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A functional relation for Tornheim's double zeta functions

by

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1. Introduction. Tornheim's double zeta function is defined as

$$\zeta(s,t;u)=\sum_{m,n=1}^{\infty}\frac{1}{m^s n^t (m+n)^u}$$

for $(s,t,u)\in\mathbb{C}^3$ with $\mathrm{Re}(s+u)>1$, $\mathrm{Re}(t+u)>1$ and $\mathrm{Re}(s+t+u)>2$. It is known, from Matsumoto [8, Theorem 1], that $\zeta(s,t;u)$ can be meromorphically continued to the whole space $\mathbb{C}^3$, and its singularities are located on the subsets of $\mathbb{C}^3$ defined by one of the equations $s+u=1-l$, $t+u=1-l$ $(l=0,1,2,\dots)$ and $s+t+u=2$. This function can be regarded as a generalization of some well-known zeta functions: the product of two Riemann zeta functions $\zeta(s)\zeta(t)=\zeta(s,t;0)$, the Euler double zeta function $\zeta(u,t)=\zeta(0,t;u)$ and the SU(3)-type Witten zeta function $\zeta_{\mathrm{SU}(3)}(s)=2^s\zeta(s,s;s)$. Euler and Tornheim [14] and many other people gave a lot of relations between the values $\zeta(s,t;u)$ for triples (s,t,u) of non-negative integers in the domain of convergence, but few relations as functions of complex variables have been found. As an exception, Tsumura [15, Theorem 4.5] represented explicitly the function

$$Z(s,t;u)=\zeta(s,t;u)+\cos(\pi t)\zeta(t,u;s)+\cos(\pi s)\zeta(u,s;t) \tag{1.1}$$

in terms of the Riemann zeta function, when $s,t\in\mathbb{Z}_{\geq 0}$, $t\geq 2$ and $u\in\mathbb{C}$, except for singularities. Afterward Nakamura [10, Theorem 1.2] gave a simpler version: for $s,t\in\mathbb{Z}_{\geq 1}$ and all $u\in\mathbb{C}$ except for the singular points,

$$\begin{aligned} Z(s,t;u)&=2\sum_{h=0}^{[s/2]}\binom{s+t-2h-1}{s-2h}\zeta(2h)\zeta(s+t+u-2h)\\ &\quad+2\sum_{k=0}^{[t/2]}\binom{s+t-2k-1}{t-2k}\zeta(2k)\zeta(s+t+u-2k), \end{aligned} \tag{1.2}$$

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where $[x]$ for $x \in \mathbb{R}$ denotes the greatest integer not exceeding x. This result seems really fascinating because it contains most of the known relations between the values $\zeta(s,t;u)$ for $s,t,u \in \mathbb{Z}_{\geq 1}$ and the Riemann zeta values (see [10, §3]). The aim of this paper is to generalize it to a relation between Tornheim's double zeta functions of three complex variables. Our main result is

Theorem. *The following relation holds on the whole space $\mathbb{C}^3$ except for the singular points of both sides:*

$$Z(s,t;u) = A(s,t;u) + A(t,s;u),$$

where, for $(s,t,u) \in \mathbb{C}^3$ with $s, -t, 1-t-u \neq 0,1,2,\dots$,

(1.3)
$$A(s,t;u) = \frac{\sin(\pi s)}{2\pi i} \int_L \cot\left(\frac{\pi(s-\eta)}{2}\right) \frac{\Gamma(t+\eta)\Gamma(-\eta)}{\Gamma(t)} \zeta(s-\eta)\zeta(t+u+\eta)\, d\eta.$$

Here, the contour L is a line from $-i\infty$ to $i\infty$ indented in such a manner as to separate the poles at $\eta = s-2n, -t-n, 1-t-u$ $(n = 0,1,2,\dots)$ from the poles at $\eta = 0,1,2,\dots$.

Remark 1.1. The singularities of $A(s,t;u)$ are located only on the subsets of $\mathbb{C}^3$ defined by the equations $t+u = 1-l$ $(l = 0,1,2,\dots)$. This can be easily seen by shifting the contour L as follows. Let K be a non-negative integer. If $\mathrm{Re}(s) < K+1/2$, $-K-1/2 < \mathrm{Re}(t)$ and $-K+1/2 < \mathrm{Re}(t+u)$, then

$$\begin{aligned}(1.4)\qquad A(s,t;u) &= 2\sum_{k=0}^{K} \frac{(t)_k}{k!} \cos^2\left(\frac{\pi(s-k)}{2}\right) \zeta(s-k)\zeta(t+u+k) \\ &\quad + \frac{\sin(\pi s)}{2\pi i} \int_{L_K} \cot\left(\frac{\pi(s-\eta)}{2}\right) \frac{\Gamma(t+\eta)\Gamma(-\eta)}{\Gamma(t)} \zeta(s-\eta)\zeta(t+u+\eta)\, d\eta,\end{aligned}$$

where $(t)_k = \Gamma(t+k)/\Gamma(t)$ and L_K is the vertical line from $K+1/2-i\infty$ to $K+1/2+i\infty$. Also, from this, it is clear that our functional relation is a generalization of (1.2) (see (4.2)).

Remark 1.2. Since

$$\begin{pmatrix} Z(s,t;u) \\ Z(t,u;s) \\ Z(u,s;t) \end{pmatrix} = \begin{pmatrix} 1 & \cos(\pi t) & \cos(\pi s) \\ \cos(\pi t) & 1 & \cos(\pi u) \\ \cos(\pi s) & \cos(\pi u) & 1 \end{pmatrix} \begin{pmatrix} \zeta(s,t;u) \\ \zeta(t,u;s) \\ \zeta(u,s;t) \end{pmatrix},$$

we can write $\zeta(s,t;u)$ in terms of the Z-function as

$$\begin{aligned}(1.5)\qquad \Delta(s,t,u)\zeta(s,t;u) &= (1-\cos^2(\pi u))Z(s,t;u) \\ &\quad + (\cos(\pi s)\cos(\pi u) - \cos(\pi t))Z(t,u;s) \\ &\quad + (\cos(\pi t)\cos(\pi u) - \cos(\pi s))Z(u,s;t),\end{aligned}$$

where

$$\varDelta(s,t,u)=1-\cos^2(\pi s)-\cos^2(\pi t)-\cos^2(\pi u)+2\cos(\pi s)\cos(\pi t)\cos(\pi u),$$

and so the Theorem gives a new integral representation of $\zeta(s,t;u)$. Some special cases will be displayed in Proposition 4.1.

