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Exceptional sets in Waring's problem: two squares and s biquadrates

by

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1. Introduction. Waring's problem for sums of mixed powers involving one or two squares has been widely investigated. In 1987–1988, Brüdern [1, 2] considered the representation of n in the form

$$n = x_1^2 + x_2^2 + y_1^{k_1} + \cdots + y_s^{k_s},$$

with $k_1^{-1} + \cdots + k_s^{-1} > 1$. Earlier, Linnik [8] and Hooley [6] investigated sums of two squares and three cubes. In 2002, Wooley [11] investigated the exceptional set related to the asymptotic formula in Waring's problem involving one square and five cubes. Recently, Brüdern and Kawada [3] established the asymptotic formula for the number of representations of the positive number n as the sum of one square and seventeen fifth powers.

Let $R_s(n)$ denote the number of representations of the positive number n as the sum of two squares and s biquadrates. Very recently, subject to the truth of the Generalised Riemann Hypothesis and the Elliott–Halberstam Conjecture, Friedlander and Wooley [4] established that $R_3(n) > 0$ for all large n under certain congruence conditions. They also showed that if one is prepared to permit a small exceptional set of natural numbers n, then the anticipated asymptotic formula for $R_s(n)$ can be obtained.

To state their results precisely, we introduce some notations. We define

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-2-s} S_2(q,a)^2 S_4(q,a)^s e(-na/q), \tag{1.1}$$

where the Gauss sum $S_k(q,a)$ is

$$S_k(q,a) = \sum_{r=1}^{q} e(ar^k/q). \tag{1.2}$$

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As in [4], we refer to a function $\psi(t)$ as being *sedately increasing* when $\psi(t)$ is a function of a positive variable t, increasing monotonically to infinity, and satisfying the condition that when t is large, one has $\psi(t) = O(t^\delta)$ for a positive number δ sufficiently small in the ambient context. Then we introduce $E_s(X,\psi)$ to denote the number of integers n with $1 \le n \le X$ such that

$$\left| R_s(n) - c_s\Gamma\left(\frac{5}{4}\right)^4 \mathfrak{S}_s(n) n^{s/4} \right| > n^{s/4}\psi(n)^{-1}, \tag{1.3}$$

where $c_3 = \frac{2}{3}\sqrt{2}$ and $c_4 = \frac{1}{4}\pi$. Friedlander and Wooley [4] established the upper bounds

$$E_3(X,\psi) \ll X^{1/2+\varepsilon}\psi(X)^2, \tag{1.4}$$

$$E_4(X,\psi) \ll X^{1/4+\varepsilon}\psi(X)^4, \tag{1.5}$$

where $\varepsilon > 0$ is arbitrarily small.

The main purpose of this note is to prove the following result.

THEOREM 1.1. *Suppose that $\psi(t)$ is a sedately increasing function. Let $E_s(X,\psi)$ be defined as above. Then for each $\varepsilon > 0$, one has*

$$E_3(X,\psi) \ll X^{3/8+\varepsilon}\psi(X)^2, \tag{1.6}$$

$$E_4(X,\psi) \ll X^{1/8+\varepsilon}\psi(X)^2, \tag{1.7}$$

where the implicit constants may depend on ε.

We establish Theorem 1.1 by means of the Hardy–Littlewood method. In order to estimate the corresponding exceptional sets effectively, we employ the method developed by Wooley [10, 11].

As usual, we write $e(z)$ for $e^{2\pi iz}$. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Note that the "value" of ε may consequently change from statement to statement. We assume that X is a large positive number, and $\psi(t)$ is a sedately increasing function.

2. Preparations. Throughout this section, we assume that $X/2 < n \le X$. For $k \in \{2,4\}$, we define the exponential sum

$$f_k(\alpha) = \sum_{1\le x\le P_k} e(\alpha x^k),$$

where $P_k = X^{1/k}$. We take s to be either 3 or 4. By orthogonality, we have

$$R_s(n) = \int_0^1 f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)\, d\alpha. \tag{2.1}$$

When Q is a positive number, we define $\mathfrak{M}(Q)$ to be the union of the intervals

$$\mathfrak{M}_Q(q,a) = \{\alpha : |q\alpha - a| \le QX^{-1}\},$$

with $1 \le a \le q \le Q$ and $(a,q) = 1$. Whenever $Q \le X^{1/2}/2$, the intervals $\mathfrak{M}_Q(q,a)$ are pairwise disjoint for $1 \le a \le q \le Q$ and $(a,q) = 1$. Let ν be a sufficiently small positive number, and let $R = P_4^\nu$. We take $\mathfrak{M} = \mathfrak{M}(R)$ and $\mathfrak{m} = (R/N, 1 + R/N] \setminus \mathfrak{M}$.

