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Beta expansion of Salem numbers approaching Pisot numbers with the finiteness property

by

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1. Introduction and basic definitions. In general, for any real number $\beta > 1$, it is possible to expand $x \in [0, 1]$ in many different ways as

$$x = \sum_{i=1}^{\infty} x_i \beta^{-i},$$

where the sequence $(x_i)_{i\in\mathbb{N}^*}$, called the *expansion* of x *in base* β, takes values in the alphabet $\mathcal{A}_\beta = \{0, 1, \ldots, \lfloor\beta\rfloor\}$.

Example: If β is the golden ratio $\phi = (1 + \sqrt{5})/2$, which is a root of $x^2 - x - 1$, then 1 can be expanded as 0.11, 0.1011, 0.101011 and so on.

If $\beta \notin \mathbb{N}$, we distinguish among all possible expansions of x in base β the lexicographically greatest one, which is called the *beta expansion* or the *greedy expansion* and it is denoted by (the zero point is omitted)

$$d_\beta(x) = x_1 x_2 \ldots.$$

In order to avoid any confusion, note that some authors use the name of "beta expansion" for all types of expansions in base β, but here it is reserved for the greedy expansion.

This beta expansion, in a more general context, was first introduced by Rényi [Ré] and the digits x_i can be computed by the following algorithm:

GREEDY ALGORITHM. Denote by $\lfloor y\rfloor$ and $\{y\}$ respectively the integer part and the fractional part of a real number y. Set $r_0 = x$ and for $i \geq 1$, $x_i = \lfloor\beta r_{i-1}\rfloor$, $r_i = \{\beta r_{i-1}\}$.

Or similarly, $x_i = \lfloor\beta T^{i-1}(x)\rfloor$ using the beta transformation $T = T_\beta$ of the unit interval given by

$$T = T_\beta : x \mapsto \{\beta x\} = \beta x - [\beta x].$$

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If the representation of $x \in [0,1]$ ends with infinitely many zeros, then the ending zeros are omitted and the representation is said to be *finite*. Note that this can happen if and only if there exists $i \geq 0$ for which $T_\beta^i(x) = 0$.

An important property of the beta expansion is its monotonicity. Namely, if $x < y$, then the beta expansion (x_n) of x is lexicographically less than the beta expansion (y_n) of y, i.e. $x_n < y_n$ for the smallest $n \geq 1$ such that $x_n \neq y_n$.

This monotonicity implies that all the information on the beta transformation is already contained in the expansion of 1. This explains why $d_\beta(1)$ plays a crucial role.

In fact, the simplest expansion of 1 in base β is 1. However, defining $d_\beta(1)$ as a sequence computed by the greedy algorithm provides useful information on the arithmetic and dynamical proprieties of β (see for instance [Par], [F] or [Bl]).

In the remainder of this paper we are concerned only with the beta expansion of 1. For simplicity, by the beta expansion of a given $\beta > 1$ we mean the beta expansion of 1 in base β.

Moreover, we will define the digits to be the sequence of integers $(c_1, c_2, \ldots)$ which are defined by:

$$\forall n \geq 1,\ c_n = \lfloor \beta\alpha_{n-1} \rfloor, \quad \text{where} \quad \alpha_0 = 1 \text{ and } \alpha_n = \{\beta\alpha_{n-1}\}.$$

In this case, we write $d_\beta(1) = c_1c_2\ldots$, where the digits $\{c_i\}_{i\geq 1}$ have many properties [F]. For instance, for all $i \geq 1$, $c_i \in \mathcal{A}_\beta = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ and

$$1 = \sum_{i=1}^{\infty} c_i\beta^{-i}.$$

Parry [Par] defined β to be a *beta number* (also called more recently a *Parry number*) if the orbit $\{\alpha_n : n \geq 1\}$ is finite.

In this case, there exist smallest $m \geq 1$ and $p \geq 1$ for which $\alpha_m = \alpha_{m+p}$; we denote $D = D(\beta) = \operatorname{card}\{\alpha_n\}$.