In this paper, to prove the Theorem, we employ Li's method [7] which gave a simple proof of (1.2). In §2, we will generalize some partial fraction decompositions used there to a form usable in our case. We will give a proof of the Theorem in §3 and exhibit its applications in §4. In the Appendix, we will prove a functional equation for Euler's double zeta function.

2. Generalized partial fraction decomposition. The following partial fraction decomposition plays a fundamental role in the theory of multiple zeta values: for two independent variables p,q and two positive integers s,t,

$$\frac{1}{p^s q^t}=\sum_{h=0}^{s-1}\frac{(t)_h}{h!}\,\frac{1}{p^{s-h}(p+q)^{t+h}}+\sum_{k=0}^{t-1}\frac{(s)_k}{k!}\,\frac{1}{q^{t-k}(p+q)^{s+k}} \tag{2.1}$$

(see [5], for instance). In this section, we will give two partial fraction decompositions in the case of s,t being complex numbers.

Lemma 2.1. *Let p,q be positive real numbers and let s,t be complex numbers whose real parts are positive. If $s,t\neq 1,2,\dots,$ then*

$$\frac{\varGamma(s)\varGamma(t)}{p^s q^t}=I(s,t;p,r)+I(t,s;q,r), \tag{2.2}$$

where $r=p+q$ and

$$I(s,t;p,r)=\frac{1}{2\pi i}\int_{L_{s,t}}\frac{\varGamma(1-s+\eta)\varGamma(-\eta)}{\varGamma(1-s)}\,\frac{\varGamma(s-\eta)}{p^{s-\eta}}\,\frac{\varGamma(t+\eta)}{r^{t+\eta}}\,d\eta. \tag{2.3}$$

Here, the contour $L_{s,t}$ is a line from $-i\infty$ to $i\infty$ indented in such a manner as to separate the points $\eta=s-1-m,-t-m$ $(m=0,1,2,\dots)$ from the points $\eta=s+n,n$ $(n=0,1,2,\dots)$.

Proof. From the usual integral representation of the gamma function, it follows that

$$I(s,t;p,r)=\frac{1}{2\pi i}\int_{L_{s,t}}\frac{\varGamma(1-s+\eta)\varGamma(-\eta)}{\varGamma(1-s)}\Big(\iint_{(\mathbb{R}_{>0})^2}e^{-p\mu-r\nu}\mu^{s-\eta-1}\nu^{t+\eta-1}\,d\mu\,d\nu\Big)\,d\eta.$$

By a suitable choice of $L_{s,t}$, it can be shown that the integrations can be interchanged. Hence,

$$I(s,t;p,r) = \iint_{(\mathbb{R}_{>0})^2} e^{-p\mu-r\nu}\mu^{s-1}\nu^{t-1}\left(\frac{1}{2\pi i}\int_{L_{s,t}} \frac{\Gamma(1-s+\eta)\Gamma(-\eta)}{\Gamma(1-s)}(\nu/\mu)^{\eta}\,d\eta\right) d\mu\,d\nu.$$

Since the innermost integral is $(1+\nu/\mu)^{s-1}$ (see [17, §14.51, Corollary]), we obtain

$$\begin{aligned} I(s,t;p,r) &= \iint_{(\mathbb{R}_{>0})^2} e^{-p\mu-r\nu}(\mu+\nu)^{s-1}\nu^{t-1}\,d\mu\,d\nu \\ &= \iint_{0<\nu<\mu} e^{-p\mu-q\nu}\mu^{s-1}\nu^{t-1}\,d\mu\,d\nu. \end{aligned}$$

We note that

$$I(t,s;q,r) = \iint_{0<\mu<\nu} e^{-p\mu-q\nu}\mu^{s-1}\nu^{t-1}\,d\mu\,d\nu,$$

and so the right hand side of (2.2) is equal to

$$\iint_{(\mathbb{R}_{>0})^2} e^{-p\mu-q\nu}\mu^{s-1}\nu^{t-1}\,d\mu\,d\nu = \frac{\Gamma(s)\Gamma(t)}{p^s q^t}.$$

This is the desired result. ■

LEMMA 2.2. *Let p,q,s,t be as in Lemma 2.1. If $p<q$ and $s,t \neq 1,2,\dots,$ then*

$$\frac{\cos(\pi s)\Gamma(s)\Gamma(t)}{p^s q^t} = J(s,t;p,q-p) + I(t,s;q,q-p), \tag{2.4}$$

where

$$J(s,t;p,q-p) = \frac{1}{2\pi i}\int_{L_{s,t}} \frac{\Gamma(1-s+\eta)\Gamma(-\eta)}{\Gamma(1-s)}\,\frac{\cos(\pi(s-\eta))\Gamma(s-\eta)}{p^{s-\eta}}\,\frac{\Gamma(t+\eta)}{(q-p)^{t+\eta}}\,d\eta.$$

Proof. For any $p,r \in \mathbb{C}^{\times}$, the integrand in (2.3) is

$$\ll |\eta|^{\mathrm{Re}(t)-1} e^{-2\pi|\eta|} e^{\mathrm{Im}(\eta)(\arg r - \arg p)} |p^{-s}|\,|r^{-t}|$$

as $\eta \to \pm i\infty$ on $L_{s,t}$, where the implied constant does not depend on p,r,η. This estimate ensures that, for any fixed $q \in \mathbb{R}_{>0}$, the right hand side of (2.2) can be continued to $\mathbb{C} \setminus \{\pm i\mathbb{R}_{\geq 0} \cup (-q \pm i\mathbb{R}_{\geq 0})\}$ as a holomorphic function in p, where the double signs correspond, and hence, if $0<p<q$, then

$$\frac{e^{\pm\pi i s}\Gamma(s)\Gamma(t)}{p^s q^t} = \frac{\Gamma(s)\Gamma(t)}{(-p)^s q^t} = I(s,t;-p,q-p) + I(t,s;q,q-p)$$

and

$$I(s,t;-p,q-p) = \frac{1}{2\pi i}\int\limits_{L_{s,t}} \frac{\Gamma(1-s+\eta)\Gamma(-\eta)}{\Gamma(1-s)} \frac{e^{\pm\pi i(s-\eta)}\Gamma(s-\eta)}{p^{s-\eta}} \frac{\Gamma(t+\eta)}{(q-p)^{t+\eta}}\,d\eta.$$

