ACTA ARITHMETICA
140.2 (2009)

Additive problems involving primes of special type

by

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1. Introduction. In 1742, in his letters to Euler, Goldbach proposed his well-known conjectures, which can be formulated in modern mathematical terms as follows:

(A) For any even integer $n \geq 4$, the equation

$$n = p_1 + p_2 \tag{1.1}$$

is solvable in primes p_1, p_2.

(B) For any odd integer $n \geq 7$, the equation

$$n = p_1 + p_2 + p_3 \tag{1.2}$$

is solvable in primes p_1, p_2, p_3.

Nowadays the best results concerning Conjectures (A) and (B) are due to Chen [2] and Vinogradov [18] respectively. In 1937 Vinogradov [18] showed that Conjecture (B) holds for any sufficiently large odd integers. As for Conjecture (A), in 1973, by adding his ingenious innovations into sieve theory, Chen [2] proved that any sufficiently large even integer n can be represented in the form

$$n = p_1 + P_2 \tag{1.3}$$

where p_1 is a prime and P_2 is an almost-prime with at most two prime factors.

In 1938, basing upon Vinogradov's work, Hua [9] showed that for sufficiently large $n \equiv 5 \pmod{24}$, the equation

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 \tag{1.4}$$

is solvable in primes p_1, p_2, p_3, p_4, p_5.

In 1939, by Vinogradov's method, van der Corput [17] proved that there exist infinitely many arithmetic progressions of three different prime terms. In 1981, Heath-Brown [8] showed that there exist infinitely many arithmetic

2010 *Mathematics Subject Classification*: 11P32, 11N36.
Key words and phrases: additive number theory, sieve, almost-prime.

DOI: 10.4064/aa140-2-6

progressions of four different terms, three of which are primes, and the fourth is P_2. In 2006, Green and Tao [3] established that there exist infinitely many arithmetic progressions consisting of three different primes $p_1<p_2<p_3$ such that $p_j+2=P_2$ for each $j=1,2,3$. Recently [4] they showed that this holds for any number $k\geq 3$ of primes.

Motivated by Heath-Brown [8], Tolev [14–16] and Peneva [12, 13] studied additive problems with primes p such that $p+2$ is an almost-prime. In [16] Tolev showed, by using the vector sieve developed in [1], that:

1) If n is sufficiently large and $n\equiv 3 \pmod 6$, then the equation (1.2) is solvable in primes p_1,p_2,p_3 such that

$$p_1+2=P_2,\quad p_2+2=P_5,\quad p_3+2=P_7.$$

2) If n is sufficiently large and $n\equiv 5 \pmod{24}$, then the equation (1.4) is solvable in primes p_1,p_2,p_3,p_4,p_5 such that

$$p_1+2=P_2,\quad p_2+2=P_2',\quad p_3+2=P_5,\quad p_4+2=P_5',\quad p_5+2=P_8.$$

In this paper, by inserting a weighted sieve approach into Tolev's argument, we obtain the following sharper results

THEOREM 1. *If n is sufficiently large and $n\equiv 5 \pmod{24}$, then the equation* (1.4) *is solvable in primes p_1,p_2,p_3,p_4,p_5 such that*

$$p_1+2=P_2,\quad p_2+2=P_2',\quad p_3+2=P_4,\quad p_4+2=P_4',\quad p_5+2=P_5.$$

THEOREM 2. *If n is sufficiently large and $n\equiv 3 \pmod 6$, then the equation* (1.2) *is solvable in primes p_1,p_2,p_3 such that*

$$p_1+2=P_2,\quad p_2+2=P_3,\quad p_3+2=P_5.$$

THEOREM 2′. *If n is sufficiently large and $n\equiv 3 \pmod 6$, then the equation* (1.2) *is solvable in primes p_1,p_2,p_3 such that*

$$p_1+2=P_2,\quad p_2+2=P_4,\quad p_3+2=P_4'.$$

2. Some preliminary lemmas. In this paper we follow the notation of Tolev [16] as closely as possible. For the convenience of the reader, we recall some of it here.

Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. Let $A\geq 10^4$ denote a constant. The constants in O-terms and $\ll$-symbols are absolute or depend only on A. Let N denote a sufficiently large integer and $X=N^{1/2}$, $Q=(\log X)^{10^3A}$. The letter p, with or without subscripts, is reserved for primes. Boldface letters denote vectors of dimension three. As usual, $\mu(n)$, $\varphi(n)$, $\tau(n)$, $\nu_2(n)$ denote the Möbius function, Euler's function, the number of divisors of n and the total number of prime factors of n respectively, and $\tau_k(n)$ denotes the number of solutions of the equation $m_1\cdots m_k=n$ in positive integers $m_1,\dots,m_k$,

$\tau_2(n) = \tau(n)$. By $(m_1, \dots, m_k)$ we denote the largest common divisor of $m_1, \dots, m_k$. If $p^l \mid m$ but $p^{l+1} \nmid m$ then we write $p^l \parallel m$. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. We denote by $\sum_{x(q)}$ and $\sum_{x(q)*}$ sums with x running over a complete system and a reduced system of residues modulo q respectively. By $\left(\frac{l}{p}\right)$ we denote the Legendre symbol. We use $\mathbb{N}$ to denote the set of positive integers. For $\mathbf{k} = \{k_1, k_2, k_3\} \in \mathbb{N}^3$ and $\mathbf{l} = \{l_1, l_2, l_3\} \in \mathbb{N}^3$, define $\mathbf{kl} = \{k_1l_1, k_2l_2, k_3l_3\}$. For an arithmetic function f we define $f(\mathbf{k}) = f(k_1)f(k_2)f(k_3)$. For a set S, we denote its cardinality by $|S|$. Set

