Asymptotic distribution and symmetric means
of algebraic numbers

by

Igor E. Pritsker (Stillwater, OK)

Dedicated to the memory of Professor Albert Baernstein II

1. Schur's problems on means of algebraic numbers. Schur [20] considered several problems for various classes of polynomials with integer coefficients and zeros restricted to certain sets. For convenience, we introduce the following notation. Let E be a subset of the complex plane $\mathbb{C}$. Consider the set $\mathbb{Z}_n(E)$ of polynomials with integer coefficients of exact degree n and all zeros in E. We denote the subset of $\mathbb{Z}_n(E)$ with simple zeros by $\mathbb{Z}_n^s(E)$. Given $M > 0$, we write $P_n = a_n z^n + \cdots \in \mathbb{Z}_n^s(E, M)$ if $|a_n| \leq M$ and $P_n \in \mathbb{Z}_n^s(E)$ (respectively $P_n \in \mathbb{Z}_n(E, M)$ if $|a_n| \leq M$ and $P_n \in \mathbb{Z}_n(E)$). In particular, Schur [20, §§4–8] studied the limit behavior of the arithmetic means of zeros for polynomials from $\mathbb{Z}_n^s(E, M)$ as $n \to \infty$, where $M > 0$ is an arbitrary fixed number. Two of his main results in this direction are stated below. Let $D := \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk, and let $\mathbb{R}_+ := [0, \infty)$. For a polynomial $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_{k,n})$, denote the arithmetic mean of its zeros by $A_n := \sum_{k=1}^n \alpha_{k,n}/n$.

THEOREM A (Schur [20, Satz XI]). *If $P_n \in \mathbb{Z}_n^s(\mathbb{R}_+, M)$ is a sequence of polynomials such that $n \to \infty$, then*

$$(1.1) \qquad \liminf_{n \to \infty} A_n \geq \sqrt{e} > 1.6487.$$

THEOREM B (Schur [20, Satz XIII]). *If $P_n \in \mathbb{Z}_n^s(D, M)$ is a sequence of polynomials such that $n \to \infty$, then*

$$(1.2) \qquad \limsup_{n \to \infty} |A_n| \leq 1 - \sqrt{e}/2 < 0.1757.$$

Schur remarked that the lim sup in (1.2) is equal to 0 for *monic* polynomials from $\mathbb{Z}_n(D)$ by Kronecker's theorem [9]. We proved [14] that $\lim_{n \to \infty} A_n$

$= 0$ for any sequence of polynomials from Schur's class $\mathbb{Z}_n^s(D, M)$, $n \in \mathbb{N}$. This result was obtained as a consequence of the asymptotic equidistribution of zeros near the unit circle. Namely, if $\{\alpha_{k,n}\}_{k=1}^n$ are the zeros of P_n, we define the zero counting measure

$$\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_{k,n}},$$

where $\delta_{\alpha_{k,n}}$ is the unit point mass at $\alpha_{k,n}$. Consider the normalized arclength measure μ_D on the unit circle, with $d\mu_D(e^{it}) := \frac{1}{2\pi} dt$. If τ_n converge to μ_D in weak* topology as $n \to \infty$ (written $\tau_n \overset{*}{\to} \mu_D$) then

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \int z \, d\tau_n(z) = \int z \, d\mu_D(z) = 0.$$

Thus the sharp version of Schur's Theorem B immediately follows from the convergence $\tau_n \overset{*}{\to} \mu_D$ as $n \to \infty$, which was originally proved in [14], and generalized in several directions in [15]–[17]. Furthermore, we found essentially sharp rates of convergence of A_n to 0 in the setting of Theorem B. A similar approach via the asymptotic distribution of algebraic numbers and limiting measures for τ_n can be used to develop Theorem A, but this problem is much more complicated because algebraic numbers are contained in the unbounded set $\mathbb{R}_+$.

We give a brief description of the history, referring to [15] for a more complete account. If $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_{k,n})$ is irreducible over the integers, then $\{\alpha_{k,n}\}_{k=1}^n$ is called a *complete set of conjugate algebraic numbers of degree n*. When $a_n = 1$, we refer to $\{\alpha_{k,n}\}_{k=1}^n$ as *algebraic integers*. If $\alpha = \alpha_{1,n}$ is one of the conjugates, then the sum of $\{\alpha_{k,n}\}_{k=1}^n$ is also called the *trace* $\text{tr}(\alpha)$ of α over the rationals. Siegel [21] improved Theorem A for totally positive algebraic integers to

$$\liminf_{n\to\infty} A_n = \liminf_{n\to\infty} \text{tr}(\alpha)/n > 1.7336105,$$

by using a refinement of the arithmetic-geometric means inequality that involves the discriminant of $\alpha_{k,n}$. Smyth [23] introduced a numerical method of "auxiliary polynomials", which was used by many authors to obtain improvements of the above lower bound. The original papers [22, 23] contain the bound 1.7719. More recent results include the bounds 1.784109 by Aguirre and Peral [1], and 1.78702 by Flammang [7]. McKee [11] designed a modification of the method that achieves the bound 1.78839, which is apparently the best currently known lower bound.

PROBLEM A (The Schur–Siegel–Smyth trace problem [4]). *Find the smallest limit point ℓ for the set of values of mean traces A_n for all totally positive and real algebraic integers.*

It was observed by Schur [20] (see also Siegel [21]) that $\ell \leq 2$. This immediately follows by considering the Chebyshev polynomials $t_n(x) := 2\cos(n\arccos((x-2)/2))$ for the interval $[0,4]$, whose zeros are symmetric about the midpoint 2. They have integer coefficients, and $t_p(x)/(x-2)$ is irreducible for any prime p, giving the above mentioned upper bound 2 (cf. [20]). In fact, for the counting measures of zeros we have $\tau_n \xrightarrow{*} dx/(\pi\sqrt{x(4-x)})$ as $n \to \infty$, so that

$$\lim_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} = \lim_{n\to\infty} \int x\, d\tau_n(x) = \int_0^4 \frac{x\,dx}{\pi\sqrt{x(4-x)}} = 2.$$

More generally, if a sequence of totally positive algebraic numbers satisfies $\tau_n \xrightarrow{*} \mu$ as $n \to \infty$, then

$$\liminf_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} = \liminf_{n\to\infty} \int x\, d\tau_n(x) \geq \int x\, d\mu(x).$$

This brings us to the problem of minimizing the centroids (first moments) of measures arising as weak* limits for counting measures of totally positive algebraic numbers. We develop and generalize this approach so that it applies to a wide range of problems on symmetric and convex means of algebraic numbers. It may also be useful for other types of problems on algebraic numbers.

The Schur–Siegel–Smyth trace problem remains a difficult open question despite all efforts. As a partial result towards its solution, we gave the sharp lower bound $\liminf_{n\to\infty} A_n \geq 2$ for sets of algebraic numbers whose polynomials do not grow exponentially fast on compact sets of $\mathbb{R}_+$ of capacity (transfinite diameter) 1: see [15, Corollary 2.6] and Theorem 2.6 here.

