On covers of abelian groups by cosets

by

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1. Introduction. As in any textbook on group theory, for a subgroup $H$ of a group $G$ with the index $[G : H]$ finite, $G$ can be partitioned into $k = [G : H]$ left cosets of $H$ in $G$, i.e., all the $k$ left cosets of $H$ form a disjoint cover of $G$.

In 1954 B. H. Neumann [N1, N2] discovered the following basic result on covers of groups.

**Theorem 1.1 (Neumann).** Let $\{a_s G_s\}_{s=1}^{k}$ be a cover of a group $G$ by (finitely many) left cosets of subgroups $G_1, \ldots, G_k$. Then $G$ is the union of those $a_s G_s$ with $[G : G_s] < \infty$. In other words, if $\{a_s G_s\}_{s=t}$ is not a cover of $G$ then $[G : G_t] < \infty$.

In 1966 J. Mycielski (cf. [MS]) posed an interesting conjecture on disjoint covers of abelian groups. Before stating the conjecture we give a definition.

**Definition 1.1.** The Mycielski function $f : \mathbb{Z}^+ = \{1, 2, \ldots\} \to \{0, 1, \ldots\}$ is given by

\[ f(n) = \sum_{p \in \mathcal{P}(n)} \text{ord}_p(n)(p - 1), \]

where $\mathcal{P}(n)$ denotes the set of prime divisors of $n$ and $\text{ord}_p(n)$ represents the largest nonnegative integer $\alpha$ such that $p^\alpha | n$.

**Remark 1.1.** Since $p \leq 2^{p-1}$ for any prime $p$, (1.1) implies that $n \leq 2^{f(n)}$ (i.e., $f(n) \geq \log_2 n$).


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Mycielski’s conjecture. Let $G$ be an abelian group, and \( \{a_s G_s\}_{s=1}^k \) be a disjoint cover of $G$ by left cosets of subgroups. Then $k \geq 1 + f([G : G_t])$ for each $t = 1, \ldots, k$.

When $G$ is the additive group $\mathbb{Z}$ of integers, Mycielski’s conjecture says that for any disjoint cover \( \{a_s(n_s)\}_{s=1}^k \) of $\mathbb{Z}$ by residue classes (where $a_s \in \mathbb{Z}$, $n_s \in \mathbb{Z}^+$ and $a_s(n_s) = a_s + n_s \mathbb{Z}$) we have $k \geq 1 + f(n_t)$ for every $t = 1, \ldots, k$.

This was first confirmed by Š. Znám [Z66]. For problems and results on covers of $\mathbb{Z}$, the reader is referred to [G04], [PS], [S03] and [S05].

Definition 1.2. For a subnormal subgroup $H$ of a group $G$ with finite index, we define
\[
d(G, H) = \sum_{i=1}^{n} ([H_i : H_{i-1}] - 1),
\]
where $H_0 = H \subset H_1 \subset \cdots \subset H_n = G$ is any composition series from $H$ to $G$.

By [S90, Theorem 6] and [S01, Theorem 3.1], for any subnormal subgroup $H$ of a group $G$ with $[G : H] < \infty$, we have $d(G, H) \geq f([G : H])$, and equality holds if and only if $G/H_G$ is solvable, where $H_G = \bigcap_{g \in G} gHg^{-1}$ is the core of $H$ in $G$ (i.e., the largest normal subgroup of $G$ contained in $H$).

The following result is stronger than Mycielski’s conjecture.

Theorem 1.2 (I. Korec, Z. W. Sun). Let $a_1 G_1, \ldots, a_k G_k$ be left cosets of subnormal subgroups $G_1, \ldots, G_k$ of a group $G$. If $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ forms an exact $m$-cover of $G$, i.e., $\mathcal{A}$ covers each element of $G$ exactly $m$ times, then $[G : \bigcap_{s=1}^k G_s] < \infty$ and
\[
k \geq m + d(G, \bigcap_{s=1}^k G_s) \geq m + f\left(\left[\bigcap_{s=1}^k G_s\right]\right),
\]
where the lower bound $m + d(G, \bigcap_{s=1}^k G_s)$ is best possible.