$$S_k(q,a) = \frac{\varphi((k,q))}{\varphi(q)} \sum_{\substack{x(q)* \\ x+2\equiv 0 \, (\mathrm{mod}\, (k,q))}} e_q(ax^2),$$

$$S_{\mathbf{k}}(q,a) = \prod_{j=1}^{3} S_{k_j}(q,a), \qquad \mathbf{k} = \{k_1, k_2, k_3\} \in \mathbb{N}^3,$$

$$t(q;n;\mathbf{k}) = \sum_{a(q)*} S_{\mathbf{k}}(q,a) e_q(-an),$$

$$\mathfrak{S}(n;Q;\mathbf{k}) = 8 \prod_{3\le p<Q} (1 + t(p;n;\mathbf{k})),$$

$$I(n;\mathbf{k}) = \sum_{\substack{p_1^2+p_2^2+p_3^2=n \\ p_j+2\equiv 0 \, (\mathrm{mod}\, k_j) \\ j=1,2,3}} \log \mathbf{p},$$

$$\mathfrak{R}(n;Q;\mathbf{k}) = I(n;\mathbf{k}) - \frac{\pi}{4} n^{1/2} \frac{\mathfrak{S}(n;Q;\mathbf{k})}{\varphi(\mathbf{k})},$$

$$h_0(p) = \begin{cases} \dfrac{\left(\frac{-n}{p}\right)p^2 + \left(3\left(\frac{n}{p}\right) + 3\left(\frac{-1}{p}\right)\right)p + 1}{(p-1)^3}, & p \nmid n, \\ \dfrac{-3\left(\frac{-1}{p}\right)p - 1}{(p-1)^2}, & p \mid n, \end{cases}$$

$$h_1(p) = \begin{cases} \dfrac{\left(-2\left(\frac{n-4}{p}\right) - \left(\frac{-1}{p}\right)\right)p - 1}{(p-1)^2}, & p \nmid (n-4), \\ \dfrac{\left(\frac{-1}{p}\right)p + 1}{p-1}, & p \mid (n-4), \end{cases}$$

$$h_2(p) = \begin{cases} \dfrac{\left(\frac{n-8}{p}\right)p + 1}{p-1}, & p \nmid (n-8), \\ -1, & p \mid (n-8), \end{cases}$$

$$h_3(p) = \begin{cases} -1, & p \nmid (n-12), \\ p-1, & p \mid (n-12). \end{cases}$$

LEMMA 1 ([16]). *For* $\mathbf{k} \in \mathbb{N}^3$ *with square-free odd components, the function* $t(q; n; \mathbf{k})$ *is multiplicative with respect to* q. *We have*

$$t(2^l; n; \mathbf{k}) = \begin{cases} 1, & l = 1, \\ 2, & l = 2, \\ 3, & l = 3, \\ 0, & l > 3. \end{cases}$$

For $p > 2$ *we have*

$$t(p^l; n; \mathbf{k}) = \begin{cases} h_j(p), & p^j \parallel k_1 k_2 k_3 \text{ and } l = 1, \\ 0, & l > 1. \end{cases}$$

LEMMA 2 ([16]). *Put*

$$K_1 = K_2 = X^{1/2} (\log X)^{-2 \cdot 10^4 A}, \qquad K_3 = X^{1/3} (\log X)^{-2 \cdot 10^4 A}$$

and let $\beta_j(k)$, $j = 1, 2, 3$, *denote complex numbers such that*

$$\beta_j(k) = 0 \quad \text{if } 2 \mid k \text{ or } \mu(k) = 0 \text{ or } k > K_j,$$
$$|\beta_j(k)| \le \tau_3(k).$$

Then

$$\sum_n{}^{*} \Big| \sum_{\substack{k_j \le K_j \\ j = 1,2,3}} \beta_1(k_1) \beta_2(k_2) \beta_3(k_3) \mathfrak{R}(n; Q; \mathbf{k}) \Big| \ll X^3 \log^{-A} X,$$

where $\sum_n^*$ *means that the summation is taken over the integers* n *satisfying*

$$N/2 \le n \le N, \quad n \equiv 3 \pmod{24} \quad \text{and} \quad n \not\equiv 0 \pmod 5.$$

LEMMA 3 ([12]). *Suppose that* $\phi(n_1, n_2, n_3)$ *is a function defined on* $\mathbb{N}^3$ *such that for any* $\{n_1, n_2, n_3\}, \{l_1, l_2, l_3\} \in \mathbb{N}^3$ *satisfying* $(n_1 n_2 n_3, l_1 l_2 l_3) = 1$ *we have* $\phi(n_1 l_1, n_2 l_2, n_3 l_3) = \phi(n_1, n_2, n_3) \phi(l_1, l_2, l_3)$. *Then the function*

$$\Phi(n) = \sum_{d_1, d_2, d_3 \mid n} \phi(d_1, d_2, d_3)$$

is multiplicative.

For fixed $D \ge 1$ we define *Rosser's weights* $\lambda^{\pm}(d)$ of order D as follows: for $d = p_1 \cdots p_r$ with $p_1 > \cdots > p_r$, let

$$\lambda^{+}(d) = \begin{cases} (-1)^r & \text{if } p_1 \cdots p_{2l} p_{2l+1}^3 < D \text{ whenever } 0 \le l \le (r-1)/2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\lambda^{-}(d) = \begin{cases} (-1)^r & \text{if } p_1 \cdots p_{2l} p_{2l}^3 < D \text{ whenever } 1 \le l \le r/2, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, put $\lambda^{\pm}(1) = 1$ and $\lambda^{\pm}(d) = 0$ if d is not square-free.