We use the *generalized Mahler measure* to measure the size of integer polynomials over a certain set. The classical Mahler measure of a polynomial $P_n(z) = a_n \prod_{k=1}^{n}(z - \alpha_{k,n})$, $a_n \neq 0$, is defined by

$$M(P_n) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(e^{it})|\,dt\right) = |a_n| \prod_{k=1}^{n} \max(1, |\alpha_{k,n}|),$$

where the last equality is a consequence of Jensen's formula. The Mahler measure was generalized to compact sets of capacity 1 by Rumely [19]. We employ a similar generalization, which was introduced in [15] to obtain an "if and only if" theorem on the equidistribution of algebraic numbers near arbitrary compact sets in the plane.

Consider an arbitrary compact set $E \subset \mathbb{C}$ with logarithmic capacity $\mathrm{cap}(E) = 1$ (see [24, p. 55]). In particular, $\mathrm{cap}(D) = 1$ and the capacity of an interval is equal to one quarter of its length [24, p. 84]. Let μ_E be the equilibrium measure of E [24, p. 55], which is a unique probability measure expressing the steady state distribution of charge on the conductor E.

Note that μ_E is supported on the boundary of the unbounded connected component Ω_E of $\overline{\mathbb{C}} \setminus E$ by [24, p. 79]. Two important examples for us are $d\mu_D(e^{it}) = \frac{1}{2\pi} dt$ and

$$d\mu_{[0,4]}(x) = \frac{dx}{\pi\sqrt{x(4-x)}}, \quad x \in (0,4).$$

Consider the Green function $g_E(z, \infty)$ for Ω_E with pole at ∞ (cf. [24, p. 14]), which is a positive harmonic function in $\Omega_E \setminus \{\infty\}$. Note that $g_D(z, \infty) = \log|z|$, $|z| > 1$, and $g_{[0,4]}(z, \infty) = \log|z - 2 + \sqrt{z^2 - 4z}| - \log 2$, $z \in \mathbb{C} \setminus [0,4]$. A natural generalization of the Mahler measure for $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_{k,n})$, $a_n \neq 0$, on an arbitrary compact set E of capacity 1 is given by

$$(1.3) \qquad M_E(P_n) := |a_n| \exp\Big(\sum_{\alpha_{k,n} \in \Omega_E} g_E(\alpha_{k,n}, \infty) \Big).$$

If no $\alpha_{k,n}$ is in Ω_E then we assume that the above (empty) sum is zero. It is clear that $M_E(P_n) \geq |a_n| \geq 1$ for any P_n with integer coefficients and $a_n \neq 0$. If $M_E(P_n)$ does not grow fast as $n \to \infty$, which is specifically expressed by condition (2.9), then the roots of P_n are equidistributed in E, meaning that $\tau_n \overset{*}{\to} \mu_E$. This allows us to obtain the lower bound (2.10) of Theorem 2.6 for the symmetric means of such algebraic numbers.

The elementary symmetric functions in the roots of a polynomial $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_{k,n}) = \sum_{k=0}^n a_k z^k$ are expressed through the coefficients by

$$(1.4) \qquad \sigma_m := \sum_{j_1 < \cdots < j_m} \alpha_{j_1,n} \cdots \alpha_{j_m,n} = (-1)^m \frac{a_{n-m}}{a_n}.$$

Thus the Schur–Siegel–Smyth problem is equivalent to a question about the growth of $|a_{n-1}|$ with n, and it is natural to consider higher symmetric functions σ_m and the asymptotic behavior of the coefficients a_{n-m} for a fixed $m \in \mathbb{N}$ when $n \to \infty$.

2. Symmetric means of algebraic numbers. For motivation and clarity of presentation, we first restrict ourselves to monic polynomials, following Siegel [21]. Thus we assume for a moment that $P_n(z) = z^n + a_{n-1,n} z^{n-1} + \cdots + a_{0,n} \in \mathbb{Z}_n^s(\mathbb{R}_+, 1)$. Observe that each σ_m has $\binom{n}{m}$ products in the defining sum (1.4). Thus it is natural to consider the symmetric means

$$S_m(x_1, \ldots, x_n) := \binom{n}{m}^{-1} \sum_{j_1 < \cdots < j_m} x_{j_1} \cdots x_{j_m}, \quad m \in \mathbb{N}.$$

The inequalities of Maclaurin (cf. [8, Section 52]) give fundamental relations between symmetric means of nonnegative numbers $\{x_i\}_{i=1}^n$:

$$(2.1) \qquad (S_n)^{1/n} \leq \cdots \leq (S_2)^{1/2} \leq S_1.$$

Note that equality holds in (2.1) if and only if all numbers x_i are equal. Thus in the context of Schur's problems and their generalizations we always have strict inequalities in (2.1). Furthermore, (2.1) shows for totally positive algebraic integers that

$$S_m \geq (S_n)^{m/n} = \Big(\prod_{k=1}^n \alpha_{k,n}\Big)^{m/n} = |a_0|^{m/n} \geq 1, \quad 1 \leq m \leq n.$$

For any sequence of polynomials $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{Z}_n^s(\mathbb{R}_+, 1)$, we obtain

$$\liminf_{n \to \infty} \frac{|a_{n-m,n}|}{\binom{n}{m}} = \liminf_{n \to \infty} \frac{\sigma_m}{\binom{n}{m}} = \liminf_{n \to \infty} S_m \geq 1,$$

where we assume that $m \in \mathbb{N}$ is fixed. This suggests a generalization of the Schur–Siegel–Smyth trace problem:

PROBLEM B. *Given a fixed $m \in \mathbb{N}$, find the smallest limit point ℓ_m for the symmetric means S_m of roots of polynomials $P_n \in \mathbb{Z}_n^s(\mathbb{R}_+, 1)$.*

Clearly, $\ell_1 = \ell$. We shall investigate relations between ℓ_m, $m \geq 2$, and ℓ in the present paper. An immediate consequence of (2.1) is that $\ell_m \leq (\ell_k)^{m/k} \leq \ell^m$ for $m \leq k \leq 1$. On the other hand, Theorem 2.7 of [17] suggests the conjecture $\ell_m = \ell^m$, $m \in \mathbb{N}$. Below we give more evidence in support of this conjecture.

We continue developing the approach to extremal problems on means of algebraic numbers via the integrals of limiting measures (for counting measures of those numbers). Any sequence of zero counting measures τ_n has a weak* convergent subsequence by Helly's selection theorem. While the classical version of Helly's theorem requires the measures to be supported in a fixed compact set, we can apply the result on each closed disk $\{z : |z| \leq n\}$, $n \in \mathbb{N}$, and construct nested subsequences to obtain a subsequence of measures weak* convergent on the union of expanding disks that fills the whole plane $\mathbb{C}$. Details of this argument may be found, for example, in [10, §1]. Note that the limiting measure τ obtained in this way may have unbounded support and may have total mass $\tau(\mathbb{C}) < 1$. One of our main goals is to understand the properties of such weak* limits that shed new light on difficult problems related to algebraic numbers. A natural class of limiting measures for these problems is described below.