Thus, we obtain Lemma 2.2. ∎

3. Proof of Theorem. For simplicity of description, we suppose that $\mathrm{Re}(s),\mathrm{Re}(t),\mathrm{Re}(u) > 2$, $s,t \neq 3,4,5,\ldots$, and the contour $L_{s,t}$ always satisfies the condition $-1/2 \le \mathrm{Re}(\eta) \le \mathrm{Re}(s) - 1/2$ for all $\eta \in L_{s,t}$. We first evaluate

$$\zeta(s,t;u) = \sum_{m,n=1}^{\infty} \frac{1}{m^s n^t (m+n)^u}.$$

Applying Lemma 2.1 with $(p,q) = (m,n)$, we see that

$$\zeta(s,t;u) = X(s,t;u) + X(t,s;u),$$

where

$$X(s,t;u) = \frac{1}{2\pi i} \sum_{m,n=1}^{\infty} \int\limits_{L_{s,t}} \frac{\Gamma(s,t;\eta)}{m^{s-\eta}(m+n)^{t+u+\eta}}\,d\eta,$$

$$\Gamma(s,t;\eta) = \frac{\Gamma(1-s+\eta)\Gamma(-\eta)\Gamma(s-\eta)\Gamma(t+\eta)}{\Gamma(1-s)\Gamma(s)\Gamma(t)}.$$

From the condition of $L_{s,t}$, it follows that the summation and integration in $X(s,t;u)$ can be interchanged. As a result,

$$\begin{aligned} X(s,t;u) &= \frac{1}{2\pi i}\int\limits_{L_{s,t}} \Gamma(s,t;\eta)\zeta(s-\eta,0;t+u+\eta)\,d\eta \\ &= \frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}\zeta(1,0;s+t+u-1) \\ &\quad + \frac{1}{2\pi i}\int\limits_{L_{s-1,t}} \Gamma(s,t;\eta)\zeta(s-\eta,0;t+u+\eta)\,d\eta. \end{aligned} \tag{3.1}$$

We next treat $\cos(\pi t)\zeta(t,u;s) + \cos(\pi s)\zeta(u,s;t)$. Set

$$a_{m,n}(s,t;u) = \frac{\cos(\pi t)}{m^t n^u (m+n)^s} + \frac{\cos(\pi s)}{n^u m^s (m+n)^t}$$

for $m,n \in \mathbb{Z}_{\ge 1}$. Applying Lemma 2.2 to each term, we obtain

$$a_{m,n}(s,t;u) = b_{m,n}(s,t;u) + b_{m,n}(t,s;u),$$

where

$$
\begin{aligned}
b_{m,n}&(s,t;u)\\
&= \frac{I(s,t;m+n,n)+J(s,t;m,n)}{\Gamma(s)\Gamma(t)n^u}\\
&= -\frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}\frac{1}{n^{s+t+u-1}}\left(\frac{1}{m}-\frac{1}{m+n}\right)\\
&\quad+\frac{1}{2\pi i}\int\limits_{L_{s-1,t}}\Gamma(s,t;\eta)\left(\frac{1}{n^{t+u+\eta}(m+n)^{s-\eta}}+\frac{\cos(\pi(s-\eta))}{m^{s-\eta}n^{t+u+\eta}}\right)d\eta.
\end{aligned}
$$

Put $Y(s,t;u)=\sum_{m,n=1}^{\infty}b_{m,n}(s,t;u)$. Then it is easily seen that

$$\cos(\pi t)\zeta(t,u;s)+\cos(\pi s)\zeta(u,s;t)=Y(s,t;u)+Y(t,s;u),$$

and

$$
\begin{aligned}
(3.2)\qquad Y(s,t;u) &= -\frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}(\zeta(1,0;s+t+u-1)+\zeta(s+t+u))\\
&\quad+\frac{1}{2\pi i}\int\limits_{L_{s-1,t}}\Gamma(s,t;\eta)\big\{\zeta(t+u+\eta,0;s-\eta)\\
&\qquad\qquad+\cos(\pi(s-\eta))\zeta(s-\eta)\zeta(t+u+\eta)\big\}\,d\eta.
\end{aligned}
$$

Combining (3.1) and (3.2), we have

$$
\begin{aligned}
(3.3)\qquad X(s,t;u)&+Y(s,t;u)\\
&= -\frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}\zeta(s+t+u)\\
&\quad+\frac{1}{2\pi i}\int\limits_{L_{s-1,t}}\Gamma(s,t;\eta)\{1+\cos(\pi(s-\eta))\}\zeta(s-\eta)\zeta(t+u+\eta)d\eta\\
&\quad-\frac{1}{2\pi i}\int\limits_{L_{s-1,t}}\Gamma(s,t;\eta)d\eta\,\zeta(s+t+u)
\end{aligned}
$$

because generally

$$(3.4)\qquad \zeta(s,0;t)+\zeta(t,0;s)=\zeta(s)\zeta(t)-\zeta(s+t).$$

The second term on the right hand side of (3.3) becomes

$$\frac{\Gamma(s+t)}{s\Gamma(s)\Gamma(t)}\zeta(s+t+u)+A(s,t;u)$$

by shifting the contour to $L_{s+1,t}$, and the third term is

$$
\begin{aligned}
\frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}\zeta(s+t+u)&-\frac{1}{2\pi i}\int\limits_{L_{s,t}}\Gamma(s,t;\eta)\,d\eta\,\zeta(s+t+u)\\
&=\frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}\zeta(s+t+u)-\frac{\Gamma(s+t)}{t\Gamma(s)\Gamma(t)}\zeta(s+t+u)
\end{aligned}
$$

by Barnes' lemma (see [17, 14.52]). Hence,

$$X(s,t;u)+Y(s,t;u)=\left(\frac{1}{s}-\frac{1}{t}\right)\frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)}\zeta(s+t+u)+A(s,t;u).$$

Thus, we have

$$\begin{aligned}Z(s,t;u)&=X(s,t;u)+X(t,s;u)+Y(s,t;u)+Y(t,s;u)\\&=A(s,t;u)+A(t,s;u),\end{aligned}$$

when $\mathrm{Re}(s),\mathrm{Re}(t),\mathrm{Re}(u)>2$ and $s,t\neq 3,4,5,\ldots$. By the theory of analytic continuation, the proof of the Theorem is complete.