LEMMA 4 ([10, 11]). *Let $\mathscr{P}$ denote a set of primes and set*

$$P(z) = \prod_{\substack{p<z\\ p\in\mathscr{P}}} p.$$

Then for Rosser's weights $\lambda^{\pm}(d)$ of order D, any integer $n \ge 1$ and real number $z \ge 2$ we have

$$\sum_{d|(n,P(z))} \lambda^{-}(d) \le \sum_{d|(n,P(z))} \mu(d) \le \sum_{d|(n,P(z))} \lambda^{+}(d). \tag{2.1}$$

Moreover, for any multiplicative function ω satisfying

$$\begin{cases} 0 < \omega(p) < p & \text{if } p \in \mathscr{P},\\ \omega(p) = 0 & \text{if } p \notin \mathscr{P},\end{cases}$$

and

$$\prod_{w_1 \le p < w_2} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \le \frac{\log w_2}{\log w_1}\left(1 + \frac{L}{\log w_1}\right)$$

(*for all $2 \le w_1 < w_2$, where L is a positive constant*), *we have*

$$V(z) \ge \sum_{d|P(z)} \lambda^{-}(d)\,\frac{\omega(d)}{d} \ge V(z)(f(s) + O(e^{\sqrt{L}-s}\log^{-1/3} D)) \tag{2.2}$$

for $2 \le z \le D^{1/2}$, and

$$V(z) \le \sum_{d|P(z)} \lambda^{+}(d)\,\frac{\omega(d)}{d} \le V(z)(F(s) + O(e^{\sqrt{L}-s}\log^{-1/3} D)) \tag{2.3}$$

for $2 \le z \le D$, where

$$V(z) = \prod_{p<z}\left(1 - \frac{\omega(p)}{p}\right), \qquad s = \frac{\log D}{\log z},$$

and $f(s)$ and $F(s)$ denote the classical functions in the linear sieve.

LEMMA 5 ([5, 6]). *For the functions $f(s)$ and $F(s)$ we have*

$$sf(s) = 2e^{\gamma}\log(s-1), \qquad 2 \le s \le 4;$$

$$sf(s) = 2e^{\gamma}\left(\log(s-1) + \int_2^{s-2} \frac{\log(t-1)}{t}\log\frac{s-1}{t+1}\,dt\right), \qquad 4 \le s \le 6;$$

$$sF(s) = 2e^{\gamma}, \qquad 1 \le s \le 3;$$

$$sF(s) = 2e^{\gamma}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t}\,dt\right), \qquad 3 \le s \le 5;$$

$$sF(s) = 2e^{\gamma}\Bigg(1 + \int_{2}^{s-1} \frac{\log(t-1)}{t}\,dt + \int_{2}^{s-3} \frac{\log(t-1)}{t}\,dt \int_{t+2}^{s-1} \log\frac{u-1}{t+1}\,\frac{du}{u}\Bigg), \qquad 5 \le s \le 7,$$

where $\gamma = 0.577\ldots$ denotes Euler's constant.

3. Some propositions. The following propositions play a central role in the proof of the theorems.

Proposition 1. *Denote by $\mathcal{K}$ the set of integers n for which the equation $n = p_1^2 + p_2^2 + p_3^2$ is solvable in primes p_1, p_2, p_3 such that*

$$p_1 + 2 = P_4, \quad p_2 + 2 = P_4', \quad p_3 + 2 = P_5,$$

and set

$$\mathcal{F} = \{N/2 \le n \le N : n \equiv 3 \pmod{24},\ n \not\equiv 0 \pmod{5}\} \setminus \mathcal{K}.$$

Let $\mathcal{Y}(N)$ denote the cardinality of $\mathcal{F}$. Then for any $B > 0$ we have

$$\mathcal{Y}(N) \ll N \log^{-B} N.$$

Proposition 2. *Denote by $\mathcal{K}_0$ the set of integers n for which the equation $n = p_1 + p_2$ is solvable in primes p_1, p_2 such that*

$$p_1 + 2 = P_3, \quad p_2 + 2 = P_5,$$

and set

$$\mathcal{F}_0 = \{N/2 \le n \le N : n \equiv 4 \pmod{6}\} \setminus \mathcal{K}_0.$$

Let $\mathcal{Y}_0(N)$ denote the cardinality of $\mathcal{F}_0$. Then for any $B > 0$ we have

$$\mathcal{Y}_0(N) \ll N \log^{-B} N.$$

Proposition 2$'$. *Denote by $\mathcal{K}_1$ the set of integers n for which the equation $n = p_1 + p_2$ is solvable in primes p_1, p_2 such that*

$$p_1 + 2 = P_4, \quad p_2 + 2 = P_4',$$

and set

$$\mathcal{F}_1 = \{N/2 \le n \le N : n \equiv 4 \pmod{6}\} \setminus \mathcal{K}_1.$$

Let $\mathcal{Y}_1(N)$ denote the cardinality of $\mathcal{F}_1$. Then for any $B > 0$ we have

$$\mathcal{Y}_1(N) \ll N \log^{-B} N.$$

4. Proof of the propositions. In this paper we present only the proof of Proposition 1. By the Proposition in [15] and similar arguments, Propositions 2 and 2$'$ follow easily. In the proof of Proposition 1 we adopt the following notation:

$$Q_0 = \log^{3/5} X, \qquad D_0 = \exp(\log^{3/5} X),$$

$$D_1 = D_2 = X^{1/2} \exp(-4\log^{3/5} X), \qquad D_3 = X^{1/3} \exp(-4\log^{3/5} X),$$

$$w_1 = w_2 = D_1^{1/5}, \qquad w_3 = D_3^{1/6}, \qquad z_1 = z_2 = D_1^{4/5}, \qquad z_3 = D_3^{5/6},$$