In particular, if $\alpha_m = 0$ (so $c_n = 0$ for $n > m$), we write

$$d_\beta(1) = c_1, \ldots, c_m$$

and call β a *simple* beta number.

Otherwise, if $\alpha_m = \alpha_{m+p} \neq 0$, then $d_\beta(1)$ is eventually periodic and we write $d_\beta(1) = (c_1, \ldots, c_m)(c_{m+1}, \ldots, c_{m+p})^w$. The values m and p are known as the *preperiod length* and the *period length*, respectively.

It can be easily checked by induction that for all $n \geq 1$, $\alpha_n = P_n(\beta)$ where

$$P_n(x) = x^n - c_1x^{n-1} - \cdots - c_n.$$

Consequently, if β is a beta number, then it must be a root of a monic polynomial R with integral coefficients, called the *companion polynomial* of β, with

$$R(x) = \begin{cases} P_m(x) & \text{if } \alpha_m = 0, \\ P_{m+p}(x) - P_m(x) & \text{if } \alpha_m = \alpha_{m+p} \neq 0. \end{cases}$$

Hence, any beta number is in particular an algebraic integer and its minimal polynomial $P(x)$ divides the companion polynomial $R(x)$. In this case, we have $R(x) = P(x)Q(x)$, where the polynomial $Q(x)$ (possibly constant) is called the *co-factor* of the beta expansion.

Moreover, Parry [Par] showed that the roots of the companion polynomial $R(x)$ other than β (called *beta-conjugates*) lie in the disk $|z| < \min(2, \beta)$, and this was improved to $|z| \leq (\sqrt{5}+1)/2$ by Solomyak [So] and independently by Flatto, Lagarias, and Poonen [FLP].

In the following definitions, we recall some interesting class of algebraic integers which constitute the classic areas for the study of beta expansion:

Definition 1.1. An algebraic integer $\beta > 1$ is called a *Perron number* if all its Galois conjugates have modulus less than β.

Definition 1.2. A *Pisot number* is an algebraic integer > 1 whose conjugates are all of modulus strictly less than 1.

Definition 1.3. A *Salem number* is an algebraic integer > 1 whose conjugates all have modulus at most 1, with at least one conjugate having modulus exactly 1. The minimal polynomial of a Salem number is also called a *Salem polynomial*.

Remark 1.4. (1) The set of Pisot numbers is extended to contain all positive integers $n > 1$, since they have minimal polynomial $P(x) = x - n$, and so have no Galois conjugate ≥ 1.

(2) Note that the last definition implies that a Salem number is an algebraic integer whose minimal polynomial is reciprocal and of even degree (≥ 4). More precisely, its minimal polynomial has only one root τ outside the unit disk, one root $1/\tau$ inside the unit disk, and all other roots on the unit circle.

There are many results which show that the sets of Salem and Pisot numbers are closely linked, for instance:

Salem [Sa1] has proved that every Pisot number is the limit from both sides of a sequence of Salem numbers.

On the other hand, it is proved in [BB] that, if R is the minimal polynomial of a Salem number, then there exists an associated Pisot number with minimal polynomial P such that $(x^2+1)R(x) = xP(x) + P^*(x)$, where P^* is the reciprocal of P.

However, despite these close links, there is no as much information in the literature concerning Salem numbers as in the case of Pisot numbers.

For example, it was proved that all Pisot numbers are beta numbers ([**?**] or [Sc]). More generally, the structure of the beta expansion of Pisot numbers is very well known since it was extensively studied in several works ([Pan], [Z], ...). But for Salem numbers this structure is still unclear.

Actually, the fact that all Salem numbers are beta numbers is just a direct consequence of a conjecture due to Schmidt [Sc]. This was proved by Boyd [Bo2] for Salem numbers of degree 4. However, the same author [Bo3] gave a heuristic argument suggesting the existence of Salem numbers of degree ≥ 8 that are not beta numbers. This clearly casts a doubt on Schmidt's conjecture.

More recently, Hare and Tweedle [HT] have determined some sufficient conditions for a Salem number to have a periodic expansion. They used this information to provide infinite families of Salem numbers with eventually periodic beta expansion.