Remark 3.1. We have used Li's method in this section, but it is possible to prove the Theorem by Nakamura's original method. In fact, all of his argument is valid here except for the property of the Bernoulli polynomial [10, (2,7)], which can be proved by the partial fractional decomposition (2.1) in a way similar to Eisenstein's proof of addition formulas for trigonometric functions (see [16, Chapter II] or [13, §2.1]).

4. Application. In this section, we will deduce some new results from the Theorem. Each of the following propositions can be proved independently of the others. However, the next lemma seems to be useful for some applications, and so we state it first.

Lemma 4.1. *Let a be an integer and b,c be non-negative integers. Set*

$$F(s,t;c)=\sum_{k=0}^{c}\binom{c}{k}\zeta(s-k)\zeta(t-c+k) \tag{4.1}$$

for $s,t\in\mathbb{C}$. Then:

(1) *For $t,u\in\mathbb{C}$ with $t+u\neq 1-l$ $(l=0,1,2,\ldots)$,*

$$A(a,t;u)=2\sum_{k=0}^{[a/2]}\binom{t+a-2k-1}{a-2k}\zeta(2k)\zeta(t+u+a-2k), \tag{4.2}$$

where the value of any empty sum is defined to be 0.

(2) *For $s,u\in\mathbb{C}$ with $u\neq b+1-l$ $(l=0,1,2,\ldots)$,*

$$A(s,-b;u)=\sum_{k=0}^{b}\binom{b}{k}(\cos(\pi s)+(-1)^k)\zeta(s-k)\zeta(u-b+k).$$

(3) *For any $s\in\mathbb{C}$,*

$$\lim_{u\to -c}A(s,-b;u)=(\cos(\pi s)-(-1)^{b+c})F(s,-c;b)+\delta_{c0}(-1)^{b+1}\zeta(s-b),$$

where δ_{ij} denotes the Kronecker symbol.

(4) *For any* $s \in \mathbb{C}$,

$$\lim_{t \to -b} A(s,t;-c)$$
$$= (\cos(\pi s) - (-1)^{b+c})\left(F(s,-c;b) + \frac{(-1)^{c+1} b!c!}{(b+c+1)!}\zeta(s-b-c-1)\right)$$
$$+ \delta_{c0}(-1)^{b+1}\zeta(s-b).$$

Proof. (1)&(2) The formulas follow immediately from (1.4).

(3) We apply (2) to get

$$\lim_{u \to -c} A(s,-b;u) = (\cos(\pi s) - (-1)^{b+c}) \sum_{k=0}^{b-1} \binom{b}{k} \zeta(s-k)\zeta(-b-c+k)$$
$$+ (\cos(\pi s) + (-1)^b)\zeta(s-b)\zeta(-c),$$

where we have used the fact that $\zeta(-b-c+k) = 0$ if $0 \le k \le b-1$ and $k \equiv b+c \pmod 2$. Thus, by a simple calculation, we obtain the result.

(4) The result follows in a similar way to the above. ∎

We give integral representations of several zeta functions.

PROPOSITION 4.1. (1) *The Euler double zeta function* $\zeta(s,t)$ *has the following representation:*

$$(4.3)\qquad (\cos(\pi t) - \cos(\pi s))\zeta(s,t)$$
$$= A(s,t;0) + A(t,s;0) - (1+\cos(\pi s))\zeta(s)\zeta(t) + \cos(\pi s)\zeta(s+t).$$

(2) *Let* n *be a non-negative integer. Then*

$$(4.4)\qquad (1+\cos(\pi s))\zeta(s)\zeta(s+2n)$$
$$= A(s,s+2n;0) + A(s+2n,s;0) + \cos(\pi s)\zeta(2s+2n)$$

and

$$(1+\cos(\pi s))\zeta(s)\zeta(-s+2n)$$
$$= A(s,-s+2n;0) + A(-s+2n,s;0) + \cos(\pi s)\zeta(2n) + \delta_n(s),$$

where

$$\delta_n(s) = \begin{cases} -\pi s \sin(\pi s)/12 & \text{if } n = 0, \\ -\pi \sin(\pi s)/(s-1) & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$(1+\cos(\pi s))\zeta(s)^2 - \cos(\pi s)\zeta(2s) = 2A(s,s;0)$$
$$= \frac{2\sin(\pi s)}{2\pi i} \int_L \cot\left(\frac{\pi(s-\eta)}{2}\right) \frac{\Gamma(s+\eta)\Gamma(-\eta)}{\Gamma(s)} \zeta(s-\eta)\zeta(s+\eta)\, d\eta.$$

(3) *The Witten zeta function of* SU(3) *can be written as*

$$2^{-s-1}(1+2\cos(\pi s))\zeta_{\mathrm{SU}(3)}(s) = A(s,s;s) = \frac{\sin(\pi s)}{2\pi i}\int_L \cot\left(\frac{\pi(s-\eta)}{2}\right)\frac{\Gamma(s+\eta)\Gamma(-\eta)}{\Gamma(s)}\zeta(s-\eta)\zeta(2s+\eta)\,d\eta. \tag{4.5}$$

REMARK 4.1. We can regard (4.4) as a generalization of the formula

$$\zeta(2l)\zeta(2m)-\frac{1}{2}\zeta(2l+2m) = \sum_{k=0}^{\max\{l,m\}}\left\{\binom{2l+2m-2k-1}{2l-1}+\binom{2l+2m-2k-1}{2m-1}\right\} \times \zeta(2k)\zeta(2l+2m-2k)$$

for $l,m\in\mathbb{Z}_{\geq 1}$. Indeed, taking $s=2l$ ($l=1,2,\ldots$) in (4.4) and putting $m=l+n$, we obtain this from (4.2).

Proof of Proposition 4.1. (1) Substituting $u=0$ in (1.1) and using (3.4), we see

$$\begin{aligned}Z(s,t;0) &= \zeta(s)\zeta(t)+\cos(\pi t)\zeta(t,0;s)+\cos(\pi s)\zeta(0,s;t)\\ &= (\cos(\pi t)-\cos(\pi s))\zeta(t,0;s)+(1+\cos(\pi s))\zeta(s)\zeta(t)-\cos(\pi s)\zeta(s+t).\end{aligned}$$

Hence, the result follows from the Theorem.