$$\theta_1 = \theta_2 = \frac{1}{2.498}, \qquad \theta_3 = \frac{1}{2.398}, \qquad s_1 = s_2 = 5, \qquad s_3 = 6,$$

$$\mathfrak{R} = \{p : p \ge 11,\, p \nmid (n-4)\} \cup \{p : p \ge 11,\, p \mid (n-4),\, p \equiv 1 \pmod 4\},$$

$$\mathcal{B}_0 = \prod_{3 \le p < Q_0} p, \qquad \mathcal{P}_0 = \prod_{\substack{Q_0 \le p < Q \\ p \in \mathfrak{R}}} p,$$

$$\mathcal{P}_j = \prod_{Q \le p < w_j} p, \qquad Q_j = \mathcal{B}_0 \mathcal{P}_0 \mathcal{P}_j, \qquad P(w_j) = \prod_{p < w_j} p, \qquad j = 1, 2, 3,$$

$$g_j'(x) = 1 - \frac{\log x}{\log z_j}, \qquad g_j(x) = \sum_{\substack{w_j \le p < z_j \\ p \mid x}} g_j'(p), \qquad j = 1, 2, 3,$$

$\lambda_j^{\pm}(d)$ Rosser's weights of order D_j, $\quad j = 0, 1, 2, 3,$

$\lambda_j^{\pm(p)}(d)$ Rosser's weights of order D_j/p, $\quad w_j \le p < z_j$, $j = 1, 2, 3,$

$$\Phi_j = \sum_{k \mid (p_j + 2, \mathcal{B}_0)} \mu(k), \qquad \Psi_j = \sum_{l \mid (p_j + 2, \mathcal{P}_0)} \mu(l), \qquad \Lambda_j = \sum_{m \mid (p_j + 2, \mathcal{P}_j)} \mu(m),$$

$$\Psi_j^{\pm} = \sum_{k \mid (p_j + 2, \mathcal{P}_0)} \lambda_0^{\pm}(k), \qquad \Lambda_j^{\pm} = \sum_{l \mid (p_j + 2, \mathcal{P}_j)} \lambda_j^{\pm}(l), \qquad j = 1, 2, 3,$$

$$\mathcal{F}^* = \{n : n \in \mathcal{F},\, \nu_2(n-4) \le A \log\log X\}.$$

For the proof of Proposition 1 we consider the sum

$$\Gamma = \sum_{n \in \mathcal{F}^*} \sum_{\substack{p_1^2 + p_2^2 + p_3^2 = n \\ (p_j + 2, Q_j) = 1 \\ j = 1,2,3}} (\log \mathbf{p}) \Big(1 - \sum_{j=1}^{3} \theta_j g_j(p_j + 2) \Big) \tag{4.1}$$

$$= \sum_{n \in \mathcal{F}^*} \sum_{\substack{p_1^2 + p_2^2 + p_3^2 = n \\ (p_j + 2, Q_j) = 1 \\ j = 1,2,3}} \log \mathbf{p} - \sum_{j=1}^{3} \theta_j \sum_{n \in \mathcal{F}^*} \sum_{\substack{p_1^2 + p_2^2 + p_3^2 = n \\ (p_j + 2, Q_j) = 1 \\ j = 1,2,3}} (\log \mathbf{p}) g_j(p_j + 2)$$

$$= \Gamma^{(0)} - \sum_{j=1}^{3} \theta_j \Gamma_j^{(1)} = \Gamma^{(0)} - \Gamma^{(1)}.$$

A) *The upper bound for Γ.* Write

$$\Gamma = \sum_{n\in\mathcal{F}^*} w(n), \qquad w(n) = \sum_{\substack{p_1^2+p_2^2+p_3^2=n\\(p_j+2,Q_j)=1\\j=1,2,3}} (\log \mathbf{p})\Big(1-\sum_{j=1}^{3}\theta_j g_j(p_j+2)\Big).$$

Let $n \in \mathcal{F}^*$ give a positive contribution to Γ. Then we have

$$p_1^2+p_2^2+p_3^2 = n, \tag{4.2}$$

$$(p_j+2, Q_j) = 1, \qquad j = 1,2,3, \tag{4.3}$$

$$\theta_j g_j(p_j+2) < 1, \qquad j = 1,2,3, \tag{4.4}$$

for some primes p_1, p_2, p_3.

The contribution from those representations satisfying (4.2)–(4.4) with some p_j+2 non-square-free is

$$\begin{aligned}
&\ll \sum_{w_3\le p<X^{1/2}} \sum_{\substack{p_3\le X\\ p_3\equiv -2\,(\mathrm{mod}\,p^2)}} \sum_{p_1^2+p_2^2\le N-p_3^2} \log^3 X \\
&\ll N \sum_{w_3\le p<X^{1/2}} \sum_{\substack{p_3\le X\\ p_3\equiv -2\,(\mathrm{mod}\,p^2)}} \log^3 X \\
&\ll N \sum_{w_3\le p<X^{1/2}} \Big(\frac{X}{p^2}+1\Big)\log^3 X \\
&\ll (X^3 w_3^{-1} + X^{5/2})\log^3 X \ll X^{59/20}.
\end{aligned} \tag{4.5}$$

For the remaining representations satisfying (4.2)–(4.4), p_j+2 is square-free for $j = 1,2,3$. If $(p_j+2, P(w_j)) = 1$ for $j = 1,2,3$, then we have

$$\nu_2(p_j+2) = \sum_{\substack{p\mid(p_j+2)\\ p\ge w_j}} 1, \qquad j = 1,2,3. \tag{4.6}$$