Define the *logarithmic energy* of a measure μ by

$$I[\mu] := \int \int \log \frac{1}{|z - t|} \, d\mu(z) \, d\mu(t)$$

(see [24, p. 54]). We state our first result under the general assumption that algebraic numbers have bounded Weil height (absolute Mahler measure).

THEOREM 2.1. *Suppose that* $P_n(z) = a_n \prod_{k=1}^{n}(z - \alpha_{k,n}) \in \mathbb{Z}_n^s(\mathbb{C})$, $n \in \mathbb{N}$, *is a sequence of polynomials with* $\tau_n \xrightarrow{*} \tau$ *along a subsequence* $\mathcal{N} \subset \mathbb{N}$. *If*

$$(2.2) \qquad\qquad H := \limsup_{\mathcal{N} \ni n \to \infty}(M(P_n))^{1/n} < \infty$$

then τ *is a unit measure with finite logarithmic energy. Furthermore,*

$$(2.3) \qquad\qquad \int \log^+ |z| \, d\tau(z) \le \log H$$

and the restriction $\tau_R := \tau|_{\overline{D}_R}$, *where* $D_R := \{z : |z| < R\}$, *satisfies*

$$(2.4) \qquad -\log 2 - 2\tau(\overline{D}_R) \log H \le I[\tau_R] \le (1 - \tau(D_R)) \log 4 + 2 \log H$$

for all but countably many $R > 1$.

Thus no loss of mass for the limiting measure occurs in this case, even though some algebraic numbers may tend to infinity as the degree n increases. Furthermore, finiteness of the energy $I[\tau]$ carries information on the distribution of algebraic numbers. In particular, it implies that τ has no point masses, reflecting the fact that the algebraic numbers are well spaced. We remark that (2.2) is equivalent to

$$(2.5) \quad \limsup_{\mathcal{N} \ni n \to \infty} |a_n|^{1/n} < \infty \quad \text{and} \quad \limsup_{\mathcal{N} \ni n \to \infty} \Big(\prod_{k=1}^{n} \max(1, |\alpha_{k,n}|) \Big)^{1/n} < \infty.$$

Theorem 2.1 has immediate applications to problems on symmetric means of algebraic numbers.

COROLLARY 2.2. *If* (2.2) *is replaced with*

$$(2.6) \quad \limsup_{\mathcal{N} \ni n \to \infty} |a_n|^{1/n} < \infty \quad \text{and} \quad \limsup_{\mathcal{N} \ni n \to \infty} S_m(|\alpha_{1,n}|, \dots, |\alpha_{n,n}|) < \infty$$

for a fixed $m \in \mathbb{N}$, *and if all other assumptions of Theorem* 2.1 *hold, then* τ *is a unit measure with finite logarithmic energy.*

On the other hand, we show that omitting assumptions may lead to essentially arbitrary limiting distribution of algebraic numbers.

THEOREM 2.3. *Given any positive Borel measure* μ, $0 \le \mu(\mathbb{C}) \le 1$, *that is symmetric about real line, there is a sequence of complete sets of conjugate algebraic integers such that their counting measures* τ_n *converge weak* to* μ.

The above irregular behavior of algebraic numbers may only occur when the height grows superexponentially and some of the conjugates escape to infinity as the degree increases, according to Theorem 2.1. As an example of conjugate algebraic integers with counting measures converging to the identically zero measure in the weak* topology, we mention the roots of $P_p(z) = z^p - p!$ for prime p.

If all algebraic numbers are uniformly bounded for all $n \in \mathbb{N}$, then we can prove the conjectured relation between the limits of symmetric means.

THEOREM 2.4. *If the numbers $\{z_{k,n}\}_{k=1}^{n} \subset \mathbb{C}$ are uniformly bounded, and their counting measures satisfy $\tau_n \overset{*}{\to} \tau$ for $n \in \mathcal{N} \subset \mathbb{N}$, then*

$$(2.7) \qquad \lim_{\mathcal{N} \ni n \to \infty} S_m(z_{1,n}, \ldots, z_{n,n}) = \lim_{\mathcal{N} \ni n \to \infty} (S_1(z_{1,n}, \ldots, z_{n,n}))^m$$

$$= \left(\int z \, d\tau(z) \right)^m, \qquad m \in \mathbb{N}.$$

In the case of unbounded numbers, we prove the following lower bound for their symmetric means. It is convenient to consider numbers located in closed sectors $V_m := \{z \in \mathbb{C} : |\mathrm{Arg}\, z| \leq \pi/(2m)\}$, where $m \in \mathbb{N}$.

THEOREM 2.5. *For $m \in \mathbb{N}$, let the sets $\{z_{k,n}\}_{k=1}^{n} \subset V_m$ be symmetric about the real line for all $n \in \mathcal{N} \subset \mathbb{N}$. If the corresponding counting measures satisfy $\tau_n \overset{*}{\to} \tau$, $n \in \mathcal{N}$, where $\tau(\mathbb{C}) = 1$, then*

$$(2.8) \qquad \liminf_{\mathcal{N} \ni n \to \infty} S_m(z_{1,n}, \ldots, z_{n,n}) \geq \left(\int z \, d\tau(z) \right)^m.$$

It is clear from weak* convergence that τ is symmetric about the real axis, and is supported in V_m by inheritance. Therefore, $\int z \, d\tau(z) \geq 0$. Note that (2.8) holds for all $m \in \mathbb{N}$ when all $\{z_{k,n}\}_{k=1}^{n}$ belong to R_+.

Let ν_m be a weak* limit of counting measures for algebraic integers that produce the smallest limit point ℓ_m of the symmetric means S_m, $m \in \mathbb{N}$. We have

$$\ell_m \geq \left(\int x \, d\nu_m(x) \right)^m, \qquad m \in \mathbb{N},$$

by (2.8). It is plausible that equality holds above for all $m \in \mathbb{N}$, but we have not been able to prove this. Furthermore, if $\ell = \int x \, d\nu_m(x)$ then the above inequality and (2.1) give $\ell_m = \ell^m$. While we cannot show this holds in general, we can handle an important case of totally positive algebraic numbers with relatively small generalized Mahler measure (low height).

THEOREM 2.6. *Let $P_n(x) = a_n \prod_{k=1}^{n} (x - \alpha_{k,n}) \in \mathbb{Z}_n^s(\mathbb{R}_+)$, $n \in \mathcal{N} \subset \mathbb{N}$, be a sequence of polynomials, and let $E \subset \mathbb{R}_+$ be a compact set of logarithmic capacity 1. If*

$$(2.9) \qquad \lim_{\mathcal{N} \ni n \to \infty} (M_E(P_n))^{1/n} = 1$$

then

$$(2.10) \qquad \liminf_{\mathcal{N} \ni n \to \infty} S_m(\alpha_{1,n}, \ldots, \alpha_{n,n}) \geq 2^m, \qquad m \in \mathbb{N}.$$

Equality holds in (2.10) *for the roots of* $t_n(x) = 2\cos(n \arccos((x-2)/2))$.

