ACTA ARITHMETICA
168.2 (2015)

Another smallest part function related to Andrews' spt function

by

Alexander E. Patkowski (Centerville, MA)

1. Introduction and main results. In [A2], we find the identity of Andrews

$$(1.1)\qquad \sum_{n\geq 1}\frac{q^n}{(1-q^n)(q^n)_\infty} = \sum_{n\geq 1} np(n)q^n + \frac{1}{(q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}.$$

Here $p(n)$ is the number of partitions of n, and the last series on the right generates $N_2(n) = \sum_{m\in\mathbb{Z}} m^2 N(m,n)$, $N(m,n)$ being the number of partitions of n with rank m (see [A1]). The largest part minus the number of parts is defined to be the *rank*. The function $\mathrm{spt}(n)$ counts the number of smallest parts among integer partitions of n. For some other functions counting smallest parts among partitions see [P1]. Lastly, we have used the familiar notation $(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ (see [GR]).

In this note we find a spt-type function that is related to the generating function in (1.1) and falls into the same class of spt-type functions as the one offered in [P2]. However, this note differs from [P2] in that we will find the "crank companion" to create a "full" sptfunction related to Andrews' spt function. Here we are also appealing to relations to $\mathrm{spt}(n)$ modulo 2, whereas in [P2] we concentrated on relations to $\mathrm{spt}(n)$ modulo 3. Lastly, the partitions involved in this study are different, and deserve a separate study.

Let $M_2(n) = \sum_{m\in\mathbb{Z}} m^2 M(m,n)$, where $M(m,n)$ is the number of partitions of n with crank m (see [AG]).

2010 *Mathematics Subject Classification*: Primary 11P81; Secondary 11P83.
Key words and phrases: partitions, q-series, smallest parts function.

DOI: 10.4064/aa168-2-1

Theorem 1.1. *We have*

$$\sum_{n\ge1}\frac{q^n(q^{2n+1};q^2)_\infty}{(1-q^n)^2(q^{n+1})_\infty}=\frac{1}{(q^2;q^2)_\infty}\sum_{n\ge1}\frac{nq^n}{1-q^n}-\frac{1}{2}\sum_{n\ge1}N_2(n)q^{2n}, \tag{1.2}$$

$$\sum_{n\ge1}\frac{q^{n(n+1)/2}(q^{2n+1};q^2)_\infty}{(1-q^n)^2(q^{n+1})_\infty}=\frac{1}{(q^2;q^2)_\infty}\sum_{n\ge1}\frac{nq^n}{1-q^n}-\frac{1}{2}\sum_{n\ge1}M_2(n)q^{2n}. \tag{1.3}$$

For our next theorem, which is a number-theoretic interpretation of Theorem 1.1, we will use the following definitions. We define a triangular partition to be of the form $\delta_l=(l-1,l-2,\dots,1)$, $l\in\mathbb{N}$. Define the smallest part of a partition π to be $s(\pi)$, and the largest part to be $l(\pi)$. We will also consider the partition pair $\sigma=(\pi,\delta_i)$, where we set $i=s(\pi)$. The latter condition yields $s(\pi)-l(\delta_{i=s(\pi)})=s(\pi)-(s(\pi)-1)=1$. If we include δ_i in a partition, we are increasing its size by $\binom{i}{2}$ and including the component $q^{1+\cdots+i-1}$ in its generating function. This has the property that all parts from 1 to $i-1$ appear exactly once and are less than i.

Theorem 1.2. *Let* $\mathrm{spt}_o^+(n)$ *count the number of smallest parts among the integer partitions* π *of* n *where odd parts greater than* $2s(\pi)$ *do not occur. Let* $\mathrm{spt}_o^-(n)$ *count the number of smallest parts among the integer partitions* $\sigma=(\pi,\delta_{s(\pi)})$ *of* n *such that* π *is a partition where odd parts greater than* $2s(\pi)$ *do not occur. Define* $\mathrm{spt}_o(n):=\mathrm{spt}_o^+(n)-\mathrm{spt}_o^-(n)$. *Then* $\mathrm{spt}_o(2n)=\mathrm{spt}(n)$.

With the above definitions, we can write the generating function. We have

$$\sum_{n\ge1}\mathrm{spt}_o(n)q^n=\sum_{n\ge1}(q^n+2q^{2n}+3q^{3n}+\cdots)\frac{(q^{2n+1};q^2)_\infty}{(q^{n+1})_\infty}(1-q^{1+2+\cdots n-1}).$$

2. Proof of Theorems 1.1 and 1.2. The proofs require the methods used in [B, G, P2] and a few more observations. A pair of sequences (α_n,β_n) is known to be a *Bailey pair* with respect to a if

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(aq;q)_{n+r}(q;q)_{n-r}}. \tag{2.1}$$

The next result is Bailey's lemma [B].