In this paper, we study the structure of the beta expansion of the sequence of Salem numbers defined by R. Salem [Sa1] that approaches a Pisot number θ. In particular, we find that there exists a marked difference between the cases when θ is a Pisot number of degree 2 and of degree 3. Finally, we give a sufficient condition for a Pisot number to be the limit of a sequence of Salem numbers which are beta numbers.

Most of these results are rather easily proved, but the author could not find them in the literature. This fact encourages us to present them here, hoping that it helps to explain a part of the structure of the beta expansion of Salem numbers.

2. Beta expansion of some sequences of Salem numbers. From now on we assume that θ is always a Pisot number with minimal polynomial P and for every integer $k \geq 1$ we denote by β_k the dominant root of the polynomial

$$R_k(x) = x^k P(x) + P^*(x).$$

According to [Sa2, Theorem 4, p. 30], and its proof, the polynomial R_k has a unique root β_k outside the unit disk and the sequence (β_k) converges to θ. Moreover, if θ is not a quadratic unit, then for every sufficiently large integer k, β_k is a Salem number.

This idea, as simple as it is powerful, was first introduced by Salem. But we can find many other further developed versions in [Bo1], [MS1], [MS2], ...

For simplicity, we will refer to this idea as the *Salem construction* and we will use the following notation:

NOTATION. (1) If a sequence $s = (c_j)_{j\geq 0}$ is eventually periodic, we write

$$s = c_1, \dots, c_n : c_{n+1}, \dots, c_{n+\ell},$$

and this means that $c_{n+i+k\ell} = c_{n+i}$ for all $k, i \in \mathbb{N}$.

(2) We write $(a)^k$ for the word $a \dots a$ (k times).

Finally, we recall the following important lemma called "Parry's criterion" and which is necessary for all the remainder of this paper:

LEMMA 2.1 (Parry). *Let $(c_1, c_2, \dots)$ be a sequence of non-negative integers which is different from $1(0)^w$ and satisfies $c_1 > 0$ and $c_k \leq c_1$ for $k \geq 1$. The unique solution $\beta > 1$ of*

$$x = c_1 + c_2 x^{-1} + c_3 x^{-2} + \cdots$$

has $c_1 c_2 \dots$ as the beta expansion of 1 if and only if

$$\forall k \geq 1, \qquad \sigma^k(c_1, c_2, \dots) <_{\text{lex}} (c_1, c_2, \dots)$$

where $\sigma(c_1, c_2, \dots) = (c_2, c_3, \dots)$.

Proof. Corollary 1 of Theorem 3 of [Par]. ∎

2.1. Case of degree 2. Pisot numbers of degree 2 are characterized in [FS] by the following lemma:

LEMMA 2.2. *The only Pisot numbers of degree 2 are the dominant roots of the integral polynomial $P = x^2 - ax - b$ with:*

(1) *first type: $a \geq b \geq 1$,*
(2) *second type: $a \geq 3$ and $-a + 2 \leq b \leq -1$.*

In this case we find that all Salem numbers obtained via Salem construction are beta numbers with small orbit size and with "quite organized" beta expansion, because always $m = 1$ and the companion polynomial is reciprocal.

Moreover, as the beta expansions of Salem numbers of degree 4 are already completely determined in [Bo2], we will consider only the case when β_k is the dominant root of $R_k = x^k P + P^*$ with $k \geq 4$.

THEOREM 2.3. *If $a \geq b \geq 1$ then β_k is a beta number and*

$$d_{\beta_k}(1) = a(b(0)^{k-3} b(a-1)^2)^w.$$

Proof. First we set

$$t = c_1 : c_2, c_3, \dots, c_{k-1}, c_k, c_{k+1}, c_{k+2}$$
$$\tag{2.1} = a : b, 0, \dots, 0, b, a-1, a-1,$$

and for any integer $m \geq 1$,

$$L_m(x) = x^m - c_1 x^{m-1} - \cdots - c_{m-1} x - c_m.$$

It is clear that the conditions of Parry's criterion follow for the sequence t from the condition $a \geq b \geq 1$. Thus, according to Lemma 2.1, the unique solution $\gamma > 1$ of

$$x = c_1 + c_2 x^{-1} + c_3 x^{-2} + \cdots$$

has beta expansion $t = d_\gamma(1) = c_1 : c_2, \ldots, c_{k+2}$.