In the case $m = 1$ and $G = \mathbb{Z}$, Theorem 1.2 was first conjectured by Znám [Z69]. When $m = 1$ and $G_1, \ldots, G_k$ are normal in $G$, Theorem 1.2 was obtained by Korec [K74] in 1974. In 1990 Sun [S90] deduced Theorem 1.2 in the case $m = 1$ by a method different from that of Korec. The current version of Theorem 1.2 was established by Sun [S01] in 2001; the proof depends heavily on the condition that $\mathcal{A}$ covers all the elements of $G$ the same number of times. Under the conditions of Theorem 1.2, Sun [S04] also showed that the indices $[G : G_s]$ (1 ≤ $s$ ≤ $k$) cannot be distinct providing $k > 1$.

Call a coset in an abelian group not containing the identity element a proper coset. In 2003 W. D. Gao and A. Geroldinger [GG] proved the following conjecture for any elementary abelian $p$-group $G$ (they did not explicitly state this conjecture in [GG]).
GAO–GEROLDINGER conjecture. Let $G$ be a finite abelian group with identity $e$. If $G \setminus \{e\}$ is a union of $k$ proper cosets $a_1G_1, \ldots, a_kG_k$ then $k \geq f(|G|)$.

With the notations of the Gao–Geroldinger conjecture, if we set $a_0 = e$ and $G_0 = \{e\}$ then $\{a_sG_s\}_{s=0}^k$ forms a cover of $G$ with $a_0G_0 \cap a_sG_s = \emptyset$ for all $s = 1, \ldots, k$. Thus, by the result of [Z69], the Gao–Geroldinger conjecture holds when $G$ is cyclic.

In this paper we aim to generalize Mycielski’s conjecture in a new direction and prove an extended version of the Gao–Geroldinger conjecture.

**Definition 1.3.** Let $G$ be a group and let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be a finite system of left cosets of subgroups $G_1, \ldots, G_k$. The covering function of $\mathcal{A}$ is given by

$$w_{\mathcal{A}}(x) = \left|\{1 \leq s \leq k : x \in a_sG_s\}\right| \quad (x \in G).$$

Let $m$ be a positive integer. We call $\mathcal{A}$ an $m$-cover of $G$ if $w_{\mathcal{A}}(x) \geq m$ for all $x \in G$. If $\mathcal{A}$ forms an $m$-cover of $G$ but none of its proper subsystems does, then $\mathcal{A}$ is said to be a minimal $m$-cover of $G$.

Now we state our main result, which (in the special case $m = 1$) implies the Gao–Geroldinger conjecture for arbitrary finite abelian groups.

**Theorem 1.3.** Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be an $m$-cover of an abelian group $G$ by left cosets. Then, for any $a \in G$ with $w_{\mathcal{A}}(a) = m$, we have

$$N_a = \left[G : \bigcap_{1 \leq s \leq k, a \in a_sG_s} G_s\right] \leq 2^{k-m} \quad \text{and furthermore} \quad k \geq m + f(N_a).$$

In particular, if $\{a_sG_s\}_{s \neq t}$ fails to be an $m$-cover of $G$, then we have the inequalities

$$[G : G_t] \leq 2^{k-m} \quad \text{and} \quad k \geq m + f([G : G_t]),$$

the bounds of which are best possible.

**Remark 1.2.** When $G = \mathbb{Z}$, Theorem 1.3 was proved by Znám [Z75] in the case $m = 1$, and we can say something stronger in Section 2. Also, in the second inequality of (1.4), $N_a$ cannot be replaced by $[G : \bigcap_{s=1}^k G_s]$ as illustrated by the following example.

**Example 1.1.** Let $G$ be the abelian group $C_p \times C_p$ where $p$ is a prime and $C_p$ is the cyclic group of order $p$. Then any element $a \neq e$ of $G$ has order $p$. Let $G_1, \ldots, G_k$ be all the distinct subgroups of $G$ with order $p$. If $1 \leq i < j \leq k$, then $G_i \cap G_j = \{e\}$. Thus $\{G_s\}_{s=1}^k$ forms a minimal 1-cover of $G$ with $\bigcap_{s=1}^k G_s = \{e\}$. Since $1 + k(p - 1) = \left|\bigcup_{s=1}^k G_s\right| = |G| = p^2$, we have

$$k = p + 1 \geq 1 + f([G : G_s]) = 1 + f(p) = p.$$
However,
\[ k = p + 1 \leq 2p - 1 = 1 + f([G : \{e\}]) = 1 + d\left(G, \bigcap_{s=1}^{k} G_s\right), \]
and the last inequality becomes strict when \( p > 2 \).