By (4.4) we have

$$\begin{aligned}
&\sum_{\substack{p\mid(p_j+2)\\ w_j\le p<z_j}} \Big(1-\frac{\log p}{\log z_j}\Big) < \frac{1}{\theta_j}, \qquad j = 1,2,3,\\
&\sum_{\substack{p\mid(p_j+2)\\ p\ge w_j}} \Big(1-\frac{\log p}{\log z_j}\Big) < \frac{1}{\theta_j}, \qquad j = 1,2,3,\\
&\sum_{\substack{p\mid(p_j+2)\\ p\ge w_j}} 1 < \frac{1}{\theta_j} + \frac{\log(p_j+2)}{\log z_j}, \qquad j = 1,2,3.
\end{aligned} \tag{4.7}$$

From (4.6)–(4.7) we get

$$\nu_2(p_j+2) \le \begin{cases} 4, & j=1,2, \\ 5, & j=3. \end{cases} \tag{4.8}$$

Now (4.2) and (4.8) contradict the fact that $n \in \mathcal{F}^*$, so we must have $(p_j+2, P(w_j)) > 1$ for some j. Without loss of generality we assume that

$$(p_1+2, P(w_1)) > 1. \tag{4.9}$$

If $p_1 = 2$ then

$$w(n) \le \sum_{m_1^2+m_2^2+4=n} \log^3 X. \tag{4.10}$$

If $p_1 > 2$ then from (4.3) and (4.9) we deduce that p_1+2 has a prime factor $p > 2$ such that $p \mid (n-4)$ and $p \equiv 3 \pmod 4$. Hence $p_2^2 + p_3^2 \equiv 0 \pmod p$, which implies that $p_2 = p_3 = p$, and we have

$$w(n) \le \sum_{p \mid (n-4)} \log^3 X. \tag{4.11}$$

From (4.5) and (4.10)–(4.11) we obtain

$$\begin{aligned} \Gamma &\ll X^{59/20} + \Big(\sum_{m_1^2+m_2^2+4 \le N} 1 + \sum_{n \le N} \tau(n-4) \Big) \log^3 X \\ &\ll X^{59/20} + X^2 \log^4 X \ll X^{59/20}. \end{aligned} \tag{4.12}$$

B) *The lower bound for Γ.* In this part we give a lower bound for Γ by applying the vector sieve in [1].

- *The lower bound for $\Gamma^{(0)}$.* By (2.1) and the inequality

$$\begin{aligned} \Psi_1\Psi_2\Psi_3\Lambda_1\Lambda_2\Lambda_3 \ge{}& \Psi_1^-\Psi_2^+\Psi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^+ + \Psi_1^+\Psi_2^-\Psi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^+ \\ &+ \Psi_1^+\Psi_2^+\Psi_3^-\Lambda_1^+\Lambda_2^+\Lambda_3^+ + \Psi_1^+\Psi_2^+\Psi_3^+\Lambda_1^-\Lambda_2^+\Lambda_3^+ \\ &+ \Psi_1^+\Psi_2^+\Psi_3^+\Lambda_1^+\Lambda_2^-\Lambda_3^+ + \Psi_1^+\Psi_2^+\Psi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^- \\ &- 5\Psi_1^+\Psi_2^+\Psi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^+ \end{aligned}$$

of [16], we get

$$\begin{aligned} \Gamma^{(0)} &= \sum_{n\in\mathcal{F}^*} \sum_{p_1^2+p_2^2+p_3^2=n} (\log \mathbf{p}) \Phi_1\Phi_2\Phi_3\Psi_1\Psi_2\Psi_3\Lambda_1\Lambda_2\Lambda_3 \\ &\ge \sum_{j=1}^{6} \Gamma_j^{(0)} - 5\Gamma_7^{(0)}, \end{aligned} \tag{4.13}$$

where

$$\Gamma_1^{(0)} = \sum_{n\in\mathcal{F}^*} \sum_{p_1^2+p_2^2+p_3^2=n} (\log \mathbf{p}) \Phi_1\Phi_2\Phi_3\Psi_1^-\Psi_2^+\Psi_3^+\Lambda_1^+\Lambda_2^+\Lambda_3^+,$$

and the definition of the other sums $\Gamma_j^{(0)}$ is clear. Let

$$\gamma_1(k) = \sum_{\substack{l|\mathcal{B}_0,\, m|\mathcal{P}_0,\, d|\mathcal{P}_1 \\ dlm=k}} \mu(l)\lambda_0^-(m)\lambda_1^+(d),$$

$$\gamma_j(k) = \sum_{\substack{l|\mathcal{B}_0,\, m|\mathcal{P}_0,\, d|\mathcal{P}_1 \\ dlm=k}} \mu(l)\lambda_0^+(m)\lambda_j^+(d), \qquad j = 2, 3.$$

Then by some routine arrangements we have

$$(4.14)\quad \begin{aligned} \Gamma_1^{(0)} &= \sum_{n\in\mathcal{F}^*} \sum_{\substack{l_j|\mathcal{B}_0,\, m_j|\mathcal{P}_0,\, d_j|\mathcal{P}_1 \\ j=1,2,3}} \mu(\mathbf{l})\lambda_0^-(m_1)\lambda_0^+(m_2)\lambda_0^+(m_3) \\ &\qquad \times \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)I(n;\mathbf{lmd}) \\ &= \sum_{n\in\mathcal{F}^*} \sum_{\substack{k_j\le\mathcal{B}_0 D_0 D_j \\ j=1,2,3}} \gamma_1(k_1)\gamma_2(k_2)\gamma_3(k_3)I(n;\mathbf{k}) \\ &= \frac{\pi}{4} \sum_{n\in\mathcal{F}^*} \sum_{\substack{k_j\le\mathcal{B}_0 D_0 D_j \\ j=1,2,3}} \gamma_1(k_1)\gamma_2(k_2)\gamma_3(k_3)n^{1/2}\, \frac{\mathfrak{S}(n;Q;\mathbf{k})}{\varphi(\mathbf{k})} \\ &\qquad + \sum_{n\in\mathcal{F}^*} \sum_{\substack{k_j\le\mathcal{B}_0 D_0 D_j \\ j=1,2,3}} \gamma_1(k_1)\gamma_2(k_2)\gamma_3(k_3)\mathfrak{R}(n;Q;\mathbf{k}) \\ &= \Gamma_{11}^{(0)} + \Gamma_{12}^{(0)}. \end{aligned}$$

