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# On covers of abelian groups by cosets

by

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**1. Introduction.** As in any textbook on group theory, for a subgroup $H$ of a group $G$ with the index $[G : H]$ finite, $G$ can be partitioned into $k = [G : H]$ left cosets of $H$ in $G$, i.e., all the $k$ left cosets of $H$ form a disjoint cover of $G$.

In 1954 B. H. Neumann [N1, N2] discovered the following basic result on covers of groups.

THEOREM 1.1 (Neumann). *Let $\{a_sG_s\}_{s=1}^k$ be a cover of a group $G$ by (finitely many) left cosets of subgroups $G_1, \ldots, G_k$. Then $G$ is the union of those $a_sG_s$ with $[G : G_s] < \infty$. In other words, if $\{a_sG_s\}_{s\neq t}$ is not a cover of $G$ then $[G : G_t] < \infty$.*

In 1966 J. Mycielski (cf. [MS]) posed an interesting conjecture on disjoint covers of abelian groups. Before stating the conjecture we give a definition.

DEFINITION 1.1. The *Mycielski function* $f : \mathbb{Z}^+ = \{1, 2, \ldots\} \to \{0, 1, \ldots\}$ is given by

$$f(n) = \sum_{p\in P(n)} \mathrm{ord}_p(n)(p-1), \tag{1.1}$$

where $P(n)$ denotes the set of prime divisors of $n$ and $\mathrm{ord}_p(n)$ represents the largest nonnegative integer $\alpha$ such that $p^\alpha \mid n$.

REMARK 1.1. Since $p \le 2^{p-1}$ for any prime $p$, (1.1) implies that $n \le 2^{f(n)}$ (i.e., $f(n) \ge \log_2 n$).

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Mycielski's conjecture. *Let $G$ be an abelian group, and $\{a_sG_s\}_{s=1}^k$ be a disjoint cover of $G$ by left cosets of subgroups. Then $k \geq 1+f([G:G_t])$ for each $t=1,\ldots,k$.*

When $G$ is the additive group $\mathbb{Z}$ of integers, Mycielski's conjecture says that for any disjoint cover $\{a_s(n_s)\}_{s=1}^k$ of $\mathbb{Z}$ by residue classes (where $a_s \in \mathbb{Z}$, $n_s \in \mathbb{Z}^+$ and $a_s(n_s)=a_s+n_s\mathbb{Z}$) we have $k \geq 1+f(n_t)$ for every $t=1,\ldots,k$. This was first confirmed by Š. Znám [Z66]. For problems and results on covers of $\mathbb{Z}$, the reader is referred to [G04], [PS], [S03] and [S05].

Definition 1.2. For a subnormal subgroup $H$ of a group $G$ with finite index, we define

$$d(G,H)=\sum_{i=1}^{n}([H_i:H_{i-1}]-1), \tag{1.2}$$

where $H_0=H\subset H_1\subset\cdots\subset H_n=G$ is any composition series from $H$ to $G$.

By [S90, Theorem 6] and [S01, Theorem 3.1], for any subnormal subgroup $H$ of a group $G$ with $[G:H]<\infty$, we have $d(G,H)\geq f([G:H])$, and equality holds if and only if $G/H_G$ is solvable, where $H_G=\bigcap_{g\in G} gHg^{-1}$ is the *core* of $H$ in $G$ (i.e., the largest normal subgroup of $G$ contained in $H$).

The following result is stronger than Mycielski's conjecture.