(2) Assume that $\mathrm{Re}(s)>1$. Comparing the limits of both sides of (4.3) as $t\to\pm s+2n$, we get

$$\begin{aligned}(1+\cos(\pi s))&\zeta(s)\zeta(\pm s+2n)\\ &= A(s,\pm s+2n;0)+A(\pm s+2n,s;0)+\cos(\pi s)\zeta(s\pm s+2n)\\ &\quad -\lim_{z\to 0}\{\cos(\pi(z\pm s))-\cos(\pi s)\}\zeta(z\pm s+2n,0;s),\end{aligned}$$

where the double signs correspond. It is clear that the limit becomes 0 unless the double signs are "−" and $n=0,1$. In the remaining cases, the last term of the right side is $\delta_n(s)$ because

$$\zeta(z-s+2n,0;s)=\frac{1}{s-1}\zeta(z+2n-1)+\frac{s}{12}\zeta(z+2n+1)+O(1)$$

as $z\to 0$ (see [8, p. 425, (4.4)]).

(3) The result follows immediately from (1.1) and the Theorem. ■

We next extend the parity result [2, Theorem 2] to the whole domain of convergence.

PROPOSITION 4.2. *Let a,b,c be integers such that $a+b+c$ is odd. Assume that $a+c\geq 2$, $b+c\geq 2$ and $a+b+c\geq 3$. If $a+b\geq 2$, then*

$$2\zeta(a,b;c)=(-1)^a\{A(c,a;b)+A(a,c;b)\}+(-1)^b\{A(c,b;a)+A(b,c;a)\},$$

where every A-value is representable in the form (4.2). *If $a+b\le 1$, then*

$$2\zeta(a,b;c) = (-1)^a\{A^*(c,a;b)+A(a,c;b)\}+(-1)^b\{A^*(c,b;a)+A(b,c;a)\} + \frac{(-1)^a 2}{(1-a-b)!}\frac{d}{ds}(s+a)_{1-a-b}\Big|_{s=0}\zeta(a+b+c-1).$$

Here

$$A^*(c,a;b) = 2\sum_k \binom{a+c-2k-1}{c-2k}\zeta(2k)\zeta(a+b+c-2k),$$

where the sum is taken over all integers $k\in[0,c/2]$, $k\neq(a+b+c-1)/2$.

Proof. Taking the limits of both sides of (1.5) as $u\to c$, $t\to b$ and $s\to a$ in order, we have

$$2\zeta(a,b;c) = (-1)^a A(a,c;b)+(-1)^b A(b,c;a) + \lim_{s\to a}((-1)^a A(c,s;b)+(-1)^b A(c,b;s)).$$

It is easily seen that the limiting value equals $(-1)^a A(c,a;b)+(-1)^b A(c,b;a)$ if $a+b\ge 2$, and

$$(-1)^a A^*(c,a;b)+(-1)^b A^*(c,b;a) + \frac{(-1)^a 2}{(1-a-b)!}\frac{d}{ds}(s+a)_{1-a-b}\Big|_{s=0}\zeta(a+b+c-1)$$

if $a+b\le 1$. Thus, we obtain Proposition 4.2. ∎

The following proposition suggests that $\zeta(s,t;u)$ can be represented as a sum of products of Riemann zeta functions, if at least two of s, t and u are non-positive integers in the sense of the coordinatewise limit.

Proposition 4.3. *Let a, b and c be non-negative integers and let s, t and u be complex numbers.*

(1) *If $s,t\neq c+1-l$ $(l=0,1,2,\dots)$ and $s+t\neq c+2$, then*

$$\zeta(s,t;-c) = F(s,t;c),$$

where $F(s,t;c)$ is defined by (4.1).

(2) *If $u\neq a+1-l, b+1-l, a+b+2$ $(l=0,1,2,\dots)$, then*

$$\zeta(-a,-b;u) = (-1)^{a+1}F(u,-a;b)+(-1)^{b+1}F(u,-b;a) + \frac{a!b!}{(a+b+1)!}\zeta(u-a-b-1)-\delta_{a0}\zeta(u-b)-\delta_{b0}\zeta(u-a).$$

(3) *For $s\in\mathbb{C}$ with $s\neq c+1-l, b+c+2$ $(l=0,1,2,\dots)$,*

$$\lim_{u\to -c}\zeta(s,-b;u) = F(s,-b;c)+\frac{(-1)^{b+1}b!c!}{(b+c+1)!}\zeta(s-b-c-1).$$

Proof. (1) This is trivial.

(2) We take the limits of both sides of (1.5) as $t \to -b$ and $s \to -a$ in order. Then the result is a direct consequence of Lemma 4.1.

(3) The result can be proved in a similar way to (2), but it also follows easily from the representation [8, (5.3)] of $\zeta(s,t;u)$. ■

COROLLARY 4.1. *Let a, b and c be non-negative integers. Then*

$$(1) \qquad \lim_{(s,t)\to(-a,-b)} \zeta(s,t;-c) = F(-a,-b;c),$$

$$(2) \qquad \lim_{u\to -c} \zeta(-a,-b;u) = F(-a,-b;c) + \left(\frac{(-1)^{a+1}a!c!}{(a+c+1)!} + \frac{(-1)^{b+1}b!c!}{(b+c+1)!}\right)\zeta(-a-b-c-1),$$

$$(3) \qquad \lim_{s\to -a}\lim_{u\to -c} \zeta(s,-b;u) = F(-a,-b;c) + \frac{(-1)^{b+1}b!c!}{(b+c+1)!}\zeta(-a-b-c-1),$$

$$(4) \qquad \lim_{t\to -b}\lim_{u\to -c} \zeta(-a,t;u) = F(-a,-b;c) + \frac{(-1)^{a+1}a!c!}{(a+c+1)!}\zeta(-a-b-c-1).$$

REMARK 4.2. Komori [3] studied Tornheim's double zeta values for coordinatewise limits at non-positive integers and gave their explicit expressions in terms of generalized Bernoulli numbers. Our formulation seems to be more concrete than his.

Okamoto [11] investigated coordinatewise limits of another double zeta function:

$$\sum_{1\le m<n} \frac{1}{m^s n^t (m+n)^u},$$

at non-positive integers.