Write

$$v_k(\beta) = \int_0^{P_k} e(\gamma^k \beta)\, d\gamma.$$

One has the estimate

$$v_k(\beta) \ll P_k(1 + X|\beta|)^{-1/k}.$$

For $\alpha \in \mathfrak{M}_{X^{1/2}/2}(q,a) \subseteq \mathfrak{M}(X^{1/2}/2)$, we define

$$f_k^*(\alpha) = q^{-1} S_k(q,a) v_k(\alpha - a/q). \tag{2.2}$$

It follows from [9, Theorem 4.1] that whenever $\alpha \in \mathfrak{M}_{X^{1/2}/2}(q,a)$, one has

$$f_k(\alpha) - f_k^*(\alpha) \ll q^{1/2}(1 + X|\alpha - a/q|)^{1/2} X^{\varepsilon}. \tag{2.3}$$

We define the multiplicative function $w_k(q)$ by

$$w_k(p^{uk+v}) = \begin{cases} kp^{-u-1/2} & \text{when } u \ge 0 \text{ and } v = 1, \\ p^{-u-1} & \text{when } u \ge 0 \text{ and } 2 \le v \le k. \end{cases}$$

Note that $q^{-1/2} \le w_k(q) \ll q^{-1/k}$. Whenever $(a,q) = 1$, we have

$$q^{-1} S_k(q,a) \ll w_k(q).$$

Therefore for $\alpha = a/q + \beta \in \mathfrak{M}_{X^{1/2}/2}(q,a) \subseteq \mathfrak{M}(X^{1/2}/2)$, one has

$$f_k^*(\alpha) \ll w_k(q) P_k (1 + X|\beta|)^{-1/k} \ll P_k q^{-1/k} (1 + X|\beta|)^{-1/k}. \tag{2.4}$$

The following conclusion is (4.1) in [4].

Lemma 2.1. *One has*

$$\int_{\mathfrak{M}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)\, d\alpha = c_s \Gamma(5/4)^4 \mathfrak{S}_s(n) n^{s/4} + O(n^{s/4-\kappa+\varepsilon})$$

for a suitably small positive number κ.

The next result provides the value of the Gauss sum $S_2(q,a)$.

Lemma 2.2. *The Gauss sum $S_2(q,a)$ has the following properties:*

(i) *If $(2a,q) = 1$, then*

$$S_2(q,a) = \left(\frac{a}{q}\right) S_2(q,1).$$

Here $\left(\frac{a}{q}\right)$ denotes the Jacobi symbol.

(ii) *If q is odd, then*

$$S_2(q,1)=\begin{cases} q^{1/2} & \text{if } q\equiv 1 \pmod 4,\\ iq^{1/2} & \text{if } q\equiv 3 \pmod 4.\end{cases}$$

(iii) *If $(2,a)=1$, then*

$$S_2(2^m,a)=\begin{cases} 0 & \text{if } m=1,\\ 2^{m/2}(1+i^a) & \text{if } m \text{ is even},\\ 2^{(m+1)/2}e(a/8) & \text{if } m>1 \text{ and } m \text{ is odd}.\end{cases}$$