Theorem 1.2 (I. Korec, Z. W. Sun). *Let $a_1G_1,\ldots,a_kG_k$ be left cosets of subnormal subgroups $G_1,\ldots,G_k$ of a group $G$. If $\mathcal{A}=\{a_sG_s\}_{s=1}^k$ forms an exact $m$-cover of $G$, i.e., $\mathcal{A}$ covers each element of $G$ exactly $m$ times, then $[G:\bigcap_{s=1}^k G_s]<\infty$ and*

$$k \geq m+d\Big(G,\bigcap_{s=1}^{k}G_s\Big)\geq m+f\Big(\Big[G:\bigcap_{s=1}^{k}G_s\Big]\Big),$$

*where the lower bound $m+d(G,\bigcap_{s=1}^k G_s)$ is best possible.*

In the case $m=1$ and $G=\mathbb{Z}$, Theorem 1.2 was first conjectured by Znám [Z69]. When $m=1$ and $G_1,\ldots,G_k$ are normal in $G$, Theorem 1.2 was obtained by Korec [K74] in 1974. In 1990 Sun [S90] deduced Theorem 1.2 in the case $m=1$ by a method different from that of Korec. The current version of Theorem 1.2 was established by Sun [S01] in 2001; the proof depends heavily on the condition that $\mathcal{A}$ covers all the elements of $G$ the same number of times. Under the conditions of Theorem 1.2, Sun [S04] also showed that the indices $[G:G_s]$ $(1\leq s\leq k)$ cannot be distinct providing $k>1$.

Call a coset in an abelian group not containing the identity element a *proper coset*. In 2003 W. D. Gao and A. Geroldinger [GG] proved the following conjecture for any elementary abelian $p$-group $G$ (they did not explicitly state this conjecture in [GG]).

Gao–Geroldinger conjecture. *Let $G$ be a finite abelian group with identity $e$. If $G \setminus \{e\}$ is a union of $k$ proper cosets $a_1G_1, \ldots, a_kG_k$ then $k \geq f(|G|)$.*

With the notations of the Gao–Geroldinger conjecture, if we set $a_0 = e$ and $G_0 = \{e\}$ then $\{a_sG_s\}_{s=0}^k$ forms a cover of $G$ with $a_0G_0 \cap a_sG_s = \emptyset$ for all $s = 1, \ldots, k$. Thus, by the result of [Z69], the Gao–Geroldinger conjecture holds when $G$ is cyclic.

In this paper we aim to generalize Mycielski's conjecture in a new direction and prove an extended version of the Gao–Geroldinger conjecture.

Definition 1.3. Let $G$ be a group and let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be a finite system of left cosets of subgroups $G_1, \ldots, G_k$. The *covering function* of $\mathcal{A}$ is given by

$$w_{\mathcal{A}}(x) = |\{1 \leq s \leq k : x \in a_sG_s\}| \quad (x \in G). \tag{1.3}$$

Let $m$ be a positive integer. We call $\mathcal{A}$ an *$m$-cover* of $G$ if $w_{\mathcal{A}}(x) \geq m$ for all $x \in G$. If $\mathcal{A}$ forms an $m$-cover of $G$ but none of its proper subsystems does, then $\mathcal{A}$ is said to be a *minimal $m$-cover* of $G$.

Now we state our main result, which (in the special case $m = 1$) implies the Gao–Geroldinger conjecture for arbitrary finite abelian groups.

Theorem 1.3. *Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be an $m$-cover of an abelian group $G$ by left cosets. Then, for any $a \in G$ with $w_{\mathcal{A}}(a) = m$, we have*

$$N_a = \Big[G : \bigcap_{\substack{1 \leq s \leq k \\ a \in a_sG_s}} G_s\Big] \leq 2^{k-m} \quad \textit{and furthermore} \quad k \geq m + f(N_a). \tag{1.4}$$

*In particular, if $\{a_sG_s\}_{s \neq t}$ fails to be an $m$-cover of $G$, then we have the inequalities*

$$[G : G_t] \leq 2^{k-m} \quad \textit{and} \quad k \geq m + f([G : G_t]), \tag{1.5}$$

*the bounds of which are best possible.*

Remark 1.2. When $G = \mathbb{Z}$, Theorem 1.3 was proved by Znám [Z75] in the case $m = 1$, and we can say something stronger in Section 2. Also, in the second inequality of (1.4), $N_a$ cannot be replaced by $[G : \bigcap_{s=1}^k G_s]$ as illustrated by the following example.