To prove (2), we have to use the following lemma and the relation

$$(-1)^{a+b+c}F(-a,-b;c) - \delta_{a0}\zeta(-b-c) - \delta_{b0}\zeta(-a-c) - \delta_{a0}\delta_{b0}\delta_{c0} = F(-a,-b;c).$$

LEMMA 4.2. *Let a, b and c be non-negative integers. Then*

$$(4.6) \qquad (-1)^{a+b}F(-a,-b;c) + (-1)^{b+c}F(-b,-c;a) + (-1)^{c+a}F(-c,-a;b) = \left(\frac{(-1)^c a!b!}{(a+b+1)!} + \frac{(-1)^a b!c!}{(b+c+1)!} + \frac{(-1)^b c!a!}{(c+a+1)!}\right)\zeta(-a-b-c-1) + \delta_{a0}\delta_{b0}\delta_{c0}.$$

This is equivalent to Theorem 2 of Chu–Wang [1]. However, their formulation is quite different from ours, and so we now prove it for the reader's convenience.

Proof. For a non-negative integer m, we set

$$\tilde{P}_m(x) = \delta_{m0} + (-1)^m \frac{m!}{x^{m+1}} + 2^{m+1} \sum_{k=0}^{\infty} (-1)^{m+k} \zeta(-m-k) \frac{(2x)^k}{k!}.$$

We calculate the value

$$R = 2^{-a-b-c-2} [x^{-1}] (\tilde{P}_a(x) - \delta_{a0})(\tilde{P}_b(x) - \delta_{b0})(\tilde{P}_c(x) - \delta_{c0})$$

in two ways, where $[x^{-1}]f(x)$ denotes the formal residue of a formal Laurent series $f(x)$. We first use the definition of $\tilde{P}_m(x)$ to obtain

$$R = (-1)^{a+b} F(-a,-b;c) + (-1)^{b+c} F(-b,-c;a) + (-1)^{c+a} F(-c,-a;b) - \left(\frac{(-1)^c a! b!}{(a+b+1)!} + \frac{(-1)^a b! c!}{(b+c+1)!} + \frac{(-1)^b c! a!}{(c+a+1)!} \right) \zeta(-a-b-c-1).$$

We next apply [12, Proposition 3.1] to get $R = \delta_{a0}\delta_{b0}\delta_{c0}$. Thus, we have (4.6). ∎

We finally describe the behavior of $\zeta_{\mathrm{SU}(3)}(s)$ at each integer.

Proposition 4.4. *Let a be a positive integer.*

(1) [2, Theorem 3]

$$\zeta_{\mathrm{SU}(3)}(a) = \frac{2^{a+2}}{1+(-1)^a 2} \sum_{k=0}^{[a/2]} \binom{2a-2k-1}{a-1} \zeta(2k)\zeta(3a-2k).$$

(2) $\zeta_{\mathrm{SU}(3)}(0) = 1/3$ *and* $\zeta'_{\mathrm{SU}(3)}(0) = \log(2^{4/3}\pi)$.

(3) *If a is odd, then $\zeta_{\mathrm{SU}(3)}(s)$ has a simple zero at $s = -a$, and*

$$\zeta'_{\mathrm{SU}(3)}(-a) = 2^{-a+2} \sum_{k=0}^{(a-1)/2} \binom{a}{2k} \zeta(-a-2k)\zeta'(-2a+2k) + \frac{2^{-a+1}(a!)^2}{(2a+1)!}\zeta'(-3a-1). \tag{4.7}$$

In particular, $\mathrm{sign}(\zeta'_{\mathrm{SU}(3)}(-a)) = (-1)^{(a-1)/2}$.

(4) *If a is even, then $\zeta_{\mathrm{SU}(3)}(s)$ has a zero of order two at $s = -a$, and*

$$\zeta''_{\mathrm{SU}(3)}(-a) = 2^{-a+2} \sum_{k=0}^{a/2} \binom{a}{2k} \zeta'(-a-2k)\zeta'(-2a+2k). \tag{4.8}$$

In particular, $\mathrm{sign}(\zeta''_{\mathrm{SU}(3)}(-a)) = (-1)^{a/2}$.

Remark 4.3. The value of Witten's zeta function $\zeta_G(s)$ of each finite group G at $s = -2$ coincides with the order of G. From this viewpoint, it is attractive to clarify the behavior of $\zeta_G(s)$ at $s = -2$ in the case of G being an infinite compact topological group. In [6], Kurokawa and Ochiai studied the values of Witten's zeta functions at negative integers, and proved that $\zeta_{\mathrm{SU}(3)}(s)$ has a zero at each negative integer. Proposition 4.4 can be regarded as a refinement of their result. Moreover, as seen below, our proof reveals that a zero of $\zeta_{\mathrm{SU}(3)}(s)$ at each negative integer comes from the gamma factors appearing on the left sides of (2.2) and (2.4).

Proof of Proposition 4.4. We use here the integral representation (4.5) of $\zeta_{\mathrm{SU}(3)}(s)$.

(1) The result is clear from (4.2).

(2) By (1.4), we see that, if K is a non-negative integer and $-K/2+1/4 < \mathrm{Re}(s) < K+1/2$, then

$$2^{-s-1}(1+2\cos(\pi s))\zeta_{\mathrm{SU}(3)}(s) = 2\sum_{k=0}^{K}\frac{(s)_k}{k!}\cos^2\left(\frac{\pi(s-k)}{2}\right)\zeta(s-k)\zeta(2s+k)+\frac{\sin(\pi s)}{\Gamma(s)}R_K(s),$$

where $R_K(s)$ is a holomorphic function. We note that every term except for the one with $k=0$ has a zero of order at least two at $s=0$. Hence, the values at $s=0$ can be immediately calculated.

(3) Set $K=2a+1$. If a is odd, then the terms with $k=0,1,\ldots,a$ satisfying $k\equiv a \pmod 2$ and the term with $k=2a+1$ have a simple zero at $s=-a$, and the others have a zero of order at least two. Hence, we can easily obtain the first part of the result. The last part follows from the functional equation of the Riemann zeta function. Indeed, we can show that the sign of each term on the right side of (4.7) coincides with $(-1)^{(a-1)/2}$.