(iv) *If $(q_1,q_2)=1$, then*

$$S_2(q_1q_2,a_1q_2+a_2q_1)=S_2(q_1,a_1)S_2(q_2,a_2).$$

Proof. These properties can be found in [5, Lemma 2]. ∎

3. The proof of Theorem 1.1. Let τ be a fixed sufficiently small positive number. Set $Y=P_4^{3/2+\tau}\psi(X)^2$. We define $\mathfrak{m}_1=\mathfrak{m}\setminus\mathfrak{M}(X^{1/2}/2)$, $\mathfrak{m}_2=\mathfrak{M}(X^{1/2}/2)\setminus\mathfrak{M}(Y)$, $\mathfrak{m}_3=\mathfrak{M}(Y)\setminus\mathfrak{M}(P_4)$ and $\mathfrak{m}_4=\mathfrak{M}(P_4)\setminus\mathfrak{M}$. Let $\eta(n)$ be sequence of complex numbers satisfying $|\eta(n)|=1$. Let $\mathcal{Z}$ be a subset of $\{n\in\mathbb{N}: X/2<n\le X\}$. We abbreviate $\mathrm{card}(\mathcal{Z})$ to Z. We introduce the exponential sum $\mathcal{E}(\alpha)$ by

$$\mathcal{E}(\alpha)=\sum_{n\in\mathcal{Z}}\eta(n)e(-n\alpha).$$

For $1\le j\le 4$, we define

$$\mathcal{I}_j=\int_{\mathfrak{m}_j}|f_2(\alpha)^2f_4(\alpha)^s\mathcal{E}(\alpha)|\,d\alpha. \tag{3.1}$$

Lemma 3.1. *Let $\mathcal{I}_1$ be defined in* (3.1). *Then*

$$\mathcal{I}_1\ll P_4^{4-1/4+s-3/2+\varepsilon}Z^{1/2}+P_4^{s-1/4+\varepsilon}Z. \tag{3.2}$$

Proof. For any $\alpha\in\mathfrak{m}_1$, there exist a and q with $1\le a\le q\le 2X^{1/2}$ and $(a,q)=1$ such that $|q\alpha-a|\le X^{-1/2}/2$. Since $\alpha\in\mathfrak{m}_1$, we conclude that $q>X^{1/2}/2$. It follows from Weyl's inequality [9, Lemma 2.4] that

$$f_2(\alpha)\ll P_2^{1/2+\varepsilon}\quad \text{for } \alpha\in\mathfrak{m}_1.$$

Thus we have

$$\begin{aligned}\mathcal{I}_1&\ll P_2^{1+\varepsilon}\int_{\mathfrak{m}_1}|f_4(\alpha)^s\mathcal{E}(\alpha)|\,d\alpha\\ &\ll P_2^{1+\varepsilon}\Big(\int_0^1|f_4(\alpha)^6|\,d\alpha\Big)^{1/2}\Big(\int_0^1|f_4(\alpha)^{2(s-3)}\mathcal{E}(\alpha)^2|\,d\alpha\Big)^{1/2}.\end{aligned}$$

By Hua's inequality [9, Lemma 2.5] and Schwarz's inequality,

$$\int_0^1 |f_4(\alpha)^6|\,d\alpha \ll \Big(\int_0^1 |f_4(\alpha)^4|\,d\alpha\Big)^{1/2}\Big(\int_0^1 |f_4(\alpha)^8|\,d\alpha\Big)^{1/2} \ll P_4^{7/2+\varepsilon}.$$

When $s = 4$, one has the bound $\int_0^1 |f_4(\alpha)^{2(s-3)}\mathcal{E}(\alpha)^2|\,d\alpha \ll P_4Z + P_4^{\varepsilon}Z^2$. Hence we get (3.2).

Indeed when $s = 3$, the estimate (3.2) holds with $P_4^{s-1/4+\varepsilon}Z$ omitted. ■

Lemma 3.2. *Let $\mathcal{I}_2$ be defined in* (3.1). *Then*

$$\mathcal{I}_2 \ll P_4^{4-1/4+(s-3)/2+\varepsilon}Z^{1/2} + P_4^{s-\tau/2+\varepsilon}\psi(X)^{-1}Z. \tag{3.3}$$

Proof. We introduce

$$\mathcal{J}_1 = \int_{\mathfrak{m}_2} |(f_2(\alpha) - f_2^*(\alpha))^2 f_4(\alpha)^s\mathcal{E}(\alpha)|\,d\alpha,$$

$$\mathcal{J}_2 = \int_{\mathfrak{m}_2} |f_2^*(\alpha)^2 f_4(\alpha)^s\mathcal{E}(\alpha)|\,d\alpha.$$