But the assumed periodicity of the sequence t implies in particular that $T_\gamma(1) = T_\gamma^{k+2}(1)$. Consequently, γ is a real root > 1 of the polynomial $L_{k+2}(x) - L_1(x) = x^{k+2} - ax^{k+1} - bx^k - bx^2 - ax + 1$, which obviously coincides with the polynomial R_k.

However, according to [Sa2, proof of Theorem, p. 30], R_k has a unique real root $\beta_k > 1$. Hence, $\beta_k = \gamma$ and

$$d_{\beta_k}(1) = a(b(0)^{k-3} b(a-1)^2)^w. \ \blacksquare$$

REMARK 2.4. Note here that if $k = 2p + 1$ is odd, then R_k is never irreducible because -1 is a root, and so $R_k = (x+1)F(x)$ with

$$F(x) = x^{2p} - (a+1)x^{2p-1} + (a-b+1)\sum_{j=2}^{2p-2}(-1)^j x^j - (a+1)x + 1.$$

Consequently, the result of Theorem 2.3 can be applied for the dominant root (which is a Salem number for all p large enough) defined by any integral polynomial of the form of F, under the condition $a \geq b \geq 1$.

For example, the Salem number $\beta \approx 5.541$ with minimal polynomial

$$x^{10} - 6x^9 + 3x^8 - 3x^7 + 3x^6 - 3x^5 + 3x^4 - 3x^3 + 3x^2 - 6x + 1$$

is a beta number satisfying

$$d_\beta(1) = 5(3(0)^6 3(4)^2)^w,$$

by applying Theorem 2.3 for $(a, b) = (5, 3)$ and $k = 9$.

Moreover, we note that no Pisot number of the first type is reciprocal. Hence we immediately get the following consequence.

COROLLARY 2.5. *Every Pisot number with minimal polynomial $P(x) = x^2 - ax - b$ with $a \geq b \geq 1$ is the limit of a sequence of Salem numbers which are beta numbers.*

For Pisot numbers of the second type (i.e. when $a \geq 3$ and $-a + 2 \leq b \leq -1$), we will give the complete beta expansion of β_k for some particular cases only, although for the missing cases many other similar results can be easily stated.

THEOREM 2.6. *If $a \geq 3$ and $-[a/2] \leq b \leq -1$, then*

$$d_{\beta_k}(1) = (a-1)\big((a+b-1)^{k-2}(a+2b)^2(a+b-1)^{k-2}(a-2)^2\big)^w.$$

Proof. We follow the same steps as in Theorem 2.3, so we start by setting

$$\begin{aligned}(2.2)\qquad t &= c_1 : c_2, \ldots, c_{k-1}, c_k, c_{k+1}, c_{k+2}, \ldots, c_{2k-1}, c_{2k}, c_{2k+1}\\ &= a-1 : s, \ldots, s, a+2b, a+2b, s, \ldots, s, a-2, a-2\end{aligned}$$

with $s = a + b - 1$. For any integer $m \geq 1$ we set

$$L_m(x) = x^m - c_1 x^{m-1} - \cdots - c_{m-1}x - c_m.$$

It is clear that the conditions of Parry's criterion follow, for the sequence t, from the conditions $a \geq 3$ and $-[a/2] \leq b \leq -1$. Thus, according to Lemma 2.1, the unique solution $\gamma > 1$ of

$$x = c_1 + c_2 x^{-1} + c_3 x^{-2} + \cdots$$

has beta expansion $t = d_\gamma(1) = c_1 : c_2, \ldots, c_{2k+1}$. But the assumed periodicity of the sequence t implies in particular that $T_\gamma(1) = T_\gamma^{2k+1}(1)$. Consequently, γ is a root > 1 of the polynomial

$$L_{2k+1}(x) - L_1(x) = (x^{k+2} - ax^{k+1} - bx^k - bx^2 - ax + 1)\frac{x^k - 1}{x - 1} = RQ$$

with $Q(x) = \frac{x^k-1}{x-1}$. However, Q is a cyclotomic polynomial, and so β_k is the unique real root > 1 of $L_{2k+1}(x) - L_1(x)$. Hence, $\beta_k = \gamma$ and

$$d_{\beta_k}(1) = (a-1)\big((a+b-1)^{k-2}(a+2b)^2(a+b-1)^{k-2}(a-2)^2\big)^w. \ \blacksquare$$