Now Lemma 2 implies that

$$(4.15)\qquad \Gamma_{12}^{(0)} \ll X^3 \log^{-A} X.$$

By Lemma 1, for $l_j \mid \mathcal{B}_0$, $m_j \mid \mathcal{P}_0$, $d_j \mid \mathcal{P}_j$, $j = 1, 2, 3$, we have

$$(4.16)\qquad \mathfrak{S}(n;Q;\mathbf{lmd}) = 8 \prod_{3\le p<Q_0} (1 + t(p;n;\mathbf{l})) \prod_{Q_0\le p<Q} (1 + t(p;n;\mathbf{m})).$$

By (4.16) we get

$$(4.17)\qquad \Gamma_{11}^{(0)} = 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}^-(n)\mathcal{G}_1^+\mathcal{G}_2^+\mathcal{G}_3^+,$$

where

$$\mathcal{J}(n) = \sum_{\substack{l_j|\mathcal{B}_0 \\ j=1,2,3}} \frac{\mu(\mathbf{l})}{\varphi(\mathbf{l})} \prod_{3\le p<Q_0} (1 + t(p;n;\mathbf{l})),$$

$$\mathcal{H}^{\pm}(n) = \sum_{\substack{m_j|\mathcal{P}_0\\ j=1,2,3}} \frac{\lambda_0^{\pm}(m_1)\lambda_0^{+}(m_2)\lambda_0^{+}(m_3)}{\varphi(\mathbf{m})} \prod_{Q_0\le p<Q} (1+t(p;n;\mathbf{m})),$$

$$\mathcal{G}_j^{\pm} = \sum_{d|\mathcal{P}_j} \frac{\lambda_j^{\pm}(d)}{\varphi(d)}, \qquad j=1,2,3.$$

By Lemma 3 it is easy to show that

$$\mathcal{J}(n) = \prod_{3\le p<Q_0} \mathcal{V}_p(n),$$

where

$$\mathcal{V}_p(n) = \sum_{l_1,l_2,l_3|p} \frac{\mu(\mathbf{l})}{\varphi(\mathbf{l})}(1+t(p;n;\mathbf{l})).$$

By (3.15)–(3.18) of [16], for $n\in\mathcal{F}^*$ we have

$$\mathcal{H}^{\pm}(n) = \mathcal{H}_0(n) + O(\log^{-2A} X), \tag{4.18}$$

$$(\log\log X)^{-9} \ll \mathcal{J}(n) \ll (\log\log X)^{9}, \tag{4.19}$$

$$(\log\log X)^{-14} \ll \mathcal{H}_0(n) \ll (\log\log X)^{14}, \tag{4.20}$$

$$\mathcal{G}_j^{\pm} \ll \log X, \qquad j=1,2,3, \tag{4.21}$$

uniformly, where

$$\mathcal{H}_0(n) = \prod_{\substack{Q_0\le p<Q\\ (p,\mathcal{P}_0)=1}} (1+h_0(p)) \prod_{p|\mathcal{P}_0} \mathcal{V}_p(n).$$

By (4.18)–(4.21) we find that

$$\Gamma_{11}^{(0)} = 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{G}_1^{+}\mathcal{G}_2^{+}\mathcal{G}_3^{+} + O(X^3\log^{-A} X). \tag{4.22}$$

By (4.14)–(4.15) and (4.22) we get

$$\Gamma_{1}^{(0)} = 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{G}_1^{+}\mathcal{G}_2^{+}\mathcal{G}_3^{+} + O(X^3\log^{-A} X). \tag{4.23}$$

In a similar manner we obtain

$$\Gamma_{j}^{(0)} = 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{G}_1^{+}\mathcal{G}_2^{+}\mathcal{G}_3^{+} + O(X^3\log^{-A} X), \quad j=2,3,7, \tag{4.24}$$

$$\Gamma_{4}^{(0)} = 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{G}_1^{-}\mathcal{G}_2^{+}\mathcal{G}_3^{+} + O(X^3\log^{-A} X), \tag{4.25}$$

$$\Gamma_{5}^{(0)} = 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{G}_1^{+}\mathcal{G}_2^{-}\mathcal{G}_3^{+} + O(X^3\log^{-A} X), \tag{4.26}$$

$$\Gamma_6^{(0)} = 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{G}_1^+\mathcal{G}_2^+\mathcal{G}_3^- + O(X^3\log^{-A} X). \tag{4.27}$$

Now, (4.23)–(4.27) and (4.13) imply that

$$\Gamma^{(0)} \ge 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{G} + O(X^3\log^{-A} X), \tag{4.28}$$

where

$$\mathcal{G} = \mathcal{G}_1^-\mathcal{G}_2^+\mathcal{G}_3^+ + \mathcal{G}_1^+\mathcal{G}_2^-\mathcal{G}_3^+ + \mathcal{G}_1^+\mathcal{G}_2^+\mathcal{G}_3^- - 2\mathcal{G}_1^+\mathcal{G}_2^+\mathcal{G}_3^+. \tag{4.29}$$