Example 1.1. Let $G$ be the abelian group $C_p \times C_p$ where $p$ is a prime and $C_p$ is the cyclic group of order $p$. Then any element $a \neq e$ of $G$ has order $p$. Let $G_1, \ldots, G_k$ be all the distinct subgroups of $G$ with order $p$. If $1 \leq i < j \leq k$, then $G_i \cap G_j = \{e\}$. Thus $\{G_s\}_{s=1}^k$ forms a minimal 1-cover of $G$ with $\bigcap_{s=1}^k G_s = \{e\}$. Since $1 + k(p-1) = |\bigcup_{s=1}^k G_s| = |G| = p^2$, we have

$$k = p + 1 \geq 1 + f([G : G_s]) = 1 + f(p) = p.$$

However,

$$k = p + 1 \le 2p - 1 = 1 + f([G : \{e\}]) = 1 + d\Big(G, \bigcap_{s=1}^{k} G_s\Big),$$

and the last inequality becomes strict when $p > 2$.

Example 1.1 also shows that we do not have an analogue of [S01, Theorem 2.1] for minimal $m$-covers of the abelian group $C_p \times C_p$ (where $p$ is a prime), thus we cannot prove our Theorem 1.3 by the method in [S01]. To obtain Theorem 1.3 we employ some tools from algebraic number theory as well as characters of abelian groups.

Corollary 1.1. *Let $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ be an $m$-cover of a group $G$ by left cosets. Provided that $a \in G$ and $w_{\mathcal{A}}(a) = m$, for any abelian subgroup $K$ of $G$ we have*

$$\begin{aligned} k - m &\ge |\{1 \le s \le k : a \notin a_s G_s \text{ and } K \not\subseteq G_s\}| \\ &\ge f\Big(\Big[K : K \cap \bigcap_{\substack{s=1 \\ a \in a_s G_s}}^{k} G_s\Big]\Big). \end{aligned} \tag{1.6}$$

*In particular, if $\{a_s G_s\}_{s \ne t}$ fails to be an $m$-cover of $G$, then for any abelian subgroup $K$ of $G$ not contained in $G_t$ we have*

$$|\{1 \le s \le k : K \not\subseteq G_s\}| \ge 1 + f([K : G_t \cap K]). \tag{1.7}$$

*Proof.* We define $J = \{1 \le s \le k : a_s G_s \cap aK \ne \emptyset\}$. For each $s \in J$, $a^{-1} a_s G_s \cap K$ is a coset of $G_s \cap K$ in $K$. Observe that $\{a^{-1} a_s G_s \cap K\}_{s \in J}$ is an $m$-cover of $K$ with $|\{s \in J : e \in a^{-1} a_s G_s \cap K\}| = |I_a| = m$ where

$$I_a = \{1 \le s \le k : a \in a_s G_s\}.$$

Applying Theorem 1.3 to the abelian group $K$ we get the inequality $|J| - m \ge f([K : \bigcap_{s \in I_a} G_s \cap K])$. If $s \in J$ and $K \subseteq G_s$, then $a^{-1} a_s G_s \cap K = K$ and hence $s \in I_a$. Thus

$$\begin{aligned} |J| - m &= |\{s \in J : e \notin a^{-1} a_s G_s \cap K\}| \\ &\le |\{1 \le s \le k : a \notin a_s G_s \text{ and } K \not\subseteq G_s\}| \le k - m \end{aligned}$$

and hence (1.6) follows.