REMARK 2.7. Note that for $b = -1$, the polynomial $P = x^2 - ax + 1$ is reciprocal and its dominant root θ is a quadratic Pisot unit. So β_k is never a Salem number since we will have $\beta_k = \theta$ for all integer $k \geq 1$. However, the result of Theorem 2.6 remains true in this case because $d_{\beta_k}(1) = d_\theta(1) = (a-1)(a-2)^w$.

To construct a sequence of Salem numbers which converge to θ and which are beta numbers it will be sufficient to perturb slightly the central coefficient of the polynomial R_{2k}, as shown in the lemma and theorem below.

LEMMA 2.8. *Let $a \geq 3$. For every integer $k \geq 1$, the dominant root β_k of the polynomial $H_k(x) = x^{2+2k} - ax^{2k+1} + x^{2k} + x^{k+1} + x^2 - ax + 1$ is a Salem number. Furthermore, the sequence (β_k) converges to the reciprocal Pisot number root of $P = x^2 - ax + 1$.*

Proof. First, we start by showing that, for every integer $k \geq 1$, the dominant root of H_k is a Salem number.

It is clear that the polynomial H_k is reciprocal, and as $a \geq 3$ we have $H_k(1) < 0$. Hence, H_k has at least two real roots $\beta_k > 1$ and $1/\beta_k < 1$.

On the other hand, we note that

$$x^{-(k+1)}H_k(x) = x^{k+1} + x^{-k-1} - a(x^k + x^{-k}) + x^{k-1} + x^{-k+1} + 1.$$

So for $x = e^{it}$ on the unit circle, we have

$$x^{-(k+1)}H_k(x) = 2\cos(k+1)t + 2\cos(k-1)t + 1 - 2a\cos kt$$

with

$$|f(t)| = |2\cos(k+1)t + 2\cos(k-1)t + 1| \le 5 < 6 \le 2a.$$

Notice also that $g(t) = 2a\cos kt$ attains alternately each of the values $2a$ and $-2a$, as t goes from 0 to 2π, at the $2k$ values $t_j = j\pi/k$ for $0 \le j \le 2k-1$. Now, as $|f(t)| < 2a$, the cosine polynomial $x^{-(k+1)}H_k(x)$, for $x = e^{it}$, will also alternate in sign at these values.

Thus, by the Intermediate Value Theorem, $x^{-(k+1))}H_k(x)$, and hence $H_k(x)$ itself, has $2k$ roots on the unit circle. Thus, β_k is a Salem number or a reciprocal Pisot number.

However, if β_k is a reciprocal Pisot number, then it satisfies

$$\beta_k^2 - b\beta_k + 1 = 0$$

for some integer b. Hence

$$H_k(\beta_k) - (\beta_k^{2k}+1)(\beta_k^2 - b\beta_k + 1) = (\beta_k^{2k}+1)(b-a)\beta_k + \beta_k^{k+1} = 0.$$

This gives $a - b = \frac{\beta_k^k}{\beta_k^{2k}+1}$, and as $\beta_k > 1$ we get $0 < a-b < 1$, which is impossible since a and b are integers. Thus, β_k is a Salem number.

Finally, the fact that $a \ge 3$ implies that $H_k(a) = a^{2k} + a^{k+1} + 1 > 0$ and $H_k(2) = (2-a)2^{2k+1} + 2^{2k} + 2^{k+1} - 2a + 3 < 0$ for all k large enough. Hence, $2 < \beta_k < a$ for all sufficiently large integers k.