Example 1.1 also shows that we do not have an analogue of [S01, Theorem 2.1] for minimal \( m \)-covers of the abelian group \( C_p \times C_p \) (where \( p \) is a prime), thus we cannot prove our Theorem 1.3 by the method in [S01]. To obtain Theorem 1.3 we employ some tools from algebraic number theory as well as characters of abelian groups.

**Corollary 1.1.** Let \( A = \{a_sG_s\}_{s=1}^{k} \) be an \( m \)-cover of a group \( G \) by left cosets. Provided that \( a \in G \) and \( w_A(a) = m \), for any abelian subgroup \( K \) of \( G \) we have

\[
(1.6) \quad k - m \geq |\{1 \leq s \leq k : a \not\in a_sG_s \text{ and } K \not\subset G_s\}| \geq f\left(\left[ K : K \cap \bigcap_{a \in a_sG_s}^{k} G_s\right]\right).
\]

In particular, if \( \{a_sG_s\}_{s \neq t} \) fails to be an \( m \)-cover of \( G \), then for any abelian subgroup \( K \) of \( G \) not contained in \( G_t \) we have

\[
(1.7) \quad |\{1 \leq s \leq k : K \not\subset G_s\}| \geq 1 + f([K : G_t \cap K]).
\]

**Proof.** We define \( J = \{1 \leq s \leq k : a_sG_s \cap aK \neq \emptyset\} \). For each \( s \in J \), \( a^{-1}a_sG_s \cap K \) is a coset of \( G_s \cap K \) in \( K \). Observe that \( \{a^{-1}a_sG_s \cap K\}_{s \in J} \) is an \( m \)-cover of \( K \) with \( |\{s \in J : e \in a^{-1}a_sG_s \cap K\}| = |I_a| = m \) where \( I_a = \{1 \leq s \leq k : a \in a_sG_s\} \).

Applying Theorem 1.3 to the abelian group \( K \) we get the inequality \( |J| - m \geq f([K : \bigcap_{s \in I_a} G_s \cap K]) \). If \( s \in J \) and \( K \subseteq G_s \), then \( a^{-1}a_sG_s \cap K = K \) and hence \( s \in I_a \). Thus

\[
|J| - m = |\{s \in J : e \not\in a^{-1}a_sG_s \cap K\}| \leq |\{1 \leq s \leq k : a \not\in a_sG_s \text{ and } K \not\subset G_s\}| \leq k - m
\]

and hence (1.6) follows.

Now suppose that \( \{a_sG_s\}_{s \neq t} \) is not an \( m \)-cover of \( G \) and \( K \) is an abelian subgroup of \( G \) with \( K \not\subset G_t \). Then \( w_A(x) = m \) for some \( x \in a_tG_t \). In light of the above,

\[
|\{1 \leq s \leq k : s \neq t \text{ and } K \not\subset G_s\}| \geq |\{1 \leq s \leq k : x \not\in a_sG_s \text{ and } K \not\subset G_s\}| \geq f([K : K \cap G_t]).
\]

This proves (1.7) and we are done. ■
Corollary 1.2. Let \( R \) be any ring. Let \( a_1, \ldots, a_k \) be elements of \( R \) and \( I_1, \ldots, I_k \) ideals of \( R \). If \( \{a_s + I_s\}_{s=1}^k \) is an \( m \)-cover of \( R \) with the coset \( a_t + I_t \) irredundant, then for the quotient ring \( R/I_t \) we have \( |R/I_t| \leq 2^{k-m} \) and furthermore \( k \geq m + f(|R/I_t|) \).

Proof. Since \( R \) is an additive abelian group, this follows from Theorem 1.3 immediately.

In the next section we will present a new approach to Mycielski’s problem on covers of \( \mathbb{Z} \). In Section 3 we are going to work with covers of abelian groups and extend some ideas from Section 2; this will lead to our proof of Theorem 1.3.

2. A new approach to Mycielski’s problem. Let \( \overline{Q} \) denote the algebraic closure of the rational field \( Q \) and \( \mathbb{Z} \) the ring of all algebraic integers in \( \overline{Q} \).

Lemma 2.1. For \( s = 1, \ldots, k \) let \( \zeta_s \in \mathbb{Z} \) be a root of unity with order \( n_s > 1 \). Then \( n \in \mathbb{Z}^+ \) divides \( \prod_{s=1}^k (1 - \zeta_s) \) in \( \mathbb{Z} \) if and only if

\[
\sum_{s=1}^k \frac{1}{\varphi(n_s)} \geq \text{ord}_p(n) \quad \text{for any prime } p,
\]

where \( \varphi \) is the well-known Euler function.