(4) Put $K=2a+1$ again. In the same way, we see that $\zeta_{\mathrm{SU}(3)}(s)$ has a zero of order at least two at $s=-a$ if a is even. In order to determine the multiplicity of the zero, we now show (4.8). Assume that $0<\varepsilon<1/2$. Set

$$f(s,\eta)=\frac{\sin(\pi s)}{\Gamma(s)}\cot\left(\frac{\pi(s-\eta)}{2}\right)\Gamma(s+\eta)\Gamma(-\eta)\zeta(s-\eta)\zeta(2s+\eta).$$

Then, by shifting the contour, we obtain the following expression of $\zeta_{\mathrm{SU}(3)}(s)$ which is valid around $s=-a$:

$$2^{-s-1}(1+2\cos(\pi s))\zeta_{\mathrm{SU}(3)}(s)=-\sum_{k=0}^{a/2}U_k(s)+\sum_{l=0}^{a/2-1}V_l(s)+W(s)+I(s),$$

where

$$U_k(s)=\operatorname*{Res}_{\eta=k}f(s,\eta),\quad V_l(s)=\operatorname*{Res}_{\eta=-s-l}f(s,\eta),\quad W(s)=\operatorname*{Res}_{\eta=1-2s}f(s,\eta)$$

and

$$I(s)=\frac{1}{2\pi i}\int_{C_\varepsilon}f(s,\eta)\,d\eta,$$

where the contour C_ε is the union of $C_\varepsilon^{(1)}$: $a/2 - i\infty \to a/2 - i\varepsilon$, $C_\varepsilon^{(2)}$: $a/2 + \varepsilon e^{i\theta}$ ($\theta : -\pi/2 \to \pi/2$) and $C_\varepsilon^{(3)}$: $a/2 + i\varepsilon \to a/2 + i\infty$. We here remark that the poles at $\eta = k, s - 2m, -s - a/2 - m$ ($k = 0, 1, \dots, a/2$; $m = 0, 1, 2, \dots$) lie on the left of the contour C_ε and the poles at $\eta = 1 - 2s$, $-s - l, a/2 + n$ ($l = 0, 1, \dots, a/2 - 1; n = 1, 2, \dots$) lie on the right. Hence,

$$2^{a-1} 3\, \zeta''_{\mathrm{SU}(3)}(-a) = -\sum_{k=0}^{a/2} U''_k(-a) + \sum_{l=0}^{a/2-1} V''_l(-a) + W''(-a) + I''(-a).$$

By a simple calculation, we first see that, for $k = 0, 1, \dots, a/2$ and $l = 0, 1, \dots, a/2 - 1$,

$$U''_k(-a) = \begin{cases} -8\dbinom{a}{k}\zeta'(-a-k)\zeta'(-2a+k) & \text{if } k \text{ is even,} \\ \pi^2\dbinom{a}{k}\zeta(-a-k)\zeta(-2a+k) & \text{if } k \text{ is odd,} \end{cases}$$

$$V''_l(-a) = \begin{cases} 4\dbinom{a}{l}\zeta'(-a-l)\zeta'(-2a+l) & \text{if } l \text{ is even,} \\ -2\pi^2\dbinom{a}{l}\zeta(-a-l)\zeta(-2a+l) & \text{if } l \text{ is odd,} \end{cases}$$

and

$$W''(-a) = \frac{3\pi^2}{2}\frac{(a!)^2}{(2a+1)!}\zeta(-3a-1).$$

We next evaluate

$$I''(-a) = \frac{1}{2\pi i}\int_{C_\varepsilon} f''(-a,\eta)\, d\eta,$$

where f'' means $(\partial/\partial s)^2 f$. Since the integrals on $C_\varepsilon^{(1)}$ and $C_\varepsilon^{(3)}$ cancel each other, we obtain

$$\begin{aligned} I''(-a) = {} & \frac{1}{2\pi i}\int_{C_\varepsilon^{(2)}} \operatorname*{Res}_{\eta=a/2} f''(-a,\eta)\frac{d\eta}{\eta - a/2} \\ & + \frac{1}{2\pi i}\int_{C_\varepsilon^{(2)}} \left(f''(-a,\eta) - \operatorname*{Res}_{\eta=a/2} f''(-a,\eta)\cdot\frac{1}{\eta - a/2}\right) d\eta. \end{aligned}$$

We note that the integrand in the second integral is holomorphic at $\eta = a/2$, and so the integral tends to zero as $\varepsilon \to 0$. Since $I(s)$ is independent of the choice of ε, we get

$$I''(-a) = \frac{1}{2}\operatorname*{Res}_{\eta=a/2} f''(-a,\eta) = D_1(a) + D_2(a),$$

where

$$D_1(a)=\begin{cases}-\dfrac{\pi^2}{2}\dbinom{a}{a/2}\zeta(-3a/2)^2 & \text{if } a\equiv 2 \pmod 4,\\ 0 & \text{if } a\equiv 0 \pmod 4,\end{cases}$$

$$D_2(a)=\begin{cases}0 & \text{if } a\equiv 2 \pmod 4,\\ -2\dbinom{a}{a/2}\zeta'(-3a/2)^2 & \text{if } a\equiv 0 \pmod 4.\end{cases}$$

Combining the above results, we have

$$-\sum_{\substack{0\le k\le a/2\\ k\,\mathrm{odd}}} U_k''(-a)+\sum_{\substack{0\le l\le a/2-1\\ l\,\mathrm{odd}}} V_l''(-a)+W''(-a)+D_1(a)$$
$$=-\frac{3\pi^2}{2}\sum_{k=0}^{a}\binom{a}{k}\zeta(-a-k)\zeta(-2a+k)+\frac{3\pi^2}{2}\frac{(a!)^2}{(2a+1)!}\zeta(-3a-1)=0,$$

where in the last step we have used (4.6) with $a=b=c$. Moreover, we see

$$-\sum_{\substack{0\le k\le a/2\\ k\,\mathrm{even}}} U_k''(-a)+\sum_{\substack{0\le l\le a/2-1\\ l\,\mathrm{even}}} V_l''(-a)+D_2(a)$$
$$=6\sum_{k=0}^{a/2}\binom{a}{2k}\zeta'(-a-2k)\zeta'(-2a+2k).$$

Thus, we obtain (4.8). In the same way as (3), we get $\mathrm{sign}(\zeta''_{\mathrm{SU}(3)}(-a))=(-1)^{a/2}$, which completes the proof of Proposition 4.4. ∎

Appendix. A functional equation for Euler's double zeta function. The A-function (1.3) has not been found in previous papers on multiple zeta functions. However, as seen in the next proposition, $A(s,t;0)$ is related to the functional equation of $\zeta(s,t)=\zeta(t,0;s)$ which was obtained by Matsumoto [9, Theorem 1].