By (2.2)–(2.3) in Lemma 4, we have

$$\mathcal{W}_j \le \mathcal{G}_j^+ \le \mathcal{W}_j(F(s_j) + O(\log^{-1/3} D_j)), \quad j = 1,2,3, \tag{4.30}$$

$$\mathcal{W}_j \ge \mathcal{G}_j^- \ge \mathcal{W}_j(f(s_j) + O(\log^{-1/3} D_j)), \quad j = 1,2,3, \tag{4.31}$$

where

$$\mathcal{W}_j = \mathcal{W}(w_j) = \prod_{Q\le p<w_j}\left(1 - \frac{1}{p-1}\right).$$

Write $\mathcal{W} = \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3$. Then by (4.29)–(4.31) we get

$$\begin{aligned}\mathcal{G} &= 2(\mathcal{G}_1^- - \mathcal{G}_1^+)\mathcal{G}_2^+\mathcal{G}_3^+ + \mathcal{G}_1^+\mathcal{G}_2^+\mathcal{G}_3^- \\ &\ge (2f(s_1)F(s_2)F(s_3) - 2F(s_1)F(s_2)F(s_3) + f(s_3) + o(1))\mathcal{W} \\ &\ge 0.99635\mathcal{W},\end{aligned} \tag{4.32}$$

where Lemma 5 and numerical integration are employed. By (4.28) and (4.32) we obtain

$$\Gamma^{(0)} \ge 0.99635\cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X). \tag{4.33}$$

- *The upper bound for $\Gamma^{(1)}$.* Write

$$\gamma_1^*(k) = \sum_{\substack{l|\mathcal{B}_0,\, m|\mathcal{P}_0,\, d|\mathcal{P}_1\\ w_1\le p<z_1,\, dlmp=k}} \mu(l)\lambda_0^+(m)\lambda_1^{+(p)}(d)g_1'(p).$$

By (2.1) we have

$$\begin{aligned}\Gamma_1^{(1)} &= \sum_{n\in\mathcal{F}^*} \sum_{\substack{p_1^2+p_2^2+p_3^2=n\\ (p_j+2,Q_j)=1\\ j=1,2,3}} (\log \mathbf{p})g_1(p_1+2) \\ &= \sum_{n\in\mathcal{F}^*}\sum_{w_1\le p<z_1} g_1'(p) \sum_{\substack{p_1^2+p_2^2+p_3^2=n,\, p_1+2\equiv 0 \,(\mathrm{mod}\, p)\\ (p_j+2,Q_j)=1\\ j=1,2,3}} \log \mathbf{p}\end{aligned} \tag{4.34}$$

$$= \sum_{n\in\mathcal{F}^*} \sum_{w_1\le p<z_1} g_1'(p) \sum_{\substack{p_1^2+p_2^2+p_3^2=n \\ p_1+2\equiv 0 \,(\mathrm{mod}\, p)}} \log \mathbf{p}$$
$$\times \varPhi_1\varPhi_2\varPhi_3\varPsi_1\varPsi_2\varPsi_3\varLambda_1\varLambda_2\varLambda_3$$
$$\le \sum_{n\in\mathcal{F}^*} \sum_{w_1\le p<z_1} g_1'(p) \sum_{\substack{p_1^2+p_2^2+p_3^2=n \\ p_1+2\equiv 0 \,(\mathrm{mod}\, p)}} \log \mathbf{p}$$
$$\times \varPhi_1\varPhi_2\varPhi_3\varPsi_1^+\varPsi_2^+\varPsi_3^+\varLambda_1^{+(p)}\varLambda_2^+\varLambda_3^+$$
$$= \sum_{n\in\mathcal{F}^*} \sum_{\substack{k_j\le\mathcal{B}_0D_0D_j \\ j=1,2,3}} \gamma_1^*(k_1)\gamma_2(k_2)\gamma_3(k_3)I(n;\mathbf{k})$$
$$= \frac{\pi}{4}\sum_{n\in\mathcal{F}^*} \sum_{\substack{k_j\le\mathcal{B}_0D_0D_j \\ j=1,2,3}} \gamma_1^*(k_1)\gamma_2(k_2)\gamma_3(k_3)n^{1/2}\,\frac{\mathfrak{S}(n;Q;\mathbf{k})}{\varphi(\mathbf{k})}$$
$$+ \sum_{n\in\mathcal{F}^*} \sum_{\substack{k_j\le\mathcal{B}_0D_0D_j \\ j=1,2,3}} \gamma_1^*(k_1)\gamma_2(k_2)\gamma_3(k_3)\mathfrak{R}(n;Q;\mathbf{k})$$
$$= \varGamma_{11}^{(1)} + \varGamma_{12}^{(1)}.$$

By Lemma 2 we find that

$$\varGamma_{12}^{(1)} \ll X^3\log^{-A} X. \tag{4.35}$$

Similar to $\varGamma_{11}^{(0)}$, by (4.16) we obtain

$$\varGamma_{11}^{(1)} = 2\pi\sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}^+(n)\mathcal{G}_1^+(g_1')\mathcal{G}_2^+\mathcal{G}_3^+, \tag{4.36}$$

where

$$\mathcal{G}_j^+(g_j') = \sum_{w_j\le p<z_j}\frac{g_j'(p)}{p-1}\sum_{d\mid\mathcal{P}_j}\frac{\lambda_j^{+(p)}(d)}{\varphi(d)}, \qquad j=1,2,3. \tag{4.37}$$