Note that $|f_2(\alpha)|^2 \ll |f_2(\alpha) - f_2^*(\alpha)|^2 + |f_2^*(\alpha)|^2$, where $f_2^*(\alpha)$ is defined in (2.2). Then

$$\mathcal{I}_2 \ll \mathcal{J}_1 + \mathcal{J}_2. \tag{3.4}$$

In view of (2.3), we know $f_2(\alpha) - f_2^*(\alpha) \ll P_2^{1/2+\varepsilon}$ for $\alpha \in \mathfrak{m}_2$. The argument leading to (3.2) also implies

$$\mathcal{J}_1 \ll P_4^{4-1/4+(s-3)/2+\varepsilon}Z^{1/2} + P_4^{s-1/4+\varepsilon}Z. \tag{3.5}$$

One has, by Schwarz's inequality,

$$\mathcal{J}_2 \le \Big(\int_{\mathfrak{m}_2} |f_4(\alpha)^6|\,d\alpha\Big)^{1/2}\mathcal{J}^{1/2} \ll P_4^{7/4+\varepsilon}\mathcal{J}^{1/2},$$

where $\mathcal{J}$ is defined as

$$\mathcal{J} = \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4 f_4(\alpha)^{2(s-3)}\mathcal{E}(\alpha)^2|\,d\alpha.$$

In order to handle $\mathcal{J}$, we need the estimate

$$\int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha)\,d\alpha = \begin{cases} O(P_4^{4+\varepsilon}Y^{-1}) & \text{when } 0 < |h| \le 2X, \\ O(P_4^{4+\varepsilon}) & \text{when } h = 0. \end{cases} \tag{3.6}$$

Recalling the definition of $f_2^*(\alpha)$, we conclude that

$$\int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha)\, d\alpha$$
$$= \sideset{}{^*}\sum_{q\le X^{1/2}/2} \int^*_{|\beta|\le 1/(2qX^{1/2})} q^{-4}\Big(\sum_{\substack{a=1\\(a,q)=1}}^{q} |S_2(q,a)|^4 e(-ha/q)\Big)|v_2(\beta)|^4 e(-h\beta)\, d\beta,$$

where the notations $\sum^*$ and $\int^*$ mean either $q > Y$ or $Xq|\beta| > Y$. Whenever $(a,q) = 1$, one finds by Lemma 2.2 that

$$|S_2(q,a)| = |S_2(q,1)| \le (2q)^{1/2}.$$

We obtain

$$\Big|\sum_{\substack{a=1\\(a,q)=1}}^{q} |S_2(q,a)|^4 e(-ha/q)\Big| = |S_2(q,1)|^4 \Big|\sum_{\substack{a=1\\(a,q)=1}}^{q} e(-ha/q)\Big|$$
$$\le 4q^2 \Big|\sum_{\substack{a=1\\(a,q)=1}}^{q} e(-ha/q)\Big| \le 4q^2(q,h),$$

whence

$$\int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha)\, d\alpha \ll P_2^4 \sideset{}{^*}\sum_{q\le X^{1/2}/2} \int^*_{|\beta|\le 1/(2qX^{1/2})} \frac{q^{-2}(q,h)}{(1+X|\beta|)^2}\, d\beta.$$

When $h = 0$, we have

$$\int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha)\, d\alpha \ll P_2^4 \sum_{q\le X^{1/2}/2} \int_{|\beta|\le 1/(2qX^{1/2})} q^{-1}(1+X|\beta|)^{-2}\, d\beta$$
$$\ll P_2^4 X^{-1} \log X.$$

When $h \ne 0$, we get

$$\int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-h\alpha)\, d\alpha \ll P_2^4 Y^{-1} \sum_{q\le X^{1/2}/2} \int_{|\beta|\le 1/(2qX^{1/2})} \frac{q^{-1}(q,h)}{1+X|\beta|}\, d\beta$$
$$\ll P_2^4 Y^{-1} X^{-1} (\log X) \sum_{q\le X^{1/2}/2} q^{-1}(q,h)$$
$$\ll P_2^4 Y^{-1} X^{-1+\varepsilon}.$$

The conclusion (3.6) is established.

