D})}(-1)\} \gg \sqrt{X}/\log X.$$

Finally, we mention that Horie and Kimura [5] recently showed that there always exist infinitely many totally imaginary quadratic extensions K over a totally real number field F such that $\lambda_3^-(K) = \mu_3^-(K) = 0$ whether $\omega_F\zeta_F(-1)$ is divisible by 3 or not.

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