By (2.3) we have

$$\sum_{d\mid\mathcal{P}_j}\frac{\lambda_j^{+(p)}(d)}{\varphi(d)} \le \mathcal{W}_j\biggl(F\biggl(\frac{\log D_jp^{-1}}{\log w_j}\biggr) + O(\log^{-1/3} D_j)\biggr), \quad j=1,2,3. \tag{4.38}$$

By (4.37)–(4.38), the prime number theorem and summation by parts we find that

$$\mathcal{G}_j^+(g_j') \le (1+o(1))\mathcal{W}_j\int_{1/s_j}^{1-1/s_j}\biggl(1-\frac{s_j}{s_j-1}\,t\biggr)\frac{F(s_j(1-t))}{t}\,dt, \tag{4.39}$$
$$j=1,2,3.$$

By (4.18)–(4.21), (4.30) and (4.39) we get

$$\Gamma_{11}^{(1)} \le (1+o(1))C_1 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X), \tag{4.40}$$

where

$$C_1 = F(5)F(6)\int_{1/5}^{4/5}\left(1-\frac{5t}{4}\right)\frac{F(5(1-t))}{t}\,dt. \tag{4.41}$$

By (4.35), (4.40)–(4.41), Lemma 5 and numerical integration, we obtain

$$\Gamma_{1}^{(1)} \le 0.77133 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X). \tag{4.42}$$

By the same arguments we get

$$\begin{aligned}\Gamma_{2}^{(1)} &\le (1+o(1))C_1 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X)\\ &\le 0.77133 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X),\end{aligned} \tag{4.43}$$

$$\begin{aligned}\Gamma_{3}^{(1)} &\le (1+o(1))C_3 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X)\\ &\le 0.89182 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X),\end{aligned} \tag{4.44}$$

where

$$C_3 = F(5)F(5)\int_{1/6}^{5/6}\left(1-\frac{6t}{5}\right)\frac{F(6(1-t))}{t}\,dt.$$

By (4.42)–(4.44) we find that

$$\begin{aligned}\Gamma^{(1)} &= \sum_{j=1}^{3}\theta_j\Gamma_j^{(1)}\\ &\le 0.98947 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X).\end{aligned} \tag{4.45}$$

By (4.1), (4.33) and (4.45) we get

$$\begin{aligned}\Gamma &= \Gamma^{(0)} - \Gamma^{(1)}\\ &\ge 0.006 \cdot 2\pi \sum_{n\in\mathcal{F}^*} n^{1/2}\mathcal{J}(n)\mathcal{H}_0(n)\mathcal{W} + O(X^3\log^{-A} X).\end{aligned} \tag{4.46}$$

C) *Proof of Proposition 1.* Upon comparing (4.12) and (4.46) we obtain

$$\mathcal{Y}^*(N) = \sum_{n\in\mathcal{F}^*} 1 \ll X^2 \log^{5-A} X, \tag{4.47}$$

where (4.19)–(4.20) and the bound

$$\mathcal{W} \gg \frac{\log^3 \log X}{\log^3 X},$$

a consequence of Mertens' product formula, have been used.

By (4.47) and the bound (see [7, Chapter 0])

$$\mathcal{Y}(N) - \mathcal{Y}^*(N) \ll X^2 (\log X)^{-A\log A - A - 1},$$

we get $\mathcal{Y}(N) \ll X^2 \log^{5-A} X$, and Proposition 1 follows.

5. Proof of the theorems. In this paper we present only the proof of Theorem 1. From Propositions 2 and 2′, Theorems 2 and 2′ follow by similar but simpler arguments (see [15] for the details). Let

$$\mathfrak{A} = \{p : p \le n^{1/2},\ p \equiv 11 \pmod{30},\ p+2 = P_2\},$$
$$\mathfrak{A}' = \{p : p \le n^{1/2},\ p \equiv 17 \pmod{30},\ p+2 = P_2\}.$$

By Chen's argument in [2], we have

$$|\mathfrak{A}| \gg n^{1/2} \log^{-2} n, \tag{5.1}$$

$$|\mathfrak{A}'| \gg n^{1/2} \log^{-2} n. \tag{5.2}$$

Case 1: $n \not\equiv 2 \pmod 5$. Let

$$\mathscr{A} = \{n - p_1^2 - p_2^2 : p_1, p_2 \in \mathfrak{A}\}, \qquad r'(k) = \sum_{\substack{p_1^2+p_2^2=k \\ p_1,p_2\in\mathfrak{A}}} 1, \qquad r(k) = \sum_{m_1^2+m_2^2=k} 1.$$

Then we have

$$\sum_{\substack{k\in\mathscr{A} \\ r'(k)>\log^5 n}} 1 \le \frac{1}{\log^5 n} \sum_{k\le n} r'(k) \le \frac{1}{\log^5 n} \sum_{k\le n} r(k) \ll \frac{n}{\log^5 n}. \tag{5.3}$$

By (5.1), (5.3) and Dirichlet's pigeon hole principle we know that $\mathscr{A}$ contains $\gg n \log^{-9} n$ distinct integers k satisfying $k \equiv 3 \pmod{24}$ and $k \not\equiv 0 \pmod 5$, and Theorem 1 follows from Proposition 1.

Case 2: $n \equiv 2 \pmod 5$. Letting

$$\mathscr{A}' = \{n - p_1^2 - p_2^2 : p_1 \in \mathfrak{A}, p_2 \in \mathfrak{A}'\},$$

and then proceeding as in Case 1, we get the proof of Theorem 1.

Acknowledgements. The authors would like to thank Professor D. I. Tolev for his assistance in the course of studying [16], and the referee for his/her patience and time in refereeing this paper.

This project was supported by the National Natural Science Foundation of China (grant no. 10171060, 10171076 and 10471104) .

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Received on 27.1.2009
and in revised form on 10.4.2009 (5923)