Now suppose that $\{a_s G_s\}_{s \ne t}$ is not an $m$-cover of $G$ and $K$ is an abelian subgroup of $G$ with $K \not\subseteq G_t$. Then $w_{\mathcal{A}}(x) = m$ for some $x \in a_t G_t$. In light of the above,

$$\begin{aligned} |\{1 \le s \le k : s \ne t \text{ and } K \not\subseteq G_s\}| &\ge |\{1 \le s \le k : x \notin a_s G_s \text{ and } K \not\subseteq G_s\}| \\ &\ge f([K : K \cap G_t]). \end{aligned}$$

This proves (1.7) and we are done. ∎

COROLLARY 1.2. *Let $R$ be any ring. Let $a_1, \dots, a_k$ be elements of $R$ and $I_1, \dots, I_k$ ideals of $R$. If $\{a_s + I_s\}_{s=1}^k$ is an $m$-cover of $R$ with the coset $a_t + I_t$ irredundant, then for the quotient ring $R/I_t$ we have $|R/I_t| \le 2^{k-m}$ and furthermore $k \ge m + f(|R/I_t|)$.*

*Proof.* Since $R$ is an additive abelian group, this follows from Theorem 1.3 immediately. ∎

In the next section we will present a new approach to Mycielski's problem on covers of $\mathbb{Z}$. In Section 3 we are going to work with covers of abelian groups and extend some ideas from Section 2; this will lead to our proof of Theorem 1.3.

**2. A new approach to Mycielski's problem.** Let $\overline{\mathbb{Q}}$ denote the algebraic closure of the rational field $\mathbb{Q}$ and $\overline{\mathbb{Z}}$ the ring of all algebraic integers in $\overline{\mathbb{Q}}$.

LEMMA 2.1. *For $s = 1, \dots, k$ let $\zeta_s \in \overline{\mathbb{Z}}$ be a root of unity with order $n_s > 1$. Then $n \in \mathbb{Z}^+$ divides $\prod_{s=1}^k (1 - \zeta_s)$ in $\overline{\mathbb{Z}}$ if and only if*

$$\sum_{\substack{s=1 \\ P(n_s)=\{p\}}}^{k} \frac{1}{\varphi(n_s)} \ge \mathrm{ord}_p(n) \quad \textit{for any prime } p, \tag{2.1}$$

*where $\varphi$ is the well-known Euler function.*

*Proof.* For each prime $p$, let $\mathbf{v}_p : \overline{\mathbb{Q}} \to \mathbb{Q}$ denote any extension of the $p$-adic valuation $\mathrm{ord}_p(\cdot)$ to $\overline{\mathbb{Q}}$, normed by $\mathbf{v}_p(p) = 1$. It is well known (cf. [W, Chap. 2]) that

$$\mathbf{v}_p(1 - \zeta_s) = \begin{cases} 1/\varphi(n_s) & \text{if } n_s \text{ is a power of } p, \\ 0 & \text{otherwise.} \end{cases}$$

Now $n$ divides $\prod_{s=1}^k (1 - \zeta_s)$ in $\overline{\mathbb{Z}}$ if and only if for each valuation $\mathbf{v} : \overline{\mathbb{Q}} \to \mathbb{Q}$ one has $\mathbf{v}(n) \le \sum_{s=1}^k \mathbf{v}(1 - \zeta_s)$. Since any valuation $\mathbf{v}$ of $\overline{\mathbb{Q}}$ is (equivalent to) an extension of $\mathrm{ord}_p(\cdot)$ for some prime $p$, we immediately obtain the desired result. ∎

COROLLARY 2.1. *Let $n > 1$ be an integer. Then $f(n)$ is the smallest positive integer $k$ such that there are roots of unity $\zeta_1, \dots, \zeta_k$ different from 1 for which $\prod_{s=1}^k (1 - \zeta_s) \in n\overline{\mathbb{Z}}$. Furthermore, this holds with $k = f(n)$ if and only if for any prime divisor $p$ of $n$ there are exactly $\mathrm{ord}_p(n)(p-1)$ of $\zeta_1, \dots, \zeta_k$ having order $p$.*

*Proof.* For $s = 1, \ldots, k$ let $\zeta_s$ be a root of unity with order $n_s > 1$. By Lemma 2.1, $n$ divides $\prod_{s=1}^{k}(1-\zeta_s)$ in $\overline{\mathbb{Z}}$ if and only if (2.1) holds. Clearly

$$\sum_{\substack{s=1\\ P(n_s)=\{p\}}}^{k} \frac{1}{\varphi(n_s)} \le \frac{|\{1 \le s \le k : P(n_s) = \{p\}\}|}{p-1} \quad \text{for every prime } p.$$