Now, if we denote by $\ell \ge 2$ an accumulation point of the sequence (β_k), using $\lim_{k\to\infty} H_k(\beta_k)/\beta_k^{2k} = 0$ we get $\ell^2 - a\ell + 1 = 0$, and so $\ell = \theta$. Consequently, (β_k) is a bounded sequence having θ as the only possible accumulation point. Thus, (β_k) converges to θ. ∎

THEOREM 2.9. *For every integer $a \ge 3$, the dominant root β_k of*

$$H_k(x) = x^{2+2k} - ax^{2k+1} + x^{2k} + x^{k+1} + x^2 - ax + 1$$

is a beta number satisfying

$$d_{\beta_k}(1) = (a-1)\big((a-2)^{k-1}(a-3)^{2k}(a-2)^{k+1}\big)^w.$$

Proof. It is clear from the proof of the previous lemma that the polynomial H_k has only one root > 1. Then the proof is similar to the case of Theorem 2.6, and it will be enough to note that the co-factor is $Q = \frac{x^{2k}-1}{x-1}$. ∎

From Lemmas 2.8, 2.2 and Theorems 2.6, 2.9, we directly get the following consequence:

COROLLARY 2.10. *Every Pisot number with minimal polynomial $P(x) = x^2 - ax - b$ with $-[a/2] \le b \le -1$ is the limit of a sequence of Salem numbers which are beta numbers.*

According to Lemma 2.2, the only case which remains is when we have $-a+2 \le b < -[a/2]$. In fact, we think that it is possible to get the complete beta expansion of β_k in the same way. The main difficulty for us was to determine a common general form of beta expansion for all values of b when $-a+2 \le b < -[a/2]$. Nevertheless, stating similar results for each fixed value of b in this case remains always possible. As an example, we give here the following theorem.

THEOREM 2.11. *Let $a \ge 5$, $k \ge 3$, and $b = -[a/2]-1$. Then*

$$d_{\beta_k}(1) = (a-1)\big((s-1)^{k-3}(s-2)(2s-1)(2s+b-1)(2s-2)^{k-3} \times (2s+b-1)(2s-1)(s-2)(s-1)^{k-3}(a-2)^2\big)^w$$

where $s = b+a$.

Proof. The same proof as for Theorems 2.3 and 2.6, with

$$Q(x) = x^{k-1} + \frac{x^{2k-1}-1}{x-1}. \ \blacksquare$$

REMARK 2.12. Note that the missing values $(a,b) = (3,-2)$ and $(a,b) = (4,-3)$ are excluded according to Lemma 2.2.

2.2. Case of degree 3. In this subsection, we assume that $\theta > 1$ is a cubic algebraic integer with

$$\mathrm{Irr}(\theta) = x^3 - ax^2 - bx - c.$$

Pisot numbers of degree 3 are characterised in [Ak] by the following theorem:

THEOREM 2.13. *θ is a Pisot number if and only if*

$$|b-1| < a+c \quad \text{and} \quad c^2 - b < \mathrm{sgn}(c)(1+ac).$$

By examining the computational results of the beta expansion of Salem numbers coming from Pisot numbers of degree 3 and of degree 2 via the Salem construction, it can be clearly seen that there is a great difference between the structure of the beta expansion in both cases (see Section 3 for more details).

One of the main differences between Pisot numbers of degree 2 and of degree 3 is the *finiteness property*, which we now recall. For any real number $\beta > 1$, let $\mathrm{FIN}(\beta)$ be the set of non-negative real numbers having finite β-expansion. Denote by $\mathbb{Z}[1/\beta]$ the minimal ring containing $\mathbb{Z}$ and $1/\beta$, and by $\mathbb{Z}[1/\beta]_{\ge 0}$ the set of non-negative elements of $\mathbb{Z}[1/\beta]$.

We say that β has the *finiteness property* or property (F) if

$$\text{(F)} \qquad \mathrm{FIN}(\beta) = \mathbb{Z}[1/\beta]_{\ge 0}.$$

This property was first introduced in [FS] where the authors showed that it implies that β is a Pisot number.

Moreover, according to [FS], all Pisot numbers of degree 2 have the finiteness property. But they gave an example to show that this is not true for degree 3.