Bailey's lemma. *If* (α_n,β_n) *form a Bailey pair with respect to* a *then*

$$\sum_{n=0}^{\infty}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n =\frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n}. \tag{2.2}$$

The following are known Bailey pairs (α_n, β_n) relative to $a = 1$:

$$\alpha_{2n+1} = 0, \tag{2.3}$$

$$\alpha_{2n} = (-1)^n q^{n(3n-1)}(1 + q^{2n}), \tag{2.4}$$

$$\beta_n = \frac{1}{(q)_n(q;q^2)_n} \tag{2.5}$$

(see [S, C(1)]), and

$$\alpha_{2n+1} = 0, \tag{2.6}$$

$$\alpha_{2n} = (-1)^n q^{n(n-1)}(1 + q^{2n}), \tag{2.7}$$

$$\beta_n = \frac{q^{n(n-1)/2}}{(q)_n(q;q^2)_n} \tag{2.8}$$

(see [S, C(5)]). In both pairs $\alpha_0 = 1$. Differentiating Bailey's lemma (putting $a = 1$) with respect to both variables ρ_1 and ρ_2 and setting each variable equal to 1 each time gives us (see [P2])

$$\sum_{n\geq 1}(q)_{n-1}^2\beta_n q^n = \alpha_0 \sum_{n\geq 1} \frac{nq^n}{1-q^n} + \sum_{n\geq 1} \frac{\alpha_n q^n}{(1-q^n)^2}. \tag{2.9}$$

Identity (1.2) follows from inserting the Bailey pair (2.3)–(2.5) into (2.9) and then multiplying through by $(q^2;q^2)_\infty^{-1}$. Identity (1.3) follows from inserting the Bailey pair (2.6)–(2.8) into (2.9) and then multiplying through by $(q^2;q^2)_\infty^{-1}$. This proves Theorem 1.1.

To get Theorem 1.2, we subtract (1.3) from (1.2), and note that $\mathrm{spt}(n) = \frac{1}{2}(M_2(n) - N_2(n))$, after observing that (see [G])

$$2\sum_{n\geq 1} np(n)q^n = \sum_{n\geq 1} M_2(n)q^n = \frac{2}{(q)_\infty}\sum_{n\geq 1}\frac{nq^n}{1-q^n}.$$

The result follows from equating the coefficients of q^{2n}.

3. More notes and concluding remarks. Naturally, it is of interest to investigate equations (1.2) and (1.3) individually. As we noted previously, the left side of (1.2) generates $\mathrm{spt}_o^+(n)$, and the left side of (1.3) generates $\mathrm{spt}_o^-(n)$.

THEOREM 3.1. *We have* $\mathrm{spt}_o^+(2n) \equiv \mathrm{spt}(n) \pmod 2$.

Proof. After noting that $\sigma(2n) = 3\sigma(n) - 2\sigma(n/2)$, $\sigma(2n) \equiv \sigma(n) \pmod 2$, and

$$\sum_{n\geq 1}\mathrm{spt}_o^+(n)q^n = \frac{1}{(q^2;q^2)_\infty}\sum_{n\geq 1}\sigma(n)q^n - \frac{1}{2}\sum_{n\geq 1} N_2(n)q^{2n},$$

we can take coefficients of q^{2n} in (1.3) to get

$$\mathrm{spt}_o^+(2n) = \sum_k p(k)\sigma(2(n-k)) - \tfrac{1}{2}N_2(n).$$

Hence, combining these, we compute

$$\begin{aligned}\mathrm{spt}_o^+(2n) &\equiv \sum_k p(k)\sigma(n-k) - \tfrac{1}{2}N_2(n) \pmod 2\\ &\equiv np(n) - \tfrac{1}{2}N_2(n) \pmod 2 \equiv \mathrm{spt}(n) \pmod 2. \ \blacksquare\end{aligned}$$

THEOREM 3.2. *We have* $\mathrm{spt}_o^-(2n) \equiv 0 \pmod 2$.

Proof. The computations are similar to Theorem 3.1. Using equation (1.3) we compute

$$\begin{aligned}\mathrm{spt}_o^-(2n) &\equiv \sum_k p(k)\sigma(n-k) - \tfrac{1}{2}M_2(n) \pmod 2\\ &\equiv np(n) - \tfrac{1}{2}M_2(n) \pmod 2 \equiv 0 \pmod 2.\end{aligned}$$

In the last line we have used $2np(n) = M_2(n)$. ∎

For an example illustrating Theorem 3.1, consider partitions of 4: (4), $(3,1)$, $(2,2)$, $(1,1,1,1)$. In a partition where odd parts greater than twice the smallest do not occur, we omit $(3,1)$. Hence $\mathrm{spt}_o^+(4) = 7$, and $\mathrm{spt}(2) = 3$ (counting smallest of (2) and $(1,1)$). Hence 2 divides $7 - 3 = 4$.