If (2.1) is valid, then

$$k \ge \sum_{p \in P(n)} |\{1 \le s \le k : P(n_s) = \{p\}\}| \ge \sum_{p \in P(n)} \mathrm{ord}_p(n)(p-1) = f(n).$$

Now assume that $k = f(n)$. When (2.1) is valid, equality holds in the last three inequalities and hence

$$|\{1 \le s \le k : n_s = p\}| = |\{1 \le s \le k : P(n_s) = \{p\}\}| = \mathrm{ord}_p(n)(p-1)$$

for any prime $p$. Conversely, (2.1) holds if $|\{1 \le s \le k : n_s = p\}| = \mathrm{ord}_p(n)(p-1)$ for all $p \in P(n)$.

Combining the above we have completed the proof. ∎

Lemma 2.2. *Suppose that $A = \{a_s(n_s)\}_{s=1}^{k}$ is an $m$-cover of $\mathbb{Z}$ by residue classes and $a \in \mathbb{Z}$ is covered by $A$ exactly $m$ times. Let $N_a$ be the least common multiple of those $n_s$ with $a \in a_s(n_s)$, and let $m_s \in \mathbb{Z}$ for $s \in J$ where $J = \{1 \le s \le k : a \notin a_s(n_s)\}$. Then for any $0 \le \alpha < 1$ we have*

$$C_0(\alpha) = C_1(\alpha) = \cdots = C_{N_a-1}(\alpha), \tag{2.2}$$

*where*

$$C_r(\alpha) = \sum_{\substack{I \subseteq J\\ \{\sum_{s\in I} m_s/n_s\} = (\alpha+r)/N_a}} (-1)^{|I|} e^{2\pi i \sum_{s\in I}(a_s-a)m_s/n_s} \tag{2.3}$$

*for every $r = 0, 1, \ldots, N_a - 1$, and we use $\{\theta\}$ to denote the fractional part of a real number $\theta$.*

*Proof.* This follows from [S99, Lemma 2]. ∎

Theorem 2.1. *Let $A = \{a_s(n_s)\}_{s=1}^{k}$ be an $m$-cover of $\mathbb{Z}$, and suppose that $a$ is an integer with $w_A(a) = m$. Then $k \ge m + f(N_a)$ where $N_a$ is the least common multiple of those $n_s$ with $a \in a_s(n_s)$. Furthermore, for any prime $p$ we have*

$$|I(p)| \ge \sum_{s \in I(p)} \frac{1}{p^{\mathrm{ord}_p(n_s) - \mathrm{ord}_p(a_s - a) - 1}} \ge \mathrm{ord}_p(N_a)(p-1), \tag{2.4}$$

*where*

$$I(p) = \left\{1 \le s \le k : \frac{n_s}{p^{\mathrm{ord}_p(n_s)}} \mid a_s - a \text{ but } n_s \nmid a_s - a\right\}. \tag{2.5}$$

*Proof.* Let $J = \{1 \le s \le k : a \notin a_s(n_s)\}$. For each $s \in J$, let $m_s$ be an integer not divisible by $n_s/(n_s, a_s - a) > 1$. Then $\zeta_s = e^{2\pi i(a_s-a)m_s/n_s}$ is a primitive $d_s$th root of unity where $d_s = n_s/(n_s, (a_s - a)m_s) > 1$.