The key fact used in the proof of this result is that the roots of P_n are equidistributed in E under assumption (2.9), that is, $\tau_n \overset{*}{\to} \mu_E$ by [15, Theorem 2.1].

3. Convex means of totally positive algebraic numbers. Suppose that $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a convex strictly increasing function. We now consider the convex ϕ-means of numbers $\{x_k\}_{k=1}^n \subset \mathbb{R}_+$ defined by

$$(3.1) \qquad C_n^\phi(x_1, \ldots, x_n) := \frac{1}{n} \sum_{k=1}^n \phi(x_k).$$

Another interesting question generalizing the original Schur–Siegel–Smyth problem is to find the smallest limit point of such convex ϕ-means for totally positively algebraic integers. This problem for means of squares of algebraic numbers was already considered in the original paper of Schur [20], and means for $\phi(x) = x^m$ were studied in the papers of Smyth [22], [23]. Since ϕ is convex increasing, one immediately finds that

$$(3.2) \qquad \liminf_{n\to\infty} C_n^\phi(\alpha_{1,n}, \ldots, \alpha_{n,n}) \geq \liminf_{n\to\infty} \phi\left(\frac{1}{n} \sum_{k=1}^n \alpha_{k,n}\right) \geq \phi(\ell)$$

for any sequence of complete sets of totally positive conjugate algebraic integers $\{\alpha_{k,n}\}_{k=1}^n$, $n \in \mathbb{N}$. The following result was proved in [15, Corollary 2.6]. It is based on the equidistribution of algebraic numbers in E expressed by $\tau_n \overset{*}{\to} \mu_E$, and the lower bound for the centroid of equilibrium measure μ_E found in Baernstein, Laugesen and Pritsker [2, Theorem 1].

THEOREM 3.1. *Let $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_{k,n}) \in \mathbb{Z}_n^s(\mathbb{R}_+)$ be a sequence of polynomials. Suppose that $E \subset \mathbb{R}_+$ is a compact set of capacity 1. Moreover assume that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\phi(x^2)$ is convex on $\mathbb{R}$. If*

$$\lim_{n\to\infty} (M_E(P_n))^{1/n} = 1$$

then

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \phi(\alpha_{k,n}) \geq \int_0^4 \frac{\phi(x)\, dx}{\pi \sqrt{x(4-x)}}.$$

In particular for $\phi(x) = x^m$, $m \in \mathbb{N}$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \alpha_{k,n}^m \geq \int_0^4 \frac{x^m\, dx}{\pi \sqrt{x(4-x)}} = 2^m \frac{1 \cdot 3 \cdot \ldots \cdot (2m-1)}{m!}.$$

Equalities hold in the above inequalities for the roots of $t_n(x) = 2\cos(n \arccos((x-2)/2))$.

From a more general perspective, we show that the limiting measures for algebraic numbers with bounded convex means have finite energy, as a consequence of Corollary 2.2.

THEOREM 3.2. *Let $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_{k,n}) \in \mathbb{Z}_n^s(\mathbb{C})$, $n \in \mathbb{N}$, be a sequence of polynomials with $\tau_n \overset{*}{\to} \tau$ along a subsequence $\mathcal{N} \subset \mathbb{N}$. Suppose*

that $\phi : \mathbb{R}_+ \to \mathbb{R}$ *is a convex strictly increasing function. If*

$$(3.3) \qquad \limsup_{\mathcal{N} \ni n \to \infty} |a_n|^{1/n} < \infty \quad and \quad \limsup_{\mathcal{N} \ni n \to \infty} C_n^\phi(|\alpha_{1,n}|, \ldots, |\alpha_{n,n}|) < \infty,$$

then τ is a unit measure with finite logarithmic energy.

4. Means of algebraic numbers located in sectors. It is also possible to obtain lower bounds for the means of zeros for polynomials with integer coefficients and zeros in sectors $W_\gamma := \{z \in \mathbb{C} : |\operatorname{Arg} z| \leq \gamma\}$, where $\gamma < \pi/2$. For the arithmetic means S_1, one can easily give the following bound:

$$\frac{1}{n} \sum_{k=1}^{n} \alpha_{k,n} = \frac{1}{n} \sum_{k=1}^{n} \Re(\alpha_{k,n}) \geq \frac{1}{n} \sum_{k=1}^{n} |\alpha_{k,n}| \cos \gamma \geq \left(\frac{|a_0|}{|a_n|} \right)^{1/n} \cos \gamma \geq \frac{\cos \gamma}{|a_n|^{1/n}}.$$

If (2.6) is satisfied, then we have a positive lower bound in the above inequality. Flammang [6] applied Smyth's method of auxiliary functions to bound the smallest limit points of mean trace for algebraic integers located in sectors, and obtained many explicit results. In fact, [6] gives exact values of smallest limit points for some sectors.

Note that Theorems 2.1, 2.4 and 2.5, as well as Corollary 2.2, are clearly applicable here. In particular, combining Corollary 2.2 with Theorem 2.5, we obtain the following result on the smallest limit points of means.

THEOREM 4.1. *Let $P_n(z) = a_n \prod_{k=1}^{n}(z - \alpha_{k,n}) \in \mathbb{Z}_n^s(W_\gamma)$, $n \in \mathcal{N} \subset \mathbb{N}$, be any sequence of polynomials, and let $\gamma < \pi/2$. If* (2.6) *holds then*

$$(4.1) \qquad \liminf_{\mathcal{N} \ni n \to \infty} S_m(\alpha_{1,n}, \ldots, \alpha_{n,n}) \geq \left(\int z \, d\tau(z) \right)^m > 0, \qquad \gamma \leq \pi/(2m),$$

where each τ is a unit measure with finite logarithmic energy, which may be different for different $m \in \mathbb{N}$.

Considering the whole family of monic polynomials from $\mathbb{Z}_n^s(W_\gamma)$, we can always select a subsequence whose symmetric means provide the smallest limit point, and for which $\tau_n \overset{*}{\to} \tau$. Then (2.6) is satisfied and (4.1) gives us a (theoretic) lower bound for that limit point.

5. Proofs

Proof of Theorem 2.1. For notational convenience, we assume that τ_n converges to τ in the weak* topology along the whole sequence $\mathbb{N}$. Our first step is to prove that τ is a unit measure. If $\tau(\mathbb{C}) = \lambda < 1$ then

$$\limsup_{n \to \infty} \tau_n(\overline{D}_R) \leq \tau(\overline{D}_R) \leq \lambda$$

for any disk $D_R := \{z : |z| < R\}$, by the weak* convergence properties of [3, Theorem 2.1, p. 16]. Hence

$$\liminf_{n\to\infty} \tau_n(\mathbb{C} \setminus \overline{D}_R) \geq 1 - \lambda > 0 \quad \text{for all } R > 0.$$

Moreover, for any $\varepsilon > 0$ and any $R > 1$ there is $N \in \mathbb{N}$ such that

$$M(P_n) \geq \prod_{k=1}^{n} \max(1, |\alpha_{k,n}|) \geq \prod_{|\alpha_{k,n}|>R} |\alpha_{k,n}| \geq R^{n(1-\lambda-\varepsilon)}, \quad n \geq N.$$

The latter inequality is clearly incompatible with (2.2) when $R \to \infty$, so that $\tau(\mathbb{C}) = 1$ follows. Note that τ may have unbounded support.