To illustrate Theorem 3.2, consider the partition pair $\sigma = (\pi^*, \delta_{s(\pi^*)})$ of 6 where $(3,2) \in \pi^*$, $(1) \in \delta_{s(\pi^*)}$, and $(3) \in \pi^*$, $(2,1) \in \delta_{s(\pi^*)}$. Hence $\mathrm{spt}_o^-(6)$ is equal to 2 plus the appearances of the smallest parts in π^* of those partition pairs $\sigma = (\pi^*, \delta_{s(\pi^*)})$ which have the empty partition $\emptyset \in \delta_i$, that is, $(3,1,1,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(4,1,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(2,1,1,1,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(1,1,1,1,1,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; and finally $(3,2,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$. This gives us $\mathrm{spt}_o^-(6) = 18 = 0 \pmod 2$.

Equating the coefficients of q^{2n+1} in Theorem 1.1 gives us a nice corollary.

THEOREM 3.3. *We have* $\mathrm{spt}_o^-(2n+1) = \mathrm{spt}_o^+(2n+1)$.

Let $t_k(n)$ be the number of representations of n as a sum of k triangular numbers. We may use a classical result of Legendre that $\sigma(2n+1) = t_4(n)$ to see that $\mathrm{spt}_o^-(2n+1)$ (and therefore also $\mathrm{spt}_o^+(2n+1)$) is generated by the product expansion

$$q\frac{(q^4;q^4)_\infty^3}{(q^2;q^4)_\infty^5}.$$

To see examples for Theorem 3.3, consider first $n = 1$. Then $\mathrm{spt}_o^-(3) = \mathrm{spt}_o^+(3) = 5$. This is because $(2,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(1,1,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; and $(2) \in \pi^*$, $(1) \in \delta_{s(\pi^*)}$, for $\mathrm{spt}_o^-(3)$. The case of $\mathrm{spt}_o^+(3)$ is clearer.

Another example is $\mathrm{spt}_o^-(5) = \mathrm{spt}_o^+(5) = 12$. We only compute $\mathrm{spt}_o^-(5)$ for the reader: $(2,2) \in \pi^*$, $(1) \in \delta_{s(\pi^*)}$; $(2,2,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(4,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; $(1,1,1,1,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$; and finally $(2,1,1,1) \in \pi^*$, $\emptyset \in \delta_{s(\pi^*)}$.

It is interesting to note that since $\mathrm{spt}(n)$ is even for almost all natural n (see [FO]), the value $\mathrm{spt}_o^+(2n)$ is even for almost all natural n in terms of arithmetic density.

We can also easily obtain congruences for $\mathrm{spt}_o(n)$ using the Ramanujan-type congruences in [A2]:

$$\begin{aligned} \mathrm{spt}_o(2(5n+4)) &\equiv 0 \pmod 5, \\ \mathrm{spt}_o(2(7n+5)) &\equiv 0 \pmod 7, \\ \mathrm{spt}_o(2(13n+6)) &\equiv 0 \pmod{13}. \end{aligned}$$

It is important to make the observation that the two Bailey pairs (2.3)–(2.5) and (2.6)–(2.8) are key in obtaining the “rank component” (1.2) and the “crank component” (1.3), respectively.

References

[A1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia Math. Appl. 2, Addison-Wesley, Reading, MA, 1976.

[A2] G. E. Andrews, *The number of smallest parts in the partitions of n*, J. Reine Angew. Math. 624 (2008), 133–142.

[AG] G. E. Andrews and F. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. (N.S.) 18 (1988), 167–171.

[B] W. N. Bailey, *Identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) 50 (1949), 1–10.

[FO] A. Folsom and K. Ono, *The spt-function of Andrews*, Proc. Nat. Acad. Sci. USA 105 (2008), 20152–20156.

[G] F. Garvan, *Higher order spt-functions*, Adv. Math. 228 (2011), 241–265.

[GR] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia Math. Appl. 35, Cambridge Univ. Press, Cambridge, 1990.

[P1] A. E. Patkowski, *Divisors, partitions and some new q-series identities*, Colloq. Math. 117 (2009), 289–294.

[P2] A. E. Patkowski, *A strange partition theorem related to the second Atkin–Garvan moment*, preprint.

[S] L. J. Slater, *A new proof of Rogers' transformations of infinite series*, Proc. London Math. Soc. (2) 53 (1951), 460–475.

Alexander E. Patkowski
1390 Bumps River Rd.
Centerville, MA 02632, U.S.A.
E-mail: alexpatk@hotmail.com

Received on 8.10.2013
and in revised form on 26.2.2015 (7606)