As far as we know, there is no simple algebraic characterisation of Pisot numbers of degree ≥ 3 having property (F) (in terms of coefficients of minimal polynomials). Nevertheless, we can find a partial answer to this problem in some particular cases as shown in the following theorem [Ak]:

THEOREM 2.14. *Let θ be a cubic Pisot unit. Then θ has property* (F) *if and only if θ is a root of the polynomial $x^3 - ax^2 - bx - 1$ with integer coefficients $a \geq 0$ and $-1 \leq b \leq a+1$.*

For this case we have the following theorem:

THEOREM 2.15. *Every cubic Pisot unit with the finiteness property is the limit of a sequence of Salem numbers which are beta numbers.*

This is a direct consequence of Theorem 2.14 and the following lemma:

LEMMA 2.16. *Let k be a positive integer and β_k be the dominant root of the polynomial $R(x) := x^k P + P^*$, where $P = x^3 - ax^2 - bx + 1$ is an irreducible Pisot polynomial with property* (F). *We have:*

(1) *If $0 \leq b \leq a$, $a \geq 1$ and $k \geq 4$, then*

$$d_{\beta_k}(1) = a\big(b, 1, (0)^{k-4}, 1, b, (a-1)^2\big)^w.$$

(2) *If $b = -1$, $a \geq 2$ and $k \geq 5$, then*

$$d_{\beta_k}(1) = (a-1)\big((a-1), 0, 1, (0)^{k-5}, 1, 0, (a-1), (a-2)^2\big)^w.$$

(3) *If $b = a+1$, $k \geq 6$ for $a \geq 1$ and $k \geq 9$ for $a = 0$, then*

$$d_{\beta_k}(1) = (a+1)\big((0)^2, a, 1, (0)^{k-6}, 1, a, (0)^2, 1, 0, (a)^2\big)^w.$$

Proof. The proof is similar to the previous ones; it is sufficient to note that the co-factor is:

(1) $Q = 1$,
(2) $Q = x + 1$,
(3) $Q = x^2 - x + 1$. ∎

REMARK 2.17. The missing values in the last lemma (i.e. $a = 0$ in (1) and $a = 0$ or 1 in (2)) are excluded since the corresponding polynomial P is not irreducible, so it does not define a cubic Pisot number.

2.3. A particular case of degree ≥ 3. We will try to generalize the result of Theorem 2.15 to some other Pisot numbers of degree ≥ 3 and with property (F). Let us first recall the following theorems which we can find respectively in [FS] and [Ho].

THEOREM 2.18 (A). *The positive root of the polynomial $P = x^d - b_1x^{d-1} - b_2x^{d-2} - \cdots - b_d \in \mathbb{Z}[x]$ with $b_1 \geq \cdots \geq b_d > 0$ is a Pisot number with property* (F).

THEOREM 2.19 (B). *The positive root of the polynomial* $P = x^d - b_1x^{d-1} - b_2x^{d-2} - \cdots - b_d \in \mathbb{Z}[x]$ *with* $b_1 > \sum_{i=2}^{d} b_i$ *and* $b_i \geq 0$ $(1 \leq i \leq d)$ *is a Pisot number with property* (F).

Now, we can state a result similar to Theorem 2.15:

THEOREM 2.20. *Every Pisot number defined in Theorem* (A) *or* (B) *is the limit of a sequence of Salem numbers which are beta numbers.*

Proof. It is clear that in Theorems (A) and (B), the polynomial P is not reciprocal. Hence the polynomials $R_k = x^k P + P^*$ define a sequence of Salem numbers for all k large enough.