Proof. For each prime \( p \), let \( v_p : \overline{Q} \to \mathbb{Q} \) denote any extension of the \( p \)-adic valuation \( \text{ord}_p(\cdot) \) to \( \overline{Q} \), normed by \( v_p(p) = 1 \). It is well known (cf. [W, Chap. 2]) that

\[
v_p(1 - \zeta_s) = \begin{cases} 1/\varphi(n_s) & \text{if } n_s \text{ is a power of } p, \\ 0 & \text{otherwise}. \end{cases}
\]

Now \( n \) divides \( \prod_{s=1}^k (1 - \zeta_s) \) in \( \mathbb{Z} \) if and only if for each valuation \( v : \overline{Q} \to \mathbb{Q} \) one has \( v(n) \leq \sum_{s=1}^k v(1 - \zeta_s) \). Since any valuation \( v \) of \( \overline{Q} \) is (equivalent to) an extension of \( \text{ord}_p(\cdot) \) for some prime \( p \), we immediately obtain the desired result.

Corollary 2.1. Let \( n > 1 \) be an integer. Then \( f(n) \) is the smallest positive integer \( k \) such that there are roots of unity \( \zeta_1, \ldots, \zeta_k \) different from \( 1 \) for which \( \prod_{s=1}^k (1 - \zeta_s) \in n\mathbb{Z} \). Furthermore, this holds with \( k = f(n) \) if and only if for any prime divisor \( p \) of \( n \) there are exactly \( \text{ord}_p(n)(p-1) \) of \( \zeta_1, \ldots, \zeta_k \) having order \( p \).
Proof. For \( s = 1, \ldots, k \) let \( \zeta_s \) be a root of unity with order \( n_s > 1 \). By Lemma 2.1, \( n \) divides \( \prod_{s=1}^{k} (1 - \zeta_s) \) in \( \mathbb{Z} \) if and only if (2.1) holds. Clearly

\[
\sum_{s=1}^{k} \frac{1}{\varphi(n_s)} p(n_s) = \frac{|\{1 \leq s \leq k : P(n_s) = \{p\}\}|}{p - 1}
\]

for every prime \( p \).

If (2.1) is valid, then

\[
k \geq \sum_{p \in P(n)} |\{1 \leq s \leq k : P(n_s) = \{p\}\}| \geq \sum_{p \in P(n)} \text{ord}_p(n)(p - 1) = f(n).
\]

Now assume that \( k = f(n) \). When (2.1) is valid, equality holds in the last three inequalities and hence

\[
|\{1 \leq s \leq k : n_s = p\}| = |\{1 \leq s \leq k : P(n_s) = \{p\}\}| = \text{ord}_p(n)(p - 1)
\]

for any prime \( p \). Conversely, (2.1) holds if \(|\{1 \leq s \leq k : n_s = p\}| = \text{ord}_p(n)(p - 1)\) for all \( p \in P(n) \).

Combining the above we have completed the proof. ■

Lemma 2.2. Suppose that \( A = \{a_s(n_s)\}_{s=1}^{k} \) is an \( m \)-cover of \( \mathbb{Z} \) by residue classes and \( a \in \mathbb{Z} \) is covered by \( A \) exactly \( m \) times. Let \( N_a \) be the least common multiple of those \( n_s \) with \( a \in a_s(n_s) \), and let \( m_s \in \mathbb{Z} \) for \( s \in J \) where \( J = \{1 \leq s \leq k : a \notin a_s(n_s)\} \). Then for any \( 0 \leq \alpha < 1 \) we have

\[
C_0(\alpha) = C_1(\alpha) = \cdots = C_{N_a - 1}(\alpha),
\]

where

\[
C_r(\alpha) = \sum_{I \subseteq J} (-1)^{|I|} e^{2\pi i \sum_{s \in I} (a_s - a)m_s/n_s} (\sum_{s \in I} m_s/n_s)^{\alpha+r}/N_a
\]

for every \( r = 0, 1, \ldots, N_a - 1 \), and we use \( \{\theta\} \) to denote the fractional part of a real number \( \theta \).