Now we are able to estimate $\mathcal{J}$. When $s = 4$,

$$\mathcal{J} = \sum_{\substack{1\le x_1,x_2\le P_4\\ n_1,n_2\in\mathcal{Z}}} \eta(n_1)\overline{\eta(n_2)} \int_{\mathfrak{m}_2} |f_2^*(\alpha)^4| e(-(x_1^4 - x_2^4 + n_1 - n_2)\alpha)\, d\alpha.$$

On applying (3.6), we can deduce that

$$\mathcal{J} \ll \sum_{\substack{1\le x_1,x_2\le P_4,\, n_1,n_2\in\mathcal{Z}\\ x_1^4-x_2^4+n_1-n_2\ne 0}} P_4^{4+\varepsilon}Y^{-1} + \sum_{\substack{1\le x_1,x_2\le P_4,\, n_1,n_2\in\mathcal{Z}\\ x_1^4-x_2^4+n_1-n_2=0}} P_4^{4+\varepsilon}$$
$$\ll P_4^{6+\varepsilon}Z^2Y^{-1} + P_4^{4+\varepsilon}Z^2 + P_4^{5+\varepsilon}Z.$$

Substituting $Y = P_4^{3/2+\tau}\psi(X)^2$, we finally obtain

$$\mathcal{J} \ll P_4^{4+1/2-\tau+\varepsilon}\psi(X)^{-2}Z^2 + P_4^{5+\varepsilon}Z,$$

whence

$$\mathcal{J}_2 \ll P_4^{4-\tau/2+\varepsilon}\psi(X)^{-1}Z + P_4^{4+1/4+\varepsilon}Z^{1/2}.$$

Similarly, when $s = 3$, one has

$$\mathcal{J} \ll P_4^{5/2-\tau+\varepsilon}\psi(X)^{-2}Z^2 + P_4^{4+\varepsilon}Z$$

whence

$$\mathcal{J}_2 \ll P_4^{3-\tau/2+\varepsilon}\psi(X)^{-1}Z + P_4^{4-1/4+\varepsilon}Z^{1/2}.$$

Therefore,

$$\mathcal{J}_2 \ll P_4^{4-1/4+(s-3)/2+\varepsilon}Z^{1/2} + P_4^{s-\tau/2+\varepsilon}\psi(X)^{-1}Z. \tag{3.7}$$

Combining (3.4), (3.5) and (3.7) leads to (3.3). ∎

Lemma 3.3. *Let $\mathcal{I}_3$ be defined in* (3.1). *Then*

$$\mathcal{I}_3 \ll P_4^{4-1/4+(s-3)/2+\varepsilon}Z^{1/2} + P_4^{s-\tau+\varepsilon}\psi(X)^{-1}Z. \tag{3.8}$$

Proof. Similarly to (3.4) and (3.5), we can derive that

$$\mathcal{I}_3 \ll P_4^{4-1/4+(s-3)/2+\varepsilon}Z^{1/2} + P_4^{s-1/4+\varepsilon}Z + \mathcal{K}, \tag{3.9}$$

where

$$\mathcal{K} = \int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^s \mathcal{E}(\alpha)|\, d\alpha.$$

One has

$$\mathcal{K} \le \sup_{\alpha\in\mathfrak{m}_3} |f_4(\alpha)| \Big(\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^4|\, d\alpha\Big)^{1/2}$$
$$\times \Big(\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2|\, d\alpha\Big)^{1/2}.$$

In view of (2.3) and (2.4), for $\alpha \in \mathfrak{m}_3$ we have

$$f_4(\alpha) \ll P_4 q^{-1/4}(1 + X|\alpha - a/q|)^{-1/4} + Y^{1/2}X^{\varepsilon} \ll P_4^{3/4+\tau/2+\varepsilon}\psi(X).$$

Since $f_2^*(\alpha) - f_2(\alpha) \ll P_2^{1/2}$ for $\alpha \in \mathfrak{m}_3$, we easily deduce that

$$\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^4|\,d\alpha \ll P_2^{1/2}\int_0^1 |f_2(\alpha)f_4(\alpha)^4|\,d\alpha + \int_0^1 |f_2(\alpha)^2 f_4(\alpha)^4|\,d\alpha \ll P_4^{4+\varepsilon}.$$