Our goal now is to show that the restriction of τ to every disk has finite energy. Given $R > 1$, we define

$$\hat{\tau}_n := \tau_n|_{\overline{D}_R} = \frac{1}{n} \sum_{|\alpha_{k,n}|\leq R} \delta_{\alpha_{k,n}}.$$

Note that $\hat{\tau}_n \overset{*}{\to} \tau_R := \tau|_{\overline{D}_R}$ as $n \to \infty$ for any $R > 0$ such that $\tau(\partial D_R) = 0$, by [10, Theorem 0.5′, p. 10]. Hence $\hat{\tau}_n \overset{*}{\to} \tau_R$ for all but countably many $R > 1$, and we consider only such R below. Furthermore, it is sufficient to consider R with $\tau(\overline{D}_R) > 0$, for otherwise $I[\tau_R] = 0$ and (2.4) holds trivially. Let $\Delta(P_n) = a_n^{2n-2}(V(P_n))^2$ be the discriminant of P_n, where

$$V(P_n) := \prod_{1\leq j<k\leq n} (\alpha_{j,n} - \alpha_{k,n})$$

is the Vandermonde determinant. Since P_n has integer coefficients, $\Delta(P_n)$ is an integer (see [13, p. 24]). As P_n has simple roots, we obtain $\Delta(P_n) \neq 0$ and $|\Delta(P_n)| \geq 1$. We now order $\alpha_{k,n}$ as follows:

$$|\alpha_{1,n}| \leq \cdots \leq |\alpha_{m_n,n}| \leq R < |\alpha_{m_n+1,n}| \leq \cdots \leq |\alpha_{n,n}|.$$

Let $\hat{P}_n(z) := a_n \prod_{k=1}^{m_n}(z-\alpha_{k,n})$ and $V(\hat{P}_n) = \prod_{1\leq j<k\leq m_n}(\alpha_{j,n}-\alpha_{k,n})$. Hence

(5.1)

$$1 \leq |\Delta(P_n)| = |a_n|^{2n-2}|V(P_n)|^2 = |a_n|^{2n-2}|V(\hat{P}_n)|^2 \prod_{\substack{1\leq j<k \\ m_n<k\leq n}} |\alpha_{j,n} - \alpha_{k,n}|^2$$

$$\leq |a_n|^{2n-2}|V(\hat{P}_n)|^2 \prod_{m_n<k\leq n} (2|\alpha_{k,n}|)^{2(n-1)}$$

$$\leq |V(\hat{P}_n)|^2 4^{(n-1)(n-m_n)} \Big(|a_n| \prod_{m_n<k\leq n} |\alpha_{k,n}|\Big)^{2(n-1)},$$

where we have used $|\alpha_{j,n} - \alpha_{k,n}| \leq 2\max(|\alpha_{j,n}|, |\alpha_{k,n}|) = 2|\alpha_{k,n}|$. Equation

(2.2) implies that

$$(5.2) \quad \limsup_{n \to \infty} \left(|a_n| \prod_{m_n < k \le n} |\alpha_{k,n}| \right)^{1/n} \le \limsup_{n \to \infty} \left(M(P_n) \right)^{1/n} = H < \infty.$$

Note that $\liminf_{n \to \infty} m_n/n = \liminf_{n \to \infty} \tau_n(\overline{D}_R) = \tau(\overline{D}_R)$ by [3, Theorem 2.1], as $\tau(\partial D_R) = 0$. Thus we deduce from (5.1)–(5.2) that

$$(5.3) \quad \liminf_{n \to \infty} |V(\hat{P}_n)|^{\frac{2}{(n-1)n}} \ge 4^{\tau(\overline{D}_R)-1} \liminf_{n \to \infty} \left(|a_n| \prod_{m_n < k \le n} |\alpha_{k,n}| \right)^{-2/n}$$

$$\ge 4^{\tau(\overline{D}_R)-1} H^{-2}.$$

Let $K_M(z,t) := \min(-\log|z-t|, M)$. It is clear that $K_M(z,t)$ is a continuous function in z and t on $\mathbb{C} \times \mathbb{C}$, and that $K_M(z,t)$ increases to $-\log|z-t|$ as $M \to \infty$. Using the monotone convergence theorem and the weak* convergence of $\hat{\tau}_n \times \hat{\tau}_n$ to $\tau_R \times \tau_R$, we find for the energy of τ_R that

$$I[\tau_R] = -\int\int \log|z-t| \, d\tau_R(z) \, d\tau_R(t)$$

$$= \lim_{M \to \infty} \left(\lim_{n \to \infty} \int\int K_M(z,t) \, d\hat{\tau}_n(z) \, d\hat{\tau}_n(t) \right)$$

$$= \lim_{M \to \infty} \left(\lim_{n \to \infty} \left(\frac{2}{n^2} \sum_{1 \le j < k \le m_n} K_M(\alpha_{j,n}, \alpha_{k,n}) + \frac{M}{n} \right) \right)$$

$$\le \lim_{M \to \infty} \left(\liminf_{n \to \infty} \frac{2}{n^2} \sum_{1 \le j < k \le m_n} \log \frac{1}{|\alpha_{j,n} - \alpha_{k,n}|} \right) = \liminf_{n \to \infty} \frac{2}{n^2} \log \frac{1}{|V(\hat{P}_n)|}$$

$$\le (1 - \tau(\overline{D}_R)) \log 4 + 2 \log H,$$

where (5.3) has been used in the last estimate. This gives the upper bound in (2.4).

We continue with the same notation to prove (2.3). Note that

$$\int \log^+ |z| \, d\hat{\tau}_n = \frac{1}{n} \log \prod_{k=1}^{m_n} \max(1, |\alpha_{k,n}|) \le \log \left(M(P_n) \right)^{1/n}.$$

Extending $\log^+ |z|$ from $\overline{D}_R$ to a continuous nonnegative function with compact support in $\mathbb{C}$, we use the weak* convergence $\hat{\tau}_n \xrightarrow{*} \tau_R$ and (2.2) to obtain

$$\int \log^+ |z| \, d\tau_R = \lim_{n \to \infty} \int \log^+ |z| \, d\hat{\tau}_n \le \limsup_{n \to \infty} \log \left(M(P_n) \right)^{1/n} = \log H.$$

Thus (2.3) follows from the above inequality by letting $R \to \infty$.