Proof. This follows from [S99, Lemma 2]. ■

Theorem 2.1. Let \( A = \{a_s(n_s)\}_{s=1}^{k} \) be an \( m \)-cover of \( \mathbb{Z} \), and suppose that \( a \) is an integer with \( w_A(a) = m \). Then \( k \geq m + f(N_a) \) where \( N_a \) is the least common multiple of those \( n_s \) with \( a \in a_s(n_s) \). Furthermore, for any prime \( p \) we have

\[
|I(p)| \geq \sum_{s \in I(p)} \frac{1}{p^{\text{ord}_p(n_s) - \text{ord}_p(a_s - a) - 1}} \geq \text{ord}_p(N_a)(p - 1),
\]

where

\[
I(p) = \left\{ 1 \leq s \leq k : \frac{n_s}{p^{\text{ord}_p(n_s)}} \right\} | a_s - a \text{ but } n_s \nmid a_s - a \right\}.
\]
Proof. Let $J = \{1 \leq s \leq k : a \notin a_s(n_s)\}$. For each $s \in J$, let $m_s$ be an integer not divisible by $n_s/(n_s,a_s-a) > 1$. Then $\zeta_s = e^{2\pi i (a_s-a)m_s/n_s}$ is a primitive $d_s$th root of unity where $d_s = n_s/(n_s,(a_s-a)m_s) > 1$.

Set

$$S = \left\{ \left\{ \frac{Na}{n_s} \sum_{s \in I} m_s \right\} : I \subseteq J \right\}.$$ 

Then

$$\prod_{s \in J} (1 - \zeta_s) = \sum_{I \subseteq J} \left( -1 \right)^{|I|} e^{2\pi i \sum_{s \in I} (a_s-a)m_s/n_s}$$

$$= \sum_{\alpha \in S} \sum_{\{I \subseteq J : \sum_{s \in I} m_s/n_s = \alpha\}} \left( -1 \right)^{|I|} e^{2\pi i \sum_{s \in I} (a_s-a)m_s/n_s}$$

$$= \sum_{\alpha \in S} \sum_{r=0}^{Na-1} C_r(\alpha) = Na \sum_{\alpha \in S} C_0(\alpha),$$

where $C_r(\alpha)$ $(0 \leq r < Na)$ are given by (2.3). So $Na$ divides $\prod_{s \in J} (1 - \zeta_s)$ in the ring $\mathbb{Z}$. By Corollary 2.1, we have $k - m = |J| \geq f(Na)$. In view of Lemma 2.1,

$$\sum_{s \in J} \frac{1}{\varphi(n_s/(n_s,a_s-a))} \geq \text{ord}_p(Na) \quad \text{for each prime } p.$$ 

Now we simply let $m_s = 1$ for all $s \in J$. By the above, for any prime $p$ we have

$$\sum_{s \in I(p)} \frac{1}{\varphi(n_s/(n_s,a_s-a))} \geq \text{ord}_p(Na),$$

which is equivalent to (2.4). This concludes the proof. 

3. Working with abelian groups. We first recall some well-known facts from the theory of characters of finite abelian groups (see, e.g., [W, pp. 22–23]).

For a finite abelian group $G$, let $\hat{G}$ denote the group of all complex-valued characters of $G$. One has $\hat{G} \cong G$. For any subgroup $H$ of $G$ let $H^\perp$ denote the group of those characters $\chi \in \hat{G}$ with $\ker(\chi) = \{x \in G : \chi(x) = 1\}$ containing $H$. Then we get a canonical isomorphism $H^\perp \cong \hat{G}/H$ by putting $\chi(aH) = \chi(a)$ for any $a \in G$ and any $\chi \in H^\perp$. Furthermore, for each $a \in G \setminus H$ there exists some $\chi \in H^\perp$ with $\chi(a) \neq 1$.

Proof of Theorem 1.3. Choose a minimal $I_* \subseteq \{1, \ldots, k\}$ such that the system $\{a_sG_s\}_{s \in I_*}$ forms an $m$-cover of $G$. As $I_a = \{1 \leq s \leq k : a \in a_sG_s\}$ has cardinality $m$, we see that $I_a$ is contained in $I_*$. So we can simply assume
that $\mathcal{A}$ is a minimal $m$-cover of $G$ (i.e., $I_s = \{1, \ldots, k\}$). By [S90, Corollary 1], $H = \bigcap_{s=1}^k G_s$ is of finite index in $G$. Instead of the minimal $m$-cover $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ of $G$, we may consider the minimal $m$-cover $\overline{\mathcal{A}} = \{\overline{a}_s G_s\}_{s=1}^k$ of the finite abelian group $\overline{G} = G/H$, where $\overline{a}_s = a_s H$ and $\overline{G}_s = G_s/H$ (hence $[\overline{G} : \overline{G}_s] = [G : G_s]$). Therefore, without any loss of generality, we can assume that $G$ is finite.