Therefore we arrive at

$$\mathcal{K} \ll P_4^{11/4+\tau/2+\varepsilon}\psi(X)\Big(\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^{2(s-3)}\mathcal{E}(\alpha)^2|\,d\alpha\Big)^{1/2}.$$

Similarly to (3.6), we have

$$\int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2| e(-h\alpha)\,d\alpha = \begin{cases} O(P_4^{\varepsilon}) & \text{when } 0 < |h| \le 2X, \\ O(P_4^{\varepsilon}Y) & \text{when } h = 0. \end{cases} \tag{3.10}$$

Note that

$$\begin{aligned}
&\int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2| e(-h\alpha)\,d\alpha \\
&= \sum_{q\le Y}\int_{|\beta|\le Y/(qX)} q^{-2}\Big(\sum_{\substack{a=1\\(a,q)=1}}^{q} |S_2(q,a)|^2 e(-ha/q)\Big)|v_2(\beta)|^2 e(-h\beta)\,d\beta \\
&\ll P_2^2 \sum_{q\le Y}\int_{|\beta|\le Y/(qX)} q^{-1}(q,h)(1+X|\beta|)^{-1}\,d\beta \\
&\ll (\log X)\sum_{q\le Y} q^{-1}(q,h).
\end{aligned}$$

The desired estimate (3.10) follows easily from the above.

For $s=4$, we derive that

$$\begin{aligned}
&\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 f_4(\alpha)^2 \mathcal{E}(\alpha)^2|\,d\alpha \le \int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2 f_4(\alpha)^2 \mathcal{E}(\alpha)^2|\,d\alpha \\
&= \sum_{\substack{n_1,n_2\in\mathcal{Z}\\ 1\le x_1,x_2\le P_4}} \eta(n_1)\overline{\eta(n_2)}\int_{\mathfrak{M}(Y)} |f_2^*(\alpha)^2| e(-(n_1-n_2+x_1^4-x_2^4)\alpha)\,d\alpha \\
&\ll P_4^{2+\varepsilon}Z^2 + P_4^{\varepsilon}Y(P_4^{\varepsilon}Z^2 + P_4 Z) \\
&\ll (P_4^{2+\varepsilon} + P_4^{3/2+\tau+\varepsilon}\psi(X)^2)Z^2 + P_4^{5/2+\tau+\varepsilon}\psi(X)^2 Z,
\end{aligned}$$

whence

$$\mathcal{K} \ll (P_4^{15/4+\tau/2+\varepsilon}\psi(X) + P_4^{7/2+\tau+\varepsilon}\psi(X)^2)Z + P_4^{4+\tau+\varepsilon}\psi(X)^2 Z^{1/2}.$$

In particular,

$$\mathcal{K} \ll P_4^{4+1/4+\varepsilon} Z^{1/2} + P_4^{4-\tau+\varepsilon}\psi(X)^{-1} Z$$

provided that $\psi(X) \ll X^{1/64-\tau}$. For $s = 3$, by (3.10) we have

$$\int_{\mathfrak{m}_3} |f_2^*(\alpha)^2 \mathcal{E}(\alpha)^2|\, d\alpha \ll P_4^{\varepsilon} Z^2 + P_4^{3/2+\tau+\varepsilon}\psi(X)^2 Z,$$

whence

$$\mathcal{K} \ll P_4^{11/4+\tau/2+\varepsilon}\psi(X) Z + P_4^{7/2+\tau+\varepsilon}\psi(X)^2 Z^{1/2}.$$

When $\psi(X) \ll X^{1/64-\tau}$, one has

$$\mathcal{K} \ll P_4^{4-1/4+\varepsilon} Z^{1/2} + P_4^{3-\tau+\varepsilon}\psi(X)^{-1} Z.$$

We conclude from the above that

$$\mathcal{K} \ll P_4^{4-1/4+(s-3)/2+\varepsilon} Z^{1/2} + P_4^{s-\tau+\varepsilon}\psi(X)^{-1} Z. \tag{3.11}$$