The lower bound of (2.4) is a consequence of (2.3) and of the estimate

$$
\begin{aligned}
I[\tau_R] &= -\int\int \log|z-t|\, d\tau_R(z)\, d\tau_R(t) \\
&\geq -\int\int \log(2\max(|z|,|t|))\, d\tau_R(z)\, d\tau_R(t) \\
&\geq -\log 2 - 2\int\Big(\int_{|z|\geq|t|} \log|z|\, d\tau_R(z)\Big)\, d\tau_R(t) \\
&\geq -\log 2 - 2\int\Big(\int_{|z|\geq\max(1,|t|)} \log|z|\, d\tau_R(z)\Big)\, d\tau_R(t) \\
&\geq -\log 2 - 2\int \log H\, d\tau_R(t) = -\log 2 - 2\tau_R(\overline{D}_R)\log H,
\end{aligned}
$$

where we also use $|z-t|\leq 2\max(|z|,|t|)$ and $\tau_R(\overline{D}_R)\leq 1$. ∎

Proof of Corollary 2.2. We arrange the numbers $\alpha_{k,n}$ as follows:

$$
|\alpha_{1,n}| \geq \cdots \geq |\alpha_{K_n,n}| \geq 1 > |\alpha_{K_n+1,n}| \geq \cdots \geq |\alpha_{n,n}|.
$$

Starting with the case $m = 1$, we apply the arithmetic-geometric mean inequality to obtain

$$
(5.4) \quad \Big(\prod_{k=1}^{K_n} |\alpha_{k,n}|\Big)^{1/n} \leq \Big(\frac{1}{K_n}\sum_{k=1}^{K_n} |\alpha_{k,n}|\Big)^{K_n/n}
$$

$$
\leq \Big(\frac{n}{K_n}\Big)^{K_n/n}\Big(\frac{1}{n}\sum_{k=1}^{n} |\alpha_{k,n}|\Big)^{K_n/n} \leq e^{1/e}\Big(\frac{1}{n}\sum_{k=1}^{n} |\alpha_{k,n}|\Big)^{K_n/n},
$$

where we have also used the fact $\max_{x\in[0,\infty)} x^{1/x} = e^{1/e}$ with $x = n/K_n$. Hence

$$
\limsup_{\mathcal{N}\ni n\to\infty} (M(P_n))^{1/n} \leq \limsup_{\mathcal{N}\ni n\to\infty} |a_n|^{1/n} \limsup_{\mathcal{N}\ni n\to\infty}\Big(\prod_{k=1}^{K_n} |\alpha_{k,n}|\Big)^{1/n} < \infty
$$

by the assumptions and (5.4), so that (2.2) holds. The conclusion of this corollary for $m = 1$ now follows from Theorem 2.1.

If the symmetric means in the assumptions are bounded for a fixed $m \geq 2$, then we consider two different cases. Suppose first that $m \leq K_n \leq n$. We argue similarly to (5.4), and apply (2.1) to estimate

$$
\Big(\prod_{k=1}^{K_n} |\alpha_{k,n}|\Big)^{1/n} \leq (S_m(|\alpha_{1,n}|,\ldots,|\alpha_{K_n,n}|))^{K_n/(mn)}
$$

$$
\leq \Big(\binom{n}{m}\Big/\binom{K_n}{m}\Big)^{K_n/(mn)} (S_m(|\alpha_{1,n}|,\ldots,|\alpha_{n,n}|))^{K_n/(mn)}.
$$

Note that

$$\left(\binom{n}{m}\Big/\binom{K_n}{m}\right)^{K_n/(mn)} \leq \left(\frac{n}{K_n - m + 1}\right)^{K_n/n} \leq \left(\frac{n}{K_n}\right)^{K_n/n} \frac{K_n}{K_n - m + 1}$$
$$\leq e^{1/e}m.$$

Hence $\limsup_{n\to\infty} (M(P_n))^{1/n} < \infty$ as before, and we apply Theorem 2.1.

The remaining case is when $K_n < m \leq n$. We can assume that the constant terms a_0 in the polynomials P_n are nonzero, for otherwise we can replace P_n with $P_n(z)/z$. Thus we assume that $|a_0| \geq 1$, which implies that

$$(5.5) \qquad \prod_{k=K_n+1}^{n} |\alpha_{k,n}| = \frac{|a_0|}{|a_n| \prod_{k=1}^{K_n} |\alpha_{k,n}|} \geq \frac{1}{|a_n| \prod_{k=1}^{K_n} |\alpha_{k,n}|}.$$

For $n \geq 2m$, we have

$$\prod_{k=K_n+1}^{n} |\alpha_{k,n}| = \prod_{k=K_n+1}^{m} |\alpha_{k,n}| \prod_{k=m+1}^{n} |\alpha_{k,n}| \leq \left(\prod_{k=K_n+1}^{m} |\alpha_{k,n}|\right)^2.$$

It follows from (5.5) and the above estimate that

$$\prod_{k=1}^{K_n} |\alpha_{k,n}| = \frac{\prod_{k=1}^{m} |\alpha_{k,n}|}{\prod_{k=K_n+1}^{m} |\alpha_{k,n}|} \leq \left(|a_n| \prod_{k=1}^{K_n} |\alpha_{k,n}|\right)^{1/2} \prod_{k=1}^{m} |\alpha_{k,n}|,$$

and therefore

$$\prod_{k=1}^{K_n} |\alpha_{k,n}| \leq |a_n| \left(\prod_{k=1}^{m} |\alpha_{k,n}|\right)^2 \leq |a_n| \left(\binom{n}{m} S_m(|\alpha_{1,n}|, \ldots, |\alpha_{n,n}|)\right)^2.$$

Finally,

$$\limsup_{\mathcal{N}\ni n\to\infty} \left(\prod_{k=1}^{K_n} |\alpha_{k,n}|\right)^{1/n}$$
$$\leq \limsup_{\mathcal{N}\ni n\to\infty} |a_n|^{1/n} \limsup_{\mathcal{N}\ni n\to\infty} (S_m(|\alpha_{1,n}|, \ldots, |\alpha_{n,n}|))^{2/n} < \infty,$$

so that $\limsup_{\mathcal{N}\ni n\to\infty} (M(P_n))^{1/n} < \infty$, and the result follows from Theorem 2.1 again. ∎

Proof of Theorem 2.3. We show how to approximate an arbitrary Borel measure μ, $0 \leq \mu(\mathbb{C}) \leq 1$, by counting measures of algebraic integers in the weak* topology. This is sketched below in several steps of reduction for the problem. The case $\mu(\mathbb{C}) = 0$ is covered by the example given after Theorem 2.3. Thus we assume that μ is supported in a compact set E and $\mu(E) > 0$ without loss of generality. Otherwise one can apply the approximations con-

structed below to the restrictions of μ on a family of expanding disks filling the plane, exactly as mentioned before Theorem 2.1, and obtain the desired weak* convergent approximations for μ in $\mathbb{C}$.