Put $H_a = \bigcap_{s \in I_a} G_s$; then $|H_a| = [G : H_a] = N_a$.

Note that $J = \{1 \leq j \leq k : a \notin a_j G_j\}$ has cardinality $k - m$. For each $j \in J$ we may choose a $\chi_j \in G_j$ with $\zeta_j := \chi_j(a^{-1}a_j) \neq 1$. For any $x \in G \setminus H_a$ we have $ax \notin \bigcap_{s \in I_a} a G_s = \bigcap_{s \in I_a} a_s G_s$. Since $\mathcal{A}$ is an $m$-cover of $G$, there exists some $j \in J$ with $ax \in a_j G_j$, and therefore $\chi_j(x) = \zeta_j$ by the choice of $\chi_j$ and the definition of $\zeta_j$.

For $x \in G$ we define

$$\Psi(x) = \prod_{j \in J} (\chi_j(x) - \zeta_j).$$

If $\chi \in H_a$ and $\chi(x) \neq 1$, then $x \notin H_a$ and hence $\Psi(x) = 0$ by the above. Thus $\Psi \chi = \Psi$ for all $\chi \in H_a$.

Observe that

$$\Psi(x) = \sum_{I \subseteq J} \left( \prod_{j \in I} \chi_j(x) \right) \prod_{j \in J \setminus I} (-\zeta_j) = \sum_{\psi \in \hat{G}} c(\psi) \psi(x),$$

where

$$c(\psi) = \sum_{I \subseteq J} \prod_{j \in J \setminus I} (-\zeta_j) \in \mathbb{Z}.$$

Let $\mathbb{C}$ be the complex field. As the set $\hat{G}$ is a basis of the $\mathbb{C}$-vector space $\mathbb{C}^G = \{g : g$ is a function from $G$ to $\mathbb{C}\}$ (cf. [J, p. 291]), for any $\chi \in H_a$ we have $c(\psi \chi) = c(\psi)$ for all $\psi \in \hat{G}$ because $\Psi \chi^{-1} = \Psi$.

Clearly,

$$\prod_{j \in J} (1 - \zeta_j) = \Psi(e) = \sum_{\psi \in \hat{G}} c(\psi) \psi(e) = \sum_{\psi \in \hat{G}} c(\psi).$$

Let $\psi_1 H_a \cup \cdots \cup \psi_l H_a$ be a coset decomposition of $\hat{G}$ where $l = [\hat{G} : H_a]$. Then

$$\sum_{\psi \in \hat{G}} c(\psi) = \sum_{r=1}^l \sum_{\chi \in H_a} c(\psi_r \chi) = \sum_{r=1}^l |H_a|^l c(\psi_r) = N_a \sum_{r=1}^l c(\psi_r).$$

(That $c(\psi_r \chi) = c(\psi_r)$ for all $\chi \in H_a$ is an analogy of Lemma 2.2.) Therefore $N_a$ divides $\prod_{j \in J} (1 - \zeta_j)$ in $\mathbb{Z}$, and Corollary 2.1 gives $k - m = |J| \geq f(N_a)$, and consequently $N_a \leq 2^{k-m}$ by Remark 1.1.
If \( \{a_s G_s\}_{s \neq t} \) is not an \( m \)-cover of \( G \), then for some \( x \in a_t G_t \) we have 
\[ w_A(x) = m, \]
hence \( k - m \geq f(N_x) \geq f([G : G_t]) \) and \( [G : G_t] \leq N_x \leq 2^{k-m} \) by the above.

By [S01, Example 1.2], for any subgroup \( H \) of \( G \) (with \( [G : H] < \infty \)) and an arbitrary element \( x \) of \( G \), the coset \( xH \) and \( m - 1 + d(G, H) = m - 1 + f([G : H]) \) other cosets of subgroups containing \( H \) form an (exact) \( m \)-cover of \( G \) with \( xH \) irredundant. Also, \( m - 1 \) copies of \( 0(1) \), together with the \( k - m + 1 \) residue classes
\[ 1(2), 2(2^2), \ldots, 2^{k-m-1}(2^{k-m}), 0(2^{k-m}), \]
clearly form an (exact) \( m \)-cover of \( \mathbb{Z} \) with the residue class \( 0(2^{k-m}) \) irredundant. So the inequalities in (1.5) are really best possible and we are done. \( \blacksquare \)

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