By (3.9) and (3.11), we obtain (3.8). ∎

Lemma 3.4. *Let $\mathcal{I}_4$ be defined in* (3.1). *Then*

$$\mathcal{I}_4 \ll Z P_4^{s-(s-2)\nu/4+\varepsilon}. \tag{3.12}$$

Proof. In view of (2.3) and (2.4), for $\alpha \in \mathfrak{M}_{P_4}(q,a)$, one has

$$\begin{aligned} f_4(\alpha) &\ll P_4 w_4(q)(1 + X|\alpha - a/q|)^{-1/4} + P_4^{1/2+\varepsilon} \\ &\ll P_4^{1+\varepsilon} w_4(q)(1 + X|\alpha - a/q|)^{-1/4}, \\ f_2(\alpha) &\ll P_2 q^{-1/2}(1 + X|\alpha - a/q|)^{-1/2}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \mathcal{I}_4 &\ll Z \sup_{\alpha\in\mathfrak{m}_4} |f_4(\alpha)|^{s-2} \int_{\mathfrak{M}(P_4)} |f_4(\alpha) f_2(\alpha)|^2\, d\alpha \\ &\ll Z P_4^{(s-2)(1-\nu/4)+\varepsilon} P_4^2 P_2^2 \sum_{q\le P_4} w_4(q)^2 \int_{|\beta|\le P_4/(qX)} (1 + X|\beta|)^{-3/2}\, d\beta \\ &\ll Z P_4^{2+(s-2)(1-\nu/4)+\varepsilon} \sum_{q\le P_4} w_4(q)^2. \end{aligned}$$

In light of Lemma 2.4 of Kawada and Wooley [7], one can conclude that

$$\mathcal{I}_4 \ll Z P_4^{2+(s-2)(1-\nu/4)+\varepsilon} \ll Z P_4^{s-(s-2)\nu/4+\varepsilon}. \ ∎$$

Proof of Theorem 1.1. We denote by $Z_s(X)$ the set of integers n with $X/2 < n \le X$ for which the lower bound

$$\left| R_s(n) - c_s \Gamma\left(\frac{5}{4}\right)^4 \mathfrak{S}_s(n) n^{s/4} \right| > n^{s/4}\psi(n)^{-1}$$

holds, and we abbreviate card$(Z_s(X))$ to Z_s. It follows from (2.1) and Lemma 2.1 that, for $n \in Z_s(X)$,

$$\Big|\int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)\,d\alpha\Big| \gg X^{s/4}\psi(X)^{-1},$$

whence

$$\sum_{n\in Z_s(X)} \Big|\int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)\,d\alpha\Big| \gg Z_s X^{s/4}\psi(X)^{-1}.$$

We choose complex numbers $\eta(n)$, with $|\eta(n)| = 1$, satisfying

$$\Big|\int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)\,d\alpha\Big| = \eta(n)\int_{\mathfrak{m}} f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)\,d\alpha.$$

Then we define the exponential sum $\mathcal{E}_s(\alpha)$ by

$$\mathcal{E}_s(\alpha) = \sum_{n\in Z_s(X)} \eta(n)e(-n\alpha).$$

One finds that

$$Z_s X^{s/4}\psi(X)^{-1} \ll \int_{\mathfrak{m}} |f_2(\alpha)^2 f_4(\alpha)^s \mathcal{E}_s(\alpha)|\,d\alpha. \tag{3.13}$$

Note that $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3 \cup \mathfrak{m}_4$. Now we conclude from Lemmata 3.1–3.4 and (3.13) that

$$Z_s X^{s/4}\psi(X)^{-1} \ll P_4^{4-1/4+(s-3)/2+\varepsilon} Z_s^{1/2} + P_4^{s-\delta}\psi(X)^{-1} Z_s$$

for some sufficiently small positive number δ. Therefore

$$Z_s X^{s/4}\psi(X)^{-1} \ll X^{1-1/16+(s-3)/8+\varepsilon} Z_s^{1/2}.$$

This estimate implies $Z_3 \ll X^{3/8+\varepsilon}\psi(X)^2$ and $Z_4 \ll X^{1/8+\varepsilon}\psi(X)^2$. The proof of Theorem 1.1 is completed by summing over dyadic intervals. ∎

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