For $k \geq d+1$ we set as usual

$$\begin{aligned} t &= c_1 : c_2, \ldots, c_d, c_{d+1}, \ldots, c_{k-1}, c_k, \ldots, c_{d+k-2}, c_{d+k-1}, c_{d+k} \\ &= b_1 : b_2, \ldots, b_d, 0, \ldots, 0, b_d, \ldots, b_2, b_1 - 1, b_1 - 1 \end{aligned} \tag{2.3}$$

and

$$L_m(x) = x^m - c_1x^{m-1} - \cdots - c_{m-1}x - c_m.$$

It is clear that the conditions of Parry's criterion follow, for the sequence t, from the condition on $(b_i)_{1 \leq i \leq d}$ in both cases (A) and (B). Thus, according to Lemma 2.1, the unique solution $\gamma > 1$ of

$$x = c_1 + c_2x^{-1} + c_3x^{-2} + \cdots$$

has beta expansion $t = d_\gamma(1) = c_1 : c_2, \ldots, c_{d+k}$. Furthermore, the assumed periodicity of the sequence t implies that $T_{\beta'}(1) = T_{\beta'}^{d+k}(1)$. Consequently, γ is a root > 1 of the polynomial $L_{d+k}(x) - L_1(x)$ which obviously coincides with the polynomial R_k.

However, according to [Sa2, proof of Theorem, p. 30], R_k has a unique real root $\beta_k > 1$. Hence, $\beta_k = \gamma$ and β_k is a beta number. Moreover,

$$d_\beta(1) = b_1\big(b_2, \ldots, b_d, (0)^{k-d-1}, b_d, \ldots, b_2, (b_1 - 1)^2\big)^w. \ \blacksquare$$

3. Some remarks. In this section, we give some examples which show clearly that there is a great difference between Salem numbers associated to Pisot numbers of degree 3 and those associated to Pisot numbers of degree 2. Mainly, in contrast to the case of degree 2:

(1) We note that m can take many values other than 1 and the co-factor can be non-reciprocal and hence non-cyclotomic.

For instance: The Salem number $\beta \simeq 6.27303$ defined by the polynomial

$$x^8 - 5x^7 - 7x^6 - 6x^5 - 6x^3 - 7x^2 - 5x + 1$$

obtained from the Pisot polynomial $P = x^3 - 5x^2 - 7x - 6$ for $k = 5$ is a beta number having $(m, p) = (7, 15)$ and as a co-factor, the following non-reciprocal polynomial:

$$x^{14} - x^{13} + x^{12} - x^{10} + 2x^9 - 2x^8 + x^7 + x^6 - 2x^5 + 2x^4 - x^3 + x - 1.$$

(2) We find many examples of Salem numbers of degree ≥ 8 which have very large (possibly infinite) orbit size. For such examples, we check numerically that, if the beta expansion is periodic then $D(\beta)$ exceeds 10^6. For example, this is the case of the polynomials listed below:

$$x^8 - 3x^7 - 5x^6 - 3x^5 - 3x^3 - 5x^2 - 3x + 1, \tag{3.1}$$

$$x^{10} - 3x^9 - 4x^8 - 4x^7 - 4x^3 - 4x^2 - 3x + 1, \tag{3.2}$$

$$x^{12} - 3x^{11} - x^{10} + 2x^9 + 2x^3 - x^2 - 3x + 1, \ldots. \tag{3.3}$$

Moreover, we find some particular Pisot numbers of degree 3 such that all Salem numbers produced by the Salem construction (for $k \geq 5$) have very large orbit size or are probably non-beta numbers. This holds for example in the case of

$$P = x^3 - 5x^2 + 4x - 3.$$

We think that these computational results strongly support the conjecture of Boyd concerning the existence of a strictly positive proportion of Salem numbers of degree greater than eight which are not beta numbers.

On the other hand, these results lead us to believe that it is important to determine the real cause of this large difference between the beta expansion of Salem numbers coming from Pisot numbers of degree 2 and those from Pisot numbers of degree 3.

Remark 3.1. The previous results suggest that if θ is Pisot number with the finiteness property, then all Salem numbers obtained from θ via the Salem construction are beta numbers. However, many examples show that this condition is far from being necessary. For instance, all Salem numbers β_k coming from the Pisot number θ with minimal polynomial $x^3 - 3x^2 + 2x + 2$ are beta numbers (for $k \geq 6$ we have $d_{\beta_k}(1) = (2)(1, 0, 2, (0)^{k-5}, 2, 0, (1)^3)^w$, although here θ does not have property (F).

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