Proposition A.1. *Set*

$$h(s,t)=\zeta(s,t)-\frac{\Gamma(1-t)}{\Gamma(s)}\Gamma(s+t-1)\zeta(s+t-1).$$

Then

$$\text{(A.1)}\quad \frac{h(s,t)}{(2\pi)^{s+t-1}\Gamma(1-t)}=\cos\Big(\frac{\pi}{2}(s+t-1)\Big)\frac{h(1-t,1-s)}{\Gamma(s)}+\sin\Big(\frac{\pi}{2}(s+t-1)\Big)\frac{\Gamma(1-s)}{\pi}A(1-s,1-t;0).$$

In particular, the second term on the right side of (A.1) *vanishes on the hyperplane* $s+t=2k+1$ $(k\in\mathbb{Z}\setminus\{0\})$ (cf. [4, Theorem 2.2]).

Remark A.1. Firstly, the function $g(u,v)$ in Matsumoto's paper coincides with $h(v,u)$. Secondly, it may seem that the singularities of $\zeta(s,t)$ are located on the hyperplanes $s=1-l$ and $s+t=2-l$ $(l=0,1,2,\dots)$. However, the singularities on $s=-l$ and $s+t=-1-2l$ $(l=0,1,2,\dots)$ are fake, namely, the singularities of $\zeta(s,t)$ are only located on the hyperplanes $s=1$, $s+t=1$ and $s+t=2-2l$ $(l=0,1,2,\dots)$. This can be confirmed, for instance, by (1.4) and (4.3). Hence, the last part of the proposition is justified.

Proof of Proposition A.1. We first recall the usual integral representation of $\zeta(s,t)$ (cf. [8, (5.2)]):

$$\zeta(s,t)=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s+\eta)\Gamma(-\eta)}{\Gamma(s)}\zeta(t-\eta)\zeta(s+\eta)\,d\eta$$

for $s,t\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ and $\mathrm{Re}(t)>1$, where $-\mathrm{Re}(s)+1<c<0$ and (c) is the line from $c-i\infty$ to $c+i\infty$. Since the residue of the integrand at $\eta=t-1$ is

$$-\frac{\Gamma(1-t)}{\Gamma(s)}\Gamma(s+t-1)\zeta(s+t-1)$$

unless $t=1,2,3,\dots$, we shift the contour to obtain

$$h(s,t)=\frac{1}{2\pi i}\int_{C}\frac{\Gamma(s+\eta)\Gamma(-\eta)}{\Gamma(s)}\zeta(t-\eta)\zeta(s+\eta)\,d\eta$$

for $s,t\in\mathbb{C}$ with $s\neq 1-k$ and $t\neq k$ $(k=0,1,2,\dots)$, where C is a line from $-i\infty$ to $i\infty$ indented in such a manner as to separate the points $\eta=-s+1-l,t-l$ $(l=0,1,2,\dots)$ from the points $\eta=0,1,2,\dots$. By the functional equation of the Riemann zeta function, the integrand is equal to $(2\pi)^{s+t-1}\Gamma(1-t)$ times

$$\begin{aligned}\cos&\left(\frac{\pi}{2}(s+t-1)\right)\frac{\Gamma(1-t+\eta)\Gamma(-\eta)}{\Gamma(s)\Gamma(1-t)}\zeta(1-s-\eta)\zeta(1-t+\eta)\\&+\sin\left(\frac{\pi}{2}(s+t-1)\right)\frac{\Gamma(1-s)}{\pi}\sin(\pi(1-s))\\&\qquad\times\cot\left(\frac{\pi}{2}(1-s-\eta)\right)\frac{\Gamma(1-t+\eta)\Gamma(-\eta)}{\Gamma(1-t)}\zeta(1-s-\eta)\zeta(1-t+\eta).\end{aligned}$$

Thus, we obtain (A.1). ∎

We now compare our result with the result of Matsumoto to obtain a new representation of $A(s,t;0)$. For $(s,t)\in\mathbb{C}^2$ with $\mathrm{Re}(s)<0$ and $\mathrm{Re}(t)>1$, set

$$F_{\pm}(s,t)=\sum_{k=1}^{\infty}\sigma_{s+t-1}(k)\Psi(t,s+t;\pm 2\pi i k),$$

where $\sigma_\nu(k)=\sum_{d|k} d^\nu$ and $\Psi(\alpha,\gamma;z)$ is the confluent hypergeometric function of the second kind. It is known that $F_\pm(s,t)$ can be continued meromorphically to the whole space $\mathbb{C}^2$.

COROLLARY A.1. *The function $A(s,t;0)$ can be represented in terms of the $F_\pm$-functions:*

$$2\Gamma(s)A(s,t;0)=(2\pi i)^{s+t}F_+(s,t)+(-2\pi i)^{s+t}F_-(s,t).$$

Proof. Propositions 1 and 2 of [9] show that

$$\frac{h(s,t)}{(2\pi)^{s+t-1}\Gamma(1-t)}=e^{\pi i(s+t-1)/2}F_+(t,s)+e^{\pi i(1-s-t)/2}F_-(t,s)$$

and

$$F_\pm(1-t,1-s)=(\pm 2\pi i)^{s+t-1}F_\pm(s,t), \tag{A.2}$$

respectively. These suggest

$$\frac{h(1-t,1-s)}{\Gamma(s)}=F_+(t,s)+F_-(t,s)$$

and so we see that

$$\begin{aligned}\frac{h(s,t)}{(2\pi)^{s+t-1}\Gamma(1-t)}&=\cos\left(\frac{\pi}{2}(s+t-1)\right)\frac{h(1-t,1-s)}{\Gamma(s)}\\&\quad+i\sin\left(\frac{\pi}{2}(s+t-1)\right)(F_+(t,s)-F_-(t,s)).\end{aligned}$$

By comparing this with (A.1), we have

$$\Gamma(1-s)A(1-s,1-t;0)=\pi i(F_+(t,s)-F_-(t,s)).$$

Now, we use (A.2) to obtain the result. ■

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