Set

$$S = \left\{ \left\{ N_a \sum_{s\in I} \frac{m_s}{n_s} \right\} : I \subseteq J \right\}.$$

Then

$$\begin{aligned}\prod_{s\in J}(1-\zeta_s) &= \sum_{I\subseteq J}(-1)^{|I|}e^{2\pi i\sum_{s\in I}(a_s-a)m_s/n_s} \\ &= \sum_{\alpha\in S}\ \sum_{\substack{I\subseteq J \\ \{N_a\sum_{s\in I}m_s/n_s\}=\alpha}} (-1)^{|I|}e^{2\pi i\sum_{s\in I}(a_s-a)m_s/n_s} \\ &= \sum_{\alpha\in S}\sum_{r=0}^{N_a-1} C_r(\alpha) = N_a\sum_{\alpha\in S} C_0(\alpha),\end{aligned}$$

where $C_r(\alpha)$ $(0 \le r < N_a)$ are given by (2.3). So $N_a$ divides $\prod_{s\in J}(1-\zeta_s)$ in the ring $\overline{\mathbb{Z}}$. By Corollary 2.1, we have $k - m = |J| \ge f(N_a)$. In view of Lemma 2.1,

$$\sum_{\substack{s\in J \\ P(d_s)=\{p\}}} \frac{1}{\varphi(d_s)} \ge \mathrm{ord}_p(N_a) \quad \text{for each prime } p.$$

Now we simply let $m_s = 1$ for all $s \in J$. By the above, for any prime $p$ we have

$$\sum_{s\in I(p)} \frac{1}{\varphi(n_s/(n_s, a_s - a))} \ge \mathrm{ord}_p(N_a),$$

which is equivalent to (2.4). This concludes the proof. ∎

**3. Working with abelian groups.** We first recall some well-known facts from the theory of characters of finite abelian groups (see, e.g., [W, pp. 22–23]).

For a finite abelian group $G$, let $\widehat{G}$ denote the group of all complex-valued characters of $G$. One has $\widehat{G} \cong G$. For any subgroup $H$ of $G$ let $H^\perp$ denote the group of those characters $\chi \in \widehat{G}$ with $\ker(\chi) = \{x \in G : \chi(x) = 1\}$ containing $H$. Then we get a canonical isomorphism $H^\perp \cong \widehat{G/H}$ by putting $\chi(aH) = \chi(a)$ for any $a \in G$ and any $\chi \in H^\perp$. Furthermore, for each $a \in G \setminus H$ there exists some $\chi \in H^\perp$ with $\chi(a) \ne 1$.

*Proof of Theorem 1.3.* Choose a minimal $I_* \subseteq \{1, \ldots, k\}$ such that the system $\{a_sG_s\}_{s\in I_*}$ forms an $m$-cover of $G$. As $I_a = \{1 \le s \le k : a \in a_sG_s\}$ has cardinality $m$, we see that $I_a$ is contained in $I_*$. So we can simply assume that $\mathcal{A}$ is a minimal $m$-cover of $G$ (i.e., $I_* = \{1,\dots,k\}$). By [S90, Corollary 1], $H = \bigcap_{s=1}^{k} G_s$ is of finite index in $G$. Instead of the minimal $m$-cover $\mathcal{A} = \{a_s G_s\}_{s=1}^{k}$ of $G$, we may consider the minimal $m$-cover $\overline{\mathcal{A}} = \{\overline{a}_s \overline{G}_s\}_{s=1}^{k}$ of the finite abelian group $\overline{G} = G/H$, where $\overline{a}_s = a_s H$ and $\overline{G}_s = G_s/H$ (hence $[\overline{G} : \overline{G}_s] = [G : G_s]$). Therefore, without any loss of generality, we can assume that $G$ is finite.

Put $H_a = \bigcap_{s\in I_a} G_s$; then $|H_a^{\perp}| = [G : H_a] = N_a$.

Note that $J = \{1 \le j \le k : a \notin a_j G_j\}$ has cardinality $k - m$. For each $j \in J$ we may choose a $\chi_j \in G_j^{\perp}$ with $\zeta_j := \chi_j(a^{-1}a_j) \ne 1$. For any $x \in G \setminus H_a$ we have $ax \notin \bigcap_{s\in I_a} aG_s = \bigcap_{s\in I_a} a_s G_s$. Since $\mathcal{A}$ is an $m$-cover of $G$, there exists some $j \in J$ with $ax \in a_j G_j$, and therefore $\chi_j(x) = \zeta_j$ by the choice of $\chi_j$ and the definition of $\zeta_j$.