The first step of our approximation scheme uses the Krein–Milman theorem [25, pp. 362–363] to express μ as the weak* limit of linear convex combinations of point masses of the form

$$\sum_{j=1}^{J} t_j \delta_{z_j}, \quad t_j > 0, \quad \sum_{j=1}^{J} t_i = \mu(E) = \mu(\mathbb{C}) \le 1,$$

where $\{z_j\}_{j=1}^{J} \subset E$ are symmetric about the real line. The symmetry requirement is achieved by applying the Krein–Milman theorem to the restriction of μ to the upper half-plane, and adding the reflection of the discrete measure thus obtained to approximate μ in the lower half-plane part of its support.

The set of coefficients $t_j \in \mathbb{R}$, $j = 1, \ldots, J$, can be simultaneously approximated by the rational numbers $k_j/L < t_j$ with the same denominator $L \in \mathbb{N}$. Therefore, we can select a set of k_j points $w_{i,j}$, $i = 1, \ldots, k_j$, on a sufficiently small circle around z_j, for each $j = 1, \ldots, J$, so that the measure $(1/L) \sum_{i=1}^{k_j} \delta_{w_{i,j}}$ gives an arbitrarily close approximation to $t_j \delta_{z_j}$. Furthermore, this can clearly be done so that all points $w_{i,j}$ are distinct and symmetric about the real axis. It remains to approximate the resulting discrete measure

$$\nu_K = \frac{1}{L} \sum_{j=1}^{J} \sum_{i=1}^{k_j} \delta_{w_{i,j}}, \quad K := \sum_{j=1}^{J} k_j,$$

by counting measures for algebraic integers in the third step.

Note that the measure ν_K is a counting (but not unit) measure for the points $w_{i,j}$ totalling $K < L$. Therefore, we can approximate it by a sequence of counting measures τ_{K+1} for the complete sets of conjugate algebraic integers $\{\alpha_{l,K+1}\}_{l=1}^{K+1}$ by using the theorem of Motzkin [12] (see also [5] for its effective version). For any $K \in \mathbb{N}$, we approximate each point $w_{i,j}$ as close as we wish by one of the conjugate algebraic integers $\alpha_{l,K+1}$, $1 \le l \le K$, obtained from Motzkin's theorem, while we let the remaining $(K + 1)$st conjugate algebraic integer $\alpha_{K+1,K+1}$ tend to ∞ as $K \to \infty$ (cf. [12, pp. 160–161] for details). It follows that the resulting measures

$$\tau_{K+1} = \frac{1}{L} \sum_{l=1}^{K+1} \delta_{\alpha_{l,K+1}}$$

converge to μ in the weak* topology as $K \to \infty$. ∎

Proof of Theorem 2.4. Assume that $\{z_{k,n}\}_{k=1}^n \subset D_R$ and that $\tau_n \overset{*}{\to} \tau$ for a subsequence $n \in \mathcal{N} \subset \mathbb{N}$. The multinomial theorem gives

$$\Big(\sum_{j=1}^n z_{j,n}\Big)^m = \sum_{k_1+\cdots+k_n=m} \binom{m}{k_1,\ldots,k_n} z_{1,n}^{k_1} \cdots z_{n,n}^{k_n}$$

$$= m!\,\sigma_m(z_{1,n},\ldots,z_{n,n}) + \sum_{\substack{k_1+\cdots+k_n=m \\ \exists k_j \geq 2}} \binom{m}{k_1,\ldots,k_n} z_{1,n}^{k_1} \cdots z_{n,n}^{k_n}.$$

Note that the sum s of multinomial coefficients in the latter summation can be found by setting $z_{j,n} = 1$, $j = 1,\ldots,n$, in the above formula, which gives

$$n^m = m!\binom{n}{m} + s = \prod_{j=0}^{m-1}(n-j) + s.$$

It follows that $s = O\left(n^{m-1}\right)$ as $n \to \infty$, and

$$\Big|\Big(\sum_{j=1}^n z_{j,n}\Big)^m - m!\,\sigma_m(z_{1,n},\ldots,z_{n,n})\Big| \leq R^m\,O(n^{m-1}) \quad \text{as } n \to \infty.$$

Since $\lim_{n\to\infty} m!\binom{n}{m}/n^m = 1$, we can divide the above equation by n^m and let $n \to \infty$ to obtain the first equality in (2.7), provided one of the limits in (2.7) exists. But

$$\lim_{\mathcal{N}\ni n\to\infty} S_1(z_{1,n},\ldots,z_{n,n}) = \int z\, d\tau(z)$$

is an easy consequence of the weak* convergence $\tau_n \overset{*}{\to} \tau$ for $n \in \mathcal{N}$. ∎

Proof of Theorem 2.5. Consider the half-plane $H_a = \{z \in \mathbb{C} : \Re z \leq a\}$. Given a fixed $a > 0$, we arrange each set $\{z_{k,n}\}_{k=1}^n$ in the order of increasing real parts:

$$\Re z_{1,n} \leq \cdots \leq \Re z_{l_n,n} \leq a < \Re z_{l_n+1,n} \leq \cdots \leq \Re z_{n,n}.$$

Define $\hat{\tau}_n := \tau_n|_{H_a}$. It follows that

$$\hat{\tau}_n \overset{*}{\to} \tau_a := \tau|_{H_a}, \quad n \in \mathcal{N}, \quad \text{and} \quad \lim_{\mathcal{N}\ni n\to\infty} l_n/n = \tau(H_a)$$

for any $a > 0$ such that $\tau(\{z : \Re z = a\}) = 0$, by [10, Theorem 0.5$'$] and [3, Theorem 2.1]. Hence the above equation holds for all but countably many $a > 0$, and we consider only such a below.

Observe that $S_m(z_{1,n},\ldots,z_{n,n}) \geq 0$ and $S_m(z_{1,n},\ldots,z_{l_n,n}) \geq 0$ for $\{z_{k,n}\}_{k=1}^n \subset V_m$ because all terms in these sums have positive real parts, and their imaginary parts cancel due to symmetry of $\{z_{k,n}\}_{k=1}^n$. For the same reason, we have $\sigma_m(z_{1,n},\ldots,z_{n,n}) \geq \sigma_m(z_{1,n},\ldots,z_{l_n,n})$, so that

$$S_m(z_{1,n},\ldots,z_{n,n}) \geq S_m(z_{1,n},\ldots,z_{l_n,n})\binom{l_n}{m}/\binom{n}{m}.$$

Since
$$\lim_{n\to\infty} \binom{l_n}{m} \bigg/ \binom{n}{m} = \lim_{n\to\infty} (l_n/n)^m = (\tau(H_a))^m,$$

the weak* convergence $\hat{\tau}_n \overset{*}{\to} \tau_a$, $n \in \mathcal{N}$, and Theorem 2.4 yield
$$\liminf_{\mathcal{N}\ni n\to\infty} S_m(z_{1,n}, \ldots, z_{n,n}) \geq (\tau(H_a))^m \liminf_{\mathcal{N}\ni n\to\infty} S_m(z_{1,n}, \ldots, z_{l_n,n})$$
$$= (\tau(H_a))^m \left(\int z \, d\tau_a(z)\right)^m.$$