For $x \in G$ we define

$$\Psi(x) = \prod_{j\in J} (\chi_j(x) - \zeta_j).$$

If $\chi \in H_a^{\perp}$ and $\chi(x) \ne 1$, then $x \notin H_a$ and hence $\Psi(x) = 0$ by the above. Thus $\Psi\chi = \Psi$ for all $\chi \in H_a^{\perp}$.

Observe that

$$\Psi(x) = \sum_{I\subseteq J} \Big(\prod_{j\in I} \chi_j(x)\Big) \prod_{j\in J\setminus I} (-\zeta_j) = \sum_{\psi\in\widehat{G}} c(\psi)\psi(x),$$

where

$$c(\psi) = \sum_{\substack{I\subseteq J\\ \prod_{j\in I}\chi_j=\psi}} \prod_{j\in J\setminus I} (-\zeta_j) \in \overline{\mathbb{Z}}.$$

Let $\mathbb{C}$ be the complex field. As the set $\widehat{G}$ is a basis of the $\mathbb{C}$-vector space

$$\mathbb{C}^G = \{g : g \text{ is a function from } G \text{ to } \mathbb{C}\}$$

(cf. [J, p. 291]), for any $\chi \in H_a^{\perp}$ we have $c(\psi\chi) = c(\psi)$ for all $\psi \in \widehat{G}$ because $\Psi\chi^{-1} = \Psi$.

Clearly,

$$\prod_{j\in J} (1 - \zeta_j) = \Psi(e) = \sum_{\psi\in\widehat{G}} c(\psi)\psi(e) = \sum_{\psi\in\widehat{G}} c(\psi).$$

Let $\psi_1 H_a^{\perp} \cup \dots \cup \psi_l H_a^{\perp}$ be a coset decomposition of $\widehat{G}$ where $l = [\widehat{G} : H_a^{\perp}]$. Then

$$\sum_{\psi\in\widehat{G}} c(\psi) = \sum_{r=1}^{l} \sum_{\chi\in H_a^{\perp}} c(\psi_r\chi) = \sum_{r=1}^{l} |H_a^{\perp}| c(\psi_r) = N_a \sum_{r=1}^{l} c(\psi_r).$$

(That $c(\psi_r\chi) = c(\psi_r)$ for all $\chi \in H_a^{\perp}$ is an analogy of Lemma 2.2.) Therefore $N_a$ divides $\prod_{j\in J}(1 - \zeta_j)$ in $\overline{\mathbb{Z}}$, and Corollary 2.1 gives $k - m = |J| \ge f(N_a)$, and consequently $N_a \le 2^{k-m}$ by Remark 1.1.

If $\{a_sG_s\}_{s\neq t}$ is not an $m$-cover of $G$, then for some $x \in a_tG_t$ we have $w_{\mathcal{A}}(x) = m$, hence $k - m \geq f(N_x) \geq f([G : G_t])$ and $[G : G_t] \leq N_x \leq 2^{k-m}$ by the above.

By [S01, Example 1.2], for any subgroup $H$ of $G$ (with $[G : H] < \infty$) and an arbitrary element $x$ of $G$, the coset $xH$ and $m - 1 + d(G,H) = m - 1 + f([G : H])$ other cosets of subgroups containing $H$ form an (exact) $m$-cover of $G$ with $xH$ irredundant. Also, $m-1$ copies of $0(1)$, together with the $k - m + 1$ residue classes

$$1(2), 2(2^2), \ldots, 2^{k-m-1}(2^{k-m}), 0(2^{k-m}),$$

clearly form an (exact) $m$-cover of $\mathbb{Z}$ with the residue class $0(2^{k-m})$ irredundant. So the inequalities in (1.5) are really best possible and we are done. ∎

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