The concluding step is to let $a \to \infty$ in the above inequality, using
$$\lim_{a\to\infty} \tau(H_a) = \tau(\mathbb{C}) = 1 \quad \text{and} \quad \lim_{a\to\infty} \int z \, d\tau_a(z) = \int z \, d\tau(z). \quad \blacksquare$$

Proof of Theorem 2.6. Equation (2.9) implies that $\tau_n \overset{*}{\to} \mu_E$ by [15, Theorem 2.1], where μ_E is the equilibrium measure of E. Hence
$$\liminf_{\mathcal{N}\ni n\to\infty} S_m(\alpha_{1,n}, \ldots, \alpha_{n,n}) \geq \left(\int z \, d\mu_E(z)\right)^m, \quad m \in \mathbb{N},$$

by Theorem 2.5. Lower bounds for the moments of equilibrium measures were established in [2]. We apply the change of variable $x = t^2$, and define the compact set $K = \{t \in \mathbb{R} : t^2 \in E\}$. Then K is symmetric about the origin, so that $\int t \, d\mu_K(t) = 0$. Furthermore, $d\mu_K(t) = d\mu_E(t^2)$, $t \in K$, and $\text{cap}(K) = 1$ (see [18, p. 134]). It now follows from [2, Theorem 1] that
$$\int x \, d\mu_E(x) = \int t^2 \, d\mu_K(t) \geq \int t^2 \, d\mu_{[-2,2]}(t) = \int_{-2}^{2} \frac{t^2 \, dt}{\pi\sqrt{4 - t^2}} = 2,$$

so that (2.10) is proved.

We discussed in Section 1 that the Chebyshev polynomials $t_n(x) := 2\cos(n \arccos((x - 2)/2))$ for $[0, 4]$ satisfy the assumptions of this theorem. Moreover, $\tau_n \overset{*}{\to} \mu_{[0,4]} = dx/(\pi\sqrt{x(4 - x)})$ as $n \to \infty$, which gives
$$\lim_{n\to\infty} S_1(\alpha_{1,n}, \ldots, \alpha_{n,n}) = \int_{0}^{4} \frac{x \, dx}{\pi\sqrt{x(4 - x)}} = 2.$$

Combining this with (2.1) and (2.10), we immediately see that equality holds in (2.10) for all $m \in \mathbb{N}$. $\blacksquare$

Proof of Theorem 3.2. Using convexity of ϕ, we have
$$\limsup_{\mathcal{N}\ni n\to\infty} \phi\left(\frac{1}{n} \sum_{k=1}^{n} |\alpha_{k,n}|\right) \leq \limsup_{\mathcal{N}\ni n\to\infty} C_n^\phi(|\alpha_{1,n}|, \ldots, |\alpha_{n,n}|) < \infty.$$

Since ϕ is also strictly increasing, we conclude that $\lim_{x\to\infty} \phi(x) = \infty$. Hence

the above inequality yields

$$\limsup_{\mathcal{N} \ni n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\alpha_{k,n}| < \infty.$$

Thus all assumptions of Corollary 2.2 are satisfied with $m = 1$, and the desired conclusion follows. ∎

Proof of Theorem 4.1. Since (2.6) is satisfied, Corollary 2.2 shows that any weak* limit of counting measures τ_n for our sequence must be a unit measure with finite logarithmic energy. The inequality $\gamma \le \pi/(2m)$ ensures that $W_\gamma \subset V_m$ for each such $m \in \mathbb{N}$. Hence all conditions of Theorem 2.5 are satisfied, and (2.8) gives us (4.1), where τ is a weak* limit of τ_n along a subsequence that attains the value of $\liminf_{\mathcal{N} \ni n \to \infty} S_m(\alpha_{1,n}, \dots, \alpha_{n,n})$. Note that τ is supported in W_γ, and is different from the point mass δ_0 at the origin, because τ has finite energy. Hence $\int z \, d\tau(z) > 0$. ∎

References

[1] J. Aguirre and J. C. Peral, *The trace problem for totally positive algebraic integers*, in: Number Theory and Polynomials (Bristol, 2006), J. McKee and C. Smyth (eds.), London Math. Soc. Lecture Note Ser. 352, Cambridge Univ. Press, Cambridge, 2008, 1–19.

[2] A. Baernstein II, R. S. Laugesen and I. E. Pritsker, *Moment inequalities for equilibrium measures in the plane*, Pure Appl. Math. Quart. 7 (2011), 51–86.

[3] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1999.

[4] P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.

[5] A. Dubickas, *Conjugate algebraic numbers close to a symmetric set*, St. Petersburg Math. J. 16 (2005), 1013–1016.

[6] V. Flammang, *On the absolute trace of polynomials having all zeros in a sector*, Experiment. Math. 17 (2008), 443–450.

[7] V. Flammang, *Trace of totally positive algebraic integers and integer transfinite diameter*, Math. Comp. 78 (2009), 1119–1125.

[8] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

[9] L. Kronecker, *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*, J. Reine Angew. Math. 53 (1857), 173–175.

[10] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer, New York, 1972.

[11] J. McKee, *Computing totally positive algebraic integers of small trace*, Math. Comp. 80 (2011), 1041–1052.

[12] Th. Motzkin, *From among n conjugate algebraic integers, n−1 can be approximately given*, Bull. Amer. Math. Soc. 53 (1947), 156–162.

[13] V. V. Prasolov, *Polynomials*, Springer, Berlin, 2004.

[14]   I. E. Pritsker, *Means of algebraic numbers in the unit disk*, C. R. Math. Acad. Sci. Paris 347 (2009), 119–122.
[15]   I. E. Pritsker, *Distribution of algebraic numbers*, J. Reine Angew. Math. 657 (2011), 57–80.
[16]   I. E. Pritsker, *Equidistribution of points via energy*, Ark. Mat. 49 (2011), 149–173.
[17]   I. E. Pritsker, *Polynomials with integer coefficients and their zeros*, J. Math. Sci. (N.Y.) 183 (2012), 810–822.
[18]   T. Ransford, *Potential Theory in the Complex Plane*, Cambridge Univ. Press, Cambridge, 1995.
[19]   R. Rumely, *On Bilu's equidistribution theorem*, in: Spectral Problems in Geometry and Arithmetic (Iowa City, IA, 1997), Contemp. Math. 237, Amer. Math. Soc., Providence, RI, 1999, 159–166.
[20]   I. Schur, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. 1 (1918), 377–402.
[21]   C. L. Siegel, *The trace of totally positive and real algebraic integers*, Ann. of Math. 46 (1945), 302–312.
[22]   C. J. Smyth, *Totally positive algebraic integers of small trace*, Ann. Inst. Fourier (Grenoble) 34 (1984), no. 3, 1–28.
[23]   C. J. Smyth, *The mean values of totally real algebraic integers*, Math. Comp. 42 (1984), 663–681.
[24]   M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea, New York, 1975.
[25]   K. Yosida, *Functional Analysis*, 5th ed., Springer, Berlin, 1978.

Igor E. Pritsker
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078, U.S.A.
E-mail: igor@math.okstate.edu